A survey of logarithmic D-modules

Luis Narváez-Macarro[1]

Departamento de Álgebra, Universidad de Sevilla

Iberian Meeting on Algebraic Analysis and Geometry
28th to 30th September, 2011

[1] Partially supported by MTM2010–19298 and FEDER.

1. A topological motivation
2. Free divisors and logarithmic D-modules
3. Explicit computations.

Local topology of irreducible plane curves

Local topology of irreducible plane curves

- Let $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible germ of holomorphic function.

Local topology of irreducible plane curves

- Let $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible germ of holomorphic function.
- Let $h : X \to \Delta$ be a "good" representative of h_0 giving the Milnor fibration $h : U = X - D \to \Delta^*$, $D = f^{-1}(0)$. The "Milnor fiber" is $F = f^{-1}(t_0)$, $t_0 \in \Delta^*$.

Local topology of irreducible plane curves

- Let $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible germ of holomorphic function.
- Let $h : X \to \Delta$ be a "good" representative of h_0 giving the Milnor fibration $h : U = X - D \to \Delta^*$, $D = f^{-1}(0)$. The "Milnor fiber" is $F = f^{-1}(t_0)$, $t_0 \in \Delta^*$.
- $1 \to L = \pi_1(F, x_0) \to G = \pi_1(U, x_0) \to \pi_1(\Delta^*, t_0) \to 1.$

Local topology of irreducible plane curves

- Let $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible germ of holomorphic function.
- Let $h : X \to \Delta$ be a "good" representative of h_0 giving the Milnor fibration $h : U = X - D \to \Delta^*$, $D = f^{-1}(0)$. The "Milnor fiber" is $F = f^{-1}(t_0)$, $t_0 \in \Delta^*$.
- $1 \to L = \pi_1(F, x_0) \to G = \pi_1(U, x_0) \to \pi_1(\Delta^*, t_0) \to 1$.
- L is a free group of rank $\mu = \mu(h)$, $\pi_1(\Delta^*, t_0)$ is infinite cyclic with "positive" generator δ.

Local topology of irreducible plane curves

- Let $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible germ of holomorphic function.
- Let $h : X \to \Delta$ be a "good" representative of h_0 giving the Milnor fibration $h : U = X - D \to \Delta^*$, $D = f^{-1}(0)$. The "Milnor fiber" is $F = f^{-1}(t_0)$, $t_0 \in \Delta^*$.
- $1 \to L = \pi_1(F, x_0) \to G = \pi_1(U, x_0) \to \pi_1(\Delta^*, t_0) \to 1$.
- L is a free group of rank $\mu = \mu(h)$, $\pi_1(\Delta^*, t_0)$ is infinite cyclic with "positive" generator δ.
- The bord F^∂ of F is a circle and $F^\partial = \pi_1(F^\partial, x_0)$ is infinite cyclic with "positive" generator γ.

Local topology of irreducible plane curves

- Let $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible germ of holomorphic function.
- Let $h : X \to \Delta$ be a "good" representative of h_0 giving the Milnor fibration $h : U = X - D \to \Delta^*$, $D = f^{-1}(0)$. The "Milnor fiber" is $F = f^{-1}(t_0)$, $t_0 \in \Delta^*$.
- $1 \to L = \pi_1(F, x_0) \to G = \pi_1(U, x_0) \to \pi_1(\Delta^*, t_0) \to 1$.
- L is a free group of rank $\mu = \mu(h)$, $\pi_1(\Delta^*, t_0)$ is infinite cyclic with "positive" generator δ.
- The bord F^∂ of F is a circle and $F^\partial = \pi_1(F^\partial, x_0)$ is infinite cyclic with "positive" generator γ.
- $\exists\ \pi_1(\Delta^*, t_0) \hookrightarrow G$ and $G = L \ltimes \pi_1(\mathbb{C}^*, 1)$.

Local topology of irreducible plane curves

- Let $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible germ of holomorphic function.
- Let $h : X \to \Delta$ be a "good" representative of h_0 giving the Milnor fibration $h : U = X - D \to \Delta^*$, $D = f^{-1}(0)$. The "Milnor fiber" is $F = f^{-1}(t_0)$, $t_0 \in \Delta^*$.
- $1 \to L = \pi_1(F, x_0) \to G = \pi_1(U, x_0) \to \pi_1(\Delta^*, t_0) \to 1.$
- L is a free group of rank $\mu = \mu(h)$, $\pi_1(\Delta^*, t_0)$ is infinite cyclic with "positive" generator δ.
- The bord F^∂ of F is a circle and $F^\partial = \pi_1(F^\partial, x_0)$ is infinite cyclic with "positive" generator γ.
- $\exists \ \pi_1(\Delta^*, t_0) \hookrightarrow G$ and $G = L \ltimes \pi_1(\mathbb{C}^*, 1)$.
- $\mathrm{int}_\delta : \alpha \in L \mapsto \delta\alpha\delta^{-1} \in L$ corresponds to the geometric monodromy of the Milnor fibration ($\delta\gamma\delta^{-1} = \gamma$).

An example

An example

- $h_0 = x^2 - y^3 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $\mu = 2$.

An example

- $h_0 = x^2 - y^3 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $\mu = 2$.
- $h = x^2 - y^3 : X = \mathbb{C}^2 \to \mathbb{C}$ is good, $D = \{h = 0\}$, $U = \mathbb{C}^2 - U$ and $F = h^{-1}(1)$.

An example

- $h_0 = x^2 - y^3 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $\mu = 2$.
- $h = x^2 - y^3 : X = \mathbb{C}^2 \to \mathbb{C}$ is good, $D = \{h = 0\}$, $U = \mathbb{C}^2 - U$ and $F = h^{-1}(1)$.
- $L = \pi_1(F, x_0) = \langle a, b \rangle$, $\gamma = aba^{-1}b^{-1}$, $\delta a \delta^{-1} = b^{-1}, \delta b \delta^{-1} = ba$.

An example

- $h_0 = x^2 - y^3 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $\mu = 2$.
- $h = x^2 - y^3 : X = \mathbb{C}^2 \to \mathbb{C}$ is good, $D = \{h = 0\}$, $U = \mathbb{C}^2 - U$ and $F = h^{-1}(1)$.
- $L = \pi_1(F, x_0) = \langle a, b \rangle$, $\gamma = aba^{-1}b^{-1}$, $\delta a \delta^{-1} = b^{-1}, \delta b \delta^{-1} = ba$.
- $G = \pi_1(U, x_0) = \langle a, b, \delta; \delta a \delta^{-1} = b^{-1}, \delta b \delta^{-1} = ba \rangle$.

Perverse sheaves with respect to plane curves

Perverse sheaves with respect to plane curves

- $h : X \to \Delta$ a good representative of $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $D = h^{-1}(0)$, $j : U = X - D \hookrightarrow X$.

Perverse sheaves with respect to plane curves

- $h : X \to \Delta$ a good representative of $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $D = h^{-1}(0)$, $j : U = X - D \hookrightarrow X$.
- (MacPherson-Vilonen, Deligne, Verdier; 1982-83) A perverse sheaf $\mathbf{K}$ on X stratified by $\{0\}, D - \{0\}, U$ is determined by

$$(\mathcal{L}, \mathbf{F}, u : R\psi_h\mathcal{L} \to \mathbf{F}, v : \mathbf{F} \to R\psi_h\mathcal{L})$$

  with $\mathcal{L} = j^*\mathbf{K}$ a local system on U, $\mathbf{F} = \phi_h\mathbf{K}$ a perverse sheaf on D (stratified w.r.t. $\{0\}, D - \{0\}$) and u, v maps of perverse sheaves such that $\mathrm{Id} + v \circ u = T_{\mathcal{L}}$.

Perverse sheaves with respect to plane curves

- $h : X \to \Delta$ a good representative of $h_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $D = h^{-1}(0)$, $j : U = X - D \hookrightarrow X$.
- (MacPherson-Vilonen, Deligne, Verdier; 1982-83) A perverse sheaf $\mathbf{K}$ on X stratified by $\{0\}, D - \{0\}, U$ is determined by

$$(\mathcal{L}, \mathbf{F}, u : R\psi_h\mathcal{L} \to \mathbf{F}, v : \mathbf{F} \to R\psi_h\mathcal{L})$$

  with $\mathcal{L} = j^*\mathbf{K}$ a local system on U, $\mathbf{F} = \phi_h\mathbf{K}$ a perverse sheaf on D (stratified w.r.t. $\{0\}, D - \{0\}$) and u, v maps of perverse sheaves such that $\mathrm{Id} + v \circ u = T_{\mathcal{L}}$.
- But $(D, 0) \simeq (\mathbb{C}, 0)$ and perverse sheaves are well known...

Explicit description (N-M, 1984)

Explicit description (N-M, 1984)

- Each of the data $(\mathcal{L}, \mathbf{F}, u : R\psi_h\mathcal{L} \to \mathbf{F}, v : \mathbf{F} \to R\psi_h\mathcal{L})$ can be explicitly described.

Explicit description (N-M, 1984)

- Each of the data $(\mathcal{L}, \mathbf{F}, u : R\psi_h\mathcal{L} \to \mathbf{F}, v : \mathbf{F} \to R\psi_h\mathcal{L})$ can be explicitly described.
- $\mathcal{L}$ is given by a complex representation $G = \pi_1(U) \to \mathrm{GL}(E)$.

Explicit description (N-M, 1984)

- Each of the data $(\mathcal{L}, \mathbf{F}, u : R\psi_h\mathcal{L} \to \mathbf{F}, v : \mathbf{F} \to R\psi_h\mathcal{L})$ can be explicitly described.
- $\mathcal{L}$ is given by a complex representation $G = \pi_1(U) \to \mathrm{GL}(E)$.
- $\mathbf{F}$ is given by a diagram of vector spaces $(C_1, C_2; p : C_1 \to C_2, q : C_2 \to C_1)$ with $\mathrm{Id} + q \circ p$ is $\simeq$.

