Functorial properties of $\mathscr{D}[[\hbar]]$-modules

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Notations

- X denotes a complex analytic manifold of dimension d_X
- $\mathscr{O}_X$ denotes the ring of holomorphic functions on X
- $\mathscr{D}_X$ denotes the ring of holomorphic differential operators on X
- Ω_X denotes the sheaf of differential forms of maximal degree on X
- $\mathrm{SS}(F)$ denotes the microsupport of a sheaf F
- $\mathrm{char}(\mathscr{M})$ denotes the characteristic variety of a coherent $\mathscr{D}_X$-module $\mathscr{M}$
- $\mathbb{C}^\hbar := \mathbb{C}[[\hbar]]$ denotes the ring of formal power series in $\hbar$
- $\mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar))$ denotes the field of Laurent series in $\hbar$

Formal extensions

Consider the left exact functor:

$$\begin{aligned}(\,\bullet\,)^{\hbar} : \mathrm{Mod}(\mathbb{C}_X) &\to \mathrm{Mod}(\mathbb{C}_X^{\hbar}),\\ F &\mapsto F^{\hbar} := \varprojlim_{n}(F \otimes \mathbb{C}_X^{\hbar}/\hbar^n\mathbb{C}_X^{\hbar}).\end{aligned}$$

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Proposition: [A. D'Agnolo, S. Guillermou, P. Schapira]

The objects belonging to the categories $\mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$, $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)$ and $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ are acyclic for the functor $(\bullet)^\hbar$.

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Lemma:

For $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$ we have an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$:

$$\mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(F, G^{\mathrm{R}\hbar}) \simeq \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(F, G)^{\mathrm{R}\hbar}.$$

Cohomologically complete objects

Definition:

One says that $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$ is cohomologically complete if $\mathrm{R}\mathscr{H}om_{\mathbb{C}_X^{\hbar}}(\mathbb{C}_X^{\hbar,\mathrm{loc}}, F)$ vanishes.

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3. If $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_X)$ is cohomologically complete and $f : X \to Y$ is a morphism of complex manifolds, then $\mathrm{R}f_*F$ is cohomologically complete.

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3. If $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$ is cohomologically complete and $f : X \to Y$ is a morphism of complex manifolds, then $\mathrm{R}f_* F$ is cohomologically complete.
4. The formal extension $F^{\mathrm{R}\hbar}$ of any $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$ is cohomologically complete.

Cohomologically complete objects

Let $f : X \to Y$ be a morphism of complex manifolds and consider the natural maps

$$f_\pi : X \times_Y T^*Y \to T^*Y, \quad f_d : X \times_Y T^*Y \to T^*X,$$

and set $T^*_X Y := f_d^{-1}(T^*_X X) \subset X \times_Y T^*Y$.
f is said to be non-characteristic for $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar{}_Y)$ if

$$T^*_X Y \cap f_\pi^{-1}(SS(F)) \subset X \times_Y T^*_Y Y.$$

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Lemma:

If f is non-characteristic for a cohomologically complete object $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}{}_Y)$, then $f^{-1}F$ is also cohomologically complete.

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If f is non-characteristic for a cohomologically complete object $F \in \mathsf{D}^{\mathrm{b}}({\mathbb{C}^\hbar}_Y)$, then $f^{-1}F$ is also cohomologically complete.

Proof.

The result follows from the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_X}(\mathbb{C}_X^{\hbar,\mathrm{loc}}, f^!\mathscr{M}) \simeq f^!\mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_Y}(\mathbb{C}_Y^{\hbar,\mathrm{loc}}, \mathscr{M}).$$

Graduate functor

Consider also the right exact functor that associates to each $F \in \mathrm{Mod}(\mathbb{C}_X^\hbar)$ the quotient $F/\hbar F \in \mathrm{Mod}(\mathbb{C}_X)$.
Its left derived functor is called the graduate functor:

$$\begin{aligned} \mathrm{gr}_\hbar : \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar) &\to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), \\ F &\mapsto F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} \mathbb{C}_X. \end{aligned}$$

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(a) The functor $\mathrm{gr}_\hbar$ is conservative when restricted to cohomologically complete objects.

