Combinatorial Remarks on Normal Flatness in Analytic Spaces

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Normal flatness was introduced as a simplification of Hironaka's original argument on desingularization.

- Geometry of singularities in characteristic zero: Hironaka (1974); Aroca, Hironaka, and Vicente (1975, 1977); Bennett (1970); all in the seventies.
- Technical study of normal flatness: Herrmann and Orbanz (1982); Orbanz and Robbiano (1984); Robbiano (1979, 1983)
- Effective approach **without normal flatness**: Bierstone and Milman (1997, 1989); Bravo, Encinas, and Villamayor (2005); Encinas and Hauser (2002); Encinas and Villamayor (1998, 2003) (just a few).

We try to show that normal flatness is "easy to compute."

Previous Work

1. Other versions of the Weierstraß-Hironaka division theorem: Aroca, Hironaka, and Vicente (1975); Galligo (1979); Herrmann, Ikeda, and Orbanz (1988).
2. The Fundamental Theorem of normal flatness: Idà and Manaresi (1983); Lejeune-Jalabert and Teissier (1971).
3. Hilbert functions and normal flatness: Bennett (1970); Orbanz and Robbiano (1984).

So, Just what *is* new here?

1. Normal flatness can be read in an easy way from combinatorics attached to a singular analytic space.
2. Using combinatorics to prove the Fundamental Theorem and relationship to the Hilbert function as a by-product.

We need a very specific statement of the Weierstraß-Hironaka theorem.

The setup I

In plain words: we look (locally) at singular embedded analytic spaces containing a smooth subspace.

$R = \mathbb{C}\{\boldsymbol{z}, \boldsymbol{w}\}$

$\boldsymbol{z} = \{z_1, ..., z_c\}, \qquad \boldsymbol{w} = \{w_1, ..., w_d\}$

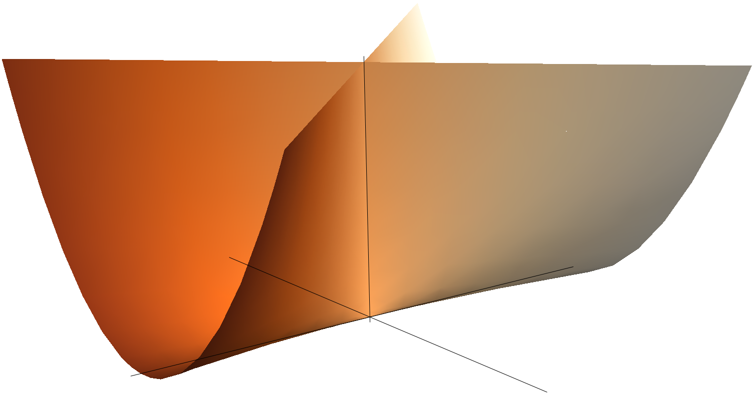
$\mathfrak{p} = (\boldsymbol{z}), \quad \mathfrak{m} = (\boldsymbol{z}, \boldsymbol{w})$

$I \subset (\boldsymbol{z})R = \mathfrak{p}$

W the analytic space defined in a neighbourhood of the origin of $\mathbb{C}^{c+d}$ by $\mathfrak{p}$, and, likewise, X defined by I.

This is not completely uncommon… Think Whitney's Umbrella.

The setup II



Whitney's Umbrella: $z_1^2 = z_2^2 w_1$

The setup III

For a series $f \in R$, we write

$$f = \sum_{A \in \mathbb{Z}_0^c} f_A(\boldsymbol{w})\, \boldsymbol{z}^A, \qquad f_A(\boldsymbol{w}) \in \mathbb{C}\{\boldsymbol{w}\} \text{ for all } A \in \mathbb{Z}_0^c$$

$$= \sum_{(A,B) \in \mathbb{Z}_0^{c+d}} f_{(A,B)}\, \boldsymbol{z}^A\, \boldsymbol{w}^B, \qquad f_{(A,B)} \in \mathbb{C} \text{ for all } (A,B) \in \mathbb{Z}_0^{c+d}$$

and we define the supports

$$\mathscr{E}_{\boldsymbol{z}}(f) = \{A \in \mathbb{Z}_0^c \text{ such that } f_A \neq 0\}$$

