Rudimentary recursion, provident sets and forcing.

A. R. D. Mathias

Many important set-theoretical functions, such as rank or transitive closure, are defined by a recursion on the epsilon relation of the form

$$F(x) = G(F \restriction x)$$

where G is a rudimentary function, and $F \restriction x$ denotes the restriction of F to (the members of) x.

Others become similarly definable when we permit parameters. If p is a set, call F *p-rud-rec* if there is a rud function G such that for all x,

$$F(x) = G(p, F \restriction x),$$

and call a set A ***provident*** if it is transitive, non-empty, closed under unordered pairing, and closed under all p-rud-rec functions with p a member of A.

If ζ is the least ordinal not in a provident set A, then ζ is infinite and indecomposable, that is, that the sum of two ordinals less than ζ is less than ζ. Conversely, if $\zeta \geqslant 1$ is indecomposable, $\eta \geqslant \zeta$ and $p \in J_\zeta$, then the Jensen set J_η is is closed under all p-rud rec functions; in particular J_ζ is provident.

Let c be a transitive set. A modification of the usual hierarchy defining the constructible closure $L(c)$ of c proves desirable: define, by a simultaneous rudimentary recursion on ordinals, sets c_ν, P^c_ν thus:

$$\begin{array}{lllllllll} c_0 & = & \varnothing & c_{\nu+1} & = & c \cap \{x \mid x \subseteq c_\nu\} & c_\lambda & = & \bigcup_{\nu<\lambda} c_\nu \\ P^c_0 & = & \varnothing & P^c_{\nu+1} & = & \{c_\nu\} \cup c_{\nu+1} \cup \mathbb{T}(P^c_\nu) & P^c_\lambda & = & \bigcup_{\nu<\lambda} P^c_\nu \end{array}$$

$\mathbb{T}$ being the rudimentary function introduced in [M] such that for each transitive set u, $u \subseteq \mathbb{T}(u) \subseteq \mathcal{P}(u)$, $u \in \mathbb{T}(u)$, and the rudimentary closure of $u \cup \{u\}$ equals $\bigcup_{n\in\omega} \mathbb{T}^n(u)$. Then $L(c) = \bigcup_{\nu \in ON} P^c_\nu$.

Proposition: *Let c be transitive and θ indecomposable and infinite. Then P^c_θ is provident.*

Let θ be indecomposable, and C a collection of transitive sets, each of rank strictly less than θ, with the property that any two elements of C are members of a third. Then $\bigcup_{c\in C} P^c_\theta$ is provident, and every provident set is of this form.

Another characterization is that a provident set is a transitive set A closed under the binary functions unordered pairing and ordinal addition and under the unary functions rank, transitive closure and $\mathbb{T}$, and with the property that whenever c is a transitive member of A and ν is an ordinal in A, the set P^c_ν is in A. This remark gives a finitely axiomatisable set theory, which we call MSF, for "minimal for set forcing", of which the transitive models are precisely the provident sets containing ω.

The following theorem suggests that the class of rudimentarily recursive functions might be expected to provide a fine analysis of forcing much as the class of rudimentary functions has provided a fine analysis of constructibility.

Theorem: *Let A be provident and $\mathbb{P}$ a separative poset which is a member of A. Then forcing over A with $\mathbb{P}$ can be defined in A, and the forcing relation for Δ_0 formulae is in a precise sense close to $\mathbb{P}$-rud-rec; if A is provident and if X is $(A, \mathbb{P})$ generic, then the generic extension $A[X]$ is provident.*

Versions of the above for certain J-like fragments were known to Hauser [H] and to Steel, cf. [S]; our more general result shows that as each of the computations required for establishing properties of forcing and building generic extensions can be done in a local J-like fragment, no global J-like structure is needed.

Finally, Gandy in [G] mentions an unsatisfactory attempt to study fragments of the infinitary language $L_{\omega_1\omega}$ using the class of rudimentary functions; it appears that the fragments $A \cap L_{\omega_1\omega}$, where A is provident, will have the properties he was seeking.

[G] R. O. Gandy, Set-theoretic functions for elementary syntax, in *Proceedings of Symposia in Pure Mathematics*, **13**, Part II, ed. T.Jech, American Mathematical Society, 1974, 103–126.

[H] K. Hauser, Generic Relativizations of Fine Structure, *Archive for Mathematical Logic* **39** (2000) 227–251.

[M] A. R. D. Mathias, Weak systems of Gandy, Jensen and Devlin, in *Set Theory: Centre de Recerca Matemàtica, Barcelona 2003-4* , edited by Joan Bagaria and Stevo Todorcevic, Trends in Mathematics, Birkhäuser Verlag, Basel, 2006, 149–224.

[S] J. Steel, Scales in $K(\mathbb{R})$, preprint dated September 11 2003.