of mathematical ideas that took place between China and the other cultural centres (p. 212). Yes, we should distinguish among claims, beliefs, and historical facts.

Sometimes Joseph does not notice that he disproves his own argumentation. He complains about Eurocentrism because it cannot bring itself to face the idea of independent developments in early Indian mathematics, even as a remote possibility. But he does not concede this possibility to the Greeks with regard to the earlier cultures of the Near East. By all means, it is a too condescending attitude to concede only "that the Greek approach to mathematics produced some [!] remarkable results" (p. 346).

Thus the reader is left with mixed feelings. While Joseph rightly rejects the hegemony of a Western version of mathematics, he is inclined to replace it by another one, although he explicitly states that "since the first edition we are no closer to gathering further definitive evidence of transmission of mathematical knowledge to Europe" (p. 354).

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# Mathematics and Music. A Diderot Mathematical Forum

*edited by G. Assayag, H.-G. Feichtinger, and J. F. Rodrigues*

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**REVIEWED BY SERGE PERRINE**

The book under review brings together sixteen contributions to the Diderot Mathematical Forum held under the auspices of the European Mathematical Society, simultaneously in Lisbon, Paris, and Vienna, with teleconference exchanges, on 3 and 4 December 1999. The conference in Lisbon covered "Historical aspects," the topic in Vienna was "Mathematical methods and calculation in music," while in Paris the Forum dealt with "Mathematical logic and musical logic in the twentieth century."

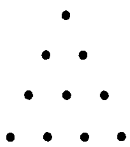

Fig. 1

These three topics are covered in a fairly balanced way in this book, five articles dealing with the first topic, seven with the second, and four with the last. All these articles are of significant interest, whether from a historical or theoretical point of view. Bringing them together in the same publication sheds magnificent light on the dialogue and mutual enrichment that Mathematics and Music have developed over the centuries [1] [2] [3] [4].

The first article, by Manuel Pedro Ferreira, deals with the musical theory constructed by Pythagoras. Two sounds from the same taut string are said to be consonant when they are pleasing to listen to simultaneously. In the Greek cultural arena of that period such sounds are produced by lengths of string that are inversely proportional to the numbers 1, 2, 3, and 4. These compose the famous *Tetraktys* ($1 + 2 + 3 + 4 = 10$), a diagram of figured numbers symbolising pure harmony, the "vertical hierarchy of relation between Unity and emerging multiplicity," also the source of the Music of Spheres that Pythagoreans referred to when required to swear (Fig. 1).

It was perhaps because he was impressed by the mathematical consistency of consonance that Pythagoras devised the idea that Number is the substance of the Universe.

Be that as it may, on an instrument consisting of a single taut string vibrating on a sounding board and fitted with keys that make it possible to select suitable lengths of the string being vibrated, one obtains with the *Tetraktys* the intervals known as octaves, fifths, and fourths. Figure 2 represents such a single-stringed instrument (e.g., the Vosges spinet, still used today by certain folk groups in Eastern France) with the corresponding modern names of the notes.

Musical instruments such as the tetrachord lyre may also be built with four strings having these same lengths (L, L/2, L/3, L/4) that produce simultaneous sounds. The respective tensions are adjusted so that the sound produced by each string is that of the string having the same length on the monochord.

The article describes the improvements brought to this theory by Philolaus and others. The chief result of that period was obtained by Archytas, who demonstrated the need for unequal divisions in order to obtain all the consonants comprised in an octave. He recognised the importance of arithmetic, geometric, and harmonic means. This

