Resolution of Singularities of Quasi-Ordinary Legendrian Surfaces

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1. Characteristic Variety

Let (x, y) be a set of coordinates on $\mathbb{C}^2\left(\mathbb{C}^2_{x,y}\right)$. Let

$$\theta = \xi dx + \eta dy = \sigma(\partial_x)dx + \sigma\left(\partial_y\right)dy$$

be the canonical 1-form on $T^*\mathbb{C}^2_{x,y}$. Let $\mathcal{M}$ be the D-module on $\mathbb{C}^2_{x,y}$ defined by

$$\begin{cases} \left(2x\partial_x + 3y\partial_y - \lambda\right)u = 0 \\ \left(\partial_x^2 - \left(\frac{3}{2}\right)^2 x\partial_y^2\right)u = 0 \end{cases}, \lambda \in \mathbb{C}.$$

$$\mathrm{Char}(\mathcal{M}) = \begin{cases} 2x\xi + 3y\eta = 0 \\ \xi^2 - \left(\frac{3}{2}\right)^2 x\eta^2 = 0 \end{cases}.$$

Let $\pi: T^*\mathbb{C}^2_{x,y} \longrightarrow \mathbb{C}^2_{x,y}$ be the canonical projection,

$$\pi\left(\mathrm{Char}(\mathcal{M})\backslash T^*_{\mathbb{C}^2_{x,y}}\mathbb{C}^2_{x,y}\right) = \{y^2 - x^3 = 0\} \text{ (Singular locus of } \mathcal{M}\text{).}$$

In fact,

$$\mathrm{Char}(\mathcal{M}) = T^*_{\{y^2-x^3=0\}}\mathbb{C}^2_{x,y} \cup T^*_{\mathbb{C}^2_{x,y}}\mathbb{C}^2_{x,y}.$$

2. Logarithmic Characteristic Variety

Let N be a normal crossings divisor of a complex manifold M. Let $\mathcal{D}_M[N]$ be the sheaf of meromorphic differential operators with poles on N. Let $\Theta_M\langle N\rangle$ be the sheaf of logarithmic vector fields along N. Let $\mathcal{D}_M\langle N\rangle$ be the sub-ring of $\mathcal{D}_M$ generated by $\Theta_M\langle N\rangle$ and $\mathcal{O}_M$. There are symbol maps

$$\sigma_{N,n}\colon \mathcal{D}_M\langle N\rangle_n \longrightarrow \mathcal{O}_{T^*\langle M/N\rangle}.$$

If $M = \mathbb{C}^2_{x,y}$ and $N = \{x = 0\}$

$$\sigma_N\left(x\partial_x + ny\partial_y\right) = \xi + ny\eta.$$

The *logarithmic characteristic variety* of $\mathcal{M}$, $\mathrm{Char}_N(\mathcal{M})$, is the closure in $T^*\langle M/N\rangle$ of $\mathrm{Char}\left(\mathcal{M}|_{M\setminus N}\right)$.

Assume $N = \{x = 0\}$. Let us compute the logarithmic characteristic variety of the example studied on section 1. Let

$$\theta_N = \xi\frac{dx}{x} + \eta dy = \sigma_N(x\partial_x)\frac{dx}{x} + \sigma_N\left(\partial_y\right)dy$$

be the canonical 1-form on $T^*\langle \mathbb{C}^2_{x,y}/N\rangle$.

$$\text{If } \mathcal{M} = \begin{cases} \left(2x\partial_x + 3y\partial_y - \lambda\right)u = 0 \\ \left((x\partial_x - 1)x\partial_x - \left(\frac{3}{2}\right)^2 x^3\partial_y^2\right)u = 0 \end{cases},$$

$$\mathrm{Char}_N(\mathcal{M}) = \begin{cases} 2\xi + 3y\eta = 0 \\ \xi^2 - \left(\frac{3}{2}\right)^2 x^3\eta^2 = 0 \end{cases}.$$

Let $\pi_N: T^*\langle\mathbb{C}^2_{x,y}/N\rangle \longrightarrow \mathbb{C}^2_{x,y}$ be the canonical projection,

$$\pi_N\left(\mathrm{Char}_N(\mathcal{M})\backslash T^*_{\mathbb{C}^2_{x,y}}\langle\mathbb{C}^2_{x,y}/N\rangle\right) = \{y^2 - x^3 = 0\}.$$

