Extension of functors for algebras of formal deformation

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- A. D'Agnolo, S. Guillermou and P. Schapira, *Regular holonomic* $\mathcal{D}[[\hbar]]$*-modules*, Publ. RIMS, Kyoto Univ. (2010),

for a detailed study of these objects.

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3. Construction of the formal extension functors

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$\mathcal{M}$ is said to be $\hbar$-complete if the canonical morphism $\mathcal{M} \to \varprojlim_{n\geq 0} \mathcal{M}_n$ is an isomorphism.

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Recall that for $\mathcal{M} \in \mathrm{Mod}(\mathcal{A})$, one has

$$H^0(gr_{\hbar}^{n}(\mathcal{M})) \simeq \mathcal{M}_n,$$

$$H^{-1}(gr_{\hbar}^{n}(\mathcal{M})) \simeq_n \mathcal{M},$$

$$H^j(gr_{\hbar}^{n}(\mathcal{M})) \simeq 0,\ \forall j \in \mathbb{Z}\backslash\{-1, 0\}.$$

K-S, *Deformation Quantization Modules*

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Let $n \geq 0$. An $\mathcal{A}_n$-module $\mathcal{M}$ is coherent as an $\mathcal{A}_n$-module if and only if it is so as an $\mathcal{A}$-module.

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The functor $gr_{\hbar} : D^b(\mathcal{A}) \to D^b(\mathcal{A}_0)$ is conservative in the category of $c\hbar c$ objects.

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The sequence

$$0 \to \mathcal{M}_{\hbar-tor} \to \mathcal{M} \to \mathcal{M}_{\hbar-tf} \to 0,$$

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Moreover, both $\mathcal{M}_{\hbar-tor}$ and $\mathcal{M}_{\hbar-tf}$ are coherent $\mathcal{A}$-modules.

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Moreover, both $\mathcal{M}_{\hbar-tor}$ and $\mathcal{M}_{\hbar-tf}$ are coherent $\mathcal{A}$-modules.

An $\mathcal{A}$-module of $\hbar$-torsion is coherent as an $\mathcal{A}$-module if and only if, locally, it is coherent as an $\mathcal{A}_n$-module for n big enough.

Review on specialization for $\mathcal{D}$-modules

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The notion of specialization for $\mathcal{D}_X$-modules (along a submanifold) was developed in the pioneering work:

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$V_Y^{\bullet}(\mathcal{D}_X)$ the V-filtration of $\mathcal{D}_X$ with respect to Y:

$$V^k(\mathcal{D}_X) = \left\{P \in \mathcal{D}_X : P(I^j) \subset I^{j+k}, \forall j,k \in \mathbb{Z}\, j, j+k \geq 0\right\}$$

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and for $k \in \mathbb{N}$,

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The graduate ring $gr_V(\mathcal{D}_X)$ is isomorphic to $\pi_* \mathcal{D}_{[T_Y X]}$, where $\mathcal{D}_{[T_Y X]}$ denotes the sheaf of homogeneous differential operators over $T_Y X$.

Definition

Let θ be the Euler field on $T_Y X$. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is specializable along Y if for every good-V filtration $V^{\bullet}(\mathcal{M})$ on $\mathcal{M}$ there is locally a non zero polynomial $b \in \mathbb{C}[s]$ such that

$$b(\theta - k)V^k(\mathcal{M}) \subset V^{k+1}(\mathcal{M}), \quad \forall k \in \mathbb{Z}.$$

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You can have in mind

$$\mathcal{D}_X/\mathcal{D}_X(t\partial_t - \alpha), \ \alpha \in \mathbb{R},$$

where $X = \mathbb{C}^n$ with coordinates $(t, x_1, ..., x_{n-1})$ and $Y = \{t = 0\}$.

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The following definition doesn't depend on the choice of G:

Definition

Let $\mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{D}_X)$ be specializable along a submanifold Y. The specialized of $\mathcal{M}$ along Y is the coherent $\mathcal{D}_{T_Y X}$-module defined by:

$$\nu_Y(\mathcal{M}) = \mathcal{D}_{T_Y X} \otimes_{\mathcal{D}_{[T_Y X]}} \pi^{-1} gr_{V_G}(\mathcal{M}).$$

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$$(\nu_Y(\mathcal{D}_X/\mathcal{D}_X(t\partial_t - \alpha)) = \mathcal{D}_{T_Y X}/\mathcal{D}_{T_Y X}(\tau\partial_\tau - \alpha))$$

Denoting by $\mathrm{Mod}_{sp}(\mathcal{D}_X)$ the full abelian subcategory of $\mathrm{Mod}_{coh}(\mathcal{D}_X)$ of specializable $\mathcal{D}_X$-modules along Y, we obtain an exact functor