$$\mathscr{E}_{\boldsymbol{z},\boldsymbol{w}}(f) = \left\{(A,B) \in \mathbb{Z}_0^{c+d} \text{ such that } f_{(A,B)} \neq 0\right\}$$

and "initial" forms

$$\overline{u}_{\boldsymbol{z}}(f) = \{A \in \mathbb{Z}_0^c \text{ such that } |A| = v_{\mathfrak{p}}(f) \text{ and } f_A(\boldsymbol{0}) \neq 0\}$$

$$\overline{u}_{\boldsymbol{z},\boldsymbol{w}}(f) = \left\{(A,B) \in \mathbb{Z}_0^{c+d} \text{ such that } |(A,B)| = v_{\mathfrak{m}}(f) \text{ and } f_{(A,B)} \neq 0\right\}$$

The setup IV

This gives us

$$u_{\mathbf{z}}(f) = \sup\nolimits_{\mathrm{lex}} \overline{u}_{\mathbf{z}}(f)$$
$$u_{\mathbf{z},\mathbf{w}}(f) = \sup\nolimits_{\mathrm{lex}} \overline{u}_{\mathbf{z},\mathbf{w}}(f)$$

Finally,

$$u_{\mathbf{z}}(I) = \{u_{\mathbf{z}}(f) \mid f \in I,\ f \neq 0\} \subset \mathbb{Z}_0^c$$
$$u_{\mathbf{z},\mathbf{w}}(I) = \{u_{\mathbf{z},\mathbf{w}}(f) \mid f \in I,\ f \neq 0\} \subset \mathbb{Z}_0^{c+d}.$$

Weierstraß-Hironaka Division—Reminder

Theorem (Weierstraß-Hironaka Theorem)

Let $\{f_i\}_{1\leq i\leq k}$ and $\{A_i\}_{i\leq i\leq k}$, $f_i \in \mathbb{C}\{\boldsymbol{z},\boldsymbol{w}\} = R$, $A_i \in \mathbb{Z}_0^c$, be two families such that $S_L\left(\{f_i, A_i\}\right) \neq \emptyset$. Then, for every $g \in R$, there exists an uniquely determined family $\{h_i\}_{0\leq i\leq k}$ of elements $h_i \in R$ such that

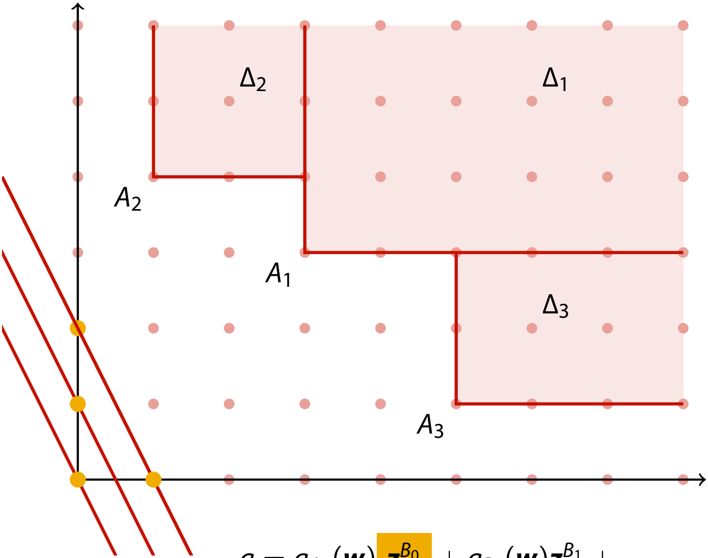
1. $g = h_0 + \sum_{i=1}^{k} h_i f_i.$
2. $\mathscr{E}_{\boldsymbol{z}}(h_i \boldsymbol{z}^{A_i}) \subset \Delta_i,\ 0 \leq i \leq k.$
3. *For all* $\Delta_L \in S_L\left(\{f_i, A_i\}\right)$,

$$v_L(h_i) \geq v_L(g) - v_L(f_i), \quad 1 \leq i \leq k.$$

If $f_1, \dots, f_k, g$ converge in a neighbourhood of $\overline{D}_\rho$, then $h_0, \dots, h_k$ converge also in the same neighbourhood.