(b) Assume that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^\hbar)$ is cohomologically complete and that $\mathrm{gr}_\hbar(\mathscr{M})$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Then, $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$.

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(c) Assume that $\mathscr{M}$ is a cohomologically complete $\mathscr{D}_X^\hbar$-module with no $\hbar$-torsion such that $\mathscr{M}/\hbar\mathscr{M}$ is flat over $\mathscr{D}_X$. Then $\mathscr{M}$ is flat over $\mathscr{D}_X^\hbar$.

Contents

Extension of functors for algebras of formal deformations

Let X and X' be complex manifolds and assume that $\mathcal{B}'$ is a *suitable* basis of compact subsets of X'. Assume also that $\mathcal{S}$ (resp. $\mathcal{S}'$) is a full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}^{\hbar}_X)$ (resp. $\mathcal{S}'$ is full abelian subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}^{\hbar}_{X'})$ satisfying Condition 1).

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Condition 1:

Each $\mathscr{M} \in \mathcal{S}'|_U$ admits for each $y \in U$ an open neighborhood V of y, such that for each submodule $\mathscr{N}$ of $\mathscr{M}|_V$, if $K \in \mathcal{B}'$ is contained in V then $H^j(K;\mathscr{N}) = 0$ for any $j > 0$.

Theorem: [A.R.Martins, T.M. Fernandes, D.R.]

Let $F : \mathcal{S} \to \mathcal{S}'$ be a right exact functor. Then F defines canonically a right exact functor

$$F^\hbar : \mathrm{Mod}_{\mathcal{S}}(\mathscr{D}_X^\hbar) \to \mathrm{Mod}(\mathscr{D}_{X'}^\hbar), \quad \mathscr{M} \mapsto F^\hbar := \varprojlim_n F(\mathscr{M}_n),$$

and if $\mathcal{S}'$ is a subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{X'}^\hbar)$ then $F^\hbar$ takes values in $\mathrm{Mod}_{\mathcal{S}'}(\mathscr{D}_X^\hbar)$. Moreover, when F is exact, $F^\hbar$ is also exact.

The inverse image extension

Let $f : X \to Y$ be a morphism of complex manifolds. Consider the right exact functor in the $\mathscr{D}$-modules framework:

$$\begin{aligned} \underline{f}^* : \mathrm{Mod}(\mathscr{D}_Y) &\to \mathrm{Mod}(\mathscr{D}_X), \\ \mathscr{M} &\mapsto \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M}. \end{aligned}$$

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For our purpose we can consider $\mathcal{S}'$ as the whole category of $\mathscr{D}_X$-modules that satisfy Condition 1.

Proposition:

If $\mathscr{M}$ is a coherent $\mathscr{D}_Y$-module, then $\underline{f}^*(\mathscr{M})$ is an object of $\mathcal{S}'$.

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Corollary:

The functor $\underline{f}^*$ extends canonically to a right exact functor:

$$\begin{aligned} \underline{f}^{*,\hbar} : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^{\hbar}) &\to \mathrm{Mod}(\mathscr{D}_X^{\hbar}), \\ \mathscr{M} &\mapsto \underline{f}^{*,\hbar}(\mathscr{M}) := \varprojlim_n(\underline{f}^*(\mathscr{M}_n)). \end{aligned}$$

Transfer module

Let $f : X \to Y$ be a morphism of complex manifolds and consider the object:

$$\mathscr{K} := \mathscr{D}_{X \to Y,\hbar} = \underline{f}^{*,\hbar}(\mathscr{D}_Y^\hbar) = \varprojlim_n (\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}(\mathscr{D}^\hbar_{Y\,n})).$$

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4. For each $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y^{\hbar})$, $\mathscr{K} \overset{\mathrm{L}}{\otimes}_{f^{-1}(\mathscr{D}_X^{\hbar})} f^{-1}\mathscr{M}$ is cohomologically complete.

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5. For each $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^{\hbar})$, $\underline{f}^{*,\hbar}(\mathscr{M})$ is cohomologically complete.