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$$\begin{array}{rccc} \nu_Y : & \mathrm{Mod}_{sp}(\mathcal{D}_X) & \to & \mathrm{Mod}_{coh}(\mathcal{D}_{T_Y X}) \\ & \mathcal{M} & \mapsto & \nu_Y(\mathcal{M}). \end{array}$$

Specialization for $\mathcal{D}_X^{\hbar}$-modules

Definition

A coherent $\mathcal{D}_X^{\hbar}$-module $\mathcal{M}$ is specializable along Y if, for each $n \geq 0$, both ${}_n\mathcal{M}$ and $\mathcal{M}_n$ are specializable $\mathcal{D}_X$-modules along Y.

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We note $\mathrm{Mod}_{sp}(\mathcal{D}_X^{\hbar})$ the category of specializable $\mathcal{D}_X^{\hbar}$-modules.

Definition

Given $\mathcal{M} \in \mathrm{Mod}_{sp}(\mathcal{D}_X^{\hbar})$, we define the specialized of $\mathcal{M}$ along Y as:

$$\nu_Y^{\hbar}(\mathcal{M}) := \varprojlim_{n \geq 0} \nu_Y(\mathcal{M}_n).$$

Specialization for $\mathcal{D}_X^\hbar$-modules.

Indeed, we may prove that the specialized defines an exact functor

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Proposition

Let $\mathcal{M}$ be an $\hbar$-torsion $\mathcal{D}_X^{\hbar}$-module. Then $\mathcal{M}$ is specializable as a $\mathcal{D}_X^{\hbar}$-module if and only if $\mathcal{M}$ is specializable in the $\mathcal{D}_X$-modules sense. Moreover, if $\mathcal{M}$ is specializable then $\nu_Y^{\hbar}(\mathcal{M}) \simeq \nu_Y(\mathcal{M})$ in $\mathrm{Mod}_{coh}(\mathcal{D}_{T_Y X})$.

Proposition

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X^{\hbar}$-module. Then the following properties are equivalent:

1. $\mathcal{M}$ is a specializable $\mathcal{D}_X^{\hbar}$-module;
2. $\mathcal{M}_0$ is a specializable $\mathcal{D}_X$-module;
3. $\mathcal{M}_n$ is specializable as a $\mathcal{D}_X$-module, for each $n \geq 0$.

Examples

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Let $X = \mathbb{C}^m$, for some $m \in \mathbb{N}$, with coordinates $(t, x_1, ..., x_{m-1})$, and $Y = \{(t, x_1, ..., x_{m-1}) \in \mathbb{C}^m : t = 0\}$.

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One has

$$\mathcal{M}_n \simeq \frac{\mathcal{D}_X^{\hbar}}{\hbar^{n+1}\mathcal{D}_X^{\hbar} + \mathcal{J}}.$$

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Suppose that $\mathcal{M} = \mathcal{D}_X^{\hbar}/\mathcal{D}_X^{\hbar} b(t\partial_t)$, where $b(s)$ is a polynomial in $\mathbb{C}^{\hbar}[s]$,

$$b(s) = \sum_{i=0}^{m} a_i(\hbar) s^i,$$

for some $m \in \mathbb{N}$ and, for $i \geq 0$, $a_i(\hbar) := \sum_{j \geq 0} a_{ij} \hbar^j \in \mathbb{C}^{\hbar}$.

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Since $\mathcal{M}_0 \simeq \mathcal{D}_X / \mathcal{D}_X b_0(t\partial_t)$, $\mathcal{M}$ is specializable if and only if $b_0(s)$ is a non zero polynomial in $\mathbb{C}[s]$.

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Example

Let $\mathcal{M} = \mathcal{D}_X^{\hbar}/\mathcal{D}_X^{\hbar}(\hbar t\partial_t + 1)$.Clearly $\mathcal{M}_0 = 0$ hence $\mathcal{M}_n = 0$ for every n, which entails $\nu_Y^{\hbar}(\mathcal{M}) = 0$.

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Example

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It is not hard to prove that $\mathcal{M}_n$ can be identified with $\mathcal{D}_X/\mathcal{D}_X(t\partial_t)^{n+1}$.

Therefore, denoting by (x, τ) the associated coordinates in T_YX, we obtain $\nu_Y(\mathcal{M}_n) \simeq \mathcal{D}_{T_YX}/\mathcal{D}_{T_YX}(\tau\partial_\tau)^{n+1}$ and it follows that

$$\nu^{\hbar}_Y(\mathcal{M}) = \varprojlim_n \nu_Y(\mathcal{M}_n) \simeq \frac{\mathcal{D}^{\hbar}_{T_YX}}{\mathcal{D}^{\hbar}_{T_YX}(\tau\partial_\tau - \hbar)}.$$

The (regular) holonomic case.

