Sheaves on subanalytic sites and $\mathcal{D}$-modules

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Satisfying the following gluing conditions. Let U be open and let $\{U_j\}_{j\in J}$ be a covering of U. We have the exact sequence

$$0 \to F(U) \to \prod_{j\in J} F(U_j) \to \prod_{j,k\in J} F(U_j \cap U_k)$$

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- if $s \in \Gamma(U; F)$ and $s|_{U_j} = 0$ for each j then $s = 0$
- if $s_j \in \Gamma(U_j; F)$ such that $s_j = s_k$ on $U_j \cap U_k$ then they glue to $s \in \Gamma(U; F)$ (i.e. $s|_{U_j} = s_j$)

Examples

Let us consider

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The correspondence
$U \mapsto \Gamma(U; \mathcal{C}_X) = \{\text{continuous real valued functions on } U\}$
defines a sheaf on X

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Let us consider

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- For example, let $X = \mathbb{R}$, $U_n = (-n, n)$, $n \in \mathbb{N}$, and $s_n \colon U_n \to \mathbb{R}$, $x \mapsto x^2$. Then s_n is bounded on U_n for each $n \in \mathbb{N}$, but $x \mapsto x^2$ is not bounded on $\mathbb{R}$.

Examples

Let us consider

$$\begin{aligned} \mathcal{C}_X^b : \text{Open sets of } X &\to \mathrm{Mod}(\mathbb{R}) \\ U &\mapsto \{\text{continuous bounded functions on } U\} \end{aligned}$$

$\Rightarrow$ The correspondence $U \mapsto \Gamma(U; \mathcal{C}_X^b) =$ $\{$continuous bounded real valued functions on $U\}$ does not define a sheaf on X.

More Examples

Sheaves: holomorphic functions, $\mathcal{C}^\infty$ functions , distributions.

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Not sheaves: L^2 functions, tempered distributions. In fact they do not satisfy gluing conditions.

If we consider “less open subsets” and “less coverings” they may become sheaves. We need the notion of site.

Topological sites

The definition of sheaf depends only on

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Then $F : \mathcal{T} \to \mathrm{Mod}(k)$ is a sheaf on $X_{\mathcal{T}}$ if for each $U \in \mathcal{T}$ and each $\{U_j\}_{j \in J} \in \mathrm{Cov}(U)$ we have the exact sequence

$$0 \to F(U) \to \prod_{j \in J} F(U_j) \to \prod_{j,k \in J} F(U_j \cap U_k)$$

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and consider the correspondence $U \mapsto \Gamma(U; \mathcal{C}_X^b)$ (continuous bounded functions).

- If $\{s_i\}$ are bounded on a finite covering $\{U_i\}$ of U, such that $s_i = s_j$ on $U_i \cap U_j$, then there exists s bounded on U with $s = s_i$ on each U_i.

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defines a sheaf on $X_{\mathcal{T}}$.

The general case

Let X be a topological space and consider a family of open subsets $\mathcal{T}$ satisfying:

$$\begin{cases} \text{(i) } U, V \in \mathcal{T} \Leftrightarrow U \cap V, U \cup V \in \mathcal{T}, \\ \text{(ii) } U \setminus V \text{ has finite numbers of connected components } \forall U, V \in \mathcal{T}, \\ \text{(iii) } \mathcal{T} \text{ is a basis for the topology of } X. \end{cases}$$

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- open subsets: elements of $\mathcal{T}$
- $\mathrm{Cov}(U)$ (coverings of $U \in \mathrm{Op}(X_\mathcal{T})$): finite coverings of U

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2. $\mathcal{T}$={open relatively compact subanalytic subsets of a real analytic manifold}, the subanalytic site X_{sa}.
3. $\mathcal{T}$={open definable subsets of N^n}, given an O-minimal structure $(N, <, \dots)$, the site X_{def}.

Construction of sheaves on $X_{\mathcal{T}}$

Let F be a presheaf on $X_{\mathcal{T}}$. Assume that

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Then F is a sheaf on $X_{\mathcal{T}}$.

Subanalytic sheaves

From now on we will consider the subanalytic site X_{sa}.