Explicit description (N-M, 1984)

- Each of the data $(\mathcal{L}, \mathbf{F}, u : R\psi_h\mathcal{L} \to \mathbf{F}, v : \mathbf{F} \to R\psi_h\mathcal{L})$ can be explicitly described.
- $\mathcal{L}$ is given by a complex representation $G = \pi_1(U) \to \mathrm{GL}(E)$.
- $\mathbf{F}$ is given by a diagram of vector spaces $(C_1, C_2; p : C_1 \to C_2, q : C_2 \to C_1)$ with $\mathrm{Id} + q \circ p$ is $\simeq$.
- (N-M) $(R\psi_h\mathcal{L}, T_{\mathcal{L}})$ is given by:

$$(E, \mathrm{Hom}_{\mathbb{C}[L]}(I(L), E), U, V) \curvearrowright (t_1, t_2) \text{ with}$$

$$U : E \to \mathrm{Hom}_{\mathbb{C}[L]}(I(L), E), \quad U(e)(g) = ge,$$

$$V : \mathrm{Hom}_{\mathbb{C}[L]}(I(L), E) \to E), \quad V(\varphi) = \varphi(\gamma - 1),$$

and $t_1(e) = \delta^{-1}e$, $t_2(\varphi)(g) = \delta^{-1}\varphi(\delta g \delta^{-1})$.

A formula for the characteristic cycle

A formula for the characteristic cycle

If our perverse sheaf **K** on X is given by $G \to \mathrm{GL}(E)$, $(C_1, C_2; p : C_1 \to C_2, q : C_2 \to C_1)$ and $u_1 : E \to C_1, v_1 : C_1 \to E$, $u_2 : \mathrm{Hom}_{\mathbb{C}[L]}(I(L), E) \to C_2, v_2 : C_2 \to \mathrm{Hom}_{\mathbb{C}[L]}(I(L), E)$ with the corresponding commutativiy conditions and $\mathrm{Id} + v_1 \circ u_1 = t_1$, $\mathrm{Id} + v_2 \circ u_2 = t_2$, then:

A formula for the characteristic cycle

If our perverse sheaf **K** on X is given by $G \to \mathrm{GL}(E)$, $(C_1, C_2; p : C_1 \to C_2, q : C_2 \to C_1)$ and $u_1 : E \to C_1, v_1 : C_1 \to E$, $u_2 : \mathrm{Hom}_{\mathbb{C}[L]}(I(L), E) \to C_2, v_2 : C_2 \to \mathrm{Hom}_{\mathbb{C}[L]}(I(L), E)$ with the corresponding commutativiy conditions and $\mathrm{Id} + v_1 \circ u_1 = t_1$, $\mathrm{Id} + v_2 \circ u_2 = t_2$, then:

$$\mathrm{CC}(\mathbf{K}) = m_2 T^*_X(X) + m_1 \overline{T^*_{D^{reg}}(X)} + m_0 T^*_0(X),$$

$m_2 = \dim E$, $m_1 = \dim C_1$,
$m_0 = \mu \dim E + (1 - e) \dim C_1 + \dim C_2$.

The case of a cusp

The case of a cusp

- $\mathcal{L} \leftrightarrow \rho : G = \langle a, b, \delta; \delta a \delta^{-1} = b^{-1}, \delta b \delta^{-1} = ba \rangle \to \mathrm{GL}(E)$:

$$\rho(a) = A, \quad \rho(b) = B, \quad \rho(\delta) = \Delta.$$

The case of a cusp

- $\mathcal{L} \leftrightarrow \rho : G = \langle a, b, \delta; \delta a \delta^{-1} = b^{-1}, \delta b \delta^{-1} = ba \rangle \to \mathrm{GL}(E)$:

$$\rho(a) = A, \quad \rho(b) = B, \quad \rho(\delta) = \Delta.$$

- $R\psi_h \mathcal{L}$ is the perverse sheaf on D given by:

$$\left(E, E^2, \begin{pmatrix} A - I \\ B - I \end{pmatrix}, (B - BAB^{-1}A^{-1}, I - BAB^{-1})\right)$$

The case of a cusp

- $\mathcal{L} \leftrightarrow \rho : G = \langle a, b, \delta ; \delta a \delta^{-1} = b^{-1}, \delta b \delta^{-1} = ba \rangle \to \mathrm{GL}(E)$:

$$\rho(a) = A, \quad \rho(b) = B, \quad \rho(\delta) = \Delta.$$

- $R\psi_h \mathcal{L}$ is the perverse sheaf on D given by:

$$\left(E, E^2, \begin{pmatrix} A - I \\ B - I \end{pmatrix}, (B - BAB^{-1}A^{-1}, I - BAB^{-1})\right)$$

and $T_{\mathcal{L}}$ is given by:

$$t_1 = \Delta^{-1}, \quad t_2 = \begin{pmatrix} 0 & -\Delta^{-1}B^{-1} \\ \Delta^{-1}B^{-1} & \Delta^{-1} \end{pmatrix}.$$

Description of intersection complexes

Description of intersection complexes

- Let $\mathcal{L}$ be a local system on $U = X - D$.

Description of intersection complexes

- Let $\mathcal{L}$ be a local system on $U = X - D$.
- The *intersection complex* $j_{!*}\mathcal{L}$ is the perverse sheaf given by

$$(\mathcal{L}, \mathrm{Im}(T_{\mathcal{L}} - 1), T_{\mathcal{L}} - 1, \text{inclusion}).$$

Description of intersection complexes

- Let $\mathcal{L}$ be a local system on $U = X - D$.
- The *intersection complex* $j_{!*}\mathcal{L}$ is the perverse sheaf given by

$$(\mathcal{L}, \mathrm{Im}(T_{\mathcal{L}} - 1), T_{\mathcal{L}} - 1, \text{inclusion}).$$

- If $\mathcal{L}$ is associated with the representation $G \to \mathrm{GL}(E)$, then

$$\mathrm{CC}(j_{!*}\mathcal{L}) = m_2 T^*_X(X) + m_1 \overline{T^*_{D^{reg}}(X)} + m_0 T^*_0(X),$$

$m_2 = \dim E,\ m_1 = \mathrm{rank}(t_1 - 1),$
$m_0 = \mathrm{rank}(t_2 - 1) - \mu m_2 + (\mathrm{mult}_0(D) - 1)m_1.$

Local systems of rank 1 on the complement of the cusp

Local systems of rank 1 on the complement of the cusp

- For $z \in \mathbb{C}^*$, let $\mathcal{L}_z$ be the local system on U of rank 1 given by $\varrho_z : G \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$, $\varrho_z(\delta) = z$, $\varrho_z(a) = \varrho_z(b) = 1$.

Local systems of rank 1 on the complement of the cusp

- For $z \in \mathbb{C}^*$, let $\mathcal{L}_z$ be the local system on U of rank 1 given by $\varrho_z : G \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$, $\varrho_z(\delta) = z, \varrho_z(a) = \varrho_z(b) = 1$.
- $t_1 = z^{-1}, \quad t_2 = \begin{pmatrix} 0 & -z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix}$

Local systems of rank 1 on the complement of the cusp

- For $z \in \mathbb{C}^*$, let $\mathcal{L}_z$ be the local system on U of rank 1 given by $\varrho_z : G \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$, $\varrho_z(\delta) = z$, $\varrho_z(a) = \varrho_z(b) = 1$.
- $t_1 = z^{-1}, \quad t_2 = \begin{pmatrix} 0 & -z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix}$
- $\mathrm{CC}(j_{!*}\mathcal{L}_z) = T^*_X(X) + m_1 \overline{T^*_{D^{reg}}(X)} + m_0 T^*_0(X)$.

Local systems of rank 1 on the complement of the cusp

- For $z \in \mathbb{C}^*$, let $\mathcal{L}_z$ be the local system on U of rank 1 given by $\varrho_z : G \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$, $\varrho_z(\delta) = z, \varrho_z(a) = \varrho_z(b) = 1$.
- $t_1 = z^{-1}, \quad t_2 = \begin{pmatrix} 0 & -z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix}$
- $\mathrm{CC}(j_{!*}\mathcal{L}_z) = T^*_X(X) + m_1 \overline{T^*_{D^{reg}}(X)} + m_0 T^*_0(X)$.
- If $z = 1$ then $\mathcal{L}_z = \mathbb{C}_U$ and $j_{!*}\mathcal{L}_z = \mathbb{C}_X$, $m_1 = m_0 = 0$.

Local systems of rank 1 on the complement of the cusp

- For $z \in \mathbb{C}^*$, let $\mathcal{L}_z$ be the local system on U of rank 1 given by $\varrho_z : G \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$, $\varrho_z(\delta) = z$, $\varrho_z(a) = \varrho_z(b) = 1$.
- $t_1 = z^{-1}, \quad t_2 = \begin{pmatrix} 0 & -z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix}$
- $\mathrm{CC}(j_{!*}\mathcal{L}_z) = T^*_X(X) + m_1 \overline{T^*_{D^{reg}}(X)} + m_0 T^*_0(X)$.
- If $z = 1$ then $\mathcal{L}_z = \mathbb{C}_U$ and $j_{!*}\mathcal{L}_z = \mathbb{C}_X$, $m_1 = m_0 = 0$.
- If $z \neq 1$ then $m_1 = 1$ and

$$\begin{cases} \text{if} \quad z^2 - z + 1 = 0 \quad \text{then} \quad m_0 = 0 \\ \text{if} \quad z^2 - z + 1 \neq 0 \quad \text{then} \quad m_0 = 1. \end{cases}$$

- $z^2 - z + 1 = 0 \leftrightarrow z = e^{\frac{2\pi i}{6}}$.

