Higher Dimensional Rigid Local Systems I

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Each irreducible singular regular linear differential equation of order 2 on $\mathbb{P}^1$, with singularities at $0, 1$ and ∞ and holomorphic solutions defined in the neighbourhood of 0 and 1 is in the family

$$t(1-t)\frac{d^2u}{dt^2} + (c - (a+b+1)t)\frac{du}{dt} + abu = 0, \quad a, b, c \in \mathbb{C}. \tag{1}$$

(Riemann's Hypergeometric differential equation)

The eigenvalues of the monodromies of (1) around $0, 1$ and ∞ are respectively

$$1, \alpha, \qquad 1, \beta, \qquad \text{and} \qquad \lambda, \mu.$$

Moreover,

$$\alpha\beta = \lambda\mu.$$

Hence the set of parameters of the coefficients of the equations of the family (1) and the sets of parameters of the local monodromies of the equations of the family (1) have both 3 **degrees of freedom.**

Hence we can recover a, b, c from $\alpha, \beta, \lambda, \mu$.

We are able to recover the **global data** a, b, c from the **local data** $\alpha, \beta, \lambda, \mu$!

The hypergeometric differential eaquation is an example of an equation **without accessory parameters.**

(In general there are more parameters on the coefficients then eigenvalues)

Linear equations without accessory parameters have a central role on the theory of Fuchsian differential equations.

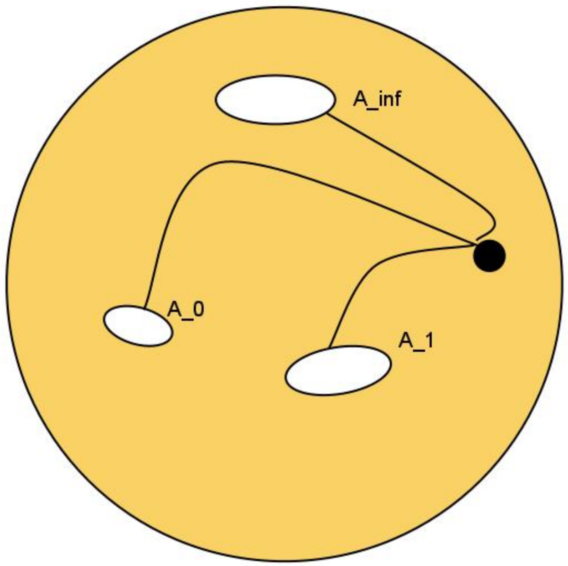
They can be presented in a modern framework:

A local system on the punctured Riemann sphere is called **rigid** if it is determined by its local monodromies.

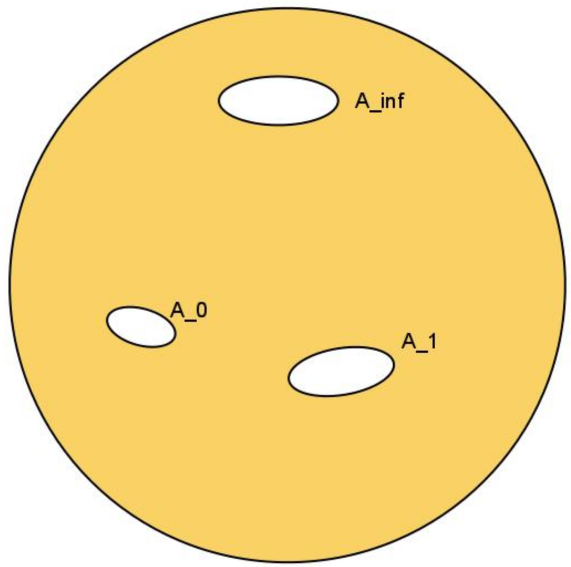
Consider invertible matrices $A_0, A_1, \ldots, A_n, A_\infty$ such that

$$A_0 A_1 \cdots A_n = A_\infty.$$

We call the tuple $A_0, A_1, \ldots, A_n, A_\infty$ **rigid** if
given matrices $B_0, B_1, \ldots, B_n, B_\infty$ such that
$B_0 B_1 \cdots B_n = B_\infty$
and B_i is conjugated to A_i, $i = 0, 1, \ldots, n, \infty$,
there is an invertible matrice X such that $B_i = X^{-1} A_i X$,
for $i = 0, 1, \ldots, n, \infty$.



$$A_0 A_1 = A_\infty$$


A_inf
A_0
A_1

Sato, Kashiwara, Kimura and Oshima introduced a very interesting higher dimensional generalization of the notion of accessory parameters. Let Y be the weighted homogeneous plane curve

$$\prod_{i=1}^{\ell}(y^k - C_i x^n) = 0. \tag{2}$$

Set $X = \mathbb{C}^2$. Let $\mathfrak{M}$ be a simple holonomic $\mathfrak{D}_X$-module with characteristic variety $T^*_Y X \cup T^*_X X$.