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Subanalytic sets

Let $U \subset \mathbb{R}^n$. Let $\mathcal{A}(U)$ denote the real-analytic valued functions. Let $S(\mathcal{A}(U))$ be the smallest set of subsets of U, containing $\{x \in U; f(x) > 0\}$ for all $f \in \mathcal{A}(U)$, and is closed under finite union, finite intersection and complement.

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Definition: A set $S \in \mathbb{R}^n$ is semianalytic if and only if for each $x \in \mathbb{R}^n$, there exists a neighborhood U of x , such that $S \cap U \in S(\mathcal{A}(U))$. It means that S is locally defined by analytic functions.

Subanalytic sets

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Definition: We say $S \in \mathbb{R}^n$ is a subanalytic set if for each $x \in \mathbb{R}^n$, there exists a relatively compact semianalytic set $X \subset \mathbb{R}^{n+m}$ and a neighborhood U of x , such that $S \cap U$ is the projection of X onto the first n coordinates.

Subanalytic sets

The family $\mathrm{Op}(X_{sa})$ of open subanalytic relatively compact satisfy

$$\begin{cases} \text{(i) } U, V \in \mathrm{Op}(X_{sa}) \Leftrightarrow U \cap V, U \cup V \in \mathcal{T}, \\ \text{(ii) } U \setminus V \text{ has finite number of connected comp. } \forall U, V \in \mathrm{Op}(X_{sa}), \\ \text{(iii) } \mathrm{Op}(X_{sa}) \text{ is a basis for the topology of } X. \end{cases}$$

Tempered functions

Let X be a real analytic manifold manifold and let $U \subset X$ be a relatively compact subanalytic open subset, f $\mathcal{C}^\infty$ on U is tempered if $\exists\, M, C > 0$ such that

$$\left|f(x)\right| \leq \frac{C}{\mathrm{dist}(x, \partial U)^M} \; .$$

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This is not a sheaf with the usual topology.
For example, let $X = \mathbb{R}$, then $e^{1/x}$ is tempered (even bounded) on $U_n = \{1/n < x < 1\}_{n \in \mathbb{N}}$ but it is not tempered on $\cup U_n = \{0 < x < 1\}$.

Tempered functions

Anyway one can show that of U, V are open subanalytic the sequence

$$0 \to \mathcal{C}_X^{\infty,t}(U \cup V) \to \mathcal{C}_X^{\infty,t}(U) \oplus \mathcal{C}_X^{\infty,t}(V) \to \mathcal{C}_X^{\infty,t}(U \cap V)$$

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is exact. This implies that $U \mapsto \mathcal{C}_X^{\infty,t}(U)$ is a sheaf on the subanalytic site X_{sa}.

Fibers

In the case of subanalytic sheaves we do not have the notion of fibers in the usual sense, i.e. if we consider

$$F_x = \varinjlim_{U \ni x} F(U)$$

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there are $F \not\simeq G$ even if $F_x \simeq G_x \ \forall x \in X$.

Example: Let $X = \mathbb{R}$ and consider the sheaves $\mathcal{C}_{\mathbb{R}}$ and $\mathcal{C}^b_{\mathbb{R}}$. Then $\mathcal{C}_{\mathbb{R},x} \simeq \mathcal{C}^b_{\mathbb{R},x} \ \forall x \in \mathbb{R}$. Indeed, any continuous function f in $(x-\varepsilon, x+\varepsilon)$, $\varepsilon > 0$ is bounded in $(x-\varepsilon/2, x+\varepsilon/2)$.

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We need to consider more points.

Spectral topology

Let us consider a countable locally finite covering $\{U_n\}_{n\in\mathbb{N}}$ of X, with $U_n \simeq \mathbb{R}^n$ relatively compact and subanalytic.

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A neighborhood of an ultrafilter α is a globally subanalitic open subset U contained in α.
We call $\widetilde{X}$ the associated topological space. In $\widetilde{X}$ any covering of a relatively compact subanalytic open subset has a finite subcover.

Example

For example, the points of $\widetilde{\mathbb{R}}$ are the following. Let $x \in \mathbb{R}$

1. $\{S \text{ subanalytic}, S \supseteq x\}$ (the point x)
2. $\{S \text{ subanalytic}, S \supseteq (x, x+\varepsilon),\ \varepsilon > 0\}$ (the point x^+)
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Thanks to these new points we can distinguish $\mathcal{C}_{\mathbb{R}}$ from $\mathcal{C}^b_{\mathbb{R}}$ on $\widetilde{\mathbb{R}}$. For example let $f = x^{-1}$. Then $f \notin \mathcal{C}^b_{\mathbb{R}}(0, \varepsilon)$ $\forall \varepsilon > 0$.