Some examples of local systems of rank 2 on the complement of the cusp

Some examples of local systems of rank 2 on the complement of the cusp

- For $s, t \in \mathbb{C}^*$ let $\mathcal{L}_{s,t}$ be $\leftrightarrow \varrho_{s,t} : G \rightarrow \mathrm{GL}(2, \mathbb{C})$:
  $\varrho_{s,t}(\delta) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^2 \end{pmatrix}, \varrho_{s,t}(a) = \varrho_{s,t}(b) = \begin{pmatrix} 0 & s \\ t & 0 \end{pmatrix}, \theta = e^{\frac{2\pi i}{3}}.$

Some examples of local systems of rank 2 on the complement of the cusp

- For $s, t \in \mathbb{C}^*$ let $\mathcal{L}_{s,t}$ be $\leftrightarrow \varrho_{s,t} : G \to \mathrm{GL}(2, \mathbb{C})$:
  $\varrho_{s,t}(\delta) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^2 \end{pmatrix}, \varrho_{s,t}(a) = \varrho_{s,t}(b) = \begin{pmatrix} 0 & s \\ t & 0 \end{pmatrix}, \theta = e^{\frac{2\pi i}{3}}.$
- The $\mathcal{L}_{s,t}$ are irreducible.

Some examples of local systems of rank 2 on the complement of the cusp

- For $s, t \in \mathbb{C}^*$ let $\mathcal{L}_{s,t}$ be $\leftrightarrow \varrho_{s,t} : G \to \mathrm{GL}(2, \mathbb{C})$:
  $\varrho_{s,t}(\delta) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^2 \end{pmatrix}, \varrho_{s,t}(a) = \varrho_{s,t}(b) = \begin{pmatrix} 0 & s \\ t & 0 \end{pmatrix}, \theta = e^{\frac{2\pi i}{3}}$.
- The $\mathcal{L}_{s,t}$ are irreducible.
- $\mathrm{CC}(j_{!*}\mathcal{L}_{s,t}) = 2T^*_X(X) + m_1 \overline{T^*_{D^{reg}}(X)} + m_0 T^*_0(X)$.

Some examples of local systems of rank 2 on the complement of the cusp

- For $s, t \in \mathbb{C}^*$ let $\mathcal{L}_{s,t}$ be $\leftrightarrow \varrho_{s,t} : G \to \mathrm{GL}(2, \mathbb{C})$:
$\varrho_{s,t}(\delta) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^2 \end{pmatrix}, \varrho_{s,t}(a) = \varrho_{s,t}(b) = \begin{pmatrix} 0 & s \\ t & 0 \end{pmatrix}, \theta = e^{\frac{2\pi i}{3}}.$
- The $\mathcal{L}_{s,t}$ are irreducible.
- $\mathrm{CC}(j_{!*}\mathcal{L}_{s,t}) = 2T^*_X(X) + m_1 \overline{T^*_{D^{reg}}(X)} + m_0 T^*_0(X)$.
- $$m_1 = \begin{cases} 2 & \text{if} \quad st \neq 1 \\ 1 & \text{if} \quad st = 1 \end{cases}, \quad m_0 = \begin{cases} 2 & \text{if} \quad st \neq 1 \\ 0 & \text{if} \quad st = 1 \end{cases}$$
- $j_{!*}\mathcal{L}_{s,1/s}$ are simple perverse sheaves with
$$Ch(j_{!*}\mathcal{L}_{s,1/s}) = T^*_X(X) \cup \overline{T^*_{D^{reg}}(X)}.$$

Can we systematically find explicit systems of linear partial differential equations in $\mathbb{C}^2$ giving rise to simple (germs at 0 of) regular holonomic D-modules stratified with respect (for instance) to a cusp with characteristic variety $T^*_X(X) \cup \overline{T^*_{D^{reg}}(X)}$?

Can we systematically find explicit systems of linear partial differential equations in $\mathbb{C}^2$ giving rise to simple (germs at 0 of) regular holonomic D-modules stratified with respect (for instance) to a cusp with characteristic variety $T^*_X(X) \cup \overline{T^*_{D^{reg}}(X)}$?

Or still, can we find explicit systems of linear partial differential equations in $\mathbb{C}^2$ (or $\mathbb{C}^d$) corresponding through the Riemann-Hilbert correspondence to "intersection complexes" as $j_{!*}\mathcal{L}$?

Logarithmic D-modules can help!

Logarithmic vector fields

Logarithmic vector fields

- From now on: X = a connected complex analytic manifold and $D \subset X$ a divisor (= hypersurface), $d = \dim X$, $U = X - D$.

Logarithmic vector fields

- From now on: X = a connected complex analytic manifold and $D \subset X$ a divisor (= hypersurface), $d = \dim X$, $U = X - D$.
- $\mathrm{Der}(\log D)$ = the $\mathcal{O}_X$-module of logarithmic vector fields along D. It is a submodule and a Lie subalgebra of $\mathrm{Der}(\mathcal{O}_X)$.

Logarithmic vector fields

- From now on: X = a connected complex analytic manifold and $D \subset X$ a divisor (= hypersurface), $d = \dim X$, $U = X - D$.
- $\mathrm{Der}(\log D)$ = the $\mathcal{O}_X$-module of logarithmic vector fields along D. It is a submodule and a Lie subalgebra of $\mathrm{Der}(\mathcal{O}_X)$.
- If $h \in \mathcal{O}_{X,p}$ is a reduced local equation of D at p, then a germ of vector field $\delta \in \mathrm{Der}(\mathcal{O}_{X,p})$ is logarithmic along D if and only if $\delta(h) \in \mathcal{O}_{X,p} \cdot h$.

Logarithmic vector fields

- From now on: X = a connected complex analytic manifold and $D \subset X$ a divisor (= hypersurface), $d = \dim X$, $U = X - D$.
- $\mathrm{Der}(\log D)$ = the $\mathcal{O}_X$-module of logarithmic vector fields along D. It is a submodule and a Lie subalgebra of $\mathrm{Der}(\mathcal{O}_X)$.
- If $h \in \mathcal{O}_{X,p}$ is a reduced local equation of D at p, then a germ of vector field $\delta \in \mathrm{Der}(\mathcal{O}_{X,p})$ is logarithmic along D if and only if $\delta(h) \in \mathcal{O}_{X,p} \cdot h$.
- If $x_1, \ldots, x_d$ is a system of local coordinates at p and $\delta = \sum a_j \frac{\partial}{\partial x_j}$, then δ is logarithmic along D if and only if $\sum_j a_j h'_{x_j} = a_0 h$, and so the stalk at p of $\mathrm{Der}(\log D)$ can be identified with the syzygies of $h, h'_{x_1}, \ldots, h'_{x_d}$.

Logarithmic vector fields

- From now on: X = a connected complex analytic manifold and $D \subset X$ a divisor (= hypersurface), $d = \dim X$, $U = X - D$.
- $\mathrm{Der}(\log D)$ = the $\mathcal{O}_X$-module of logarithmic vector fields along D. It is a submodule and a Lie subalgebra of $\mathrm{Der}(\mathcal{O}_X)$.
- If $h \in \mathcal{O}_{X,p}$ is a reduced local equation of D at p, then a germ of vector field $\delta \in \mathrm{Der}(\mathcal{O}_{X,p})$ is logarithmic along D if and only if $\delta(h) \in \mathcal{O}_{X,p} \cdot h$.
- If $x_1, \ldots, x_d$ is a system of local coordinates at p and $\delta = \sum a_j \frac{\partial}{\partial x_j}$, then δ is logarithmic along D if and only if $\sum_j a_j h'_{x_j} = a_0 h$, and so the stalk at p of $\mathrm{Der}(\log D)$ can be identified with the syzygies of $h, h'_{x_1}, \ldots, h'_{x_d}$.
- $\mathrm{Der}(\log D)$ it is a coherent $\mathcal{O}_X$-module.

Logarithmic vector fields

- From now on: $X =$ a connected complex analytic manifold and $D \subset X$ a divisor ($=$ hypersurface), $d = \dim X$, $U = X - D$.
- $\mathrm{Der}(\log D) =$ the $\mathcal{O}_X$-module of logarithmic vector fields along D. It is a submodule and a Lie subalgebra of $\mathrm{Der}(\mathcal{O}_X)$.
- If $h \in \mathcal{O}_{X,p}$ is a reduced local equation of D at p, then a germ of vector field $\delta \in \mathrm{Der}(\mathcal{O}_{X,p})$ is logarithmic along D if and only if $\delta(h) \in \mathcal{O}_{X,p} \cdot h$.
- If $x_1, \ldots, x_d$ is a system of local coordinates at p and $\delta = \sum a_j \frac{\partial}{\partial x_j}$, then δ is logarithmic along D if and only if $\sum_j a_j h'_{x_j} = a_0 h$, and so the stalk at p of $\mathrm{Der}(\log D)$ can be identified with the syzygies of $h, h'_{x_1}, \ldots, h'_{x_d}$.
- $\mathrm{Der}(\log D)$ it is a coherent $\mathcal{O}_X$-module.
- $\mathrm{Der}(\log D)$ is reflexive and its dual is $\Omega_X(\log D)$.

Free divisors

Free divisors

Definition (Saito)

We say that D is a *free* divisor if $\mathrm{Der}(\log D)$ (or $\Omega_X(\log D)$) is a locally free $\mathcal{O}_X$-module (necessarily of rank d).

Free divisors

Definition (Saito)

We say that D is a *free* divisor if $\mathrm{Der}(\log D)$ (or $\Omega_X(\log D)$) is a locally free $\mathcal{O}_X$-module (necessarily of rank d).

Example

The following classes of divisors are free:

Free divisors

Definition (Saito)

We say that D is a *free* divisor if $\mathrm{Der}(\log D)$ (or $\Omega_X(\log D)$) is a locally free $\mathcal{O}_X$-module (necessarily of rank d).

