Monodromy Zeta Function Formula for Embedded **Q**-Resolutions

Jorge Martín-Morales

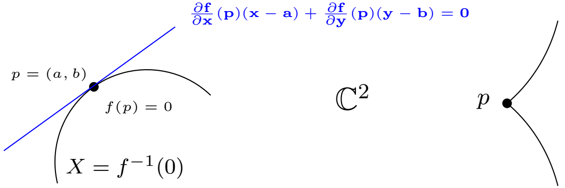
Centro Universitario de la Defensa Zaragoza

Iberian Meeting on Algebraic Analysis and Geometry
Universidade de Lisboa, Portugal, 28–30 September 2011

Introduction and Motivation

Invariants in Singularity Theory

- Let $f \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_0, \ldots, x_n]$ be a polynomial.
- $p \in \mathbb{C}^{n+1}$ is said to be *singular* if $p \in V(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})$.



- In order to try to study the singular points $\rightsquigarrow$ invariants.
- Two hypersurfaces $X = V(f)$ e $Y = V(g)$ are said to be *topologically equivalent* (at $p \in \mathbb{C}^{n+1}$) if there exists U an open neighborhood of p such that as pairs $(U, X \cap U) \cong (U, Y \cap U)$.

Our interest

1. Monodromy of Picard-Lefschetz $\varphi : H^q(F,\mathbb{C}) \to H^q(F,\mathbb{C})$.
2. The zeta function $Z(f;t)$ provides partial information.
3. Jordan blocks of the monodromy (with the semistable reduction).

$$\Downarrow$$

Milnor fibration

Ehresmann's Fibration Theorem

Theorem ([Ehresmann])

Let $f : M \to N$ be a smooth mapping between two smooth manifolds satisfying:

1. f is a surjective submersion.
2. f is a proper map, (in particular if M is compact).

Then f is a locally trivial fibration.

Milnor Fibration

- Let $(H,0) \subset (\mathbb{C}^{n+1},0)$ be a germ of hypersurface singularity defined by a holomorphic function $f : U \to \mathbb{C}$. This means that

$$H = \{\mathbf{x} \in U \mid f(\mathbf{x}) = 0\}.$$

- By Ehresmann's Fibration Theorem, the restriction $(0 < \eta << \epsilon)$

$$f| : f^{-1}(\overline{D_\eta} \setminus \{0\}) \cap \overline{B_\epsilon^{2n+2}} \longrightarrow \overline{D_\eta} \setminus \{0\}$$

is a locally trivial fibration for all ϵ small enough.

Definition

The previous fibration is called the *Milnor fibration* of f and any of its fibers $F := \{\mathbf{x} \in \mathbb{C}^{n+1} : ||\mathbf{x}|| \leq \epsilon, f(\mathbf{x}) = \eta\}$ is called the *Milnor fiber*.

Monodromy Zeta Function

- Let $h : F \to F$, $\widetilde{\alpha}(0) \mapsto \widetilde{\alpha}(1)$ be the geometric monodromy. It is just well defined up to homotopy.

(FIGURE $\to$ blackboard)

- Denoted by $\varphi := H^q(h) : H^q(F, \mathbb{C}) \to H^q(F, \mathbb{C})$ the induced automorphisms on the complex cohomology groups.

Definition (*Lefschetz Numbers* & *Monodromy Zeta Function*)

$$\Lambda(h^k) := \sum_{q \geq 0} (-1)^q \operatorname{tr} H^q(h^k) \in \mathbb{Q}, \quad (h^k := h \circ \cdots \circ h)$$

$$Z(f; t) := \prod_{q \geq 0} \underbrace{\det(\mathrm{id}^* - t \cdot H^q(h))}_{\text{char. poly. of } H^q(h)}{}^{(-1)^q} \in Q(\mathbb{Q}[t])$$

Embedded Resolution

Definition

An *embedded resolution* of $(H, 0) \subset (\mathbb{C}^{n+1}, 0)$ is a proper analytic map $\pi : X \to (\mathbb{C}^{n+1}, 0)$ such that:

1. X is a (smooth) manifold.
2. π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
3. $\pi^{-1}(H)$ is a hypersurface with normal crossings on X.