Once again,

$$\mathrm{Char}_N(\mathcal{M}) = T^*_{\{y^2-x^3=0\}}\langle\mathbb{C}^2_{x,y}/N\rangle \cup T^*_{\mathbb{C}^2_{x,y}}\langle\mathbb{C}^2_{x,y}/N\rangle.$$

3. A Quick Recap on the Resolution of Singularities of Plane Curves

The blow up of $\mathbb{C}^2_{x,y}$ at the origin, $\widetilde{\mathbb{C}^2_{x,y}}$, is the gluing of the charts $\mathbb{C}^2_{x_1,y_1}$ and $\mathbb{C}^2_{x_2,y_2}$ by the change of coordinates

$$\begin{cases} x_2 = {y_1}^{-1} \\ y_2 = x_1 y_1 \end{cases} .$$

The restrictions of the blow up map $\sigma\colon \widetilde{\mathbb{C}^2_{x,y}} \longrightarrow \mathbb{C}^2_{x,y}$ are given by

$$\sigma|_{\mathbb{C}^2_{x_1,y_1}}(x_1, y_1) = (x_1, x_1 y_1),$$

$$\sigma|_{\mathbb{C}^2_{x_2,y_2}}(x_2, y_2) = (x_2 y_2, y_2).$$

The exceptional divisor of the blow up, $E = \sigma^{-1}(0)$, is defined by

$$E \cap \mathbb{C}^2_{x_1,y_1} = \{x_1 = 0\},$$

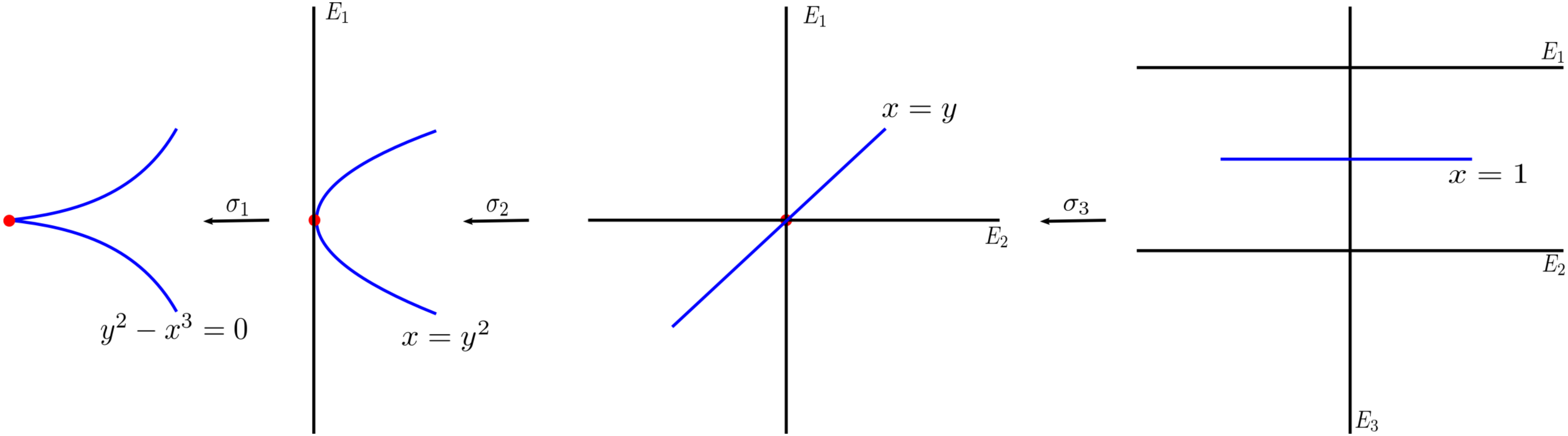
$$E \cap \mathbb{C}^2_{x_2,y_2} = \{y_2 = 0\},$$

and isomorphic to a complex projective space of dimension one.

The blow up map is a isomorphism when restricted to $\widetilde{\mathbb{C}^2_{x,y}} \backslash E$.

