Sheaves on subanalytic sites and \mathcal{D} -modules

Luca Prelli

Lisboa, 28 september 2011

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2 Sheaves on subanalytic sites





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What is a sheaf?

Let X be a topological space and let k be a field.

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What is a sheaf?

Let X be a topological space and let k be a field. Definition: A sheaf of k-vector spaces is the data of:

 $\begin{array}{rcl} \text{Open sets of } X & \to & \operatorname{Mod}(k) \\ & U & \mapsto & \Gamma(U;F) & (=F(U)) \end{array}$

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Satisfying the following gluing conditions. Let *U* be open and let $\{U_j\}_{j \in J}$ be a covering of *U*. We have the exact sequence

$$0 o F(U) o \prod_{j \in J} F(U_j) o \prod_{j,k \in J} F(U_j \cap U_k)$$

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• if $s \in \Gamma(U; F)$ and $s|_{U_i} = 0$ for each *j* then s = 0

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It means that

- if $s \in \Gamma(U; F)$ and $s|_{U_i} = 0$ for each *j* then s = 0
- if $s_j \in \Gamma(U_j; F)$ such that $s_j = s_k$ on $U_j \cap U_k$ then they glue to $s \in \Gamma(U; F)$ (i.e. $s|_{U_j} = s_j$)

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Let us consider

$$\mathcal{C}_X : \text{Open sets of } X \rightarrow \operatorname{Mod}(\mathbb{R})$$

 $U \mapsto \{\text{continuous real valued functions on } U\}$

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$$\mathcal{C}_X : \text{Open sets of } X \rightarrow \operatorname{Mod}(\mathbb{R})$$

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The correspondence $U \mapsto \Gamma(U; C_X) = \{\text{continuous real valued functions on } U\}$ defines a sheaf on *X*

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Let us consider

$$\mathcal{C}^b_X : \text{Open sets of } X \to \operatorname{Mod}(\mathbb{R})$$

 $U \mapsto \{\text{continuous bounded functions on } U\}$

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Let us consider

 \mathcal{C}^b_X : Open sets of $X \to \operatorname{Mod}(\mathbb{R})$ $U \mapsto \{ \text{continuous bounded functions on } U \}$

• For example, let $X = \mathbb{R}$, $U_n = (-n, n)$, $n \in \mathbb{N}$, and $s_n : U_n \to \mathbb{R}$, $x \mapsto x^2$. Then s_n is bounded on U_n for each $n \in \mathbb{N}$, but $x \mapsto x^2$ is not bounded on \mathbb{R} .

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Let us consider

$$\mathcal{C}^b_X : \text{Open sets of } X \to \operatorname{Mod}(\mathbb{R})$$

 $U \mapsto \{\text{continuous bounded functions on } U\}$

⇒ The correspondence $U \mapsto \Gamma(U; C_X^b) =$ {continuous bounded real valued functions on U} does not define a sheaf on X.

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Sheaves: holomorphic functions, \mathcal{C}^{∞} functions , distributions.

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Sheaves: holomorphic functions, C^{∞} functions , distributions. Not sheaves: L^2 functions, tempered distributions.

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Sheaves: holomorphic functions, C^{∞} functions, distributions. Not sheaves: L^2 functions, tempered distributions. In fact they do not satisfy gluing conditions.

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Sheaves: holomorphic functions, C^{∞} functions, distributions. Not sheaves: L^2 functions, tempered distributions. In fact they do not satisfy gluing conditions.

If we consider "less open subsets" and "less coverings" they may become sheaves.

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Sheaves: holomorphic functions, C^{∞} functions , distributions. Not sheaves: L^2 functions, tempered distributions. In fact they do not satisfy gluing conditions.

If we consider "less open subsets" and "less coverings" they may become sheaves. We need the notion of site.

Topological sites

The definition of sheaf depends only on

- open subsets
- coverings



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Topological sites

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One can generalize this notion by choosing a subfamily of open subsets \mathcal{T} of X and for each U a subfamily Cov(U) of coverings if U satisfying suitable hypothesis (defining a site $X_{\mathcal{T}}$).

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Topological sites

One can generalize this notion by choosing a subfamily of open subsets \mathcal{T} of X and for each U a subfamily Cov(U) of coverings if U satisfying suitable hypothesis (defining a site $X_{\mathcal{T}}$).