Theorem There are complex numbers $\lambda, \lambda_{r,s}$, $r, s \geq 0$, $ns - (n-k)(\ell+1) \leq r \leq (n-k)s - 1$, and $\widehat{C}_i$, $1 \leq i \leq \ell$, such that $\mathfrak{M}$ is isomorphic to the system

$$(kx\partial_x + ny\partial_y - \lambda)u = Pu = 0, \tag{3}$$

where

$$P = \prod_{i=1}^{\ell} (\partial_x^k + \widehat{C}_i x^{n-k} \partial_y^k) +$$

$$+ \sum_{r,s} \lambda_{r,s} x^r \partial_x^{j-nr+(n-k)(\ell+1)} \partial_y^{ns}.$$

Set $p =$ number of parameters $\lambda, \lambda_{r,s}$ that occur in the system $\mathfrak{M}$.

Set $q =$ number of irreducible components of Y.

SKKO showed that $p \geq q$.

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The two definitions of acessory parameters are connected:

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Hence there is a multivalued holomorphic function φ on the Riemann sphere, ramified along $0, \infty, C_i, 1 \leq i \leq \ell$, such that $u(x,y) = y^{\lambda/n}\varphi(y^k/x^n)$.

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Moreover, there is a Fuchsian differential operator H such that

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SKKO system is without acessory parameters iff $H\varphi = 0$ is without acessory parameters.

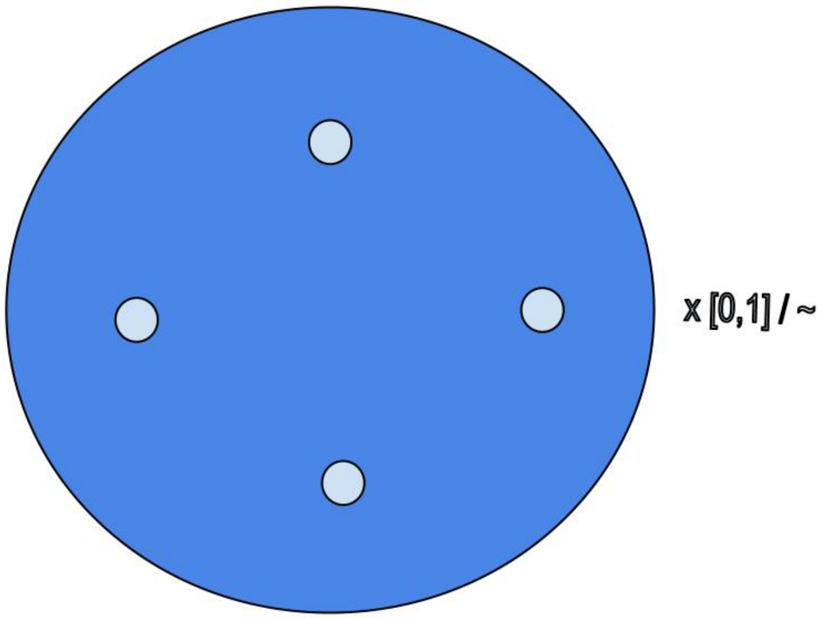
Is there a higher dimensional generalization of the notion of rigid local system that covers the SKKO example?

SKKO showed that when the characteristic variety has several irreducible components the systems are never without acessory parameters.

If we consider multiplicity one systems instead of simple characteristic systems, can we find more systems without accessory parameters?

Yes we can!

Let us recall some 3 dim topology: **Seifert Manifold**



Le M be a 3-fold. Let $\mathfrak{T} = \cup_{1 \leq i \leq k} T_i$ be the union of a familiy of torii (surfaces $\simeq S^1 \times S^1$).

$\mathfrak{T}$ defines a **Waldhausen decomposition** of M if $M \setminus \mathfrak{T}$ is a union of Seifert manifolds.