The strict transform of an analytic subset of $\mathbb{C}^2_{x,y}$ is the closure in $\widetilde{\mathbb{C}^2_{x,y}}$ of the set

$$\sigma^{-1}(S) \backslash E.$$


E_1
E_1
E_1
$x = y$
$x = 1$
σ_1
σ_2
σ_3
E_2
E_2
$y^2 - x^3 = 0$
$x = y^2$
E_3

4. Blow Up of a *D*-module

Let us now study the evolution of our example along the resolution of $\{y^2 - x^3 = 0\}$. Set $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{M}_i$ as the blow up of $\mathcal{M}_{i-1}$, $i = 1,2,3$. Set N_i as the system of exceptional divisors after σ_i, $i = 1,2,3$. From the blow up map we have the following relations:

- On $\mathbb{C}^2_{x_1,y_1}$

$$x = x_1,$$
$$y = x_1 y_1,$$
$$x\partial_y = \partial_{y_1}, \qquad \partial_y \longmapsto \frac{1}{x_1}\partial_{y_1},$$
$$x\partial_x = x_1\partial_{x_1} - y_1\partial_{y_1}.$$

- On $\mathbb{C}^2_{x_2,y_2}$

$$x = x_2 y_2,$$
$$y = y_2,$$
$$y\partial_x = \partial_{x_2}, \qquad \partial_x \longmapsto \frac{1}{y_2}\partial_{x_2},$$
$$y\partial_y = y_2\partial_{y_2} - x_2\partial_{x_2}.$$

We have a morphism

$$\sigma^{-1}\mathcal{D}_{\mathbb{C}^2_{x,y}} \longrightarrow \mathcal{D}_{\widetilde{\mathbb{C}^2_{x,y}}}[E]$$

For simplicity's sake we will "reset" the coordinates after the blow up.

First blow up: $\mathcal{M}_1$ is defined by

$$\begin{cases} (2x\partial_x + y\partial_y - \lambda)u = 0 \\ \left((x\partial_x - y\partial_y - 1)(x\partial_x - y\partial_y) - \left(\frac{3}{2}\right)^2 x\partial_y^2 \right) u = 0 \end{cases},$$

$$\pi_{N_1}\left(\mathrm{Char}_{N_1}(\mathcal{M}_1) \backslash T^*_{\mathbb{C}^2_{x,y}}\langle \mathbb{C}^2_{x,y}/N_1 \rangle\right) = \{x - y^2 = 0\},$$

$$\mathrm{Char}_{N_1}(\mathcal{M}_1) = T^*_{\{x-y^2=0\}}\langle \mathbb{C}^2_{x,y}/N_1 \rangle \cup T^*_{\mathbb{C}^2_{x,y}}\langle \mathbb{C}^2_{x,y}/N_1 \rangle.$$

Second blow up: $\mathcal{M}_2$ is defined by

$$\begin{cases} \left(x\partial_x + y\partial_y - \lambda\right)u = 0 \\ \left(y\left(2x\partial_x - y\partial_y - 1\right)\left(2x\partial_x - y\partial_y\right) - \left(\frac{3}{2}\right)^2 x\left(y\partial_y - x\partial_x - 1\right)\left(y\partial_y - x\partial_x\right) \right) u = 0 \end{cases},$$

$$\pi_{N_2}\left(\mathrm{Char}_{N_2}(\mathcal{M}_2) \backslash T^*_{\mathbb{C}^2_{x,y}}\langle \mathbb{C}^2_{x,y}/N_2 \rangle\right) = \{x - y = 0\},$$

$$\mathrm{Char}_{N_2}(\mathcal{M}_2) = T^*_{\{x-y=0\}}\langle \mathbb{C}^2_{x,y}/N_2 \rangle \cup T^*_{\mathbb{C}^2_{x,y}}\langle \mathbb{C}^2_{x,y}/N_2 \rangle.$$