See Aroca, Hironaka, and Vicente (1975).

Weierstraß-Hironaka Division–Reminder



$$g = g_{A_0}(\boldsymbol{w})\, \boldsymbol{z}^{B_0} + g_{B_1}(\boldsymbol{w})\boldsymbol{z}^{B_1} + \cdots$$

Lemma (WHD 1)

Suppose that

$$u_{\mathbf{z}}(I) = \bigcup_{i=1}^{r} (A_i + \mathbb{Z}_0^c)$$

and let $\{g_1, \dots, g_r\}$ *be a family such that* $u_{\mathbf{z}}(g_i) = A_i$*. Then,* $\{g_1, \dots, g_r\}$ *is a division basis, i.e., it verifies the hypothesis of WH Division Theorem. Moreover, W-H division*

$$g = h_0 + \sum_{i=1}^{r} h_i g_i$$

verifies

$$v_{\mathbf{z}}(h_i) \geq v_{\mathbf{z}}(g) - |A_i|.$$

Proof.

The trick is choosing a very special linear form for WH division. Set

$$L_c(x_1, ..., x_c) = \sum_{i=1}^{c} \left(1 - \frac{1}{10^{i(m+t)}}\right) x_i$$

$$L_{c,d}(x_1, ..., x_c; y_1, ..., y_d) = L_c(x_1, ..., x_c) + \frac{1}{10^m} \sum_{j=1}^{d} y_j.$$

Then,

$$v_{c,d}(g_i) = L_c(A_i) \quad \text{and} \quad v_{c,d}\left(g_i - \boldsymbol{z}^{A_i}\right) > L_c(A_i).$$

The rest of the proof is induction.
(Hint: the condensed form of the induction is 4 pages long.)

Lemma (WHD 2)

For every $A \in u_{\boldsymbol{z}}(I)$ there exists a series $h_{A,0}$ verifying the following properties:

$$\mathscr{E}_{\boldsymbol{z}}(h_{A,0}) \subset \Delta_0, \quad \boldsymbol{z}^A - h_{A,0} \in I, \quad v_{\boldsymbol{z}}(h_{A,0}) \geq |A|.$$

Moreover, all series $h_{A,0}$ have a common convergence disk, independently of A.

Proof.

Choose $\{g_1, \ldots, g_r\}$ as per WHD 1, and apply division to $\boldsymbol{z}^A$, to get

$$\boldsymbol{z}^A = h_{A,0} + \sum_{i=1}^{r} h_{A,i} g_i.$$

□

Definition

We will call the family $\mathbf{z}^A - h_{A,0}$, with $A \in \mathbb{Z}_0^c$ given by WHD 2 a specially prepared family. The finite subset

$$\{\mathbf{z}^{A_1} - h_{A_1,0}, \dots, \mathbf{z}^{A_r} - h_{A_r,0}, \quad A_i \text{ a vertex of } u(I)\},$$

will be called a specially prepared set.

These series will play a central role in everything that follows.

Normal flatness I

We introduce some notations: the global graded ring w.r.t. $\mathfrak{p}$,

$$\mathrm{gr}_{\mathfrak{p}}(R) = \bigoplus_{i \geq 0} \mathfrak{p}^i/\mathfrak{p}^{i+1} = \mathbb{C}\{\boldsymbol{w}\}[\widetilde{\boldsymbol{z}}], \quad \widetilde{z}_j = z_j + \mathfrak{p}^2$$

and

$$\tilde{f} = \overline{u}_{\boldsymbol{z}}(f) + \mathfrak{p}^2 \in \mathrm{gr}_{\mathfrak{p}}(R).$$

Note that $\widetilde{f}$ can be identified with the homogeneous part of f having degree $v_{\boldsymbol{z}}(f)$ and hence we will also call it the initial form of f w.r.t. $\boldsymbol{z}$, or the $\boldsymbol{z}$–initial form. We will also write, given the ideal $I \subset R$,

$$\mathrm{In}_{\mathfrak{p}}(I) = \left\{\widetilde{f} \mid f \in I\right\} \subset \mathrm{gr}_{\mathfrak{p}}(R),$$

called the initial ideal of I w.r.t. $\mathfrak{p}$.