Transfer module

Proof:

1. In fact the components of the projective limits coincide, that is, for each n we have: $\mathscr{D}_{X\longrightarrow Y}\otimes_{\mathbb{C}_X}\mathbb{C}^{\hbar}_X/\hbar^n\mathbb{C}^{\hbar}_X\simeq\mathscr{O}_X\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}(\mathscr{D}^{\hbar}_{Y\,n})$.

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4. We can replace $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^{\hbar}_Y)$ by a local free resolution of finite rank: $\mathscr{D}^{\hbar,\bullet}_Y\overset{\mathrm{qis}}{\to}\mathscr{M}$. Then we obtain a quasi-isomorphism:

$$\mathscr{K}^{\bullet}\overset{\mathrm{qis}}{\to}\mathscr{K}\overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}^{\hbar}_Y}f^{-1}\mathscr{M}$$

and the result follows from 3.

5. Since $\mathscr{M}$ is a coherent $\mathscr{D}_Y^{\hbar}$-module we may consider a finite presentation of $\mathscr{M}$:

$$\mathscr{D}_Y^{\hbar,\oplus M} \to \mathscr{D}_Y^{\hbar,\oplus N} \to \mathscr{M} \to 0.$$

Applying the right exact functor $\underline{f}^{*,\hbar}$ we obtain the exact sequence below:

$$\mathscr{K}^{\oplus M} \to \mathscr{K}^{\oplus N} \to \underline{f}^{*,\hbar}(\mathscr{M}) \to 0.$$

Since the category of cohomologically complete modules is thick, the result follows from 3.

The inverse image in the derived category

Proposition:

For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar})$, we have an isomorphism in $\mathrm{Mod}(\mathscr{D}_Y^{\hbar})$:

$$\mathscr{K} \otimes_{f^{-1}(\mathscr{D}_X^{\hbar})} f^{-1}\mathscr{M} \xrightarrow{\sim} \underline{f}^{*,\hbar}(\mathscr{M}). \qquad (1)$$

Proof:

Step 1: first we construct morphisms

$$\mathscr{K} \otimes_{f^{-1}\mathscr{O}_Y^{\hbar}} f^{-1}\mathscr{M} \to \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M}_n.$$

Applying the universal property of projective limits we get the canonical morphism (1).

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Applying the universal property of projective limits we get the canonical morphism (1).

Step 2: if $\mathscr{M}$ is a free $\mathscr{D}_X^{\hbar}$-module of finite rank, say $\mathscr{D}_X^{\hbar,\oplus M}$, then (1) is an isomorphism since both sides are isomorphic to $\mathscr{K}^{\oplus M}$.

The inverse image in the derived category

Step 3: to prove the general case we take a finite presentation of $\mathscr{M}$:

$$\mathscr{D}_Y^{\hbar,\oplus M} \to \mathscr{D}_Y^{\hbar,\oplus N} \to \mathscr{M} \to 0.$$

Applying the right exact functors $\mathscr{K} \otimes_{f^{-1}\mathscr{D}_Y^{\hbar}} f^{-1}\bullet$ and $\underline{f}^{*,\hbar}$, we get a commutative diagram:

$$\begin{array}{ccccccc}
\mathscr{K}^{\oplus M} & \longrightarrow & \mathscr{K}^{\oplus N} & \longrightarrow & \mathscr{K} \otimes_{f^{-1}\mathscr{D}_Y^{\hbar}} f^{-1}\mathscr{M} & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr \\
\mathscr{K}^{\oplus M} & \longrightarrow & \mathscr{K}^{\oplus N} & \longrightarrow & \underline{f}^{*,\hbar}(\mathscr{M}) & \longrightarrow & 0
\end{array}$$

The conclusion follows by the five lemma.