Third blow up: $\mathcal{M}_3$ is defined by

$$\begin{cases} \left(y\partial_y - \lambda\right)u = 0 \\ y\left(\left(3x\partial_x - y\partial_y - 1\right)\left(3x\partial_x - y\partial_y\right) - \left(\frac{3}{2}\right)^2 x\left(y\partial_y - 2x\partial_x - 1\right)\left(y\partial_y - 2x\partial_x\right)\right)u = 0 \end{cases}.$$

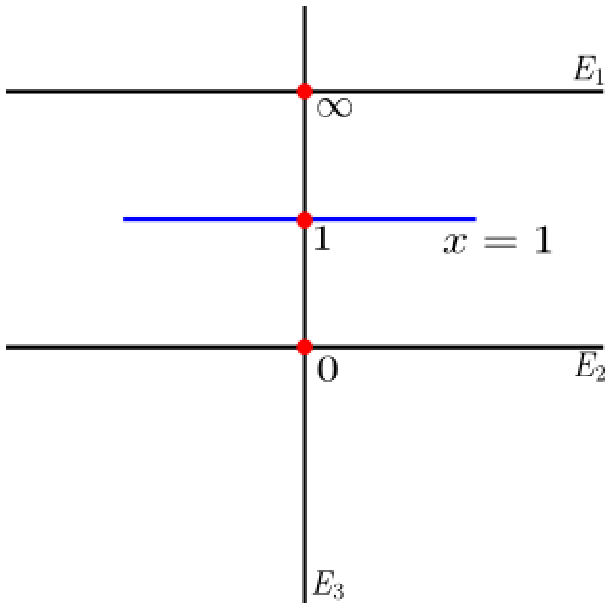
This system can be simplified into

$$\begin{cases} \left(y\partial_y - \lambda\right)u = 0 \\ \left((9x^2\partial_x^2 - 6(\lambda - 1)x\partial_x + \lambda^2 + \lambda) - \left(\frac{3}{2}\right)^2 x(4x^2\partial_x^2 - (4\lambda - 6)x\partial_x + \lambda^2 - \lambda)\right)u = 0 \end{cases}.$$

We obtain a system where each operator depends only on one variable!

$$\begin{cases} (y\partial_y - \lambda)u = 0 \\ P(x, \partial_x)u = 0 \end{cases}.$$

P is an hypergeometric differential operator defined along E_3 with singularities at 0,1 and ∞.



Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_M[N]$-module, $\widetilde{\mathcal{M}}$ the blow up of $\mathcal{M}$ with center L and $\widetilde{N}$ the inverse image of N by the blow up. We have that $\mathrm{Char}_{\widetilde{N}}(\widetilde{\mathcal{M}})$ is the strict transform of $\mathrm{Char}_N(\mathcal{M})$ by the blow up of $\mathbb{P}^*\langle M/N\rangle$ with center Λ associated to L.

When we desingularize legendrian varieties we are "desingularizing the characteristic variety of $\mathcal{D}_M$-modules".

5. Desingularization of Legendrian Curves

Given a plane curve Y of $\left(\mathbb{C}^2_{x,y}, 0\right)$ with multiplicity k and tangent cone transversal to $\{x = 0\}$, there are a positive integer n and $\alpha \in \mathbb{C}\{t\}$ such that $\alpha(0) \neq 0$, $(n, k) = 1$ and Y admits the local parameterization