We will use analogous notations and definitions for:

- the graded ring w.r.t. $\mathfrak{m}$ (noted $\operatorname{gr}_{\mathfrak{m}}(R)$),
- the ordinary initial form of f or the $\{\boldsymbol{z}, \boldsymbol{w}\}$–initial form (noted $\bar{f}$), and
- the initial ideal of I w.r.t. $\mathfrak{m}$ (noted $\operatorname{In}_{\mathfrak{m}}(I)$).

The most interesting graded ring by far, is the local graded ring w.r.t. the situation $I \subset \mathfrak{p}$,

$$\operatorname{gr}_{(\mathfrak{p}/I)}(R/I) = \mathbb{C}\{\boldsymbol{w}\}[\widetilde{\boldsymbol{z}}]/\operatorname{In}_{\mathfrak{p}}(I) = \mathbb{C}\{\boldsymbol{w}\}[\boldsymbol{z}^*],$$

where $z_j^* = \widetilde{z}_j + \operatorname{In}_{\mathfrak{p}}(I)$, for $1 \leq j \leq c$.

Finally, define

$$\Gamma = \left\{(\boldsymbol{z}^*)^A \text{ such that } A \notin u_{\boldsymbol{z}}(I)\right\}.$$

Normal flatness III

Proposition

The set Γ *is a minimal generating system for* $\mathrm{gr}_{(\mathfrak{p}/I)}(R/I)$ *as an* $(R/\mathfrak{p})$*-module.*

Definition

If $f \in R$ is a non zero, non unit, we will say that *f is* $\mathfrak{p}$*-equimultiple* (or simply ***equimultiple***) if $v_{\mathbf{z}}(f) = v_{\mathbf{z},\mathbf{w}}(f)$.

A basis $\{g_1, \ldots, g_s\}$ of I will be called ***equimultiple*** if every g_i is equimultiple. A basis of I will be called ***standard*** if the ordinary inital forms $\{\bar{g}_1, \ldots, \bar{g}_s\}$ generate the initial ideal $\mathrm{In}_{\mathfrak{m}}(I)$.

We say that X ***is normally flat along*** W ***at the origin*** if $\mathrm{gr}_{(\mathfrak{p}/I)}(R/I)$ is a free (equivalently flat) $(R/\mathfrak{p})$–module.

Combinatorial Characterization Theorem

Theorem (Characterisation of Normal Flatness)

With the notations we used throughout previous sections, the following conditions are equivalent:

1. *X is normally flat along W at $\mathbf{0}$.*
2. *I has an equimultiple standard basis.*

Proof.

$1 \Longrightarrow 2$: Take a specially prepared set $\{f_1, \ldots, f_r\}$ given by

$$f_i = \boldsymbol{z}^{A_i} - h_{A_i,0},$$

and, using that Γ is a basis of $\mathrm{gr}_{(\mathfrak{p}/I)}(R/I)$, show that $\{f_1, \ldots, f_r\}$ is actually a basis of I, then check equimultiplicity. □

Combinatorial Characterization Theorem

For the other implication, we need:

Lemma (Min-lex)

Let $\{g_1, \dots, g_s\}$ be an equimultiple standard basis of I. For every non zero $f \in I$, if $A = \min_{\text{lex}}(\mathcal{E}_{\mathbf{z}}(\bar{f}))$, then $A \in u_{\mathbf{z}}(I)$ and there exists $g \in I$ equimultiple such that $u_{\mathbf{z}}(g) = A$.

Then, we can easly show that for every degree n,

$$\Gamma_n = \left\{ (\mathbf{z}^*)^A \text{ such that } A \notin u_{\mathbf{z}}(I), |A| = n \right\}$$

is linearly independent over $\mathbb{C}\{\mathbf{w}\}$.

Sketch of the proof: any non-trivial linear combination

$$\sum_{A \in \Gamma_n} a_A(\mathbf{w})(\mathbf{z}^*)^A = 0$$

“lifts” to some initial form

$$\tilde{f} = \sum_{A \in \Gamma_n} a_A(\boldsymbol{w})\tilde{\boldsymbol{z}}^A, \qquad f \in I.$$

But this is a contradiction with the Min-lex Lemma.