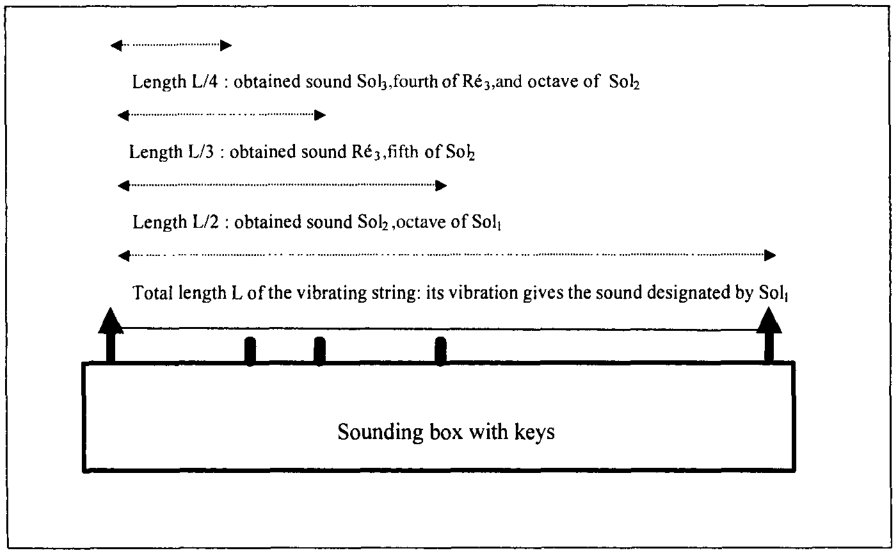


Fig. 2

| $G_1$ L | | $G_2$ L/2 | | $D_3$ L/3 | | $G_3$ L/4 | | $B_3$ L/5 | | $D_4$ L/6 |
|---|---|---|---|---|---|---|---|---|---|---|
| | Octave 1/2 | | Perfect fifth 2/3 | | Perfect fourth 3/4 | | Major third 4/5 | | Minor third 5/6 | |

Fig. 3

allowed him to enrich the range of sounds used and their associated intervals (Fig. 3).

The reference made to Aristoxenus—a pupil of Aristotle who totally rejected pythagorean harmony in favour of a musical theory based on the continuous sounds perceived by the ear, as well as on the tensions of the strings and their relaxation time—shows the rich diversity of musical thinking in Ancient Greece.

However, the most interesting aspect of this article, whose numerous references provide ample scope for digging deeper, is certainly the description of the rich musical evolution flowing from the Greek roots into the Latin world and right up to the fourteenth century of our era. In St. Augustine's *De Musica*, written at the end of the fourth century, rhythms are also classified according to their proportions (the proportional notation used today came much later). Then in the ninth century, Carolingian policy in educational and ecclesiastical matters defined new practices. It encouraged the use of neumes that indicate the inflexions of the voice, but not the pitch of the sounds. The names Do, Re, Mi, Fa, Sol, etc., appeared with Guido d'Arezzo in the eleventh century, deriving from the syllables at the beginning of the stanzas (voces) of a hymn addressed to St John the Baptist, written around 770 A.D. The notes (claves) are also designated by letters, a practice that is still in use today in English-speaking countries (La = A, Ti = B, Do = C, . . . ) and in Germany (with some specificities). Finally, polyphony created new needs for harmonic mastery, the response coming from Philippe de Vitry in the fourteenth century with his *Ars Nova*: in this work he defined new musical notations as well as new ways of combining rhythms. However, this culmination of the pythagorean musical base that had developed over many centuries eventually degenerated in the following century because it proved to be inadequate for responding to the new aesthetic trends that were appearing as well as the practical needs of musicians. Those who were concerned with the tuning of their keyboard instrument were led to consider the problem of temperament [5].

Having had one's mind brilliantly stimulated by such an article, one is led to wonder about the Byzantine evolution, geographically so close to the Greek source; regrettably, however, this aspect is not touched upon in the article. One wonders too what was the contribution of the ancient manuscripts passed on by the "sons of the Greeks," as the Arabs of the time called themselves. Fortunately the article refers to the influence of Arabian and Persian music in the *Cantigas* of the Iberian Peninsula, and the contribution of the reading of the ancients, thanks to the translation in the twelfth century of the musical treatise by Al Farabi.