Normal Crossing Divisor

The third condition above means that $\pi^{-1}(H) = (f \circ \pi)^{-1}(0)$ is locally given by an equation of the form

$$x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} = 0.$$

A'Campo's Formula

- Let $\pi : X \to (\mathbb{C}^{n+1}, 0)$ be an embedded resolution of $(H, 0)$.
- Total transform: $\pi^*(H) = \underbrace{\widehat{H}}_{\text{strict transform}} + \underbrace{\sum_{i=1}^{r} m_i E_i}_{\text{exceptional divisor}}$.
- Now, define

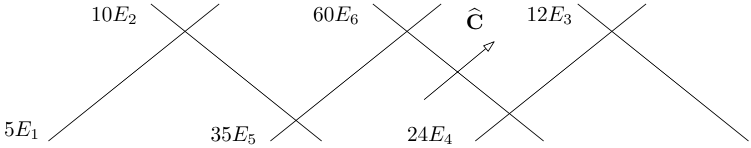
$$\check{E}_i := E_i \setminus \left(E_i \cap \left(\bigcup_{j \neq i} E_j \cup \widehat{H} \right) \right).$$

Theorem ([A'Campo])

$$\Lambda(h^k) = \sum_{i=1,\ m_i | k}^{r} m_i \chi(\check{E}_i), \qquad Z(f;t) = \prod_{i=1}^{r} (1 - t^{m_i})^{\chi(\check{E}_i)}.$$

Example of a Plane Curve

Let us compute the characteristic polynomial of the complex monodromy of $f = x^5 + y^{12}$ using A' Campo's formula. The following picture represents an embedded resolution:



- $E_i \simeq \mathbb{P}^1 \Longrightarrow \chi(E_i) = 2$
- $\chi(\check{E}_2) = \chi(\check{E}_5) = \chi(\check{E}_4) = 0$
- $\chi(\check{E}_1) = \chi(\check{E}_3) = 1$
- $\chi(\check{E}_6) = -1$

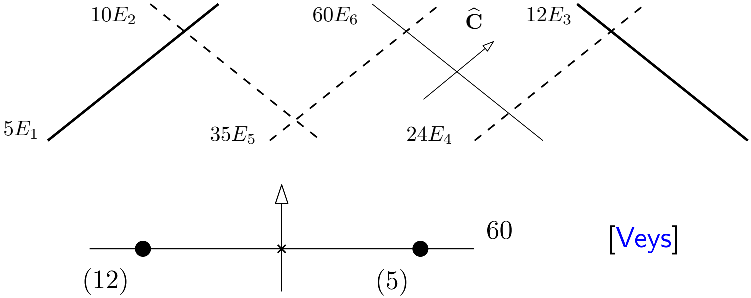
$$Z(f;t) = \prod_{i=1}^{r} (1 - t^{m_i})^{\chi(\check{E}_i)}$$

Example of a Plane Curve

This implies:

$$Z(f;t) = (1-t^{60})^{-1}(1-t^5)(1-t^{12}) = \frac{(1-t^5)(1-t^{12})}{(1-t^{60})}.$$

The following picture provides the same information:



[Veys]

The Main Aim

Questions:

1. How to formalize this idea of "contraction" of the exceptional divisors which do not contribute to the monodromy zeta function?
2. Can one directly compute the simplified resolution (without computing the standard one and then perform the contractions)?
3. Does there exist a formula for calculating $Z(f;t)$ using this new kind of resolutions even in higher dimension?

The answer is YES in all cases

1
Embedded $\mathbf{Q}$-Resolution

Embedded Q-Resolution

Definition

An *embedded* **Q***-resolution* of $(H,0) \subset (\mathbb{C}^{n+1},0)$ is a proper analytic map $\pi : X \to (\mathbb{C}^{n+1},0)$ such that:

1. X is a V-manifold with abelian quotient singularities.
2. π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
3. $\pi^{-1}(H)$ is a hypersurface with $\mathbb{Q}$-normal crossings on X.

$\mathbb{Q}$-Normal Crossing Divisor ([Steenbrink])

The third condition above means that $\pi^{-1}(H) = (f \circ \pi)^{-1}(0)$ is locally given by function of the form $x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C}$.