Theorem (Le Dung Trang 2003)
Let Y be a germ of plane curve.
Let S^ε be the Milnor sphere.
There is a minimal Waldhausen decomposition of $S^\varepsilon \backslash Y$.

Example

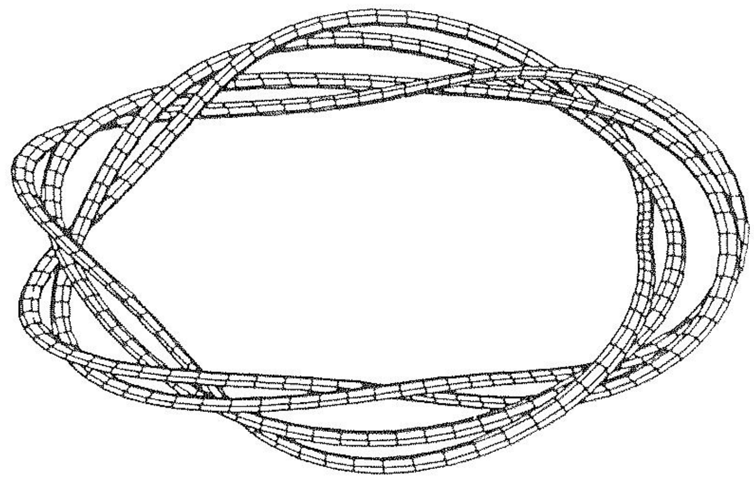
Consider the plane curve Y parametrized by

$$x = t^4, \quad y = t^{10} + t^{11}.$$

The intersection of Y with the Milnor sphere $\mathbb{S}^\varepsilon$ defined by

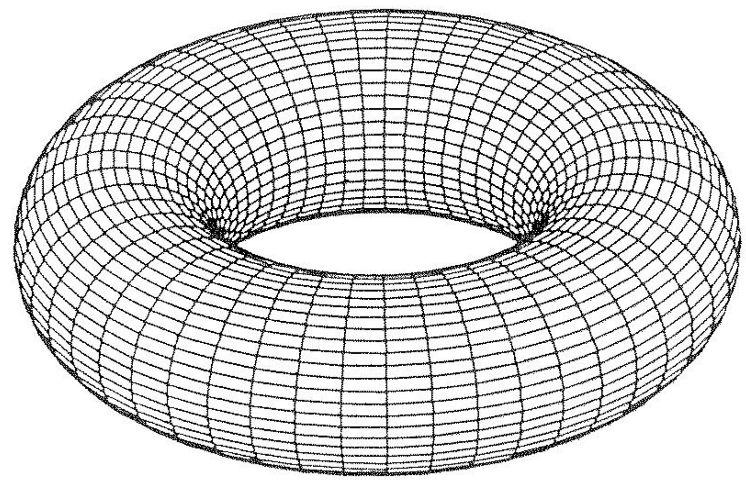
$$|x|^2 + |y|^2 = \varepsilon^2$$

equals the knot:

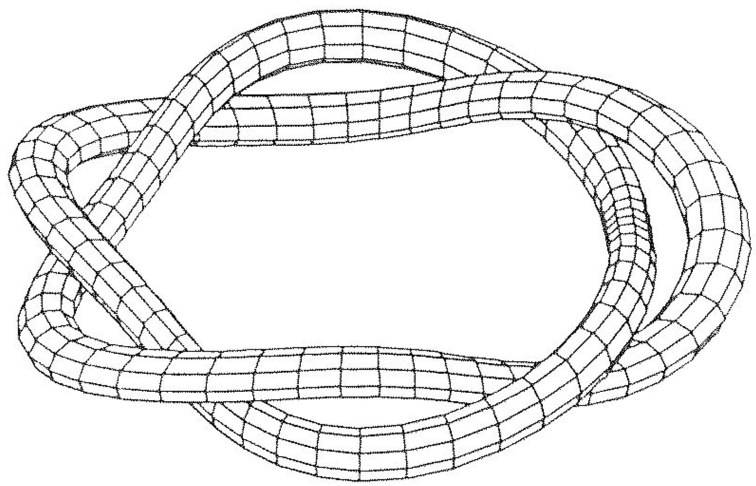


The Waldhausen decomposition of $\mathbb{S}^{\varepsilon} \setminus Y$ is defined by three torii T_0, T_1, T_2 associated to the curves with Puiseux expansions $y = 0$, $y = x^{\frac{5}{2}}$, $y = x^{\frac{5}{2}} + x^{\frac{11}{4}}$.