$$x = t^k, y = t^n \alpha(t).$$

The plane curve $\{y^2 - x^3 = 0\}$ admits the parameterization

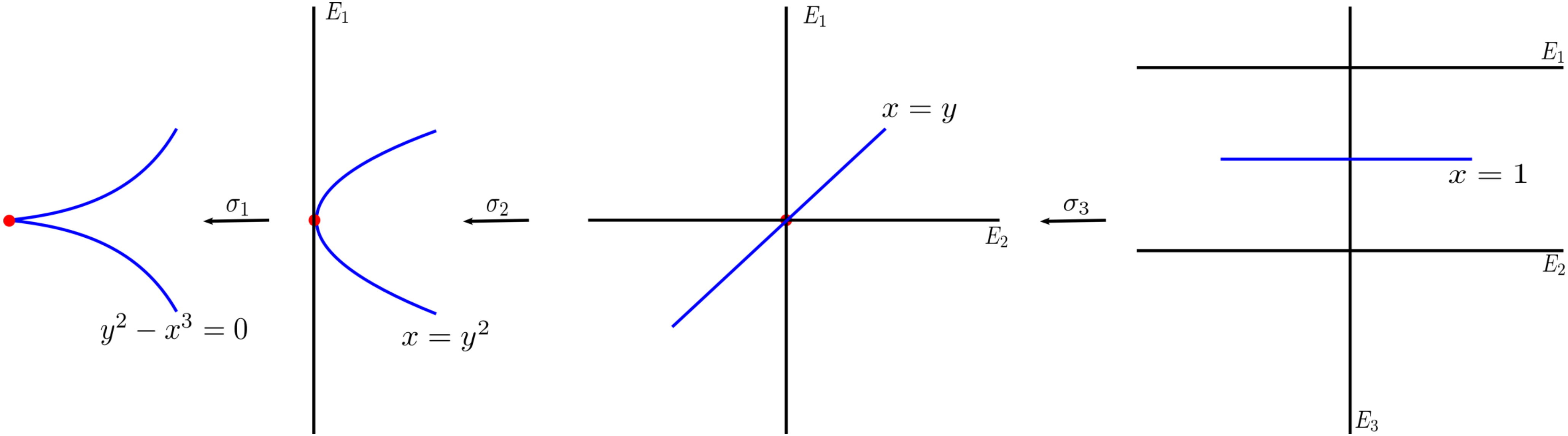
$$\begin{cases} x = t^2 \\ y = t^3 \end{cases}.$$

Let $\pi: \mathbb{P}^*\mathbb{C}^2_{x,y} \longrightarrow \mathbb{C}^2_{x,y}$ be the canonical projection. Using the parameterization of Y we obtain a parameterization of $\mathbb{P}^*_Y\mathbb{C}^2_{x,y}$ defined on neighbourhood of the point

$$\mathbb{P}^*_Y\mathbb{C}^2_{x,y} \cap \pi^{-1}(0).$$

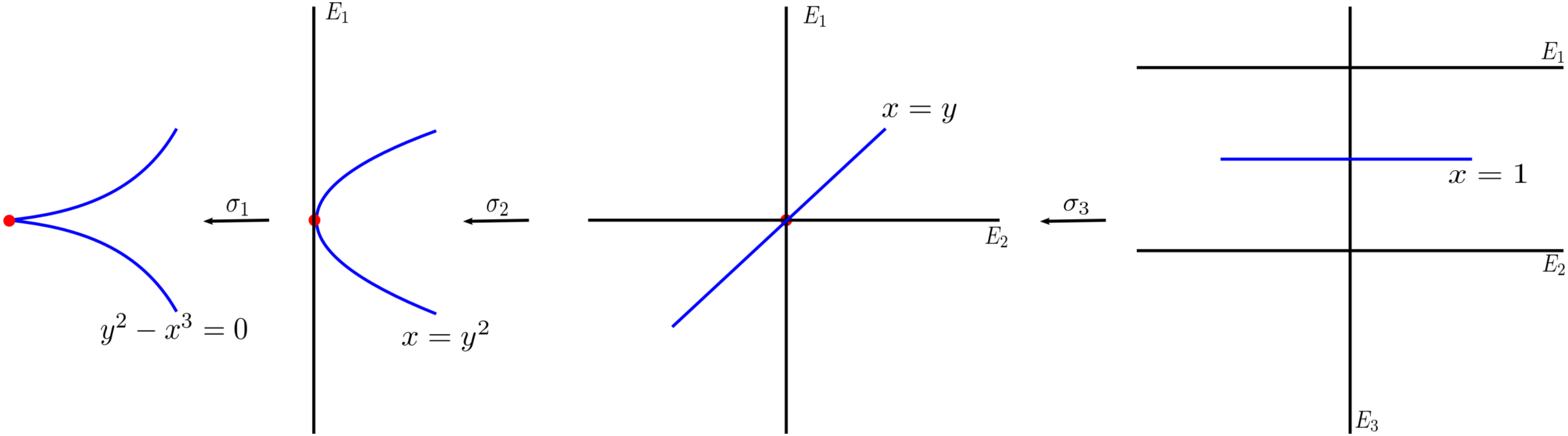
In the case of $\{y^2 - x^3 = 0\}$, we have on the chart $\left(\mathbb{C}^3_{x,y,p}, \langle dy - pdx \rangle\right)$ of $\mathbb{P}^*\mathbb{C}^2_{x,y}$

$$\begin{cases} x = t^2 \\ y = t^3 \\ p = \dfrac{y'(t)}{x'(t)} = \dfrac{3}{2}t \end{cases}.$$