Recall that following [DGS] an object $\mathcal{M} \in D^b_{coh}(\mathcal{D}^{\hbar}_X)$ is holonomic (resp. regular holonomic) if $gr_{\hbar}(\mathcal{M})$ is an object of $D^b_{hol}(\mathcal{D}_X)$ (resp. $D^b_{rh}(\mathcal{D}_X)$).

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Similarly, we have:

Corollary

Any holonomic $\mathcal{D}^{\hbar}_X$-module $\mathcal{M}$ is specializable along any submanifold Y. Moreover $\nu^{\hbar}_Y(\mathcal{M})$ is a holonomic $\mathcal{D}^{\hbar}_{T_YX}$-module. If $\mathcal{M}$ is regular holonomic, so is $\nu^{\hbar}_Y(\mathcal{M})$.

Comparison Theorems

Recall that Kashiwara constructed for a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ (or, more generally, for an object of $D^b_{rh}(\mathcal{D}_X)$), canonical isomorphisms in $D^b(\mathbb{C}_{T_Y X})$

$$Sol_{\mathcal{D}_{T_Y X}}(\nu_Y(\mathcal{M})) \simeq \nu_Y(Sol_{\mathcal{D}_X}(\mathcal{M})),$$

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Theorem

For $\mathcal{M}$ a regular holonomic $\mathcal{D}^{\hbar}_X$-module, there are canonical isomorphisms in $D^b_{\mathbb{C}-c}(\mathbb{C}^{\hbar}_{T_Y X})$:

1. $DR_{\hbar}(\nu^{\hbar}_Y(\mathcal{M})) \simeq \nu_Y(DR_{\hbar}(\mathcal{M}))$;
2. $Sol_{\hbar}(\nu^{\hbar}_Y(\mathcal{M})) \simeq \nu_Y(Sol_{\hbar}(\mathcal{M}))$.

Tools

Our main tools were:

Lemma

Let $0 \to \mathcal{M}' \xrightarrow{\phi} \mathcal{M} \xrightarrow{\psi} \mathcal{M}'' \to 0$ be an exact sequence in $\mathrm{Mod}(\mathcal{D}_X^{\hbar})$ and suppose that $\mathcal{M}''$ has no $\hbar$-torsion. Then, for each $n \geq 0$, the sequence of $(\mathcal{D}_X^{\hbar})_n$-modules:

$$0 \to \mathcal{M}'_n \xrightarrow{\phi_n} \mathcal{M}_n \xrightarrow{\psi_n} \mathcal{M}''_n \to 0. \tag{3.1}$$

is exact.

Corollary

Let $\mathcal{M}$ be a $\mathcal{D}_X^{\hbar}$-module. Then, for each $n \geq 0$, the following sequence is exact:

$$0 \to \mathcal{M}_{\hbar-tor_n} \to \mathcal{M}_n \to \mathcal{M}_{\hbar-tf_n} \to 0.$$

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- the exactness of $\Gamma(K, \cdot)$ for K belonging to the family of Stein compact subsets of X;
- the exactness of projective limit on the category of projective systems satisfying Mittag-Lefler's condition;
- the subcategory of specializable $\mathcal{D}_X$-modules is a Serre subcategory of $\mathrm{Mod}_{coh}(\mathcal{D}_X)$;
- the family $\{{}_n\mathcal{M}\}_n$ is locally stationary for coherent $\mathcal{D}_X^{\hbar}$-modules, that is, locally there exists $N \geq 1$ such that $\hbar^N \mathcal{M}_{\hbar-tor} = 0$.

Extension Functors

Let $\mathcal{A}$ be a $\mathbb{K}[[\hbar]]$-algebra satisfying either (i), (ii) and (iii) or (i), (ii) and (iv) of the following Assumption:

Assumption 1

Consider the following conditions:

(i) $\mathcal{A}$ has no $\hbar$-torsion and is $\hbar$-complete,

(ii) $\mathcal{A}_0$ is a left Noetherian ring,

(iii) there exists a basis $\mathcal{B}$ of open subsets of X such that for any $U \in \mathcal{B}$ and any coherent $(\mathcal{A}_0|_U)$-module $\mathcal{F}$ we have $H^n(U;\mathcal{F}) = 0$ for any $n > 0$,

Assumption 1

(iv) there exists a basis $\mathcal{B}$ of compact subsets of X and a prestack $U \mapsto \mathrm{Mod}_{good}(\mathcal{A}_0|_U)$ (U open in X) such that