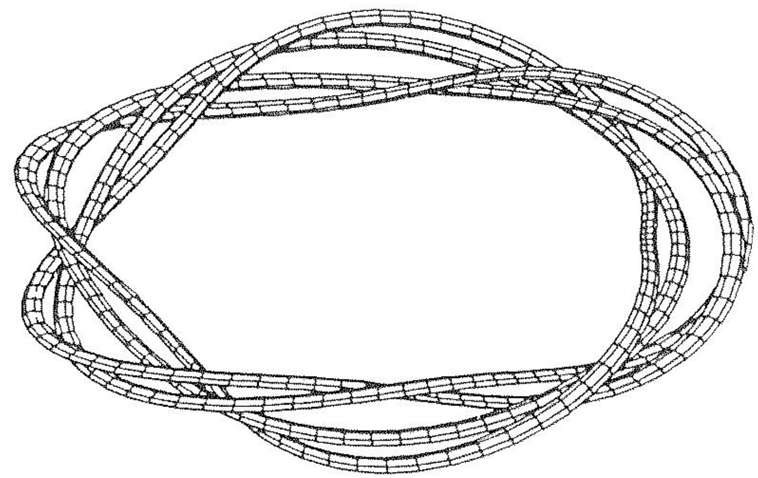
- $T_0 = \mathbb{S}^{\varepsilon} \cap \{y = a : |a| < \varepsilon_0\}$
- $T_1 = \mathbb{S}^{\varepsilon} \cap \{y = ax^{\frac{5}{2}} : |a| < \varepsilon_1\}$
- $T_2 = \mathbb{S}^{\varepsilon} \cap \{y = x^{\frac{5}{2}} + ax^{\frac{11}{4}} : |a| < \varepsilon_2\}$



$\mathbf{T_0}$



$\mathbf{T_1}$



$\mathbf{T_2}$

Definition

Let Y be a germ of plane curve.

Let X be an open neighboorhhod of the origin.

A local system $\mathcal{L}$ on $X \setminus Y$ is **rigid** if $\mathcal{L}$ is determined by $\mathcal{L} \mid_{\mathfrak{T}}$, where $\mathfrak{T}$ defines a minimal Waldhausen decomposition of $S^{\varepsilon} \setminus Y$.

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There is a higher dimensional generalization of the notion of rigidity.

Who cares?

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Connections with $\mathfrak{D}$-modules?

Definition N

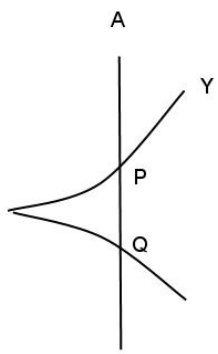
Let (Y, o) be the germ of an hypersurface of $\mathbb{C}^n$.

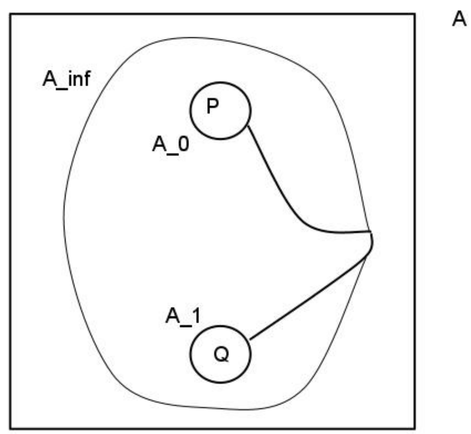
Let X be an open neighbourhood of o.

Let $\tau : \mathbb{C}^n \to \mathbb{C}$ be a linear projection with fibers transversal to the tangent cone of Y.

Let $\mathcal{L}$ be a local system on $X \setminus Y$.

$\mathcal{L}$ is **Pochhammer** if $R\tau_*(\mathcal{L})$ vanishes.