- $X(\mathbf{d}; A) := \mathbb{C}^{n+1}/\mu_{\mathbf{d}}$, $\quad G \subset GL(n+1,\mathbb{C})$ abelian group acting diagonally

Classical Blow-up of $\mathbb{C}^2$

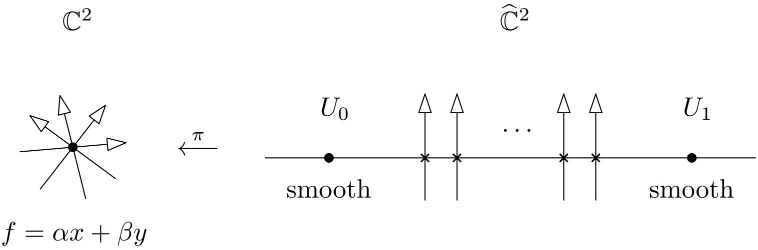
- Consider

$$\widehat{\mathbb{C}}^2 := \{((x,y),[u:v]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid (x,y) \in \overline{[u:v]}\}.$$

- Then $\pi : \widehat{\mathbb{C}}^2 \to \mathbb{C}^2$ is an isomorphism over $\widehat{\mathbb{C}}^2 \setminus \pi^{-1}(0)$.
- The *exceptional divisor* $E := \pi^{-1}(0)$ is identified with $\mathbb{P}^1$.
- The space $\widehat{\mathbb{C}}^2 = U_1 \cup U_2$ can be covered by 2 charts each of them isomorphic to $\mathbb{C}^2$.

$$\begin{array}{ccccc} \mathbb{C}^2 & \xrightarrow{\simeq} & U_1 = \{u \neq 0\} & \subset & \widehat{\mathbb{C}}^2 \\ (x,y) & \mapsto & \big((x,xy),[1:y]\big). & & \end{array}$$

Behavior of $\pi : \widehat{\mathbb{C}}^2 \to \mathbb{C}^2$



Weighted (p,q)-Blow-ups of $\mathbb{C}^2$

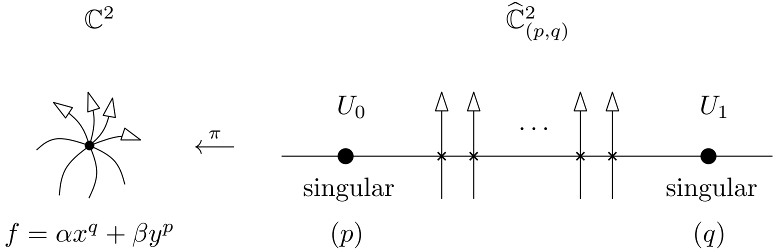
- Let $\omega = (p,q)$ be a weight vector with coprime entries. As above, consider the space

$$\widehat{\mathbb{C}}^2_\omega := \left\{ ((x,y),[u:v]_\omega) \in \mathbb{C}^2 \times \mathbb{P}^1_\omega \mid (x,y) \in \overline{[u:v]_\omega} \right\}.$$

- Then $\pi : \widehat{\mathbb{C}}^2_\omega \to \mathbb{C}^2$ is an isomorphism over $\widehat{\mathbb{C}}^2_\omega \setminus \pi^{-1}(0)$.
- The *exceptional divisor* $E := \pi^{-1}(0)$ is identified with $\mathbb{P}^1_\omega$.
- The space $\widehat{\mathbb{C}}^2_\omega = U_1 \cup U_2$ can be covered by 2 charts. For instance, the first chart is given by

$$\text{1st chart} \left| \begin{array}{ccc} X(p;-1,q) & \xrightarrow{\simeq} & U_1 = \{u \neq 0\} \subset \widehat{\mathbb{C}}^2_\omega, \\ [(x,y)] & \mapsto & ((x^p, x^q y), [1:y]_\omega). \end{array} \right.$$

