Optimal Regularity for the Obstacle Problem

John Andersson (KTH, Stockholm, Sweden)

In this talk we will consider the regularity for the obstacle problem without sign restriction. That is, we assume that $u \in W^{2,2}(B_1(0))$ and that $\Delta u(x) = f(x)\chi_{\{u\neq 0\}}$ where χ_A is the indicator function for the set A and $f(x)$ is a given, say Lipschitz, function.

Since Δu is bounded it follows, from classical theory, that $u \in C^{1,\alpha}$ for every $\alpha < 1$. But for the obstacle problem it has been known since the 60ies that the solution is actually in $C^{1,1}$ if f is smooth and $u \geq 0$. In 2000 L.A. Caffarelli, L. Karp and H. Shahgholian (Ann. Math. 2000) showed that $u \in C^{1,1}$ even when u changes sign. This is an important condition needed to investigate the regularity of the set $\{n \neq 0\}$. Their proof was easily extendable to f being Lipschitz.

This is an important result but it is unpleasing in certain ways that will be explained during the seminar. We will also sketch an elementary proof for a stronger result: that $u \in C^{1,1}$ even if f is (just slightly better than) continuous as well as explain why this result is optimal. In describing the proof we will to touch upon some issues in singular integral operators at a basic level.