Example

The following classes of divisors are free:

- Normal crossing divisors. Plane curves.

Free divisors

Definition (Saito)

We say that D is a *free* divisor if $\mathrm{Der}(\log D)$ (or $\Omega_X(\log D)$) is a locally free $\mathcal{O}_X$-module (necessarily of rank d).

Example

The following classes of divisors are free:

- Normal crossing divisors. Plane curves.
- Discriminants (Arnold, Zakalyukin, Saito, Terao) and bifurcation sets (Bruce) of versal unfoldings of germs of holomorphic functions with an isolated critical point (and some generalizations by Buchweitz, Ebeling, Graf von Bothmer, Looijenga, Damon,..).

Free divisors

Definition (Saito)

We say that D is a *free* divisor if $\mathrm{Der}(\log D)$ (or $\Omega_X(\log D)$) is a locally free $\mathcal{O}_X$-module (necessarily of rank d).

Example

The following classes of divisors are free:

- Normal crossing divisors. Plane curves.
- Discriminants (Arnold, Zakalyukin, Saito, Terao) and bifurcation sets (Bruce) of versal unfoldings of germs of holomorphic functions with an isolated critical point (and some generalizations by Buchweitz, Ebeling, Graf von Bothmer, Looijenga, Damon,..).
- Linear free divisors (Buchweitz, Mond, Granger, Nieto, Schulze, Damon, Pike).

Free divisors

Definition (Saito)

We say that D is a *free* divisor if $\mathrm{Der}(\log D)$ (or $\Omega_X(\log D)$) is a locally free $\mathcal{O}_X$-module (necessarily of rank d).

Example

The following classes of divisors are free:

- Normal crossing divisors. Plane curves.
- Discriminants (Arnold, Zakalyukin, Saito, Terao) and bifurcation sets (Bruce) of versal unfoldings of germs of holomorphic functions with an isolated critical point (and some generalizations by Buchweitz, Ebeling, Graf von Bothmer, Looijenga, Damon,..).
- Linear free divisors (Buchweitz, Mond, Granger, Nieto, Schulze, Damon, Pike).
- New examples by adding “adjoint divisors” (Mond, Schulze).

Examples

Examples

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.

Examples

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.
- D is a free divisor: a basis of $\mathrm{Der}(\log D)$ is $\{\chi, \delta\}$ with $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$.

Examples

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.
- D is a free divisor: a basis of $\mathrm{Der}(\log D)$ is $\{\chi, \delta\}$ with $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$.
- $\chi(h) = 6h$, $\delta(h) = 0$, $[\chi, \delta] = \delta$.

Integrable logarithmic connections

Integrable logarithmic connections

- $D \subset X$ a free divisor: $\mathcal{D}_X(\log D) := \mathcal{O}_X[\mathrm{Der}(\log D)] \subset \mathcal{D}_X$.

Integrable logarithmic connections

- $D \subset X$ a free divisor: $\mathcal{D}_X(\log D) := \mathcal{O}_X[\mathrm{Der}(\log D)] \subset \mathcal{D}_X$.
- An *integrable logarithmic connection* (ILC) (with respect to D) is a left $\mathcal{D}_X(\log D)$-module which is locally free** of finite rank over $\mathcal{O}_X$.

Integrable logarithmic connections

- $D \subset X$ a free divisor: $\mathcal{D}_X(\log D) := \mathcal{O}_X[\mathrm{Der}(\log D)] \subset \mathcal{D}_X$.
- An *integrable logarithmic connection* (ILC) (with respect to D) is a left $\mathcal{D}_X(\log D)$-module which is locally free** of finite rank over $\mathcal{O}_X$.
- $\mathcal{O}_X(kD)$, $k \in \mathbb{Z}$, are ILC.

Integrable logarithmic connections

- $D \subset X$ a free divisor: $\mathcal{D}_X(\log D) := \mathcal{O}_X[\mathrm{Der}(\log D)] \subset \mathcal{D}_X$.
- An *integrable logarithmic connection* (ILC) (with respect to D) is a left $\mathcal{D}_X(\log D)$-module which is locally free** of finite rank over $\mathcal{O}_X$.
- $\mathcal{O}_X(kD)$, $k \in \mathbb{Z}$, are ILC.Locally it is given by the action of a local basis of $\mathrm{Der}(\log D)$ on a local basis of $\mathcal{E}$ with some integrability conditions.

Integrable logarithmic connections

- $D \subset X$ a free divisor: $\mathcal{D}_X(\log D) := \mathcal{O}_X[\mathrm{Der}(\log D)] \subset \mathcal{D}_X$.
- An *integrable logarithmic connection* (ILC) (with respect to D) is a left $\mathcal{D}_X(\log D)$-module which is locally free** of finite rank over $\mathcal{O}_X$.
- $\mathcal{O}_X(kD)$, $k \in \mathbb{Z}$, are ILC.Locally it is given by the action of a local basis of $\mathrm{Der}(\log D)$ on a local basis of $\mathcal{E}$ with some integrability conditions.
- $\mathcal{E}, \mathcal{E}'$ ILC $\Rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}'$, $Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}')$ (in particular $\mathcal{E}^*$) ILC.

Integrable logarithmic connections

- $D \subset X$ a free divisor: $\mathcal{D}_X(\log D) := \mathcal{O}_X[\mathrm{Der}(\log D)] \subset \mathcal{D}_X$.
- An *integrable logarithmic connection* (ILC) (with respect to D) is a left $\mathcal{D}_X(\log D)$-module which is locally free** of finite rank over $\mathcal{O}_X$.
- $\mathcal{O}_X(kD)$, $k \in \mathbb{Z}$, are ILC.Locally it is given by the action of a local basis of $\mathrm{Der}(\log D)$ on a local basis of $\mathcal{E}$ with some integrability conditions.
- $\mathcal{E}, \mathcal{E}'$ ILC $\Rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}'$, $Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}')$ (in particular $\mathcal{E}^*$) ILC.
- $\mathcal{E}(kD) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(kD)$.

Integrable logarithmic connections

- $D \subset X$ a free divisor: $\mathcal{D}_X(\log D) := \mathcal{O}_X[\mathrm{Der}(\log D)] \subset \mathcal{D}_X$.
- An *integrable logarithmic connection* (ILC) (with respect to D) is a left $\mathcal{D}_X(\log D)$-module which is locally free** of finite rank over $\mathcal{O}_X$.
- $\mathcal{O}_X(kD)$, $k \in \mathbb{Z}$, are ILC.Locally it is given by the action of a local basis of $\mathrm{Der}(\log D)$ on a local basis of $\mathcal{E}$ with some integrability conditions.
- $\mathcal{E}, \mathcal{E}'$ ILC $\Rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}'$, $Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}')$ (in particular $\mathcal{E}^*$) ILC.
- $\mathcal{E}(kD) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(kD)$.
- Any ILC determines a local system on U: its horizontal sections.

Integrable logarithmic connections

- $D \subset X$ a free divisor: $\mathcal{D}_X(\log D) := \mathcal{O}_X[\mathrm{Der}(\log D)] \subset \mathcal{D}_X$.
- An *integrable logarithmic connection* (ILC) (with respect to D) is a left $\mathcal{D}_X(\log D)$-module which is locally free** of finite rank over $\mathcal{O}_X$.
- $\mathcal{O}_X(kD)$, $k \in \mathbb{Z}$, are ILC.Locally it is given by the action of a local basis of $\mathrm{Der}(\log D)$ on a local basis of $\mathcal{E}$ with some integrability conditions.
- $\mathcal{E}, \mathcal{E}'$ ILC $\Rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}'$, $Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}')$ (in particular $\mathcal{E}^*$) ILC.
- $\mathcal{E}(kD) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(kD)$.
- Any ILC determines a local system on U: its horizontal sections. It does not change by the twistings $\mathcal{E}(kD)$.

Examples of ILC with respect to the cusp

Examples of ILC with respect to the cusp

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.

Examples of ILC with respect to the cusp

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.
- A basis of Der(log D) is: $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$.

Examples of ILC with respect to the cusp

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.
- A basis of $\mathrm{Der}(\log D)$ is: $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$.
- For $\alpha \in \mathbb{C}$, $\mathcal{O}_X h^\alpha$ is a ILC with: $\chi h^\alpha = 6\alpha h^\alpha$, $\delta h^\alpha = 0$.

Examples of ILC with respect to the cusp

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.
- A basis of $\mathrm{Der}(\log D)$ is: $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$.
- For $\alpha \in \mathbb{C}$, $\mathcal{O}_X h^\alpha$ is a ILC with: $\chi h^\alpha = 6\alpha h^\alpha$, $\delta h^\alpha = 0$.
- $\mathcal{O}_X h^\alpha \simeq \mathcal{D}_X(\log D)/\mathcal{D}_X(\log D)(\chi - 6\alpha, \delta)$.

Examples of ILC with respect to the cusp

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.
- A basis of Der(log D) is: $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$.
- For $\alpha \in \mathbb{C}$, $\mathcal{O}_X h^\alpha$ is a ILC with: $\chi h^\alpha = 6\alpha h^\alpha$, $\delta h^\alpha = 0$.
- $\mathcal{O}_X h^\alpha \simeq \mathcal{D}_X(\log D)/\mathcal{D}_X(\log D)(\chi - 6\alpha, \delta)$.
- Any ILC of rank 1 with respect to D is isomorphic to $\mathcal{O}_X h^\alpha$ fo some $\alpha \in \mathbb{C}$.

Examples of ILC with respect to the cusp

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.
- A basis of Der(log D) is: $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$.
- For $\alpha \in \mathbb{C}$, $\mathcal{O}_X h^\alpha$ is a ILC with: $\chi h^\alpha = 6\alpha h^\alpha$, $\delta h^\alpha = 0$.
- $\mathcal{O}_X h^\alpha \simeq \mathcal{D}_X(\log D)/\mathcal{D}_X(\log D)(\chi - 6\alpha, \delta)$.
- Any ILC of rank 1 with respect to D is isomorphic to $\mathcal{O}_X h^\alpha$ fo some $\alpha \in \mathbb{C}$. For instance, $\mathcal{O}_X(kD) \simeq \mathcal{O}_X h^{-k}$, $k \in \mathbb{Z}$.