Behavior of $\pi_{(p,q)} : \widehat{\mathbb{C}}^2_{(p,q)} \to \mathbb{C}^2$



2
The Main Result

Generalized A'Campo's Formula

Teorema

Let $X_0 = \pi^{-1}(H)$ be the total transform and $S = \pi^{-1}(0)$ the exceptional divisor. Consider $S_{m,d}$ to be the set

$$\left\{ s \in S \;\middle|\; \begin{array}{l} \text{the local equation of } X_0 \text{ in } s \text{ is given by the well-defined} \\ \text{function } x_i^m : X(d; a_0, \ldots, a_n) \to \mathbb{C}. \end{array} \right\}.$$

Then, the monodromy zeta function of the complex monodromy of the hypersurface $(H, 0)$ is

$$Z(f;t) = \prod_{m,d} (1 - t^{m/d})^{\chi(S_{m,d})}.$$

Comments about the Proof of the Theorem

1. One way to proceed is to rebuild A'Campo's paper, thus giving a model of the Milnor fibration in our setting. This method is very natural but perhaps a bit long and tedious.
2. In [GLM], the authors give a generalization of A'Campo's formula for the monodromy zeta function via partial resolution but the ambient space considered there is smooth and the proof can not be generalized to an arbitrary analytic variety.
3. That is why a very general result by Dimca is used instead.

Comments about Proof of the Theorem

Theorem ([Dimca])

1. Assume $\pi : X \to U$ is a proper analytic map such that π induces an isomorphism between $X \setminus \pi^{-1}(H)$ and $U \setminus H$.
2. Let $g = f \circ \pi$ denote the composition and $j : X \setminus \pi^{-1}(H) \hookrightarrow X$ the inclusion.
3. Let $\mathcal{S}$ be a finite stratification of the exceptional divisor $\pi^{-1}(0)$ such that $\psi_g \left(Rj_* \mathbb{C}_{X \setminus \pi^{-1}(H)} \right)$ is equivariantly $\mathcal{S}$-constructible with respect to the semisimple part of M.

Then,

$$Z(f) = \prod_{S \in \mathcal{S}} Z(g, x_S)^{\chi(S)},$$

where x_S is an arbitrary point in the stratum S and $Z(g, x_S)$ is the zeta function of the germ g at x_S.

Monodromy Zeta Function of a $\mathbb{Q}$-Normal Crossing Divisor

Lemma

The monodromy zeta function of a normal crossing divisor given by $x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C}$, $k \geq 1$, is

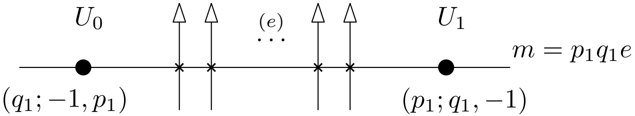
$$Z\left(x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C};\ t\right) = \begin{cases} 1 - t^{\frac{m_1}{l_1}} & k = 1; \\ 1 & k \geq 2, \end{cases}$$

where $l_1 = \operatorname{lcm}\left(\dfrac{d_1}{\gcd(d_1, a_{11})}, \ldots, \dfrac{d_r}{\gcd(d_r, a_{r1})}\right)$.

3
Applications and Examples

Examples 1

Let $f = x^p + y^q$ and assume that $e = \gcd(p,q)$, $p = p_1 e$ and $q = q_1 e$. Consider $\pi : \widehat{\mathbb{C}}^2_{(q_1,p_1)} \to \mathbb{C}^2$ the weighted blow-up at the origin.



The set $S_{m,d}$ is not empty for $(m,d) = (p_1 q_1 e, 1)$, $(p_1 q_1 e, q_1)$, $(p_1 q_1 e, p_1)$. Their Euler characteristics are

$$\chi(S_{p_1 q_1 e,1}) = 2 - (e+2) = -e, \qquad \chi(S_{p_1 q_1 e,q_1}) = \chi(S_{p_1 q_1 e,p_1}) = 1.$$

Now, we apply A'Campo's formula and obtain

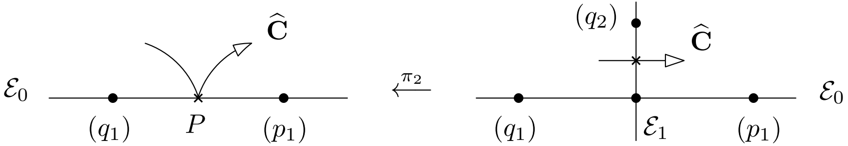
$$Z(f;t) = \frac{(1-t^p)(1-t^q)}{(1-t^{\frac{pq}{e}})^e}.$$