(a) for any $K \in \mathcal{B}$ and an open subset U such that $K \subset U$, there exists $K' \in \mathcal{B}$ such that $K \subset Int(K') \subset K' \subset U$,

(b) $U \mapsto \mathrm{Mod}_{good}(\mathcal{A}_0|_U)$ is a full subprestack of $U \mapsto \mathrm{Mod}_{coh}(\mathcal{A}_0|_U)$,

(c) for an open subset U and $\mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{A}_0|_U)$, if $\mathcal{M}|_V$ belongs to $\mathrm{Mod}_{good}(\mathcal{A}_0|_V)$ for any relatively compact open subset V of U, then $\mathcal{M}$ belongs to $\mathrm{Mod}_{good}(\mathcal{A}_0|_U)$,

(d) for any U open in X, $\mathrm{Mod}_{good}(\mathcal{A}_0|_U)$ is stable by subobjects (and hence, by quotient) in $\mathrm{Mod}_{coh}(\mathcal{A}_0|_U)$,

(e) for any $K \in \mathcal{B}$, any open set U containing K, any $\mathcal{M} \in \mathrm{Mod}_{good}(\mathcal{A}_0|_U)$ and any $j > 0$, one has $H^j(K; \mathcal{M}) = 0$,

(f) for any $\mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{A}_0|_U)$, there exists an open covering $U = \bigcup_i U_i$ such that $\mathcal{M}|_{U_i} \in \mathrm{Mod}_{good}(\mathcal{A}_0|_{U_i})$,

(g) $\mathcal{A}_0 \in \mathrm{Mod}_{good}(\mathcal{A}_0)$.

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Definition

We denote by $\mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$ the full subcategory of $\mathrm{Mod}_{coh}(\mathcal{A})$ consisting of $\mathcal{A}$-modules $\mathcal{M}$ such that, for each $n \geq 0$, the complex $gr_{\hbar}^{n}(\mathcal{M})$ belongs to $D^{b}_{\mathcal{S}}(\mathcal{A})$, that is, both ${}_{n}\mathcal{M}$ and $\mathcal{M}_{n}$ are objects of $\mathcal{S}_{n}$.

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Proposition

Let $\mathcal{M}$ be an $\hbar$-torsion $\mathcal{A}$-module. Then the following properties are equivalent:

1. $\mathcal{M} \in \mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$;
2. Locally there exists an integer $N \in \mathbb{N}$ such that $\mathcal{M}$ is an $\mathcal{A}_{N}$-module and $\mathcal{M} \in \mathcal{S}_{N}$.

Proposition

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Proposition

$\mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$ is a Serre subcategory of $\mathrm{Mod}_{coh}(\mathcal{A})$ and, in particular, it is abelian.

Extension of right exact functors

Let Y denote another complex analytic manifold and $\mathcal{A}'$ a $\mathbb{K}[[\hbar]]$-algebra on Y satisfying Assumption 1.

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Assume moreover that $\mathcal{S}'$ satisfies the following Assumption:

Assumption 2

Each $\mathcal{M} \in \mathcal{S}'|_U$ admits, for each $y \in U$, an open neighborhood V of y, such that for each submodule $\mathcal{N}$ of $\mathcal{M}|_V$, if $K \in \mathcal{B}'$ is contained in V, then $H^j(K; \mathcal{N}) = 0$, for any $j > 0$.

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Recall that this condition is satisfied when the objects of $\mathcal{S}'$ are coherent $\mathcal{A}$-modules.

Let us consider a $\mathbb{K}[[\hbar]]$-linear functor $F : \mathcal{S} \to \mathcal{S}'$.

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Let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$.

Since $\mathrm{Mod}(\mathcal{A}'_n)$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathcal{A}')$ consisting of modules $\mathcal{N}$ satisfying $\hbar^{n+1}\mathcal{N} = 0$, the structure of $\mathcal{A}'$-module on $F_n(\mathcal{M}_n)$ is compatible with the transition morphisms.

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Therefore $(F_n(\mathcal{M}_n), F_n(\rho_{k,n}))$ is a projective system of $\mathcal{A}'$-modules which provides a structure of $\mathcal{A}'$ module on $\varprojlim_{n\geq 0} F_n(\mathcal{M}_n)$.