Corollary

Whenever I has an equimultiple standard basis, the specially prepared family is also a standard basis. In other words,

X is normally flat along W at $\mathbf{0}$ $\iff$

the specially prepared set is an equimultiple standard basis.

If normal flatness does not hold, we cannot assert that the specially prepared set $\{f_1, \ldots, f_r\}$ is even a basis of I.

However, there exists a basis $I = (f_1, \ldots, f_r, g_1, \ldots, g_s)$ and an open polydisk $K \subset \mathbb{C}^c \times \mathbb{C}^d$ where f_i, g_j and $\boldsymbol{z}^A - h_{A,0}$ are all convergent, for every $A \in u_{\boldsymbol{z}}(I)$. **Let K' be the projection of K onto $\mathbb{C}^d$.**

For every $\alpha \in K'$ write $I' = (f'_1, \ldots, f'_r, g'_1, \ldots, g'_s)$ where

$$f'_i = f_i(\boldsymbol{z}, \boldsymbol{w}' + \alpha), \quad g'_j = g_j(\boldsymbol{z}, \boldsymbol{w}' + \alpha).$$

I' is an ideal in the ring $R' = \mathbb{C}\{\boldsymbol{z}, \boldsymbol{w}'\}$.

If we set $\mathfrak{p} = (\boldsymbol{z}) \cdot R'$ and denote with tilde the initial forms of elements in

$$\operatorname{gr}_{\mathfrak{p}}(R') = \bigoplus_{n \geq 0} \mathfrak{p}^n / \mathfrak{p}^{n+1},$$

then $\operatorname{In}_{\mathfrak{p}}(I')$ is an ideal of $\operatorname{gr}_{\mathfrak{p}}(R')$. We will denote with stars the classes of elements modulo I'.

Beware

These notations are actually the same as for R and I, but there is no danger of confusion. If $a = \mathbf{0}$ we will of course put $\boldsymbol{w}' = \boldsymbol{w}$.

Fundamental Theorem

Definition

For every integer $n \geq 0$, consider the $\mathbb{C}\{\boldsymbol{w}'\}$–module of finite type

$$M_n = \mathrm{gr}_{(\mathfrak{p}/I')}(R'/I')_n,$$

and the function $T_\alpha \colon \mathbb{Z}_0 \to \mathbb{Z}_0$ given by

$$\begin{aligned} T_\alpha(n) &= \dim_{\mathbb{C}}\big(M_n/(\boldsymbol{w}')\cdot M_n\big) \\ &= \text{minimal number of generators of } M_n/(\boldsymbol{w}')M_n \quad \text{(Nakayama)} \end{aligned}$$

Fundamental Theorem

Theorem (Fundamental Theorem of normal flatness)

Let X be normally flat along W at $\mathbf{0}$. Then,

1. *For every $\alpha \in K'$, X is normally flat along W at α.*
2. *The function T_α is constant in K.*

Conversely, if there exists a polydisk $K'' \subset K'$ such that T_α is constant over K'', then X is normally flat along W at $\mathbf{0}$.

To prove the first part, even without normal flatness, we have

$$\boldsymbol{z}^A - h_{A,0}(\boldsymbol{z}, \boldsymbol{w}' + \alpha) = \sum_{i=1}^{r} h_{A,i}(\boldsymbol{z}, \boldsymbol{w}' + \alpha) f_i(\boldsymbol{z}, \boldsymbol{w}' + \alpha), \qquad \forall A \in u(I),$$

which gives

$$(\boldsymbol{z}^*)^A - \tilde{h}_{A,0}(\boldsymbol{z}^*, \boldsymbol{w}' + \alpha) = 0 \quad \text{in } \operatorname{gr}_{(\mathfrak{p}/I')}(R'/I').$$

Fundamental Theorem

This proves that Γ is a homogeneus generating system of $\mathrm{gr}_{(\mathfrak{p}/I')}(R'/I')$.