Eberhard Knobloch's article provides a novel answer by introducing the concepts of Athanasius Kircher, who in 1650 wrote *Musurgia Universalis*. Kircher quotes Hermes Trismegistos, the mystical author who was so loved by the Medicis and Pico della Mirandola: "Music is nothing else than to know the order of all things." This very pythagorean concept postulates that Music is a part of Mathematics (and consequently a science). For Kircher, this is a relevant concept when seeking to help someone having virtually no knowledge of the mastery of sounds to acquire an in-depth knowledge of musical composition. Pythagoras doubtless would have disowned such a project. Yet, once rid of its absurd objective, this statement aptly sums up the concept of music prevailing in Renaissance and Baroque times, founded on number and its symbolism, a source of beauty and harmony. It actually sets it in an oriental tradition considerably older than the Greeks, that considered number as the handiwork of God who ordered all things in measure, number, and weight (Wisdom of Solomon 11.17) and on which all the work of Man rests. Kronecker's well-known phrase "God created number, all the rest is the work of Man," draws its inspiration from the same source.

In fact, Kircher's book develops the musical ideas of the minim monk Marinus Mersennus (Marin Mersenne), in particular the combinatory approach contained in his *Harmonia Universalis*, written in 1636. The article unfortunately does not speak of Mersenne's activity as the science correspondent of the whole of Europe, nor of his creation in 1635 of the Academia Parisiensis, the ancestor of the future Académie des Sciences; nor does it mention the measurement of the speed of sound that he obtained in 1636, nor his discovery of the higher harmonics of a string. No mention is made either of his systematic use of the notion of frequency, introduced at the time by Galileo Galilei. Mersenne was a student of the latter's work and was familiar with the law that gives the frequency of the fundamental vibration $f$ of a vibrating string having a length L with a linear mass $\rho$ and with tension F:

$$f = \frac{1}{2L}\sqrt{\frac{F}{\rho}}.$$

This formula is merely mentioned in the *Discorsi*, written in 1638 by Galileo, the son of the musician Vincenzo Galilei, and it was written in this modern form only in 1715, by Brook Taylor. The limited part of Mersenne's work mentioned in the book is nonetheless of major interest and sets the record straight regarding a number of misconceptions as to the history of science at that time.

In his *Harmonia Universalis*, Mersenne sets out the table of all the values of the number of permutations with $n$ elements up to $n = 64$. He discusses non-repetitive arrangements $P(n, p) = n(n-1) \ldots (n-p+1)$ and combinations $C(n, p) = P(n, p) = P(n, p)/P(p, p)$. He solves the problem of calculating the number of combinations presented by a given type of repetitions. This he does thirty years before Leibniz succeeds in obtaining, with a few errors, the same results in his "schoolboy's essay" *De Arte Combinatoria*, and well before the combinatorial work of Fermat and Pascal. If $n$ is the maximum number possible of notes for a song composed with $p$ different notes, of which $r_1$ distinct notes appear once, $r_2$ distinct notes appear twice, . . . , and using in fact $r = r_1 + r_2 + \ldots + r_m$ distinct notes in all, Mersenne gives the total number of possibilities for the corresponding songs:

$$\frac{n!}{r_1! r_2! \ldots r_m! (n-r)!}.$$

For 22 possible notes, of which 7 distinct ones are repeated according to the type 2, 2, 1, 1, 1, 1, 1, Mersenne shows that there are 3,581,424 possible songs. Is it therefore not understandable that it was the analogy with the combinatorics derived from gaming that led Mozart to devise a musical game allowing the players to produce waltzes by throwing dice [6]? This then poses the question of the link between musical creativity and chance. One may indeed wonder if certain of Haydn's compositions were not inspired by similar methods. This point is not mentioned, even though his 41st piano sonata is quoted in the article by Wilfrid Hodges and Robin J. Wilson dedicated to musical forms. Speaking of combinatorics raises the possibility of using the group of permutations of objects that one arranges and combines. . . . From there to seeing Galois's theory in the practice of sixteenth century bell-ringers and the rules laid down by Fabian Stedman [7] is but a short step, but one which none of the articles in this book dares to take. On the other hand, the approach taken does shed light on musical analysis, as may be seen in the article by Laurent Fichet, and makes it possible to extend one's horizon, as in Marc Chemillier's article dedicated to ethnomusicology.

