Limits of tangents of quasi-ordinary hypersurfaces

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Motivation

Kashiwara and Kaway proved ("Regular Holonomic Systems III") that, given a Legendrian (conic Lagrangean) variety $\Gamma \subset \mathbb{P}^*\mathbb{C}^n$ and a point $o \in \Gamma$, there is a contact transformation φ defined around o such that $\varphi(\Gamma)$ is the conormal of a hypersurface Y of $\mathbb{C}^n$ such that $\varphi(\Gamma) \cap \pi^{-1}(o)$ is a single point of $\mathbb{P}^*\mathbb{C}^n$.
This means that $\varphi(\Gamma)$ is in generic position, or equivalently, that Y has trivial limits of tangents.

These types of hypersurfaces occur in several key theorems of D-module theory:
The characteristic Variety of a holonomic D-module $\mathcal{M}$ is a conic Lagrangean variety Γ of $T^*\mathbb{C}^n$ (or Legendrian variety of $\mathbb{P}^*\mathbb{C}^n$).

Results

What can we say about a germ of hypersurface with trivial limits of tangents? Curiously, very little is known.
We present a complete characterization when Y is quasi-ordinary (actually we do a bit more).

- We obtain a method to calculate the limits of tangents of any quasi-ordinary hypersurface, and use it to prove that
  - The limit of tangents of a quasi-ordinary hypersurface is ("mostly") a topological invariant.
  - The limit of tangents of a quasi-ordinary hypersurface when the tangent cone is a hyperplane is a topological invariant.
  - The triviality of the limit of tangents of a quasi-ordinary hypersurface is a topological invariant.

Open problems:
Are the last two statements true for a general hypersurface?

background

- Le Dung Trang, Bernard Teissier (1988): Limits of tangent spaces of a complex hypersurface singularity
- Chunsheng Ban (1994): Calculated the limits of tangents of a quasi-ordinary singularity with a single special monomial.

1- Quasi-ordinary singularities

Y germ at the origin of a hypersurface of $\mathbb{C}^{n+1}$

$$Y = \{(x_1, \ldots, x_n, y) : f(x, y) = 0\}, f \in \mathbb{C}\{x_1, \ldots, x_n, y\}$$

Apparent contour of Y relative to y

$$Z = \{(x, y) : df/dy = f = 0\} \supset Sing(Y)$$

Discriminant of Y

$$\begin{array}{rcl} p : \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C}^n \\ (x, y) & \longmapsto & x \end{array}$$

$$\Delta = p(Z)$$

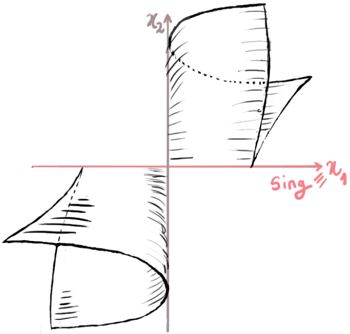
Definition

Let Y be a hypersurface of $\mathbb{C}^{n+1}$, $Y = \{f = 0\}$, $f \in \mathbb{C}\{x_1, \ldots, x_n, y\}$. We say that Y is a quasi-ordinary hypersurface if its discriminant (relative to y) is a divisor with normal crossings:

$$\Delta = \{x_1 \cdots x_l = 0\}, \quad l \leq n.$$

example

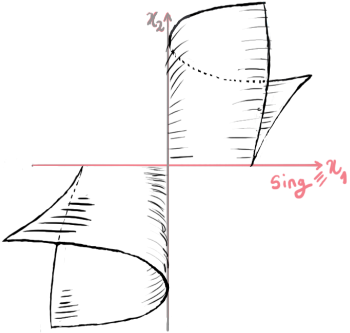
$$y^2 - x_1 x_2^3 = 0$$



The apparent contour contains the singular points and the points where the surface "turns"relative to the projection fibers. The discriminant is the shadow of the apparent contour. It is the set of points of the base over which the surface has a non-generic number of points.