To see that it is free, any non trivial linear combination

$$\sum_{A\in\Delta_0,\ |A|=n} \varphi'_A(\boldsymbol{w}')(\boldsymbol{z}^*)^A = 0,$$

gives, by lifting, truncating and change of variables, a non trivial combination

$$\sum_{A\in\Delta_0,\ |A|=n} \varphi_A(\boldsymbol{w}-\alpha)(\boldsymbol{z}^*)^A,$$

This proves 1 and 2.

To prove the converse, it is enough to show again that Γ is free: We know it is a minimal system of generators of $\mathrm{gr}_{(\mathfrak{p}/I)}(R/I)$, so

$$T_\alpha(n) = \mathrm{card}(\Gamma_n).$$

We also know it is a minimal system of generators of $\mathrm{gr}_{(\mathfrak{p}/I')}(R'/I')$. It remains to prove that it is $\mathbb{C}\{\boldsymbol{w}\}$-free.

Fundamental Theorem

Suppose a non trivial relation

$$\sum_{A \in \Delta_0,\ |A|=n} \varphi_A(\boldsymbol{w})(\boldsymbol{z}^*)^A = 0.$$

Γ minimal over $\mathrm{gr}_{(\mathfrak{p}/I)}(R/I)$ means $\varphi_A(\mathbf{0}) = 0$ for $A \in \Delta_0$. Pick an index B in the sum such that $\varphi_B(\boldsymbol{w}) \neq 0$ and α near $\mathbf{0}$ such that $\varphi_B(\alpha) \neq 0$. Then

$$\sum_{A \in \Delta_0,\ |A|=n} \varphi_A(\boldsymbol{w}' + \alpha)(\boldsymbol{z}^*)^A = 0$$

gives $(\boldsymbol{z}^*)^B$ as a linear combination of the other exponents, hence

$$T_\alpha(n) < \mathrm{card}(\Gamma_n).$$

Relationship to Hilbert functions

Definition

Recall that if $\mathcal{O}_X$ is the structure sheaf of X, $\mathcal{O}_{X,\boldsymbol{x}}$ is the stalk at $\boldsymbol{x}$ and $\mathfrak{m}_{X,\boldsymbol{x}}$ is the maximal ideal of $O_{X,\boldsymbol{x}}$, the Hilbert function

$$\mathcal{H}^{(0)}_{X,\boldsymbol{x}} \colon \mathbb{Z}_0 \to \mathbb{Z}_0$$

is defined by

$$\mathcal{H}^{(0)}_{X,\boldsymbol{x}}(n) = \dim_{\mathbb{C}} \left(\mathfrak{m}^n_{X,\boldsymbol{x}} / \mathfrak{m}^{n+1}_{X,\boldsymbol{x}} \right).$$

Also,

$$\mathcal{H}^{(1)}_{X,\boldsymbol{x}}(n) = \sum_{i=0}^{n} \mathcal{H}^{(0)}_{X,\boldsymbol{x}}(i) = \sum_{i=0}^{n} \dim_{\mathbb{C}} \left(\mathfrak{m}^i_{X,\boldsymbol{x}} / \mathfrak{m}^{i+1}_{X,\boldsymbol{x}} \right)$$
$$= \dim_{\mathbb{C}} \left(\mathcal{O}_{X,\boldsymbol{x}} / \mathfrak{m}^{n+1}_{X,\boldsymbol{x}} \right)$$

Relationship to Hilbert functions

Theorem (Normal flatness and Hilbert functions)

Suppose W is locally positive dimensional at $\mathbf{x}$. The following conditions are equivalent:

1. *X is normally flat along W at $\mathbf{x}$.*
2. $u_{\mathbf{z},\mathbf{w}}(I) = u_{\mathbf{z}}(I) \times \mathbb{Z}_0^d$.
3. *$\mathcal{H}^{(0)}_{X,\mathbf{x}}$ is constant over $W \cap \Delta$, where Δ is a polydisk in $\mathbb{C}^{c+d}$ centered at $\mathbf{x}$ such that every series in a certain basis of I are convergent.*
4. *$\mathcal{H}^{(1)}_{X,\mathbf{x}}$ is constant over $W \cap \Delta$, where Δ is a polydisk in $\mathbb{C}^{c+d}$ centered at $\mathbf{x}$ such that every series in a certain basis of I are convergent.*

$3 \iff 4$ goes for free.