$y^2 - x^3 = 0$
σ_1
E_1
$x = y^2$
σ_2
E_1
$x = y$
E_2
σ_3
E_1
$x = 1$
E_2
E_3

First blow up: Set $Y_0 = \{y^2 - x^3 = 0\}$, $Y_1 = \{x - y^2 = 0\}$. The following diagram commutes

$$\begin{array}{ccc}
\mathbb{P}^*_{Y_0}\mathbb{C}^2_{x,y} \subset \mathbb{P}^*\mathbb{C}^2_{x,y} & \xleftarrow{\rho_{\pi^{-1}(0)}} & \widetilde{\mathbb{P}^*\mathbb{C}^2_{x,y}} \simeq \mathbb{P}^*\left\langle \widetilde{\mathbb{C}^2_{x,y}}/N_1 \right\rangle \supset \widetilde{\mathbb{P}^*_{Y_0}\mathbb{C}^2_{x,y}} \simeq \mathbb{P}^*_{Y_1}\left\langle \widetilde{\mathbb{C}^2_{x,y}}/N_1 \right\rangle \\
\downarrow \pi & & \downarrow \pi_{N_1} \\
Y_0 \subset \mathbb{C}^2_{x,y} & \xleftarrow{\sigma_1} & \widetilde{\mathbb{C}^2_{x,y}} \supset Y_1
\end{array}$$