example

$$y^2 - x_1 x_2^3 = 0$$



$Sing = \{(x_1, x_2, y) : f = \partial f/\partial x_1 = \partial f/\partial x_2 = \partial f/\partial y = 0\}$
$= \{x_2 = y = 0\}$
Apparent contour relative to y:
$\{(x_1, x_2, y) : f = \frac{\partial f}{\partial y} = 0\} = \{x_1 x_2 = y = 0\}$
Discriminant: $\{(x_1, x_2) : x_1 x_2 = 0\}$ (Normal Crossings Divisor)

We note that

$$y^2 - x_1 x_2^3 = 0$$

admits the parametrization:

$$y = x_1^{\frac{1}{2}} x_2^{\frac{3}{2}}.$$

This is true in general:

Quasi-ordinary hypersurfaces always admit parametrizations that are power series with rational exponents (ramified parametrizations).

Ramified parametrizations

Let $Y = \{f = 0\}, f \in \mathbb{C}\{x_1, \ldots, x_m, y\}$ be a quasi-ordinary hypersurface (relative to the projection that forgets y).
Then there is a positive integer n and a power series $H \in \mathbb{C}\{x_1^n, \ldots, x_m^n\}$ such that

$$f(x_1, \ldots, x_m, H(x_1^{1/n}, \ldots, x_m^{1/n})) = 0$$

around o. We say that the series with rational exponents $\varphi = H(x_1^{1/n}, \ldots, x_m^{1/n}) \in \mathbb{C}\{x_1^{1/n}, \ldots, x_m^{1/n}\}$ that parametrizes Y is a quasi-ordinary ramified parametrization and we say that n is the ramification order.

Special monomials

A quasi-ordinary parametrization $y = \phi(x)$ of Y features a finite number of monomials that are topological invariants of Y. They are called the special monomials of Y. They are those where the order of ramification changes (or in particular, where a new variable first appears). For example:

$$y = x_1^{2/5} + x_1^{1/2} + x_1^{3/5} + x_1^{6/10} x_2^{1/2} + x_1^{6/10} x_2^{3/2}$$

Special monomials :

$$N_1 = x_1^{2/5}, N_2 = x_1^{1/2} = x_1^{5/10}, N_3 = x_1^{6/10} x_2^{6/10}$$

For example, $x_1^{3/5}$ is not special because $x_1^{3/5} = N_1 x^{1/5}$.

The special monomials are totally ordered:

$$(N_3) \subset (N_2) \subset (N_1)$$

The q.o. parametrization can be written as

$$y = f_1 N_1 + \ldots + f_s N_s$$

where f_i are units determined uniquely by the ramification order.

Example:

$$y = x_1^{2/5} + x_1^{1/2} + x_1^{3/5} + x_1^{6/10} x_2^{1/2} + x_1^{6/10} x_2^{3/2}$$

$$= x_1^{2/5}(1 + x_1^{1/5}) + x_1^{1/2} + x_1^{6/10} x_2^{1/2}(1 + x_2^{1/10}).$$

Very Special Monomials

A special monomial N_i is very special if $\{N_i = 0\} \neq \{N_{i-1} = 0\}$, that is, if it is a monomial where a new variable first appears.
example:

$$y = x_1^{2/5}(1 + x_1^{1/5}) + x_1^{1/2} + x_1^{6/10} x_2^{1/2}(1 + x_2^{1/10}).$$

$$y = x_1^{2/5}(1 + x_1^{1/5}) + x_1^{1/2} + x_1^{6/10} x_2^{1/2}(1 + x_2^{1/10}).$$

Very special: $M_1 = x_1^{2/5}, M_2 = x_1^{6/10} x_2^{1/2}$

Notation

They are so special that they demand from us a change in notation:

$$N_k = \prod_{i=1}^{k} \prod_{j=1}^{m_k} x_{ij}^{a_{kij}}, 1 \leq k \leq g$$

example:

$$y = x_1^{2/5} + x_1^{1/2} + x_1^{6/10} x_2^{1/2} \rightarrow y = x_{11}^{2/5} + x_{11}^{1/2} + x_{11}^{6/10} x_{21}^{1/2}$$

limits of tangents

Let $Y \subset \mathbb{C}^{n+1}$ be q.o. hypersurface, let o be a point of Y.