The formula mentioned earlier relating to the fundamental frequency of a string, in turn allows a better understanding of the problem of temperament. It consists of seeking to divide an octave into twelve equal intervals, and therefore to identify rational numbers that simultaneously come as close as possible to the irrational real numbers $2^{(1/12)}, 2^{(2/12)}, 2^{(3/12)}, \ldots, 2^{(11/12)}$, being aware that a trained ear will perceive any deviation that is too significant. This leaves plenty of margin for numerous systems, and the remarkable article by Benedetto Scimeni presents the choice proposed by Gioseffo Zarlino in his work *Le Istitutioni harmonicae* [8], published in 1558:

10/9, 9/8, 6/5, 5/4, 4/3,
3/2, 8/5, 5/3, 16/9, 9/5.

Galileo's father quarrelled with Zarlino because he preferred 18/17 to 10/9. But of course, whatever choice one makes, the practical issue is the tuning of instruments, in particular harpsichords with several octaves and the largest possible number of tones. The article referred to here mentions the remarkable work undertaken on these questions by Giuseppe Tartini, Daniel Strähle, and Christoph Gottlieb Schröter. One of the most fascinating aspects is the connection with the solution to Pell/Fermat's equation in Tartini's *Trattatto di Musica*:

$$x^2 - 2y^2 = 1.$$

In fact, this becomes obvious when one realises that the above also leads to coming as close as possible to the irrational $2^{(6/12)} = \sqrt{2}$ with a rational number, a classic problem of diophantine analysis, which is much simpler than the previous problem of simultaneous approximation of the twelfth roots of 2.

The book is incomplete if one considers it from the point of view of the history of acoustics, a word invented by Joseph Sauveur, who professed mathematics at the Collège de France from 1686. Dumb until the age of seven and deaf for the whole of his life, it is he who looked more closely at the observation made by Mersenne that there exist higher harmonics: a string may vibrate in several parts around nodes that remain fixed. The book makes scant mention of the work of Bernoulli or Euler. It remains almost completely silent concerning the discovery in 1747 by d'Alembert of the partial differential equation of vibrating strings:

$$\frac{\partial^2 y}{\partial t^2} = \nu^2 \frac{\partial^2 y}{\partial x^2}.$$

It is in fact the solution of this equation that makes Sauveur's discoveries understandable. However, research activity on sound was so extensive at the time that to describe it would be an almost impossible task. We would need to mention Wallis, Newton, La Hire, not forgetting Bach, Rousseau, and so many others; one would necessarily have to be selective. The selection made in the book is particularly relevant, but makes one want a new Forum, to take the question deeper by referring to the activities of other authors who have been left out.

The article by Jean Dhombres explores another major historical milestone, by referring to the interest shown by Lagrange around 1760 in musical texts and the theory of instruments. In his *Recherches sur la nature et la propagation du son* he gives a definition of the integral of a function as a limit. No more number theory and geometry. He shows that the same differential equation appears in the vibrations of strings and those of air. He discovers the orthogonal relationship of sine and cosine. Yet Lagrange cannot be considered to be the inventor either of series or of the Fourier analysis. It was indeed Fourier who recognised the universality of the calculus discovered by Lagrange in his study of musical sounds. From a mathematical point of view, the next stages in this millennial adventure, which are not covered in this book, are the march towards distributions [10] and the deeper understanding of spectral analysis [11] and group representations [12].

Today the Music of corpuscles and solitons is taking the place of the Music of spheres and mermaids. Considerations of the multiple infinitely small (chaos?) are replacing those on the single infinitely great (the cosmos?). The bifurcation took place at the end of the eighteenth century, at the very moment when musicians were being pushed into the category of artists, whose role was to provide pleasure for the present, and mathematicians into the category of scientists, building the society of the future.