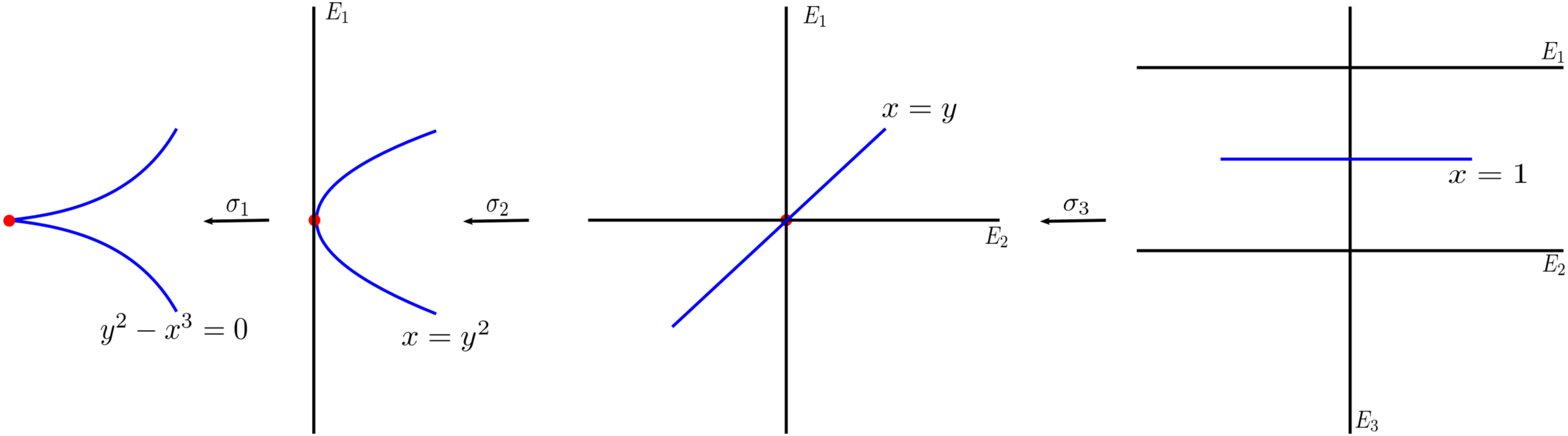

E_1
σ_1
$y^2 - x^3 = 0$
$x = y^2$
σ_2
E_1
$x = y$
E_2
σ_3
E_1
$x = 1$
E_2
E_3

Second blow up: Set $Y_2 = \{x - y = 0\}$. The following diagram commutes

$$\begin{array}{ccccccc}
\Lambda = \{x = y = p = 0\} \subset \mathbb{P}^* \left\langle \mathbb{C}^2_{x,y}/N_1 \right\rangle & \xleftarrow{\rho_\Lambda} & \mathbb{P}^* \left\langle \widetilde{\mathbb{C}^2_{x,y}}/N_1 \right\rangle & \hookleftarrow & \Omega & \xhookrightarrow{\varphi} & \mathbb{P}^* \left\langle \widetilde{\mathbb{C}^2_{x,y}}/N_2 \right\rangle \\
\downarrow \pi_{N_1} & & & & & & \downarrow \pi_{N_2} \\
\{x = y = 0\} \subset \mathbb{C}^2_{x,y} & & & \xleftarrow{\sigma_2} & & & \widetilde{\mathbb{C}^2_{x,y}}
\end{array}$$

Also

$$\varphi\left(\mathbb{P}^*_{Y_1}\langle\widetilde{\mathbb{C}^2_{x,y}}/N_1\rangle\right) = \mathbb{P}^*_{Y_2}\langle\widetilde{\mathbb{C}^2_{x,y}}/N_2\rangle.$$


$y^2 - x^3 = 0$
σ_1
E_1
$x = y^2$
σ_2
E_1
$x = y$
E_2
σ_3
E_1
$x = 1$
E_2
E_3

Third blow up: Set $Y_3 = \{x - 1 = 0\}$. The following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(L) \subset \mathbb{P}^* \left\langle \mathbb{C}^2_{x,y}/N_2 \right\rangle & \xleftarrow{\rho_{\pi^{-1}(L)}} & \mathbb{P}^* \left\langle \widetilde{\mathbb{C}^2_{x,y}/N_2} \right\rangle \simeq \mathbb{P}^* \left\langle \widetilde{\mathbb{C}^2_{x,y}}/N_3 \right\rangle \\ \downarrow \pi_{N_2} & & \downarrow \pi_{N_3} \\ L = E_1 \cap E_2 \subset \mathbb{C}^2_{x,y} & \xleftarrow{\sigma_3} & \widetilde{\mathbb{C}^2_{x,y}} \end{array}$$

Once again,

$$\mathbb{P}^*_{Y_2} \langle \widetilde{\mathbb{C}^2_{x,y}}/N_2 \rangle \simeq \mathbb{P}^*_{Y_3} \langle \widetilde{\mathbb{C}^2_{x,y}}/N_3 \rangle.$$

6. Desingularization of Singularities of Quasi-Ordinary Surfaces

Lipman proved a desingularization theorem for quasi-ordinary surfaces.

Ban and Mcewan gave an embedded version of Lipman´s result.

One can completely describe the combinatory of the constructive desingularization theorem for quai-ordinary surfaces.

Let S be a quasi-ordinary surface defined by

$$z = x^{\lambda} y^{\mu} H\left(x^{\frac{1}{m}}, y^{\frac{1}{m}}\right), (\lambda, \mu) \in \mathbb{Q}^2 \backslash \mathbb{Z}^2.$$

We will blow up with center $\{x = y = z = 0\}$ or $\{x = z = 0\}$ or $\{y = z = 0\}$, depending on:

the exceptional divisors, the global structure (reflected locally in the "history" of the divisors) and the values of λ and μ

7. Resolution of Singularities of Quasi-Ordinary Legendrian Surfaces

Theorem (6.1): Let N be a normal crossings divisor of a complex manifold M. Let L be the intersection of at least two irreducible components of N. Let $\sigma: \widetilde{M} \longrightarrow M$ be the blow up of M along L.