Examples of ILC with respect to the cusp

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.
- A basis of Der(log D) is: $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$.
- For $\alpha \in \mathbb{C}$, $\mathcal{O}_X h^\alpha$ is a ILC with: $\chi h^\alpha = 6\alpha h^\alpha$, $\delta h^\alpha = 0$.
- $\mathcal{O}_X h^\alpha \simeq \mathcal{D}_X(\log D)/\mathcal{D}_X(\log D)(\chi - 6\alpha, \delta)$.
- Any ILC of rank 1 with respect to D is isomorphic to $\mathcal{O}_X h^\alpha$ fo some $\alpha \in \mathbb{C}$. For instance, $\mathcal{O}_X(kD) \simeq \mathcal{O}_X h^{-k}$, $k \in \mathbb{Z}$.
- For $\lambda, e \in \mathbb{C}$, $\mathcal{E}_{\lambda,e} = \mathcal{O}_X^2$ with:

$$\chi \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \delta \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ ey & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Examples of ILC with respect to the cusp

- $X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.
- A basis of Der(log D) is: $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$.
- For $\alpha \in \mathbb{C}$, $\mathcal{O}_X h^\alpha$ is a ILC with: $\chi h^\alpha = 6\alpha h^\alpha$, $\delta h^\alpha = 0$.
- $\mathcal{O}_X h^\alpha \simeq \mathcal{D}_X(\log D)/\mathcal{D}_X(\log D)(\chi - 6\alpha, \delta)$.
- Any ILC of rank 1 with respect to D is isomorphic to $\mathcal{O}_X h^\alpha$ fo some $\alpha \in \mathbb{C}$. For instance, $\mathcal{O}_X(kD) \simeq \mathcal{O}_X h^{-k}$, $k \in \mathbb{Z}$.
- For $\lambda, e \in \mathbb{C}$, $\mathcal{E}_{\lambda,e} = \mathcal{O}_X^2$ with:

$$\chi \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \ \delta \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ ey & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

- $\mathcal{E}_{\lambda,e} = \mathcal{D}_X(\log D)e_1 = \mathcal{D}_X(\log D)/\mathcal{D}_X(\log D)(\chi - \lambda, \delta^2 - ey)$.

Logarithmic and meromorphic de Rham complexes

Logarithmic and meromorphic de Rham complexes

- Any ILC $\mathcal{E}$ has a *logarithmic de Rham complex* $\Omega_X^\bullet(\log D)(\mathcal{E})$.

Logarithmic and meromorphic de Rham complexes

- Any ILC $\mathcal{E}$ has a *logarithmic de Rham complex* $\Omega_X^\bullet(\log D)(\mathcal{E})$.
- We also have the meromorphic de Rham complex $\Omega_X^\bullet(\mathcal{E}[\star D])$.

Logarithmic and meromorphic de Rham complexes

- Any ILC $\mathcal{E}$ has a *logarithmic de Rham complex* $\Omega_X^\bullet(\log D)(\mathcal{E})$.
- We also have the meromorphic de Rham complex $\Omega_X^\bullet(\mathcal{E}[\star D])$.
- The meromorphic connection $\mathcal{E}[\star D]$ is regular and $\Omega_X^\bullet(\mathcal{E}[\star D]) \xrightarrow{\sim} Rj_*\mathcal{L}$, where $\mathcal{L}$ is the local system of horizontal sections of $\mathcal{E}$ on $U = X - D$.

Logarithmic and meromorphic de Rham complexes

- Any ILC $\mathcal{E}$ has a *logarithmic de Rham complex* $\Omega^\bullet_X(\log D)(\mathcal{E})$.
- We also have the meromorphic de Rham complex $\Omega^\bullet_X(\mathcal{E}[\star D])$.
- The meromorphic connection $\mathcal{E}[\star D]$ is regular and $\Omega^\bullet_X(\mathcal{E}[\star D]) \xrightarrow{\sim} Rj_*\mathcal{L}$, where $\mathcal{L}$ is the local system of horizontal sections of $\mathcal{E}$ on $U = X - D$.
- (LCP): Is the inclusion $\Omega^\bullet_X(\log D)(\mathcal{E}) \hookrightarrow \Omega^\bullet_X(\mathcal{E}[\star D])$ a quasi-isomorphism?

Logarithmic and meromorphic de Rham complexes

- Any ILC $\mathcal{E}$ has a *logarithmic de Rham complex* $\Omega^\bullet_X(\log D)(\mathcal{E})$.
- We also have the meromorphic de Rham complex $\Omega^\bullet_X(\mathcal{E}[\star D])$.
- The meromorphic connection $\mathcal{E}[\star D]$ is regular and $\Omega^\bullet_X(\mathcal{E}[\star D]) \xrightarrow{\sim} Rj_*\mathcal{L}$, where $\mathcal{L}$ is the local system of horizontal sections of $\mathcal{E}$ on $U = X - D$.
- (LCP): Is the inclusion $\Omega^\bullet_X(\log D)(\mathcal{E}) \hookrightarrow \Omega^\bullet_X(\mathcal{E}[\star D])$ a quasi-isomorphism?
- In other words, is the canonical map $\Omega^\bullet_X(\log D)(\mathcal{E}) \to Rj_*\mathcal{L}$ an isomorphism in the derived category?

The logarithmic comparison theorem (LCT)

The logarithmic comparison theorem (LCT)

- When D is a NCD and $\mathcal{E} = \mathcal{O}_X$, the (LCT) is true (easy).

The logarithmic comparison theorem (LCT)

- When D is a NCD and $\mathcal{E} = \mathcal{O}_X$, the (LCT) is true (easy).
- (Deligne) When D is a NCD,
  $\Omega_X^\bullet(\log D)(\mathcal{E}(kD)) \simeq \Omega_X^\bullet(\mathcal{E}[\star D])$ for $k >> 0$.

The logarithmic comparison theorem (LCT)

- When D is a NCD and $\mathcal{E} = \mathcal{O}_X$, the (LCT) is true (easy).
- (Deligne) When D is a NCD, $\Omega_X^\bullet(\log D)(\mathcal{E}(kD)) \simeq \Omega_X^\bullet(\mathcal{E}[\star D])$ for $k >> 0$.
- (Castro-Jiménez, Mond, N-M; 1994): If D is a *locally quasi-homogeneous (**LQH**)* free divisor and $\mathcal{E} = \mathcal{O}_X$, then the (LCT) is true.

The logarithmic comparison theorem (LCT)

- When D is a NCD and $\mathcal{E} = \mathcal{O}_X$, the (LCT) is true (easy).
- (Deligne) When D is a NCD, $\Omega_X^\bullet(\log D)(\mathcal{E}(kD)) \simeq \Omega_X^\bullet(\mathcal{E}[\star D])$ for $k >> 0$.
- (Castro-Jiménez, Mond, N-M; 1994): If D is a *locally quasi-homogeneous (**LQH**)* free divisor and $\mathcal{E} = \mathcal{O}_X$, then the (LCT) is true.
- **Question**: is (LCT) true for any free divisor and any ILC?

The logarithmic comparison theorem (LCT)

- When D is a NCD and $\mathcal{E} = \mathcal{O}_X$, the (LCT) is true (easy).
- (Deligne) When D is a NCD, $\Omega^\bullet_X(\log D)(\mathcal{E}(kD)) \simeq \Omega^\bullet_X(\mathcal{E}[\star D])$ for $k >> 0$.
- (Castro-Jiménez, Mond, N-M; 1994): If D is a *locally quasi-homogeneous* (**LQH**) free divisor and $\mathcal{E} = \mathcal{O}_X$, then the (LCT) is true.
- **Question**: is (LCT) true for any free divisor and any ILC?
- Can we understand the LCT with D-module Theory?

D-module theory enters the scene

D-module theory enters the scene

- Esnault-Viehweg (1986), for NCD: they used the sheaf of rings $\mathcal{D}_X(\log D)$ to express logarithmic de Rham complexes and to prove a duality formula.

D-module theory enters the scene

- Esnault-Viehweg (1986), for NCD: they used the sheaf of rings $\mathcal{D}_X(\log D)$ to express logarithmic de Rham complexes and to prove a duality formula.
- Calderón-Moreno (1996-97), for arbitrary free divisors and arbitrary ILC:

D-module theory enters the scene

- Esnault-Viehweg (1986), for NCD: they used the sheaf of rings $\mathcal{D}_X(\log D)$ to express logarithmic de Rham complexes and to prove a duality formula.
- Calderón-Moreno (1996-97), for arbitrary free divisors and arbitrary ILC: $V_0^D(\mathcal{D}_X) = \mathcal{D}_X(\log D)$,

D-module theory enters the scene

- Esnault-Viehweg (1986), for NCD: they used the sheaf of rings $\mathcal{D}_X(\log D)$ to express logarithmic de Rham complexes and to prove a duality formula.
- Calderón-Moreno (1996-97), for arbitrary free divisors and arbitrary ILC: $V_0^D(\mathcal{D}_X) = \mathcal{D}_X(\log D)$,

$$\Omega_X^\bullet(\log D)(\mathcal{E}) = \mathbf{R}\, Hom_{V_0}(\mathcal{O}_X, \mathcal{E}),$$

D-module theory enters the scene

- Esnault-Viehweg (1986), for NCD: they used the sheaf of rings $\mathcal{D}_X(\log D)$ to express logarithmic de Rham complexes and to prove a duality formula.
- Calderón-Moreno (1996-97), for arbitrary free divisors and arbitrary ILC: $V_0^D(\mathcal{D}_X) = \mathcal{D}_X(\log D)$,

$$\Omega_X^\bullet(\log D)(\mathcal{E}) = \mathbf{R}\, Hom_{V_0}(\mathcal{O}_X, \mathcal{E}),$$

and he introduced* the notion of Koszul free divisor and proved that: Koszul free $\Rightarrow$ $\Omega_X^\bullet(\log D)$ is perverse.