Then $F : \mathcal{T} \to Mod(k)$ is a sheaf on $X_{\mathcal{T}}$ if for each $U \in \mathcal{T}$ and each $\{U_j\}_{j \in J} \in Cov(U)$ we have the exact sequence

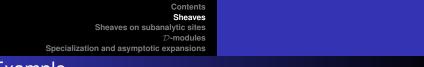
$$0 o F(U) o \prod_{j \in J} F(U_j) o \prod_{j,k \in J} F(U_j \cap U_k)$$



For example, let us consider the site X_T where

- *T*=open subsets of *X*
- Cov(*U*)={finite coverings of *U*}

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For example, let us consider the site X_T where

- *T*=open subsets of *X*
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and consider the correspondence $U \mapsto \Gamma(U; \mathcal{C}_{\chi}^{b})$ (continuous bounded functions).

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If {*s_i*} are bounded on a finite covering {*U_i*} of *U*, such that *s_i* = *s_j* on *U_i* ∩ *U_j*, then there exists *s* bounded on *U* with *s* = *s_i* on each *U_i*.

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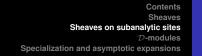


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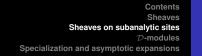
defines a sheaf on $X_{\mathcal{T}}$.



The general case

Let *X* be a topological space and consider a family of open subsets \mathcal{T} satisfying:

 $\begin{cases} \text{(i) } U, V \in \mathcal{T} \Leftrightarrow U \cap V, U \cup V \in \mathcal{T}, \\ \text{(ii) } U \setminus V \text{ has finite numbers of connected components } \forall U, V \in \mathcal{T}, \\ \text{(iii) } \mathcal{T} \text{ is a basis for the topology of } X. \end{cases}$



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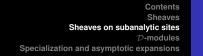
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Definition: The site X_T is defined by:

- open subsets: elements of ${\mathcal T}$
- Cov(U) (coverings of $U \in Op(X_T)$): finite coverings of U

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- \mathcal{T} ={open semialgebraic subsets of \mathbb{R}^n }
- **2** T ={open relatively compact subanalytic subsets of a real analytic manifold}, the subanalytic site X_{sa} .

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- 2 T={open relatively compact subanalytic subsets of a real analytic manifold}, the subanalytic site X_{sa} .
- T={open definable subsets of Nⁿ}, given an O-minimal structure (N, <, ...), the site X_{def}.

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Construction of sheaves on X_T

Let *F* be a presheaf on X_T . Assume that

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Construction of sheaves on X_T

Let *F* be a presheaf on X_T . Assume that

• $\forall U, V \in \mathcal{T}$ the sequence

 $0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$

is exact.

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Then *F* is a sheaf on X_T .

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From now on we will consider the subanalytic site X_{sa} .

open subsets: relatively compact subanalytic open subsets

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- open subsets: relatively compact subanalytic open subsets
- Cov(U) (coverings of $U \in Op(X_{sa})$): finite coverings of U

Subanalytic sets

Let $U \subset \mathbb{R}^n$. Let $\mathcal{A}(U)$ denote the real-analytic valued functions. Let $S(\mathcal{A}(U))$ be the smallest set of subsets of U, containing $\{x \in U; f(x) > 0\}$ for all $f \in \mathcal{A}(U)$, and is closed under finite union, finite intersection and complement.

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Definition: A set $S \in \mathbb{R}^n$ is semianalytic if and only if for each $x \in \mathbb{R}^n$, there exists a neighborhood U of x, such that $S \cap U \in S(\mathcal{A}(U))$. It means that S is locally defined by analytic functions.



Unlike for semialgebraic sets, projections of semianalytic sets are in general not semianalytic.

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Unlike for semialgebraic sets, projections of semianalytic sets are in general not semianalytic.

Definition: We say $S \in \mathbb{R}^n$ is a subanalytic set if for each $x \in \mathbb{R}^n$, there exists a relatively compact semianalytic set $X \subset \mathbb{R}^{n+m}$ and a neighborhood U of x, such that $S \cap U$ is the projection of X onto the first n coordinates.