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2. Given a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$,

$$F^{\hbar}(f) : F^{\hbar}(\mathcal{M}) \to F^{h}(\mathcal{N})$$

is the morphism associated to the morphisms

$$F_n(\mathcal{M}_n) \xrightarrow{F_n(f_n)} F_n(\mathcal{N}_n).$$

Proposition

Let $\mathcal{M}$ be an $\hbar$-torsion $\mathcal{A}$-module in $\mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$. Then, for any integer N such that locally $\hbar^{N+1}\mathcal{M} = 0$, one has $F^{\hbar}(\mathcal{M}) \simeq F_N(\mathcal{M})$ in $\mathrm{Mod}(\mathcal{A}'_N)$.

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Let us denote by $\varrho_n : F^{\hbar}(\mathcal{M}) = \varprojlim_{m\geq 0} F_m(\mathcal{M}_m) \to F_n(\mathcal{M}_n)$ the canonical projection.

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Since $\hbar^{n+1}F_n(\mathcal{M}_n) = 0$, ϱ_n induces a morphism

$$\mathcal{P}_n : F^{\hbar}(\mathcal{M})_n = F^{\hbar}(\mathcal{M})/\hbar^{n+1}F^{\hbar}(\mathcal{M}) \to F_n(\mathcal{M}_n).$$

Proposition

Let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$. Then:

(i) $\forall n \in \mathbb{N}_0, \mathcal{P}_n : F^{\hbar}(\mathcal{M})/\hbar^{n+1}F^{\hbar}(\mathcal{M}) \xrightarrow{\sim} F_n(\mathcal{M}/\hbar^{n+1}\mathcal{M})$ is an isomorphism. In particular, $F^{\hbar}(\mathcal{M})$ is an $\hbar$-complete $\mathcal{A}'$-module.

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Theorem

The functor $F^{\hbar}$ is right exact.

Corollary

For $\mathcal{M} \in \operatorname{Mod}_{\mathcal{S}}(\mathcal{A})$ the sequence below is exact:

$$F^{\hbar}(\mathcal{M}_{\hbar-tor}) \to F^{\hbar}(\mathcal{M}) \to F^{\hbar}(\mathcal{M}_{\hbar-tf}) \to 0.$$

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For $\mathcal{M} \in \mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$ the sequence below is exact:

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Proposition

Let us assume that $\mathcal{S}'$ is a Serre subcategory of $\mathrm{Mod}_{coh}(\mathcal{A}')$.
Then for every $\mathcal{M} \in \mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$, $F^{\hbar}(\mathcal{M})$ belongs to $\mathrm{Mod}_{\mathcal{S}'}(\mathcal{A}')$.

Extension of exact functors

In this case that F is exact we have the following:

Lemma

Given $\mathcal{M} \in \mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$, if $\mathcal{M}$ is $\hbar$-torsion free then so is $F^{\hbar}(\mathcal{M})$.

Extension of exact functors

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Theorem

$F^{\hbar}$ is also an exact functor.

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The sequence of $\mathcal{A}'$-modules

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Lemma

For any $\mathcal{M} \in \mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$, $F^{\hbar}(\mathcal{M})$ is $c\hbar c$.

Corollary

For every $\mathcal{M} \in Mod_{\mathcal{S}}(\mathcal{A})$ and $n \geq 0$ one has

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For every $\mathcal{M} \in \mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$ and $n \geq 0$, we have a family of isomorphisms in $D^b(\mathcal{A}'_n)$:

$$H^j gr^n_{\hbar}(F^{\hbar}(\mathcal{M})) \simeq F_n(H^j gr^n_{\hbar}(\mathcal{M})), \ \forall j.$$

Given a functor $\mathcal{G} : \mathrm{Mod}_{\mathcal{S}}(\mathcal{A}) \to \mathrm{Mod}(\mathcal{A}')$ and a functor $F : \mathcal{S} \to \mathcal{S}'$, we shall say that $\mathcal{G}$ extends F if

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We obtain a canonical morphism for each $\mathcal{M} \in \mathrm{Mod}_{\mathcal{S}}(\mathcal{A})$:

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Therefore, such an extension, if it exists and takes values in the category of $c\hbar c$ objects, is unique up to isomorphism.

Theorem

Let X and Y be complex manifolds, let $\mathcal{A}$ (resp. $\mathcal{A}'$) be a formal deformation algebra on X (resp. on Y) and let $\mathcal{S}$ (resp. $\mathcal{S}'$) be a full Serre subcategory of $\mathrm{Mod}_{coh}(\mathcal{A})$ (resp. a full abelian subcategory of $\mathrm{Mod}(\mathcal{A}')$ satisfying the Assumption 2).

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Moreover, when F is exact, $F^{\hbar}$ is an exact functor $\mathrm{Mod}_{\mathcal{S}}(\mathcal{A}) \to \mathrm{Mod}(\mathcal{A}')$ and is the unique extension of F.