Example 2

- Assume $p_1/q_1 < p_2/q_2$ are two irreducible fractions and $\gcd(q_1, q_2) = 1$. Let $\mathbf{C}$ be the complex plane curve with Puiseux expansion

$$y = x^{\frac{p_1}{q_1}} + x^{\frac{p_2}{q_2}}.$$

- Consider $\pi_1 : \widehat{\mathbb{C}}^2(q_1, p_1) \to \mathbb{C}^2$ the weighted blow-up at the origin of type (q_1, p_1).
- Let π_2 be the weighted blow-up at P of type $(q_2, p_2 q_1 - p_1 q_2)$. Then the composition $\pi_1 \circ \pi_2$ defines an embedded $\mathbf{Q}$-resolution.



Example 2

- The corresponding Euler characteristics are

$$\chi(\mathcal{E}_0 \setminus \{3 \text{ points}\}) = \chi(\mathcal{E}_1 \setminus \{3 \text{ points}\}) = -1$$

  and $\chi = 1$ for the three singular points.

- Note that the singular point of type $(p_2 q_1 - p_1 q_2)$ does not contribute to the monodromy zeta function, since it belongs to more than one divisor.

$$\Delta(t) = \frac{\left(t-1\right)\left(t^{p_1 q_1 q_2}-1\right)\left(t^{q_2(p_1 q_1 q_2 + p_2 q_1 - p_1 q_2)}-1\right)}{\left(t^{p_1 q_2}-1\right)\left(t^{q_1 q_2}-1\right)\left(t^{p_1 q_1 q_2 + p_2 q_1 - p_1 q_2}-1\right)}$$

- One can find a formula for the characteristic polynomial of an irreducible plane curve with arbitrary Puiseux pairs.

Why Abelian?

D_4 as a Quotient Singularity

Let $\mathbb{C}^2$ with coordinate (x, y) and consider the subgroup of $GL(2, \mathbb{C})$ generated by the matrices

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

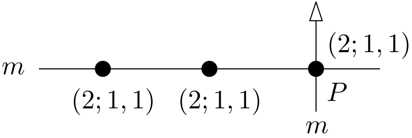
Thus $A^2 = B^2 = (AB)^2 = -Id_2$.

Definition

This group of order 8, often denoted by BD_8, is called the *binary dihedral group*. The quotient singularity $\mathbb{C}^2/\mathrm{BD}_8$ is denoted by D_4.

Zeta Function of $f := (xy)^m : D_4 \to \mathbb{C}$

The action BD_8 on $\mathbb{C}^2$ extends naturally to an action on $\widehat{\mathbb{C}}^2$ such that the induced map $\bar{\pi} : \widehat{\mathbb{C}}^2/\mathrm{BD}_8 \to \mathbb{C}^2/\mathrm{BD}_8 =: D_4$ defines an embedded **Q**-resolution of $\{f = 0\} \subset D_4$.



$$\Longrightarrow \quad Z(t) = \frac{(1 - t^{m/2})^2}{1 - t^m} = \frac{1 - t^{m/2}}{1 + t^{m/2}}$$

In particular, $Z(t)$ is not trivial although f defines a "double point" on D_4.

Conclusion

Finite abelian groups are the largest family for which the generalized A'Campo's formula can be applied.

Work in Progress

1. Give a method for computing the mixed Hodge structure on the cohomology of the Milnor fiber in terms of the semistable reduction of an embedded $\mathbf{Q}$-resolution (following Steenbrink's ideas).
2. Calculate an embedded $\mathbf{Q}$-resolution for (weighted) Yomdin-Lê surface singularities so as to apply the previous item and compute the MHS associated with this family of singularities, as in [Artal].
3. In relation to the monodromy conjecture, it could be interesting to find a formula for the topological zeta function in terms of an embedded $\mathbf{Q}$-resolution, see [Veys].

Thank You Very Much !!

J. Martín-Morales (jorge@unizar.es)

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