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Subanalytic sets

The family $Op(X_{sa})$ of open subanalytic relatively compact satisfy

$$\begin{cases} \text{(i) } U, V \in \operatorname{Op}(X_{sa}) \Leftrightarrow U \cap V, U \cup V \in \mathcal{T}, \\ \text{(ii) } U \setminus V \text{ has finite number of connected comp. } \forall U, V \in \operatorname{Op}(X_{sa}), \\ \text{(iii) } \operatorname{Op}(X_{sa}) \text{ is a basis for the topology of } X. \end{cases}$$

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Tempered functions

Let *X* be a real analytic manifold manifold and let $U \subset X$ be a relatively compact subanalytic open subset, $f C^{\infty}$ on *U* is tempered if $\exists M, C > 0$ such that

$$|f(x)| \leq \frac{C}{\operatorname{dist}(x,\partial U)^M}$$

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$$|f(x)| \leq \frac{C}{\operatorname{dist}(x,\partial U)^M}$$

This is not a sheaf with the usual topology. For example, let $X = \mathbb{R}$, then $e^{1/x}$ is tempered (even bounded) on $U_n = \{1/n < x < 1\}_{n \in \mathbb{N}}$ but it is not tempered on $\cup U_n = \{0 < x < 1\}$.

Tempered functions

Anyway one can show that of U, V are open subanalytic the sequence

$$0 \to \mathcal{C}^{\infty,t}_X(U \cup V) \to \mathcal{C}^{\infty,t}_X(U) \oplus \mathcal{C}^{\infty,t}_X(V) \to \mathcal{C}^{\infty,t}_X(U \cap V)$$

is exact.

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is exact. This implies that $U \mapsto \mathcal{C}_X^{\infty,t}(U)$ is a sheaf on the subanalytic site X_{sa} .

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In the case of subanalytic sheaves we do not have the notion of fibers in the usual sense, i.e. if we consider

$$F_x = \varinjlim_{U \ni x} F(U)$$

there are $F \not\simeq G$ even if $F_x \simeq G_x \ \forall x \in X$.

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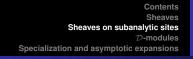
Example: Let $X = \mathbb{R}$ and consider the sheaves $\mathcal{C}_{\mathbb{R}}$ and $\mathcal{C}_{\mathbb{R}}^{b}$. Then $\mathcal{C}_{\mathbb{R},x} \simeq \mathcal{C}_{\mathbb{R},x}^{b} \, \forall x \in \mathbb{R}$. Indeed, any continuous function *f* in $(x - \varepsilon, x + \varepsilon), \varepsilon > 0$ is bounded in $(x - \varepsilon/2, x + \varepsilon/2)$.

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Hence if we consider only the fibers associated to the points of x we loose informations about $F \in Mod(k_{X_{sa}})$.

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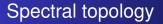




Hence if we consider only the fibers associated to the points of x we loose informations about $F \in Mod(k_{X_{sa}})$.

We need to consider more points.

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Let us consider a countable locally finite covering $\{U_n\}_{n\in\mathbb{N}}$ of X, with $U_n \simeq \mathbb{R}^n$ relatively compact and subanalytic.

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Spectral topology

Let us consider a countable locally finite covering $\{U_n\}_{n\in\mathbb{N}}$ of X, with $U_n \simeq \mathbb{R}^n$ relatively compact and subanalytic. In U_n consider the ultrafilters of globally subanalytic subsets (i.e. subanalytic in X).

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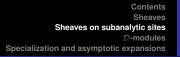
Spectral topology

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In U_n consider the ultrafilters of globally subanalytic subsets (i.e. subanalytic in X).

A neighborhood of an ultrafilter α is a globally subanalitic open subset *U* contained in α .

We call \widetilde{X} the associated topological space. In \widetilde{X} any covering of a relatively compact subanalytic open subset has a finite subcover.

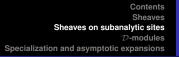




For example, the points of $\widetilde{\mathbb{R}}$ are the following. Let $x \in \mathbb{R}$

- {*S* subanalytic, $S \supseteq x$ } (the point *x*)
- **2** {*S* subanalytic, $S \supseteq (x, x + \varepsilon), \varepsilon > 0$ } (the point x^+)
- **③** {*S* subanalytic, *S* ⊇ (*x* − ε , *x*), ε > 0} (the point *x*[−])

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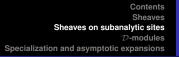


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- **2** {*S* subanalytic, $S \supseteq (x, x + \varepsilon), \varepsilon > 0$ } (the point x^+)
- **3** {*S* subanalytic, $S \supseteq (x \varepsilon, x)$, $\varepsilon > 0$ } (the point x^{-})

Thanks to these new points we can distinguish $C_{\mathbb{R}}$ from $C_{\mathbb{R}}^{b}$ on $\widetilde{\mathbb{R}}$. For example let $f = x^{-1}$. Then $f \notin C_{\mathbb{R}}^{b}(0, \varepsilon) \forall \varepsilon > 0$.