Let Σ be the set of singular points of Y.

Let (p_i) be a sequence of points of $Y - \Sigma$ converging to o. This induces a sequence of tangent spaces $T_{p_i}Y \subset T\mathbb{C}^{n+1}$.

Suppose $T_{p_i}Y$ converges to $T \in T\mathbb{C}^{n+1}$. Then we say that T is a limit of tangents of Y at o.

Let Λ_o be the set of all limits of tangents of Y at o. We call Λ_o the limit of tangents of Y at o.

Our purpose will be to calculate Λ_o.

Since Y is a hypersurface defined by $f = 0$ for a certain $f \in \mathbb{C}^{n+1}\{x_1, \ldots, x_n, y\}$, we can identify each $T_{p_i} Y$ with the kernel of $df_{p_i} \in T^*\mathbb{C}^{n+1}$. But multiplying df by $\alpha \in \mathbb{C}^*$ preserves the kernel so we identify

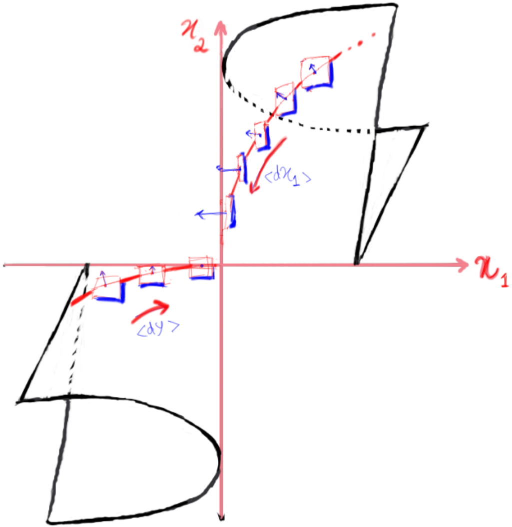
$$T_{p_i} Y = \langle df_{p_i} \rangle \in \mathbb{P}^*\mathbb{C}^{n+1}$$

where

$$\mathbb{P}^*\mathbb{C}^{n+1} = \left(T^*\mathbb{C}^{n+1} \setminus \mathbb{C}^{n+1} \right) / \mathbb{C}^*$$

is the projectivized cotangent space. We'll look at the limits of tangents as elements of $\mathbb{P}^*\mathbb{C}^{n+1}$.

How to calculate Λ? Is it enough to calculate limits of curves?



We need to establish an upper bound for Λ.

Conormal

The set of limits of tangents of $\{\Lambda_p, p \in Y\}$ can be bundled together into a variety, through the folowing construction:

Y hipersurface de $\mathbb{C}^{n+1}$.

We say that

$$\mathbb{P}^*_Y\mathbb{C}^{n+1} = \overline{\{(p, < df_p >) : p \in Y_{\text{reg}}\}} \subset \mathbb{P}^*\mathbb{C}^{n+1}$$

is the Conormal of Y. We have

$$\Lambda_p = \pi^{-1}(p) \cap \mathbb{P}^*_Y\mathbb{C}^{n+1}$$

where $\pi : \mathbb{P}^*_Y\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the canonical projection.
So the fiber above each p holds the limits of tangents of Y at p.

It can be shown that the conormal of Y is a legendrian (or conic lagrangean) subvariety of $\mathbb{P}^*\mathbb{C}^{n+1}$.

Ley γ be a point of the conormal. We say that the conormal is in generic position if $\pi^{-1}(\pi(\gamma)) \cap \mathbb{P}^*_Y\mathbb{C}^{n+1}$ is a single point.

So the conormal of Y is a lagrangean variety in generic position at γ if and only if Y has trivial limits of tangents at $\pi(\gamma)$.