The remainder of the book presents four articles by Giovanni De Poli and Davide Rocchesso, by Erich Neuwirth, by Xavier Serra, and by Jean-Claude Risset on the application of modern digital sound technology. There are new acoustic domains being explored, such as the impact of non-linearity, the hearer's perception, the use of computerised toolboxes to produce sounds, texture compositions, and acoustic illusions. The proliferation is huge and shows how that mathematical machine par excellence—the computer—is invading music. Far from slackening, the interaction between the two fields is continuing to develop and is as strong as ever. The major change seems to have been that mathematics now has its instruments—computers—whereas classical musical instruments are left standing. Experimental practice seems to have provisionally changed sides, but the process of mutual enrichment is continuing [13].

It is consequently natural to ask about the logic and meaning behind this evolution of the two fields [14]. Logic has always been essential to mathematics, but in the recent period it would appear to be less natural in music. Of course, one may consider the computerised machine learning music, as do Shlomo Dubnov and Gérard Assayag. But does this have anything to do with logic? Another article by Marie-José Durand Richard, which retraces the history of logic, shows that the issue isn't clear. It refers us to the article by François Nicolas dealing with just that question: What is the logic in music? The answer given by Nicolas is as anti-pythagorean as could be, because it results in the impossibility of defining this concept today, and hence leads to falling back on the study of the practices involved in musical production, free from mathematical, acoustic (physical), and psycho-physiological tutelage. Such a loss of meaning is in total contradiction of the tradition of a relationship between music and logic, as illustrated in the double organisation of ancient knowledge of liberal arts in the form of trivium (grammar, rhetoric, logic) and mathematical quadrivium (arithmetic or the number in itself, geometry or the number in space, music or the number in time, astronomy or the number in space and time). Yet it is thoroughly contemporary. It also sets itself completely apart from the theories that Marin Mersenne proclaimed in his *Traité de l'harmonie universelle* published in October 1627 under the pseudonym François de Sermes [15]. Theorem 1: Music is a part of mathematics and consequently a science, capable of showing the causes, effects, and properties of sounds, songs, concerts, and anything related thereto. Theorem 4: Music is both a speculative and practical science, and an art, and consequently is a virtue of understanding, which it leads to the knowledge of the truth. In this understanding, which one may consider to be outdated (wrongly, for the joint evolution of the two fields is continuing, as the present book shows), the logic of music finds profound meaning, which the author of the article submits to the meditation of its readers.

To this end, the book contains one last article that I have not yet mentioned. It is by Guerino Mazzola and is titled *The Topos Geometry of Musical Logic*. (See the review by Shlomo Dubrov, this issue.) On the mathematical side, he relies on the theory of categories and Grothendieck constructions; on the musical side, on Riemann's harmony (not G. F. Bernhard but K. W. J. Hugo, i.e., not the mathematician but the author of *Mathematische Logik* published in 1873!). He develops a Galois theory of musical concepts which locks Beauty and Truth into the same kingdom. So might there after all still be some pythagoreans in our day and age, lost among our contemporaries, Guerino being one of them? At any rate, his article is fascinating from an intellectual point of view. He confirms that the new alliance between music and mathematics announced by Pierre Lamothe in 2000 on his Web site is forging ahead, though using paths other than those he had envisaged [16].

This new alliance between pleasure and science cannot but enrich both parties. It might even constitute a remedy for the desertion from mathematical studies observed in our times, when knowledge and work are parcelled out piecemeal. The pleasure derived from reading this remarkable work is very great. The reviewer is convinced that other *Mathematics and Music* initiatives need to be taken, and that there is no lack of topics to be covered.

**Acknowledgement**

I am indebted to the non-mathematician Andrew Wiles for the English translation.