A
Y
P
Q


A
A_inf
P
A_0
A_1
Q

$$A_i := \begin{bmatrix} 1 & & & (\lambda_1 - \eta_1)\eta_1^{-1} & & & \\ & \ddots & & \vdots & & & \\ & & 1 & (\lambda_{i-1} - \eta_1)\eta_1^{-1} & & & \\ & & & \lambda_i & & & \\ & & & \lambda_{i+1} - \eta_1 & 1 & & \\ & & & \vdots & & \ddots & \\ & & & \lambda_n - \eta_1 & & & 1 \end{bmatrix}, \qquad i = 1, \ldots, n, \tag{6}$$

Let X be an open neighbourood of the origin 0 of $\mathbb{C}^n$. Let $\mathfrak{M}$ be (the germ at 0) of a honomic $\mathfrak{D}_X$-module with characteristic variety in **generic position.** Assume that $\mathcal{C}\mathrm{har}(\mathfrak{M}) \cap \pi^{-1}(0) = \langle \partial_{x_n} \rangle$.

Definition

$\mathfrak{M}$ **comes form a $\mathfrak{D}$-module** if :
$\mathcal{C}\mathrm{har}(\mathfrak{M})$ is in generic position and
$\partial_{x_n} : \mathfrak{M}_0 \to \mathfrak{M}_0$ is bijective.

Let $\mathfrak{M}$ be a honomic $\mathfrak{D}_X$-module.
We call **ramification locus** of $\mathfrak{M}$ to
$\pi(\mathcal{C}\text{ar}(\mathfrak{M}) \setminus T^*_M M)$.

The multivalued holomorphic functions of $\mathfrak{M}$ ramify along the ramification locus of $\mathfrak{M}$.

Theorem N

Let Y be an hypersurface of $\mathbb{C}^n$.

The de Rham functor

$\mathcal{DR}(\mathfrak{M}) = \Omega^* \otimes \mathfrak{M} = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}}(\mathcal{O}_X, \mathfrak{M})$ defines an equivalence of categories between the categories of

- $\mathfrak{D}$-modules that come from a $\mathcal{E}$ module and have ramification locus Y.

- Pochammer local systems on $X \setminus Y$.

Definition

We call a germ of $\mathfrak{D}$-module $\mathfrak{M}$ a **Special $\mathfrak{D}$-module** if

- $\mathfrak{M}$ and its adjoint $\mathfrak{M}^*$ come from a $\mathcal{E}$-module;
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$\mathfrak{M}$ is special if:

- $\mathfrak{M}$ is regular holonomic;
- $\mathcal{C}$har($\mathfrak{M}$) is in generic position;
- Its Pohhammer local system is rigid;
- $\mathcal{DR}(\mathfrak{M})$ and $\mathcal{S}$ol($\mathfrak{M}$) are concentrated in degree 0.

Remark

The condition
"$\mathcal{DR}(\mathfrak{M})$ and $\mathcal{S}\mathrm{ol}(\mathfrak{M})$ are concentrated in degree 0"
is a genericity condition on the system $\mathfrak{M}$.

Let $\mathfrak{M}_\lambda$ be the $\mathfrak{D}$-module defined by the relation

$$t\partial_t u = \lambda u.$$

The system $\mathfrak{M}_\lambda$ verifies the conditions above if and only if $\lambda \notin \mathbb{Z}$.

Theorem Kashiwara and Kawai
If a $\mathfrak{D}$-module $\mathfrak{M}$ comes from a $\mathcal{E}$-module
it is l generated by a holomorphic microfunction.

Example
The $\mathfrak{D}$-module $\mathfrak{M}$ with presentation $t\partial_t u = 0$
is generated by $\log t + \mathcal{O}_{\mathbb{C}}$.

$\mathfrak{M}$ is not generated by a multivalued holomorphic function.

"There is no general integral".

Theorem

If $\mathfrak{M}$ and $\mathfrak{M}^*$ come from an $\mathcal{E}$-module,
$\mathfrak{M}$ is generated by a multivalued holomorphic function.

Pochammer Local system with special eigenvalues $\neq 1 \Rightarrow$ Special $\mathfrak{D}$-module $\Rightarrow$ Special function.

Theorem

Given $\lambda_1, \ldots, \lambda_\ell \in \mathbb{C} \setminus \{0\}$
there is a special $\mathfrak{D}$-module of multiplicity 1
with characteristic variety equal to the conormal of $\{\prod_{i=1}^{\ell}(y^k - C_i x^n) = 0\}$
such that its monodromy around $\{y^k - C_i x^n = 0\}$
has eigenvalues $\lambda, 1, \ldots, 1$.