Parametrization of the conormal of a q.o. hypersurface

Let $(x_1, \ldots, x_n, y, \xi_1, \ldots, \xi_n, \zeta)$ be homogeneous coordinates of $\mathbb{P}^*\mathbb{C}^{n+1}$ and let $(x_1, \ldots, x_n, y, p_1, \ldots, p_n)$, where $p_i = \xi_i/\zeta$ be projective coordinates in the chart $\zeta \neq 0$.

If the hypersurface Y has a parametrization

$$y = \varphi(x), \varphi \in \mathbb{C}\{x^{1/n}\},$$

then its conormal $\mathbb{P}^*_Y\mathbb{C}^n$ is parametrized

$$y = \varphi(x), \quad p_i = \frac{dy}{dx_i}, 1 \leq i \leq n.$$

where

$$\frac{dy}{dx_{ij}} = a_{iij}\frac{M_i}{x_{ij}}\sigma_{ij}, \;\; \sigma_{ij} \text{ unit of } \mathbb{C}\{x^{1/n}\}.$$

From this parametrization we can obtain equations for analytic sets that bound the conormal.

Strategy:

1-Use the equations of the conormal to find an analytic set $\overline{\Lambda}$ that is an 'upper bound' for the limit of tangents Λ.

2-'Fill' the upper bound with an adequate family of limits of curves.

Example:

$$y = x_1^{1/2} x_2^{2/3}$$

The conormal is parametrized by

$$y = x_1^{1/2} x_2^{2/3}, \quad p_1 = \frac{dy}{dx_1} = \frac{1}{2} x_1^{-1/2} x_2^{2/3}, \quad p_2 = \frac{dy}{dx_2} = \frac{3}{2} x_1^{1/2} x_2^{-1/3}$$

Then $p_1 p_2 = \frac{1}{3} x_2^{1/3}$. The conormal is within the analytic set $p_1^3 p_2^3 = \frac{1}{27} x_2$. Hence, in the fiber over the origin of $\mathbb{C}^2$, we have that $p_1 p_2 = 0$.

So $\Lambda \subset \overline{\Lambda} = \{p_1 p_2 = 0\}$. We have an upper bound for the limit of tangents over 0.

$\overline{\Lambda}$ has two irreducible components, $p_1 = 0$ and $p_2 = 0$.

Now we fill up the component $\{p_2 = 0\}$.
Take the family of curves $t \mapsto (x(y), y(t)) = (\varepsilon_1 t^{\alpha_1}, \varepsilon_2 t^{\alpha_2})$.
These curves of $\mathbb{C}^2$ induce curves on the conormal.

Choosing $\alpha_1 = 2, \alpha_2 = 3$, we get

$$p_1 = \frac{2}{3}\varepsilon_1^{-1/2}\varepsilon_2^{2/3}, \quad p_2 = \frac{2}{3}\varepsilon_1^{1/2}\varepsilon_2^{-1/3}t$$

Then

$$\lim_{t\to 0}(p_1(t), p_2(t)) = \left(\frac{2}{3}\varepsilon_1^{-1/2}\varepsilon_2^{2/3}, 0\right).$$

By choosing adequate $\varepsilon_i \in \mathbb{C}^*$ e obtain all the possible limits in the component $p_2 = 0$. The other component can be filled up in a similar way. Hence $\Lambda = \overline{\Lambda} = \{p_1 p_2 = 0\}$. (the conormal is not in generic position)

In general: The choice of the α_i isolates an irreducible component; the choice of the ε_i fills up a dense set of the component. Can we always do this?(yes)

Suppose we have s very special monomials:
$y = f_1 M_1 + \ldots + f_s M_s$, where f_i are units.
For each very special monomial M_k, $k \geq 2$,

$$M_k = \prod_{s=1}^{k-1} \prod_{j=1}^{m_s} x_{sj}^{a_{ksj}} x_{k1}^{a_{k1}} \cdots x_{km_k}^{a_{km_k}}$$

we check the exponents of the new variables for subsets

$$J \in \mathcal{P}(\{(k,1), \ldots, (k, m_k)\})$$

such that

$$\sum_{j \in J} a_{kkj} \geq 1.$$

For each such set J we obtain

$$\prod_{j \in J} \xi_{kj} = 0.$$

The case of the first special monomial is different:

- If $\sum_{i=1}^{m_1} a_{11i} > 1$ then $\xi_{11} \cdots \xi_{1m_1} = 0$.
- If $\sum_{i=1}^{m_1} a_{11i} < 1$ then $\zeta = 0$.