D-module theory enters the scene

- Esnault-Viehweg (1986), for NCD: they used the sheaf of rings $\mathcal{D}_X(\log D)$ to express logarithmic de Rham complexes and to prove a duality formula.
- Calderón-Moreno (1996-97), for arbitrary free divisors and arbitrary ILC: $V_0^D(\mathcal{D}_X) = \mathcal{D}_X(\log D)$,

  $$\Omega_X^\bullet(\log D)(\mathcal{E}) = \mathbf{R}\, Hom_{V_0}(\mathcal{O}_X, \mathcal{E}),$$

  and he introduced* the notion of Koszul free divisor and proved that: Koszul free $\Rightarrow$ $\Omega_X^\bullet(\log D)$ is perverse.
- **Definition**: A free divisor D is *Koszul* at p if the symbols of any (or some) local basis $\{\delta_1, \ldots, \delta_d\}$ of $\mathrm{Der}(\log D)_p$ form a regular sequence in $\mathrm{gr}\, \mathcal{D}_{X,p} = \mathcal{O}_{X,p}[\xi_1, \ldots, \xi_d]$. We say that D is *Koszul* if it is so at any point $p \in D$.

What we need to "compute" $j_{!*}\mathcal{L}$?

What we need to "compute" $j_{!*}\mathcal{L}$?

$j_{!*}\mathcal{L}$ = image of $j_!\mathcal{L} \to Rj_*\mathcal{L}$ in the category of perverse sheaves.

What we need to "compute" $j_{!*}\mathcal{L}$?

$j_{!*}\mathcal{L}$ = image of $j_!\mathcal{L} \to Rj_*\mathcal{L}$ in the category of perverse sheaves.

$j_!\mathcal{L} = (Rj_*\mathcal{L}^\vee)^\vee$.

What we need to "compute" $j_{!*}\mathcal{L}$?

$j_{!*}\mathcal{L}$ = image of $j_!\mathcal{L} \to Rj_*\mathcal{L}$ in the category of perverse sheaves.

$j_!\mathcal{L} = (Rj_*\mathcal{L}^\vee)^\vee$.

If $\mathcal{L}$ is the local system of horizontal sections of an ILC $\mathcal{E}$, then $\mathcal{L}^\vee$ is the local system of horizontal sections of the dual ILC $\mathcal{E}^*$.

What we need to "compute" $j_{!*}\mathcal{L}$?

$j_{!*}\mathcal{L}$ = image of $j_!\mathcal{L} \to Rj_*\mathcal{L}$ in the category of perverse sheaves.

$j_!\mathcal{L} = (Rj_*\mathcal{L}^\vee)^\vee$.

If $\mathcal{L}$ is the local system of horizontal sections of an ILC $\mathcal{E}$, then $\mathcal{L}^\vee$ is the local system of horizontal sections of the dual ILC $\mathcal{E}^*$.

So, what we need is to understand:

$$\mathcal{E} \quad \rightsquigarrow \quad Rj_*\mathcal{L} \quad \text{and} \quad (\quad)^\vee.$$

.

Understanding duality

Understanding duality

THEOREM. *[Calderón-Moreno, N-M, 2004] Let $D \subset X$ be a free divisor and $\mathcal{E}$ a ILC w.r.t. D. There is a canonical isomorphism in the derived category*

$$\mathbb{D}\left(\mathcal{D}_X \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}\right) \simeq \mathcal{D}_X \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}^*(D)$$

where $\mathbb{D}$ stands for the duality in D-module theory and * for the duality of ILC.

Understanding duality

THEOREM. *[Calderón-Moreno, N-M, 2004] Let $D \subset X$ be a free divisor and $\mathcal{E}$ a ILC w.r.t. D. There is a canonical isomorphism in the derived category*

$$\mathbb{D}\left(\mathcal{D}_X \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}\right) \simeq \mathcal{D}_X \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}^*(D)$$

where $\mathbb{D}$ stands for the duality in D-module theory and * for the duality of ILC.The "Spencer" case and $\mathcal{E} = \mathcal{O}_X$: Castro, Ucha.

Understanding duality

THEOREM. *[Calderón-Moreno, N-M, 2004] Let $D \subset X$ be a free divisor and $\mathcal{E}$ a ILC w.r.t. D. There is a canonical isomorphism in the derived category*

$$\mathbb{D}\left(\mathcal{D}_X \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}\right) \simeq \mathcal{D}_X \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}^*(D)$$

where $\mathbb{D}$ stands for the duality in D-module theory and * for the duality of ILC. The "Spencer" case and $\mathcal{E} = \mathcal{O}_X$: Castro, Ucha.

COROLLARY. *Under the above hypotheses, we have a canonical isomorphism in the derived category*

$$\Omega_X^\bullet(\log D)(\mathcal{E}) = \mathrm{DR}\left(\mathcal{D}_X \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}(D)\right).$$

Understanding $Rj_*\mathcal{L} \equiv$ (LCP)

Understanding $Rj_*\mathcal{L} \equiv$ (LCP)

THEOREM. *[Calderón-Moreno, N-M, 2006] Let $D \subset X$ be a free divisor of linear Jacobian type and $\mathcal{E}$ a ILC w.r.t. D. Then, for $k >> 0$ the canonical map $\Omega_X^\bullet(\log D)(\mathcal{E}(kD)) \to Rj_*\mathcal{L}$ is an isomorphism in the derived category.*

Understanding $Rj_*\mathcal{L} \equiv$ (LCP)

THEOREM. *[Calderón-Moreno, N-M, 2006] Let $D \subset X$ be a free divisor of linear Jacobian type and $\mathcal{E}$ a ILC w.r.t. D. Then, for $k >> 0$ the canonical map $\Omega_X^\bullet(\log D)(\mathcal{E}(kD)) \to Rj_*\mathcal{L}$ is an isomorphism in the derived category.*

COROLLARY. *Under the above hypotheses, for $k >> 0$ the canonical map $j_!\mathcal{L} \to \Omega_X^\bullet(\log D)(\mathcal{E}(kD))$ is an isomorphism in the derived category.*

Understanding $Rj_*\mathcal{L} \equiv$ (LCP)

THEOREM. *[Calderón-Moreno, N-M, 2006] Let $D \subset X$ be a free divisor of linear Jacobian type and $\mathcal{E}$ a ILC w.r.t. D. Then, for $k >> 0$ the canonical map $\Omega_X^\bullet(\log D)(\mathcal{E}(kD)) \to Rj_*\mathcal{L}$ is an isomorphism in the derived category.*

COROLLARY. *Under the above hypotheses, for $k >> 0$ the canonical map $j_!\mathcal{L} \to \Omega_X^\bullet(\log D)(\mathcal{E}(kD))$ is an isomorphism in the derived category.*

COROLLARY. *Under the above hypotheses,*

$$j_{!*}\mathcal{L} = \mathrm{DR}\left(\mathrm{Im}\left(\mathcal{D}_X \overset{(\mathbf{L})}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}(-kD) \to \mathcal{D}_X \overset{(\mathbf{L})}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}(kD)\right)\right)$$

for $k >> 0$.

Examples

Examples

$$X = \mathbb{C}^2,\ h = x^2 - y^3,\ D = \{h = 0\}.$$

Examples

$X = \mathbb{C}^2,\ h = x^2 - y^3,\ D = \{h = 0\}.$

$(\mathcal{O}_X h^{\alpha})^* \simeq \mathcal{O}_X h^{-\alpha}.$

Examples

$$X = \mathbb{C}^2,\ h = x^2 - y^3,\ D = \{h = 0\}.$$
$$(\mathcal{O}_X h^\alpha)^* \simeq \mathcal{O}_X h^{-\alpha}.$$
$$\mathcal{E}_{\lambda,e} \otimes_{\mathcal{O}_X} (\mathcal{O}_X h^\alpha) \simeq \mathcal{E}_{\lambda+6\alpha,e},\ \alpha, \lambda, e \in \mathbb{C}.$$

Examples

$X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.

$(\mathcal{O}_X h^{\alpha})^* \simeq \mathcal{O}_X h^{-\alpha}$.

$\mathcal{E}_{\lambda,e} \otimes_{\mathcal{O}_X} (\mathcal{O}_X h^{\alpha}) \simeq \mathcal{E}_{\lambda+6\alpha,e}$, $\alpha, \lambda, e \in \mathbb{C}$.

$\mathcal{E}_{\lambda,e}(kD) \simeq \mathcal{E}_{\lambda-6k,e}$, $k \in \mathbb{Z}$.

Examples

$X = \mathbb{C}^2$, $h = x^2 - y^3$, $D = \{h = 0\}$.

$(\mathcal{O}_X h^\alpha)^* \simeq \mathcal{O}_X h^{-\alpha}$.

$\mathcal{E}_{\lambda,e} \otimes_{\mathcal{O}_X} (\mathcal{O}_X h^\alpha) \simeq \mathcal{E}_{\lambda+6\alpha,e}$, $\alpha, \lambda, e \in \mathbb{C}$.

$\mathcal{E}_{\lambda,e}(kD) \simeq \mathcal{E}_{\lambda-6k,e}$, $k \in \mathbb{Z}$.

$(\mathcal{E}_{\lambda,e})^* = Hom_{\mathcal{O}_X}(\mathcal{E}_{\lambda,e}, \mathcal{O}_X) \simeq \mathcal{E}_{-\lambda-1,e}$.