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Topological and subanalytic sheaves

Theorem:

Let X be a real analytic manifold. The categories $\mathrm{Mod}(k_{X_{sa}})$ and $\mathrm{Mod}(k_{\widetilde{X}})$ are equivalent.

Hence, if we want to work on fibers on X_{sa}, we have to consider the topological space $\widetilde{X}$.

Operations

Theorem:

Let $f : X \to Y$ be a morphism of real analytic manifolds. The six Grothendieck operations $\mathcal{H}om$, $\otimes$, f_*, f^{-1}, $f_{!!}$, $f^!$ are well defined for subanalytic sheaves.

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The ring of differential operators

Let X be a complex analytic manifold. We denote by $\mathcal{D}_X$ the sheaf of rings of differential operators. Locally, a section of $\Gamma(U; \mathcal{D}_X)$ may be written as $P = \sum_{|\alpha| \leq m} a_\alpha(z) \partial_z^\alpha$ with $a_\alpha(z)$ holomorphic on U.
We denote by $\mathrm{Mod}(\mathcal{D}_X)$ the category of $\mathcal{D}_X$-modules.

Subanalytic sheaves and solutions

The subanalytic sheaf $\mathcal{O}_X^t$ of tempered holomorphic functions has a structure of $\rho_!\mathcal{D}_X$-module. ($\Gamma(U; \rho_!\mathcal{D}_X)$ are differential operators $\sum_{|\alpha|\leq m} a_\alpha \partial_z^\alpha$ with a_α holomorphic in $\overline{U}$)

Characteristic variety

Let $T^*X \xrightarrow{\pi} X$ be the cotangent bundle and let $\mathcal{S} \subset \pi^{-1}\mathcal{D}_X$

$$\mathcal{S}_{(x,\xi)} = \{P \in \pi^{-1}(\mathcal{D}_X)\ ;\ \sigma^{-1}(P)(x,\xi) \neq 0\}$$

where $\sigma(P)$ is the principal symbol of P. Set $\mathcal{E}_X = \mathcal{S}^{-1}(\pi^{-1}\mathcal{D}_X)$.

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In $\mathcal{E}_X$ every P with $\sigma(P)(x,\xi) \neq 0$ is (locally) invertible

Characteristic variety

Definition: The characteristic variety $\mathrm{Char}(\mathcal{M})$ of a $\mathcal{D}_X$-module $\mathcal{M}$ is the support of $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$.

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Exemple: If P is a differential operator and $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ (i.e. $\mathcal{M} = \mathrm{coker}(\mathcal{D}_X \xrightarrow{P} \mathcal{D}_X)$), $\mathrm{Char}(\mathcal{M})$ is the zero locus of the principal symbol $\sigma(P)$ of P.
This is because if $\sigma(P) \neq 0$ then P (locally) has an inverse in $\mathcal{E}_X$

Characteristic variety

Let $f : X \to Y$ be a morphism of complex analytic manifolds and let $f_\pi : X \times_Y T^*Y \to T^*Y$ be the base change map.
Definition: f is non characteristic for $\mathcal{M}$ if

$$f_\pi^{-1}(\mathrm{Char}(\mathcal{M})) \cap T_X^*Y \subseteq X \times_Y T_Y^*Y$$

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Example: If $Y = \mathbb{C}^n$, $X = \{z_1 = 0\}$ and $f : X \hookrightarrow Y$, $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X P$ with $P = \sum_{|\alpha| \leq m} a_\alpha(z) \partial_z^\alpha$ then f is non characteristic if the coefficient of $\partial_{z_1}^m$ is $\neq 0$ on X

Complex of solutions

The sheaf of tempered holomorphic functions has a structure of $\rho_! \mathcal{D}_X$-module. ($\Gamma(U; \rho_! \mathcal{D}_X)$ are differential operators $\sum_{|\alpha| \leq m} a_\alpha \partial_z^\alpha$ with a_α holomorphic in $\overline{U}$)