Up to now all this is topological invariant since it only depends on the special exponents (which is kinda weird, actually).

Here is the exception: Suppose

$$\sum_{i=1}^{m_1} a_{11i} = 1.$$

Example:

$$y = a x_1^{1/2} x_2^{1/2} + x_1^{1/2} x_2^{1/2} x_3^{1/2}, \quad a \in \mathbb{C}^*.$$

Then

$$\begin{aligned} p_1 &= (1/2) x_1^{-1/2} x_2^{1/2} (a + x_3^{1/2}), \\ p_2 &= (1/2) x_1^{1/2} x_2^{-1/2} (a + x_3^{1/2}) \end{aligned}$$

$$p_1 p_2 = (1/4)(a^2 + 2a x_3^{1/2} + x_3).$$

$$\xi_1 \xi_2 = \zeta^2 (1/4)(a^2 + 2a x_3^{1/2} + x_3).$$

$$\Lambda \subset \{\xi_1 \xi_2 = (a^2/4)\zeta^2\}.$$

The cone we obtained depends not only on the special exponents but also on the coefficient a. Hence it is not a topological invariant.

Main Results

Theorem

Suppose that $\sum_{i=1}^{m_1} a_{11i} > 1$.

Then the tangent cone of Y *at* o *is* $\{y = 0\}$ *and* Λ *is determined by the equations*

$$\prod_{j \in J} \xi_{kj} = 0 \text{ for all } k, J \text{ such that } \sum_{j \in J} a_{kkj} \geq 1.$$

Main Results

Theorem

Suppose that $\sum_{i=1}^{m_1} a_{11i} < 1$.

Then the tangent cone of Y *at* o *is* $\{x_{11} \cdots x_{1m_1} = 0\}$ *and* Λ *is determined by the equations*

$$\begin{cases} \zeta = 0 \\ \prod_{j \in J} \xi_{kj} = 0 \text{ for all } k, J \text{ such that } \sum_{j \in J} a_{kkj} \geq 1. \end{cases}$$

Main Results

Theorem

Suppose that $\sum_{i=1}^{m_1} a_{11i} = 1$*.*

Λ *is determined by the equations*

$$\begin{cases} \prod_{j\in J} \xi_{kj} = 0 \text{ for all } k, J \text{ such that } k > 1 \text{ and } \sum_l a_{kkj} \geq 1. \\ \prod_{i=1}^{m_1} \xi^{c_i} - s\zeta_{n+1}^{c} = 0, \end{cases}$$

where c_i *are integers and* s *is a complex number determined by the coefficient* $f_1(0)$ *of* M_1 *in the ramified series of* Y*.*
The tangent cone,

$$y^c - s_1 \prod_{i=1}^{m_1} x_{1i}^{d_i} = 0$$

also depends on the value of $f_1(0)$ *(through* s_1*).*

Corollary

The set of limits of tangents of Y only depends on the tangent cone of Y and the topology of Y.

Corollary

If the tangent cone of Y is a hyperplane (or $a_{111} + \ldots + a_{11m_1} \neq 1$), the set of limits of tangents of Y only depends on the topology of Y.

Corollary

The triviality of the set of limits of tangents of Y is a topological invariant of Y.

(The set of limits of tangents of Y is trivial if and only if all the exponents of all the special monomials of Y are greater or equal than 1.)

Open problems:

Can we do this for other surfaces?