Examples of ILC of rank 1 w.r.t. the cusp

Examples of ILC of rank 1 w.r.t. the cusp

For $z \in \mathbb{C}^*$: $\mathcal{L}_z \leftrightarrow \varrho_z : G \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$,
$\varrho_z(\delta) = z, \varrho_z(a) = \varrho_z(b) = 1$.

Examples of ILC of rank 1 w.r.t. the cusp

For $z \in \mathbb{C}^*$: $\mathcal{L}_z \leftrightarrow \varrho_z : G \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$,
$\varrho_z(\delta) = z$, $\varrho_z(a) = \varrho_z(b) = 1$.
For $\alpha \in \mathbb{C}$:
$\mathcal{M}_\alpha := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} (\mathcal{O}_{\mathbb{C}^2} h^\alpha) = \mathcal{D}_{\mathbb{C}^2} / \mathcal{D}_{\mathbb{C}^2}(\chi - 6\alpha, \delta)$.

Examples of ILC of rank 1 w.r.t. the cusp

For $z \in \mathbb{C}^*$: $\mathcal{L}_z \leftrightarrow \varrho_z : G \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$,
$\varrho_z(\delta) = z$, $\varrho_z(a) = \varrho_z(b) = 1$.
For $\alpha \in \mathbb{C}$:
$\mathcal{M}_\alpha := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} (\mathcal{O}_{\mathbb{C}^2} h^\alpha) = \mathcal{D}_{\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}^2}(\chi - 6\alpha, \delta)$.

If $z \neq 1, e^{\pm\frac{2\pi i}{6}}$: $\mathrm{DR}(\mathcal{M}_\alpha) \simeq Rj_*\mathcal{L}_z \simeq j_!\mathcal{L}_z \simeq j_{!*}\mathcal{L}_z$ for any $\alpha \in \mathbb{C}$ with $e^{2\pi i\alpha} = z$.

Examples of ILC of rank 1 w.r.t. the cusp

For $z \in \mathbb{C}^*$: $\mathcal{L}_z \leftrightarrow \varrho_z : G \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$,
$\varrho_z(\delta) = z$, $\varrho_z(a) = \varrho_z(b) = 1$.
For $\alpha \in \mathbb{C}$:
$\mathcal{M}_\alpha := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} (\mathcal{O}_{\mathbb{C}^2} h^\alpha) = \mathcal{D}_{\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}^2}(\chi - 6\alpha, \delta)$.

If $z \neq 1, e^{\pm\frac{2\pi i}{6}}$: $\mathrm{DR}(\mathcal{M}_\alpha) \simeq Rj_*\mathcal{L}_z \simeq j_!\mathcal{L}_z \simeq j_{!*}\mathcal{L}_z$ for any $\alpha \in \mathbb{C}$ with $e^{2\pi i\alpha} = z$.
If $z = 1$: $\mathrm{DR}(\mathcal{M}_{-k}) \simeq Rj_*\mathbb{C}_U$ for any integer $k \geq 1$, $\mathrm{DR}(\mathcal{M}_k) \simeq j_!\mathbb{C}_U$ for any integer $k \geq 0$ and $\mathrm{DR}(\mathcal{O}_X) = \mathbb{C}_X \simeq j_{!*}\mathbb{C}_U$.

Examples of ILC of rank 1 w.r.t. the cusp

For $z \in \mathbb{C}^*$: $\mathcal{L}_z \leftrightarrow \varrho_z : G \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$,
$\varrho_z(\delta) = z$, $\varrho_z(a) = \varrho_z(b) = 1$.
For $\alpha \in \mathbb{C}$:
$\mathcal{M}_\alpha := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} (\mathcal{O}_{\mathbb{C}^2} h^\alpha) = \mathcal{D}_{\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}^2}(\chi - 6\alpha, \delta)$.

If $z \neq 1, e^{\pm\frac{2\pi i}{6}}$: $\mathrm{DR}\,(\mathcal{M}_\alpha) \simeq Rj_*\mathcal{L}_z \simeq j_!\mathcal{L}_z \simeq j_{!*}\mathcal{L}_z$ for any $\alpha \in \mathbb{C}$ with $e^{2\pi i\alpha} = z$.
If $z = 1$: $\mathrm{DR}\,(\mathcal{M}_{-k}) \simeq Rj_*\mathbb{C}_U$ for any integer $k \geq 1$, $\mathrm{DR}(\mathcal{M}_k) \simeq j_!\mathbb{C}_U$ for any integer $k \geq 0$ and $\mathrm{DR}(\mathcal{O}_X) = \mathbb{C}_X \simeq j_{!*}\mathbb{C}_U$.
If $z = e^{\pm\frac{2\pi i}{6}}$: $\mathrm{DR}\left(\mathcal{M}_{\pm\frac{1}{6}-k}\right) \simeq Rj_*\mathcal{L}_z$ for any integer $k \geq 1$, $\mathrm{DR}(\mathcal{M}_{\pm\frac{1}{6}+k}) \simeq j_!\mathcal{L}_z$ for any integer $k \geq 0$ and $\mathrm{DR}\left(\mathcal{D}_X h^{\pm 1/6}\right) \simeq j_{!*}\mathcal{L}_z$.

Examples of ILC of rank 1 w.r.t. the cusp

For $z \in \mathbb{C}^*$: $\mathcal{L}_z \leftrightarrow \varrho_z : G \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$,
$\varrho_z(\delta) = z$, $\varrho_z(a) = \varrho_z(b) = 1$.
For $\alpha \in \mathbb{C}$:
$\mathcal{M}_\alpha := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} (\mathcal{O}_{\mathbb{C}^2} h^\alpha) = \mathcal{D}_{\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}^2}(\chi - 6\alpha, \delta)$.

If $z \neq 1, e^{\pm\frac{2\pi i}{6}}$: $\mathrm{DR}\,(\mathcal{M}_\alpha) \simeq Rj_*\mathcal{L}_z \simeq j_!\mathcal{L}_z \simeq j_{!*}\mathcal{L}_z$ for any $\alpha \in \mathbb{C}$ with $e^{2\pi i\alpha} = z$.
If $z = 1$: $\mathrm{DR}\,(\mathcal{M}_{-k}) \simeq Rj_*\mathbb{C}_U$ for any integer $k \geq 1$, $\mathrm{DR}(\mathcal{M}_k) \simeq j_!\mathbb{C}_U$ for any integer $k \geq 0$ and $\mathrm{DR}(\mathcal{O}_X) = \mathbb{C}_X \simeq j_{!*}\mathbb{C}_U$.
If $z = e^{\pm\frac{2\pi i}{6}}$: $\mathrm{DR}\left(\mathcal{M}_{\pm\frac{1}{6}-k}\right) \simeq Rj_*\mathcal{L}_z$ for any integer $k \geq 1$, $\mathrm{DR}(\mathcal{M}_{\pm\frac{1}{6}+k}) \simeq j_!\mathcal{L}_z$ for any integer $k \geq 0$ and $\mathrm{DR}\left(\mathcal{D}_X h^{\pm 1/6}\right) \simeq j_{!*}\mathcal{L}_z$.
$-1, -\frac{5}{6}, -\frac{7}{6}$ are the roots of the Bernstein polynomial of h.

Examples of ILC of rank 2 w.r.t. the cusp

Examples of ILC of rank 2 w.r.t. the cusp

For $\lambda, e \in \mathbb{C}$: $\mathcal{E}_{\lambda,e} = \mathcal{D}_{\mathbb{C}^2}(\log D)/\mathcal{D}_{\mathbb{C}^2}(\log D)(\chi - \lambda, \delta^2 - ey)$.

Examples of ILC of rank 2 w.r.t. the cusp

For $\lambda, e \in \mathbb{C}$: $\mathcal{E}_{\lambda,e} = \mathcal{D}_{\mathbb{C}^2}(\log D)/\mathcal{D}_{\mathbb{C}^2}(\log D)(\chi - \lambda, \delta^2 - ey)$.
$\mathcal{M}_{\lambda,e} := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} \mathcal{E}_{\lambda,e}$.

Examples of ILC of rank 2 w.r.t. the cusp

For $\lambda, e \in \mathbb{C}$: $\mathcal{E}_{\lambda,e} = \mathcal{D}_{\mathbb{C}^2}(\log D)/\mathcal{D}_{\mathbb{C}^2}(\log D)(\chi - \lambda, \delta^2 - ey)$. $\mathcal{M}_{\lambda,e} := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} \mathcal{E}_{\lambda,e}$. $\mathcal{L}_{\lambda,e}$ the local system of horizontal sections of $\mathcal{E}_{\lambda,e}$ on U.

Examples of ILC of rank 2 w.r.t. the cusp

For $\lambda, e \in \mathbb{C}$: $\mathcal{E}_{\lambda,e} = \mathcal{D}_{\mathbb{C}^2}(\log D)/\mathcal{D}_{\mathbb{C}^2}(\log D)(\chi - \lambda, \delta^2 - ey)$. $\mathcal{M}_{\lambda,e} := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} \mathcal{E}_{\lambda,e}$. $\mathcal{L}_{\lambda,e}$ the local system of horizontal sections of $\mathcal{E}_{\lambda,e}$ on U.

For $\lambda = -2, e = 2$, $\mathcal{E}_{-2,2}(D) = \mathcal{E}_{-8,2}$ and $\mathcal{M}_{-8,2} = \mathcal{M}_{-8,2}[\star D]$, $\mathbb{D}\,\mathcal{M}_{-2,2} = (\mathbb{D}\,\mathcal{M}_{-2,2})\,[\star D]$.

Examples of ILC of rank 2 w.r.t. the cusp

For $\lambda, e \in \mathbb{C}$: $\mathcal{E}_{\lambda,e} = \mathcal{D}_{\mathbb{C}^2}(\log D)/\mathcal{D}_{\mathbb{C}^2}(\log D)(\chi - \lambda, \delta^2 - ey)$. $\mathcal{M}_{\lambda,e} := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} \mathcal{E}_{\lambda,e}$. $\mathcal{L}_{\lambda,e}$ the local system of horizontal sections of $\mathcal{E}_{\lambda,e}$ on U.