The inverse image in the derived category

Step 3: to prove the general case we take a finite presentation of $\mathscr{M}$:

$$\mathscr{D}_Y^{\hbar,\oplus M} \to \mathscr{D}_Y^{\hbar,\oplus N} \to \mathscr{M} \to 0.$$

Applying the right exact functors $\mathscr{K} \otimes_{f^{-1}\mathscr{D}_Y^{\hbar}} f^{-1}\bullet$ and $\underline{f}^{*,\hbar}$, we get a commutative diagram:

$$\begin{array}{ccccccc} \mathscr{K}^{\oplus M} & \longrightarrow & \mathscr{K}^{\oplus N} & \longrightarrow & \mathscr{K} \otimes_{f^{-1}\mathscr{D}_Y^{\hbar}} f^{-1}\mathscr{M} & \longrightarrow & 0 \\ \downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr \\ \mathscr{K}^{\oplus M} & \longrightarrow & \mathscr{K}^{\oplus N} & \longrightarrow & \underline{f}^{*,\hbar}(\mathscr{M}) & \longrightarrow & 0 \end{array}$$

The conclusion follows by the five lemma.

Corollary:

The functor $\underline{f}^{*,\hbar}$ is left derivable and for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar})$ we have:

$$\mathbb{L}\underline{f}^{*,\hbar}(\mathscr{M}) \simeq \mathscr{K} \overset{\mathrm{L}}{\otimes}_{f^{-1}(\mathscr{D}_X^{\hbar})} f^{-1}\mathscr{M}.$$

The case of non-characteristic modules

Recall that one says that f is non-characteristic for a coherent $\mathscr{D}_Y$-module $\mathscr{M}$ if:

$$T^*_X Y \cap f_\pi^{-1} \operatorname{char}(\mathscr{M}) \subset X \times_Y T^*_Y Y.$$

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Denote by $NC(f)$ the full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$ consisting of non-characteristic $\mathscr{D}_Y$-modules for f.
It is well-known that the functor $\underline{f}^*$ restricted to $NC(f)$ is an exact functor that takes values in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$.

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It is well-known that the functor $\underline{f}^*$ restricted to $NC(f)$ is an exact functor that takes values in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$.

Corollary:

We have an exact functor $\underline{f}^{*,\hbar} : \mathrm{Mod}_{NC(f)}(\mathscr{D}^{\hbar}_Y) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}^{\hbar}_X)$.

Cauchy-Kowalevsky-Kashiwara theorem

Theorem: [A.R. Martins, T. Monteiro Fernandes, D.R.]

Assume that $\mathscr{M}$ is a non-characteristic $\mathscr{D}_Y^{\hbar}$-module. Then one has a natural isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$:

$$f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y^{\hbar}}(\mathscr{M}, \mathscr{O}_Y^{\hbar}) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar}}(\underline{f}^{*,\hbar}(\mathscr{M}), \mathscr{O}_X^{\hbar}). \tag{2}$$

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Proof:

Step 1: we define the morphism (2) using the canonical morphism

$$f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y^{\hbar}}(\mathscr{M}, \mathscr{O}_Y^{\hbar}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar}}(\mathscr{K}\overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y^{\hbar}} f^{-1}\mathscr{M}, \mathscr{K}\overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y^{\hbar}} f^{-1}\mathscr{O}_Y^{\hbar})$$

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and proving that the inverse image of $\mathscr{O}_Y^{\hbar}$ is $\mathscr{O}_X^{\hbar}$. Step 2: both sides of (2) are cohomologically complete. Step 3: when we apply $\mathrm{gr}_{\hbar}$ to the morphism (2) we obtain the CKK isomorphism in the $\mathscr{D}_X$-modules framework. Hence, (2) is also an isomorphism.

Extraordinary inverse image

Similarly to the case of $\mathscr{D}$-modules, one considers the duality functor:

$$\begin{aligned} \mathbb{D}_{\hbar} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^{\hbar}_X)^{\mathrm{op}} &\to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^{\hbar}_X), \\ \mathscr{M} &\mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}^{\hbar}_X}(\mathscr{M}, \mathscr{D}^{\hbar}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})[d_X]. \end{aligned}$$

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We define the so called extraordinary inverse image:

$$^{\mathbb{L}}\underline{f}^{!,\hbar} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^\hbar_Y) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}^\hbar_X), \quad \mathscr{M} \mapsto \mathbb{D}_\hbar({}^{\mathbb{L}}\underline{f}^{*,\hbar}(\mathbb{D}_\hbar(\mathscr{M}))).$$

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Recall that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^{\hbar}_X)$ is said to be holonomic if $\mathrm{gr}_{\hbar}(\mathscr{M})$ is holonomic in the $\mathscr{D}$-module sense. Hence, the functors $\mathbb{D}_{\hbar}$, ${}^{\mathbb{L}}\underline{f}^{*,\hbar}$, and ${}^{\mathbb{L}}\underline{f}^{\hbar,!}(\mathscr{M})$ preserve holonomicity as a consequence of the analogous properties for $\mathscr{D}$-modules.