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Thanks to these new points we can distinguish $C_{\mathbb{R}}$ from $C_{\mathbb{R}}^{b}$ on $\widetilde{\mathbb{R}}$. For example let $f = x^{-1}$. Then $f \notin C_{\mathbb{R}}^{b}(0,\varepsilon) \forall \varepsilon > 0$. Hence $f \notin C_{\mathbb{R},0^{+}}^{b}$, but $f \in C_{\mathbb{R},0^{+}}$, this implies $C_{\mathbb{R},0^{+}}^{b} \not\simeq C_{\mathbb{R},0^{+}}$.

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Topological and subanalytic sheaves

Theorem:

Let *X* be a real analytic manifold. The categories $Mod(k_{X_{sa}})$ and $Mod(k_{\tilde{X}})$ are equivalent.

Hence, if we want to work on fibers on X_{sa} , we have to consider the topological space \widetilde{X} .



Theorem:

Let $f : X \to Y$ be a morphism of real analytic manifolds. The six Grothendieck operations $\mathcal{H}om$, \otimes , f_* , f^{-1} , $f_{!!}$, $f^!$ are well defined for subanalytic sheaves.

L. PRELLI Sheaves on subanalytic sites, Rendiconti del Seminario Matematico dell'Università di Padova Vol. 120 (2008).

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The ring of differential operators

Let *X* be a complex analytic manifold. We denote by \mathcal{D}_X the sheaf of rings of differential operators. Locally, a section of $\Gamma(U; \mathcal{D}_X)$ may be written as $P = \sum_{|\alpha| \le m} a_{\alpha}(z) \partial_z^{\alpha}$ with $a_{\alpha}(z)$ holomorphic on *U*.

We denote by $Mod(\mathcal{D}_X)$ the category of \mathcal{D}_X -modules.

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Subanalytic sheaves and solutions

The subanalytic sheaf \mathcal{O}_X^t of tempered holomorphic functions has a structure of $\rho_! \mathcal{D}_X$ -module. ($\Gamma(U; \rho_! \mathcal{D}_X)$ are differential operators $\sum_{|\alpha| < m} a_{\alpha} \partial_z^{\alpha}$ with a_{α} holomorphic in \overline{U})

Characteristic variety

Let $T^*X \xrightarrow{\pi} X$ be the cotangent bundle and let $\mathcal{S} \subset \pi^{-1}\mathcal{D}_X$

$$S_{(x,\xi)} = \{ P \in \pi^{-1}(\mathcal{D}_x) ; \sigma^{-1}(P)(x,\xi) \neq 0 \}$$

where $\sigma(P)$ is the principal symbol of *P*. Set $\mathcal{E}_X = \mathcal{S}^{-1}(\pi^{-1}\mathcal{D}_X)$.

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where $\sigma(P)$ is the principal symbol of *P*. Set $\mathcal{E}_X = \mathcal{S}^{-1}(\pi^{-1}\mathcal{D}_X)$.

In \mathcal{E}_X every *P* with $\sigma(P)(x,\xi) \neq 0$ is (locally) invertible

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Characteristic variety

Definition: The characteristic variety $\operatorname{Char}(\mathcal{M})$ of a \mathcal{D}_X -module \mathcal{M} is the support of $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$.

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Exemple: If *P* is a differential operator and $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X P$ (i.e. $\mathcal{M} = \operatorname{coker}(\mathcal{D}_X \xrightarrow{P} \mathcal{D}_X)$), $\operatorname{Char}(\mathcal{M})$ is the zero locus of the principal symbol $\sigma(P)$ of *P*. This is because if $\sigma(P) \neq 0$ then *P* (locally) has an inverse in \mathcal{E}_X

Characteristic variety

Let $f : X \to Y$ be a morphism of complex analytic manifolds and let $f_{\pi} : X \times_Y T^*Y \to T^*Y$ be the base change map. Definition: *f* is non characteristic for \mathcal{M} if

$$f_{\pi}^{-1}(\operatorname{Char}(\mathcal{M}))\cap \mathcal{T}_X^*Y\subseteq X imes_Y\mathcal{T}_Y^*Y$$

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Characteristic variety

Let $f : X \to Y$ be a morphism of complex analytic manifolds and let $f_{\pi} : X \times_Y T^*Y \to T^*Y$ be the base change map. Definition: *f* is non characteristic for \mathcal{M} if

$$f_{\pi}^{-1}(\operatorname{Char}(\mathcal{M}))\cap T_X^*Y\subseteq X imes_Y T_Y^*Y$$