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Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We denote by $\mathcal{S}ol^t(\mathcal{M})$ the complex $R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M},\mathcal{O}^t_X)$

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In the case $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_X P$ and U convex $\mathcal{S}ol^t(\mathcal{M})$ on U is the complex

$$\Gamma(U;\mathcal{O}^t_X)\xrightarrow{P}\Gamma(U;\mathcal{O}^t_X).$$

$$\begin{aligned} H^0(U;\mathcal{S}ol^t(\mathcal{M})) &= \{s\in\Gamma(U;\mathcal{O}^t_X),\ Ps=0\}=\ker P \\ H^1(U;\mathcal{S}ol^t(\mathcal{M})) &= \Gamma(U;\mathcal{O}^t_X)/P\Gamma(U;\mathcal{O}^t_X)=\operatorname{coker}P \end{aligned}$$

Cauchy-Kowaleskaya-Kashiwara

Theorem:

Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module and suppose that f is non characteristic for $\mathcal{M}$. Then $f^{-1}\mathcal{S}ol^t(\mathcal{M}) \simeq \mathcal{S}ol^t(\underline{f}^{-1}\mathcal{M})$.

The case of 1 operator

Let $Y = \mathbb{C}^n$, $X = \{z_1 = 0\}$ and let $f : X \hookrightarrow Y$. Suppose that U is a convex open subanalytic subset of X and $\mathcal{M} = \mathcal{D}_Y/\mathcal{D}_Y P$, with $P = \sum_{|\alpha| \leq m} a_\alpha(z)\partial_z^\alpha$ with $a_{(m,0,\dots,0)}(z) \neq 0$ on X.

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In this case $\underline{f}^{-1}\mathcal{M} \simeq \mathcal{D}_X^m$.

The case of 1 operator

We are reduced to the isomorphism

$$\begin{array}{rcl}\Gamma(U; f^{-1}\mathcal{S}ol^t(\mathcal{M})) & \xrightarrow{\sim} & (\Gamma(U; \mathcal{O}^t_X))^m \\ s & \mapsto & (s|_X, \partial_{z_1} s|_X, \ldots, \partial_{z_1}^{m-1} s|_X)\end{array}$$

where $s \in \Gamma(V; \mathcal{O}^t_Y)$ with $V \in \mathrm{Op}(Y_{sa})$, $V \cap X = U$

The case of 1 operator

i.e. to the existence and uniqueness of the solution of

$$\begin{cases} Ps = 0 \\ \partial^k s|_X = g_k \quad k = 0, \ldots, m-1 \end{cases}$$

for any $(g_k)_{k=0}^{m-1} \in (\Gamma(U; \mathcal{O}_X^t))^m$.

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Normal deformation

X: real n-dimensional analytic manifold
M: closed submanifold of codimension ℓ
$T_M X$: normal bundle.
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$$\text{such that} \begin{cases} p^{-1}(X \setminus M) & \text{isomorphic to } (X \setminus M) \times (\mathbb{R} \setminus \{0\}), \\ t^{-1}(c) & \text{isomorphic to } X \text{ for each } c \neq 0, \\ t^{-1}(0) & \text{isomorphic to } T_M X. \end{cases}$$

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$$\begin{aligned} p : \widetilde{X}_0 \simeq \mathbb{C} \times \mathbb{R} &\to \mathbb{C} \\ (z, r) &\mapsto zr \end{aligned}$$

Specialization

Let $\Omega = \mathbb{C} \times \mathbb{R}^+$
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Let x be a point of $T_0\mathbb{C} \simeq \mathbb{C}$. We have

$$\nu_0 F_x = \varinjlim_{U \in x} F(U)$$

where U is a sector containing x.

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where $\mathcal{A}$ is the sheaf (on the blow-up at 0) of holomorphic functions asymptotically developable at the origin.

Multi-specialization

It is possible to construct a normal deformation in $\mathbb{C}^n$ with respect to $D = \{z_1 \cdots z_\ell = 0\}$, $\ell \leq n$, and a specialization functor ν_D.

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Applying this functor to the sheaf of Whitney holomorphic functions we obtain Majima's sheaf $\mathcal{A}$ of functions strongly asymptotically developable.

Sheaves on subanalytic sites and $\mathcal{D}$-modules

Luca Prelli

Lisboa, 28 september 2011