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# The Topos of Music: Geometric Logic of Concepts, Theory, and Performance

*by Guerino Mazzola, with Stefan Göller and Stefan Müller*

BIRKHAUSER VERLAG, BASEL, BOSTON, BERLIN, 2002
1368 PP. Hardcover, incl. CD-ROM. €128
ISBN 3-7643-5731-2

**REVIEWED BY SHLOMO DUBNOV**

In the context of classical Greek philosophy, a *topos* (literally "a place") referred to a method of constructing and presenting an argument, being part of *rhetoric*, the art of oration or persuasion. Plato greatly opposed rhetoric, claiming that it values style or manner of persuasion over the discussion of substance. Then came Aristotle's "reconciliation" of rhetoric and dialectic saying that while dialectical methods are necessary to find truth, rhetorical methods are required to construct an argument in order to communicate it. The concept of topos was extended later to literature by a German scholar, Curtius, as a study of ways to compile knowledge by selecting and indexing important phrases, lines, and/or passages from texts and writing them down into notebooks known as "commonplace books." These notebooks were commonly indexed and arranged for easier reference, and maybe it is not surprising that the book by Mazzola opens by placing music in a new "encyclospace," a space where human knowledge production is assumed to be coupled to navigation in a topologically arranged concept space.

In mathematics, category theory is known as a study of abstract mathematical structures and relationships. Groups are often used to describe symmetries of objects, and they were used by music theoreticians for describing musical properties such as scales, pitch classes, or rhythms. Every element of the group creates a correspondence to some other set of objects, and Cayley's theorem states that every group $G$ is isomorphic to the group of its symmetric operations. The Yoneda lemma in category theory is a generalization of Cayley's theorem that allows the embedding of any category into a category of mappings (called functors) defined on that category. Using denotators to describe musical objects, categories of musical compositions are defined as elementary objects of music. Then, describing the Yoneda Perspective, Mazzola claims that in relation to the arts, "understanding painting and music is a synthesis of perspective variations." Therefore, by considering functors as the representations, one is led to defining art and music as a set of operations (symmetries) that leave the object invariant. Music composition becomes the "invariant" or the "essence" of a set of performances, an idea related also to Adorno's esthetic principle in music. The emphasis is on the rhetoric function "as a means to express understanding, and in this respect performance is not only a perspective of action but instantiation of understanding, of interpretation given structures."

Mazzola further assumes that mathematical study in the context of art will lead to objects which are "meant to describe beauty and truth." This brings topos theory to being a way for combining logic and geometry. In topos theory one replaces the set by a category, function by a morphism, and truth-values by a "subobject classifer," which is something more general than the Boolean algebra of True and False. This, together with the work of Grothendieck on algebraic geometry, allows Mazzola to define complex musical structures of Global Music compositions, with earlier categories being embedded as "patchworks" of local objects in a global theory, leading to categorization of music as constructions on geometric manifolds.

The book opens with a very general, philosophical-historical motivation, which seems vague or somewhat ambiguous, to provide an intuitive basis for dealing with the forthcoming formalisms. In Part II the author goes from concrete examples of note representations to very abstract concepts of forms and denotators, assuming prior knowledge of advanced concepts of categories, topoi, and logic. Asking the reader to "recall" these concepts from appendix G would probably require also "recalling" earlier concepts from appendices C–F on set theory, rings, algebras, and algebraic geometry, and so on.

Part III of the book deals with the next level of describing musical constructs, such as scales and chords, terming them "local compositions." This brings up a discussion of musical symmetries in the local composition objects, such as Messiaen modi and serial techniques. But there appears to be a deeper aspect of local composition related to the use of functors and their concatenations, needed in preparation for Part IV. This aspect (culminating in the Yoneda Perspective) employs the fact that in the denotator representation one has the mathematical structure of a topos, which offers properties such as unions, products, or limits (somewhat as in set theory), and allows for enumeration or classification. Musical or visual examples could help in clarifying these developments, but the author offers rather general claims about the utility of the mathematical methods to analysis of an Escher drawing or appreciation of the fractal Julia set shape, without much detail. American Music Set Theory is called "thoroughly out of date from the point of view of 20th-century mathematical conceptualization."