Example: If $Y = \mathbb{C}^n$, $X = \{z_1 = 0\}$ and $f : X \hookrightarrow Y$, $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X P$ with $P = \sum_{|\alpha| \le m} a_{\alpha}(z) \partial_z^{\alpha}$ then *f* is non characteristic if the coefficient of ∂_z^m is $\neq 0$ on *X*

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Complex of solutions

The sheaf of tempered holomorphic functions has a structure of $\rho_! \mathcal{D}_X$ -module. ($\Gamma(U; \rho_! \mathcal{D}_X)$ are differential operators $\sum_{|\alpha| \leq m} a_\alpha \partial_z^\alpha$ with a_α holomorphic in \overline{U})

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Complex of solutions

The sheaf of tempered holomorphic functions has a structure of $\rho_! \mathcal{D}_X$ -module. ($\Gamma(U; \rho_! \mathcal{D}_X)$ are differential operators $\sum_{|\alpha| \le m} a_\alpha \partial_z^\alpha$ with a_α holomorphic in \overline{U})

Let \mathcal{M} be a coherent \mathcal{D}_X -module. We denote by $Sol^t(\mathcal{M})$ the complex $R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M},\mathcal{O}_X^t)$

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Complex of solutions

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Let \mathcal{M} be a coherent \mathcal{D}_X -module. We denote by $Sol^t(\mathcal{M})$ the complex $R\mathcal{H}om_{\rho_1\mathcal{D}_X}(\rho_!\mathcal{M},\mathcal{O}_X^t)$

In the case $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X P$ and U convex $Sol^t(\mathcal{M})$ on U is the complex

$$\Gamma(U; \mathcal{O}_X^t) \xrightarrow{P} \Gamma(U; \mathcal{O}_X^t).$$

 $H^{0}(U; Sol^{t}(\mathcal{M})) = \{s \in \Gamma(U; \mathcal{O}_{X}^{t}), Ps = 0\} = \ker P$ $H^{1}(U; Sol^{t}(\mathcal{M})) = \Gamma(U; \mathcal{O}_{X}^{t}) / P\Gamma(U; \mathcal{O}_{X}^{t}) = \operatorname{coker} P$

Cauchy-Kowaleskaya-Kashiwara

Theorem:

Let \mathcal{M} be a coherent \mathcal{D}_{Y} -module and suppose that f is non characteristic for \mathcal{M} . Then $f^{-1}Sol^{t}(\mathcal{M}) \simeq Sol^{t}(\underline{f}^{-1}\mathcal{M})$.

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The case of 1 operator

Let $Y = \mathbb{C}^n$, $X = \{z_1 = 0\}$ and let $f : X \hookrightarrow Y$. Suppose that U is a convex open subanalytic subset of X and $\mathcal{M} = \mathcal{D}_Y / \mathcal{D}_Y P$, with $P = \sum_{|\alpha| \le m} a_{\alpha}(z) \partial_z^{\alpha}$ with $a_{(m,0,...,0)}(z) \ne 0$ on X.

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In this case $\underline{f}^{-1}\mathcal{M} \simeq \mathcal{D}_X^m$.

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Contents Sheaves Sheaves Sheaves on subanalytic sites $\mathcal{D} ext{-modules}$ Specialization and asymptotic expansions

The case of 1 operator

We are reduced to the isomorphism

$$\begin{array}{rcl} \Gamma(U; f^{-1}\mathcal{S}ol^t(\mathcal{M})) & \stackrel{\sim}{\to} & (\Gamma(U; \mathcal{O}_X^t))^m \\ & \boldsymbol{s} & \mapsto & (\boldsymbol{s}|_X, \partial_{z_1}\boldsymbol{s}|_X, \dots, \partial_{z_1}^{m-1}\boldsymbol{s}|_X) \end{array}$$

where $s \in \Gamma(V; \mathcal{O}_Y^t)$ with $V \in \operatorname{Op}(Y_{sa}), V \cap X = U$

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The case of 1 operator

i.e. to the existence and uniqueness of the solution of

$$\left\{egin{array}{l} Pm{s}=0\ \partial^km{s}|_X=m{g}_k \ \ k=0,\ldots,m-1 \end{array}
ight.$$

for any $(g_k)_{k=0}^{m-1} \in (\Gamma(U; \mathcal{O}_X^t))^m$.

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The case of 1 operator

Moreover we have $H^1(U; f^{-1}Sol^t(\mathcal{D}_Y/\mathcal{D}_Y P)) = 0$.