$1 \iff$ the specially prepared set is an equimultiple standard basis (Remember Corollary?) $\iff 2$.

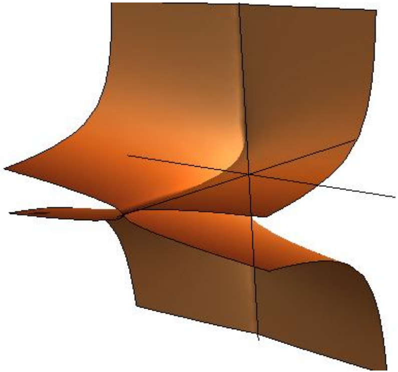
Relationship to Hilbert functions

$2 \implies 3$:

$$\mathscr{H}^{(0)}_{X,\boldsymbol{x}}(n) = \operatorname{card}\left\{C \in \mathbb{Z}_0^{c+d} \text{ such that } |C| = n, C \notin u_{\boldsymbol{z},\boldsymbol{w}}(I)\right\}$$
$$= \sum_{m=0}^{n}\left[\sum_{A \notin u_{\boldsymbol{z}}(I),\ |A|=m} \binom{n-m-d-1}{n-m}\right]$$
$$= \sum_{m=0}^{n} T_{\boldsymbol{x}}(n)\binom{n-|A|-d-1}{n-m}.$$

$3 \implies 1$: More involved, but it is some form of Gaussian elimination.

Consider the projective variety $H : z_1^2 w_1^5 + z_2^3 w_2^4 = 0 \in \mathbb{P}^3$.

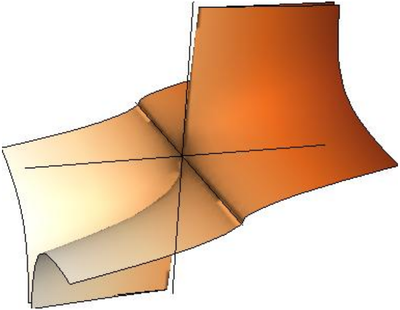


1. In the chart $A_1 : w_1 \neq 0$, the equation of H is

$$f_1 : z_1^2 + z_2^3 w_2^4 = 0,$$

which is normally flat along $W \cap A_1$ at $(0,0,0) = [0 : 0 : 1 : 0] = P_1$.

Cheater's example II



2. In the chart $A_2 : w_2 \neq 0$, the equation of H is

$$f_2 : z_1^2 w_1^5 + z_2^3 = 0$$

which **is not** normally flat along $W \cap A_2$ at $(0, 0, 0) = [0 : 0 : 0 : 1] = P_2$.

The reason for this behaviour is that $\mathrm{Sing}(H) = r_1 \cup r_2 \cup r_3$, with $r_1 \cap r_2 = P_1$, and $r_1 \cap r_3 = P_2$, but there's extra tangency at $P_2 \notin A_1$. (See pictures.)

Example

Consider the germ of analytic function around the origin given by its parametric Puiseux series

$$\begin{cases} z_1 = w^{9/5} z_3^{6/5} \\ z_2 = w^{6/5} z_3^{4/5} \end{cases}$$

or X given by the ideal

$$I = (z_1^5 - w^9 z_3^4, z_2^5 - w^6 z_3^4, z_1^2 - z_2^3).$$

After some computations, we get an equimultiple standard basis (also a specially prepared set)

$$I = (-z_1^2 + z_2^3, w^3 z_3^2 z_1^4 - z_2 z_1^5, -z_1^5 + w^3 z_3^2 z_2^2 z_1^2, w^6 z_3^4 - z_2^2 z_1^2).$$

Hence, X is normally flat along W at the origin.

Combinatorial is not (yet) effective!!!

But there's some useful work: Beck (2009).

Many things about normal flatness, specially as for its behaviour in the resolution process is concerned are still unknown.

Some interesting questions:

- What is a sufficient condition for normal flatness to hold after a blowing up?
- What happens when normal flatness does not hold after a blowing up?
- How do the Newton diagrams and Hilbert functions evolve after blowing ups?

An interesing project: glue computation of series and combinatorics.

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