Essentials of the proof:

- Good invariants (very special monomials and their exponents)
- Total order of the special monomials

and, at a more gritty level

$$\frac{dy}{dx_{ij}} = a_{iij} \frac{M_i}{x_{ij}} \sigma_{ij}, \sigma_{ij} \text{ unit.}$$

(a good working relationship between the conormal and the invariants we may have)

This is what we need to supply.

This is not the end of the talk because...

...I lied about my motivations (Sorry)

What we (me, Joao Cabral, Orlando Neto) really wanted to do was dessingularize Lagrangean Varieties (Joao will talk about this joint work later today).

And for that what we really needed was to characterize the triviality of logarithmic limits of tangents.

What is that?

Logarithmic limits of tangents

M complex manifold, N normal crossings divisor.

T^*M - vector bundle with sheaf of sections Ω^1_M.

$T^*\langle M/N\rangle$ - vector bundle with sheaf of sections $\Omega^1_M\langle N\rangle$. (logarithmic differential forms with poles along N).

$T^*\langle M/N\rangle$ is endowed with a canonical logarithmic differential form

$$\theta \in \Omega^1_{T^*\langle M/N\rangle}\langle \pi^{-1}(N)\rangle$$

where $\pi : T^*\langle M/N\rangle \to M$ is the canonical projection

If $M = \mathbb{C}^n_{x_1,\ldots,x_n}$ and $N = \{x_1 \cdots x_k = 0\}$

$$\theta_n = \sum_{i=1}^{k} \xi_i \frac{dx_i}{x_i} + \sum_{i=k+1}^{n} \xi_i dx_i$$

S hypersurface of M, $S \not\subset M \setminus N$.

$T^*_S\langle M/N\rangle$: closure of $T^*_{S\setminus N}(M \setminus N)$ in $T^*\langle M/N\rangle$.

Given $o \in S$, the **logarithmic limit of tangents** of S at o equals

$$T^*_S\langle M/N\rangle \cap \pi^{-1}(o).$$

We can calculate the limits of tangents of S with poles along N at o by similar methods to the ones discussed.

example

Recall the example $y = x_1^{1/2} x_2^{3/2}$.

Λ is equal to $p_2 = 0$.

Logarithmic limit of tangents for $N = \{x_1 = 0\}$:

The canonical 1-form is now

$$\theta = \xi_1 \frac{dx_1}{x_1} + \xi_2 dx_2 + \zeta dy$$

, so

$p_1 = x_1 \dfrac{\partial y}{\partial x_1} = x_1^{1/2} x_2^{3/2}$. Hence $p_1 = 0$

Hence in the logarithmic case the limit becomes trivial (in general what you get is lots more trivial limits of tangents in the logarithmic case).

example

Again, $y = x_1^{1/2} x_2^{3/2}$.

Logarithmic limit of tangents for $N = \{y = 0\}$:

The canonical 1-form is now

$$\theta = \xi_1 dx_1 + \xi_2 dx_2 + \zeta \frac{dy}{y}$$

, so

$p_1 = \dfrac{\frac{\partial y}{\partial x_1}}{y} = \dfrac{1}{x_1}$. Hence $x_1 \xi_1 = \zeta$.

Hence the logarithmic limit of tangents is given by $\zeta = 0$.

By similar arguments we can solve the general three-dimensional case:

$$z = \varphi(x^{1/n}, y^{1/n}) = x^{\lambda} y^{\mu} + x^{a} y^{b} + \ldots, \lambda \geq \mu$$

1- If $N = \{xy = 0\},\ N = \{yz = 0\},\ N = \{xyz = 0\}$ the logarithmic limits of tangents are always trivial.

2- If $N = \{z = 0\}$, Λ is trivial iff $\mu = 0$ and $b \geq 1$.

(etc)

Araujo, Neto, *Limits of Tangents of Quasi-Ordinary Hypersurfaces*, Proceedings of the American Mathematical Society (to appear) (http://arxiv.org/abs/1003.4668)

Araujo, Cabral, Neto, *Desingularization of Legendrian surfaces - The quasi-ordinary case*, (in preparation)