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The case of 1 operator

Moreover we have $H^1(U; f^{-1}Sol^t(\mathcal{D}_Y/\mathcal{D}_Y P)) = 0$. For a neighborhood *V* of *U* in *Y* and $s \in \Gamma(V; \mathcal{O}_Y^t)$ there exist $U \subset V' \subseteq V$ and $s' \in \Gamma(V'; \mathcal{O}_Y^t)$ such that $Ps' = s|_{V'}$.

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Normal deformation

X: real *n*-dimensional analytic manifold M: closed submanifold of codimension ℓ $T_M X$: normal bundle. Normal deformation of X:

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such that $\begin{cases} p^{-1}(X \setminus M) & \text{isomorphic to } (X \setminus M) \times (\mathbb{R} \setminus \{0\}), \\ t^{-1}(c) & \text{isomorphic to } X \text{ for each } c \neq 0, \\ t^{-1}(0) & \text{isomorphic to } T_M X. \end{cases}$

 $\begin{array}{c} \text{Contents} \\ \text{Sheaves} \\ \text{Sheaves on subanalytic sites} \\ \mathcal{D}\text{-modules} \\ \end{array}$



$$X = \mathbb{C}$$
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$$T_0 \mathbb{C} \simeq \mathbb{C}$$

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 $\begin{array}{c} \text{Contents} \\ \text{Sheaves} \\ \text{Sheaves on subanalytic sites} \\ \mathcal{D}\text{-modules} \\ \end{array}$



$$X = \mathbb{C}$$
$$M = \{0\}$$
$$T_0 \mathbb{C} \simeq \mathbb{C}$$
$$\widetilde{X}_0 \simeq \mathbb{C} \times \mathbb{R}$$
$$p : \widetilde{X}_0 \simeq \mathbb{C} \times \mathbb{R} \rightarrow (z, r) \mapsto$$

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Specialization

Let $\Omega = \mathbb{C} \times \mathbb{R}^+$ and $s : T_0 \mathbb{C} \simeq \mathbb{C} \hookrightarrow \mathbb{C} \times \mathbb{R}$.



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$$\Omega = \mathbb{C} \times \mathbb{R}^+$$

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$$\nu_0 \boldsymbol{F} = \rho^{-1} \boldsymbol{s}^{-1} \boldsymbol{\Gamma}_{\Omega} \boldsymbol{p}^{-1} \boldsymbol{F}.$$

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Let *x* be a point of $T_0\mathbb{C}\simeq\mathbb{C}$. We have

$$\nu_0 F_x = \varinjlim_{U \in x} F(U)$$

where U is a sector containing x.

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Let $\mathcal{O}_{\mathbb{C}}^t$ be the subanalytic sheaf of tempered holomorphic functions. In that case we have

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Let $\mathcal{O}_{\mathbb{C}}^t$ be the subanalytic sheaf of tempered holomorphic functions. In that case we have

$$u_0 \mathcal{O}_{\mathbb{C}}^t \simeq \mathcal{A}^{\leq 0}$$

where $\mathcal{A}^{\leq 0}$ is the sheaf (on the blow-up at 0) of holomorphic functions with moderate growth at the origin.



Let $\mathcal{O}^w_{\mathbb{C}}$ be the subanalytic sheaf of Whitney holomorphic functions. In that case we have

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Let $\mathcal{O}^w_{\mathbb{C}}$ be the subanalytic sheaf of Whitney holomorphic functions. In that case we have

$$\nu_0 \mathcal{O}^w_\mathbb{C} \simeq \mathcal{A}$$

where A is the sheaf (on the blow-up at 0) of holomorphic functions asymptotically developable at the origin.

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Multi-specialization

It is possible to construct a normal deformation in \mathbb{C}^n with respect to $D = \{z_1 \cdots z_\ell = 0\}, \ell \leq n$, and a specialization functor ν_D .

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This is done by recursively perform the normal deformation construction with respect to each divisor $D_i = \{z_i = 0\}, 1 \le i \le \ell$.

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This is done by recursively perform the normal deformation construction with respect to each divisor $D_i = \{z_i = 0\}, 1 \le i \le \ell$.

Applying this functor to the sheaf of Whitney holomorphic functions we obtain Majima's sheaf A of functions strongly asymptotically developable.

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Sheaves on subanalytic sites and \mathcal{D} -modules

Luca Prelli

Lisboa, 28 september 2011

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