1. The blow up of $\mathbb{P}^*\langle M/N\rangle$ along $\pi_N^{-1}(L)$ is a logarithmic contact manifold isomorphic to $\mathbb{P}^*\langle \widetilde{M}/\sigma^{-1}(N)\rangle$ and the diagram commutes

$$\begin{array}{ccc} \mathbb{P}^*\langle M/N\rangle & \longleftarrow & \mathbb{P}^*\left\langle \widetilde{M}/\sigma^{-1}(N)\right\rangle \\ \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi_{\sigma^{-1}(N)}} \\ M & \xleftarrow{\ \sigma\ } & \widetilde{M} \end{array}$$

2. If S is a hypersurface of M that is not an irreducible component of N and $\tilde{S}$ is the strict transform of S by σ,

$$\mathbb{P}^*_S\widetilde{\langle M/N\rangle} \simeq \mathbb{P}^*_{\tilde{S}}\langle \widetilde{M}/\sigma^{-1}(N)\rangle.$$

When blowing up $\mathbb{P}^*\langle M/N\rangle$ along the conormal of a center on M, a problem arises:

The strict transform of the conormal of a hypersurface that has not trivial logarithmic limits of tangents might not be a legendrian hypersurface!

Theorem (6.2): Let N be a normal crossings divisor of a complex manifold M of dimension 3. Let S be a quasi-ordinary surface of M. Let L be an admissible center of M that is not the intersection of irreducible components of N. Let Let $\sigma\colon \widetilde{M} \longrightarrow M$ be the blow up of M along L and $\widetilde{N} = \sigma^{-1}(N)$. Let $\rho\colon \mathbb{P}^*\widetilde{\langle M/}N\rangle \longrightarrow \mathbb{P}^*\langle M/N\rangle$ be the blow up of $\mathbb{P}^*\langle M/N\rangle$ along $\mathbb{P}^*_L\langle M/N\rangle$. The following diagram commutes:

$$\begin{array}{ccccccc}
\mathbb{P}^*_L\langle M/N\rangle \subset \mathbb{P}^*\langle M/N\rangle & \xleftarrow{\rho} & \mathbb{P}^*\widetilde{\langle M/N\rangle} & \hookleftarrow & \Omega & \overset{\varphi}{\hookrightarrow} & \mathbb{P}^*\left\langle \widetilde{M}/\widetilde{N}\right\rangle \\
\downarrow{\scriptstyle \pi_N} & & & & & & \downarrow{\scriptstyle \pi_{\widetilde{N}}} \\
L, S \subset M & & & \xleftarrow{\quad\quad\sigma\quad\quad} & & & \widetilde{M}
\end{array}$$

Let $\tilde{S}$ be the strict transform of S by σ. If S trivial logarithmic limits of tangents and the normal cone of $\mathbb{P}^*_S\langle M/N\rangle$ along $\mathbb{P}^*_L\langle M/N\rangle$ is "well behaved",

$$\mathbb{P}^*_S\widetilde{\langle M/N\rangle} \subset \Omega$$

and

$$\varphi\left(\mathbb{P}^*_S\widetilde{\langle M/N\rangle}\right) = \mathbb{P}^*_{\tilde{S}}\langle \widetilde{M}/\widetilde{N}\rangle.$$

Moreover, $\mathbb{P}^*_{\tilde{S}}\langle \widetilde{M}/\widetilde{N}\rangle$ has trivial logarithmic limits of tangents and its normal cone is "well behaved".

Let $\tilde{S}$ be the strict transform of S by σ. If S trivial logarithmic limits of tangents and the normal cone of $\mathbb{P}_S^*\langle M/N\rangle$ along $\mathbb{P}_L^*\langle M/N\rangle$ is “well behaved”,

$$\mathbb{P}_S^*\widetilde{\langle M/N\rangle} \subset \Omega$$

and

$$\varphi\left(\mathbb{P}_S^*\widetilde{\langle M/N\rangle}\right) = \mathbb{P}_{\tilde{S}}^*\langle\widetilde{M}/\widetilde{N}\rangle.$$

Moreover, $\mathbb{P}_{\tilde{S}}^*\langle\widetilde{M}/\widetilde{N}\rangle$ has trivial logarithmic limits of tangents and its normal cone is “well behaved”… Most of the time…

Problems and generalizations: At the moment we only know how to compute the limits of tangents of quasi-ordinary hypersurfaces. If we solve the problem for an arbitrary hypersurface, we could generalize the result for non quasi-ordinary hypersurfaces.