For $\lambda = -2, e = 2$, $\mathcal{E}_{-2,2}(D) = \mathcal{E}_{-8,2}$ and
$\mathcal{M}_{-8,2} = \mathcal{M}_{-8,2}[\star D]$, $\mathbb{D}\,\mathcal{M}_{-2,2} = (\mathbb{D}\,\mathcal{M}_{-2,2})\,[\star D]$.
$j_{!*}\mathcal{L}_{-2+6\mathbb{Z},2} = \mathrm{DR}\,(\mathrm{Im}\,(\mathcal{M}_{-2,2} \to \mathcal{M}_{-8,2}))$.

Examples of ILC of rank 2 w.r.t. the cusp

For $\lambda, e \in \mathbb{C}$: $\mathcal{E}_{\lambda,e} = \mathcal{D}_{\mathbb{C}^2}(\log D)/\mathcal{D}_{\mathbb{C}^2}(\log D)(\chi - \lambda, \delta^2 - ey)$. $\mathcal{M}_{\lambda,e} := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} \mathcal{E}_{\lambda,e}$. $\mathcal{L}_{\lambda,e}$ the local system of horizontal sections of $\mathcal{E}_{\lambda,e}$ on U.

For $\lambda = -2, e = 2$, $\mathcal{E}_{-2,2}(D) = \mathcal{E}_{-8,2}$ and $\mathcal{M}_{-8,2} = \mathcal{M}_{-8,2}[\star D]$, $\mathbb{D}\,\mathcal{M}_{-2,2} = (\mathbb{D}\,\mathcal{M}_{-2,2})\,[\star D]$.

$j_{!*}\mathcal{L}_{-2+6\mathbb{Z},2} = \mathrm{DR}\left(\mathrm{Im}\left(\mathcal{M}_{-2,2} \to \mathcal{M}_{-8,2}\right)\right)$.

We can compute the above image:

$$j_{!*}\mathcal{L}_{-2+6\mathbb{Z},2} = \mathrm{DR}\left(\mathcal{D}_{\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}^2}(\chi + 2, 9y\partial_x^2 - 4\partial_y^2)\right).$$

Examples of ILC of rank 2 w.r.t. the cusp

For $\lambda, e \in \mathbb{C}$: $\mathcal{E}_{\lambda,e} = \mathcal{D}_{\mathbb{C}^2}(\log D)/\mathcal{D}_{\mathbb{C}^2}(\log D)(\chi - \lambda, \delta^2 - ey)$. $\mathcal{M}_{\lambda,e} := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} \mathcal{E}_{\lambda,e}$. $\mathcal{L}_{\lambda,e}$ the local system of horizontal sections of $\mathcal{E}_{\lambda,e}$ on U.

For $\lambda = -2$, $e = 2$, $\mathcal{E}_{-2,2}(D) = \mathcal{E}_{-8,2}$ and $\mathcal{M}_{-8,2} = \mathcal{M}_{-8,2}[\star D]$, $\mathbb{D}\,\mathcal{M}_{-2,2} = (\mathbb{D}\,\mathcal{M}_{-2,2})\,[\star D]$.
$j_{!*}\mathcal{L}_{-2+6\mathbb{Z},2} = \mathrm{DR}\left(\mathrm{Im}\left(\mathcal{M}_{-2,2} \to \mathcal{M}_{-8,2}\right)\right)$.
We can compute the above image:

$$j_{!*}\mathcal{L}_{-2+6\mathbb{Z},2} = \mathrm{DR}\left(\mathcal{D}_{\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}^2}(\chi + 2, 9y\partial_x^2 - 4\partial_y^2)\right).$$

$$\mathrm{Ch}(j_{!*}\mathcal{L}_{-2+6\mathbb{Z},2}) = T^*_{\mathbb{C}^2}(\mathbb{C}^2) \cup \overline{T^*_{D^{reg}}(\mathbb{C}^2)}.$$

Examples of ILC of rank 2 w.r.t. the cusp

For $\lambda, e \in \mathbb{C}$: $\mathcal{E}_{\lambda,e} = \mathcal{D}_{\mathbb{C}^2}(\log D)/\mathcal{D}_{\mathbb{C}^2}(\log D)(\chi - \lambda, \delta^2 - ey)$. $\mathcal{M}_{\lambda,e} := \mathcal{D}_{\mathbb{C}^2} \otimes_{\mathcal{D}_{\mathbb{C}^2}(\log D)} \mathcal{E}_{\lambda,e}$. $\mathcal{L}_{\lambda,e}$ the local system of horizontal sections of $\mathcal{E}_{\lambda,e}$ on U.

For $\lambda = -2, e = 2$, $\mathcal{E}_{-2,2}(D) = \mathcal{E}_{-8,2}$ and $\mathcal{M}_{-8,2} = \mathcal{M}_{-8,2}[\star D]$, $\mathbb{D}\,\mathcal{M}_{-2,2} = (\mathbb{D}\,\mathcal{M}_{-2,2})\,[\star D]$.
$j_{!*}\mathcal{L}_{-2+6\mathbb{Z},2} = \mathrm{DR}\,(\mathrm{Im}\,(\mathcal{M}_{-2,2} \to \mathcal{M}_{-8,2}))$.
We can compute the above image:

$$j_{!*}\mathcal{L}_{-2+6\mathbb{Z},2} = \mathrm{DR}\left(\mathcal{D}_{\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}^2}(\chi + 2, 9y\partial_x^2 - 4\partial_y^2)\right).$$

$\mathrm{Ch}(j_{!*}\mathcal{L}_{-2+6\mathbb{Z},2}) = T^*_{\mathbb{C}^2}(\mathbb{C}^2) \cup \overline{T^*_{D^{reg}}(\mathbb{C}^2)}$.
It remains to see whether $\mathcal{L}_{\lambda,e}$ is irreducible or not. . .

References

R.-O. Buchweitz and D. Mond.
Linear free divisors and quiver representations.
In *Singularities and computer algebra*, vol. 324 of *London Math. Soc. Lecture Note Ser.*, pages 41–77. Cambridge Univ. Press, 2006.

F. J. Calderón-Moreno.
Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor.
Ann. Sci. École Norm. Sup. (4), 32(5) (1999), 701–714. (math.AG/9807047).

F. J. Calderón-Moreno, D. Q. Mond, L. Narváez-Macarro and F. J. Castro-Jiménez.
Logarithmic Cohomology of the Complement of a Plane Curve.
Comment. Math. Helv., 77(1) (2002), 24–38.

F. J. Calderón-Moreno and L. Narváez-Macarro.
The module $\mathcal{D}f^s$ for locally quasi-homogeneous free divisors.
Compositio Math., 134(1) (2002), 59–74.
(math.AG/0206262).

F. J. Calderón Moreno and L. Narváez Macarro.
Dualité et comparaison sur les complexes de de Rham logarithmiques par rapport aux diviseurs libres.
Ann. Inst. Fourier (Grenoble), 55(1) (2005), 47–75.
(math.AG/0411045).

F. J. Calderón Moreno and L. Narváez Macarro.
On the logarithmic comparison theorem for integrable logarithmic connections.
Proc. London Math. Soc., (3) 98 (2009), 585-606.
(math.AG/0603003).

F. J. Castro-Jiménez, D. Mond, and L. Narváez-Macarro.
Cohomology of the complement of a free divisor.

Trans. Amer. Math. Soc., 348 (1996), 3037–3049.

F. J. Castro-Jiménez and J. M. Ucha-Enríquez.
Free divisors and duality for $\mathcal{D}$-modules.
Proc. Steklov Inst. Math., 238 (2002), 88–96.
(math.AG/0103085).

P. Deligne.
Equations Différentielles à Points Singuliers Réguliers, *Lect. Notes in Math.* 163
Springer-Verlag, Berlin-Heidelberg, 1970.

H. Esnault and E. Viehweg.
Logarithmic De Rham complexes and vanishing theorems.
Invent. Math., 86 (1986), 161–194.

M. Granger, D. Mond, A. Nieto and M. Schulze.
Linear free divisors and the global logarithmic comparison theorem.
Ann. Inst. Fourier (Grenoble), 59(2) (2009), 811–850.

(math.AG/0607045).

R. MacPherson and K. Vilonen.
Elementary construction of perverse sheaves.
Invent. Math., 84 (1986), 403..436.

L. Narváez-Macarro.
Cycles évanescents et faisceaux pervers: cas des courbes planes irréductibles.
Compositio Math., 65 (1988), 321–347.

L. Narváez Macarro.
Linearity conditions on the Jacobian ideal and logarithmic-meromorphic comparison for free divisors.
Singularities I, Algebraic and Analytic Aspects (International Conference in Honor of the 60th Birthday of Lê Dũng Tráng, January 8–26, 2007, Cuernavaca, Mexico), 245–269. Edited by J.P. Brasselet, J.L. Cisneros-Molina, D. Massey, J. Seade and B. Teissier. Contemporary Mathematics, 474, AMS, 2008.
(arXiv.org: arXiv:0804.2219v1).

K. Saito.
Theory of logarithmic differential forms and logarithmic vector fields.
J. Fac. Sci. Univ. Tokyo, 27 (1980), 265–291.

T. Torrelli.
Logarithmic comparison theorem and D-modules: an overview.
In *Proc. of the 2005 Marseille Singularity School and Conference, dedicated to J.-P. Brasselet on his 60^{th} birthday*, pages 995-1009. World Scientific, 2007.
(math.AG/0510430).

J.L. Verdier.
Extension of a perverse sheaf over a closed subset.
Astérisque, 30 (1985), 210–217.