Contents

Direct image

Still denote by $f : X \to Y$ a morphism of complex manifolds.
We introduce the functors of direct image and proper direct image of $\mathscr{D}^\hbar$-modules associated to f, respectively denoted by $\mathrm{R}\underline{f}_*^\hbar$, $\mathrm{R}\underline{f}_!^\hbar : \mathsf{D}^\mathrm{b}({\mathscr{D}_X^\hbar}^{\mathrm{op}}) \to \mathsf{D}^\mathrm{b}({\mathscr{D}_Y^\hbar}^{\mathrm{op}})$, and defined by:

$$\mathrm{R}\underline{f}_*^\hbar(\mathscr{M}) := Rf_*(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}),$$

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Theorem: [A.R.Martins, T.Monteiro Fernandes,D.R.]

Suppose that $\mathscr{M} \in \mathsf{D}^\mathrm{b}_{\mathrm{good}}({\mathscr{D}_X^\hbar}^{\mathrm{op}})$ (resp. $\mathscr{M} \in \mathsf{D}^\mathrm{b}_{\mathrm{hol}}({\mathscr{D}_X^\hbar}^{\mathrm{op}})$) and that f is proper when restricted to $\mathrm{supp}(\mathscr{M})$. Then,

$$\mathrm{R}\underline{f}_*^\hbar(\mathscr{M}) \in \mathsf{D}^\mathrm{b}_{\mathrm{good}}({\mathscr{D}_Y^\hbar}^{\mathrm{op}})$$

(resp. $\mathrm{R}\underline{f}_*^\hbar(\mathscr{M}) \in \mathsf{D}^\mathrm{b}_{\mathrm{hol}}({\mathscr{D}_Y^\hbar}^{\mathrm{op}})$.)

The case of a closed immersion

Consider the case of a closed submanifold $i : X \hookrightarrow Y$. In this case $\mathscr{D}_{X\to Y}$ is flat over $\mathscr{D}_X$. Hence, $\mathscr{K} \simeq \mathscr{D}^{\hbar}_{X\to Y}$ is flat over $\mathscr{D}^{\hbar}_X$.

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$$\underline{i}_* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\mathrm{op}}) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^{\mathrm{op}}), \quad \mathscr{M} \mapsto \underline{i}_*(\mathscr{M}) := i_*(\mathscr{M} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}).$$

We can extend $\underline{i}_*$ to an exact functor

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Proposition:

The extension $\underline{i}^{\hbar}_*$ coincides with the direct image previously defined.

Contents

Elliptic pairs: a generalization of elliptic systems

Definition:
Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$ and $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. The pair $(\mathscr{M}, F)$ is an f-elliptic pair (over $\mathbb{C}$) if:

$$\mathrm{char}_f(\mathscr{M}) \cap \mathrm{SS}(F) \subset T^*_X X.$$

Let $a_X : X \to \{\mathrm{pt}\}$ denote the constant map. One says that $(\mathscr{M}, F)$ is an elliptic pair if it is an a_X-elliptic pair.

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Definition:

Let M be a real analytic manifold and X a complexification of M. One says that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ is elliptic if $(\mathscr{M}, \mathbb{C}_M)$ is an elliptic pair.

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Example:
For $X = \mathbb{C}^n$ and $M = \mathbb{R}^n$, the Laplace operator Δ yields an elliptic $\mathscr{D}_X$-module $\mathscr{D}_X / \mathscr{D}_X \Delta$.

Regularity, finiteness and duality for f-elliptic pairs

Theorem: [P. Schapira, J.P. Schneiders]
Let $(\mathscr{M}, F)$ be an f-elliptic pair. Then the canonical morphism below is an isomorphism:

$$F \otimes_{\mathbb{C}_X} (\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Y}) \to \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(\mathrm{D}' F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Y}).$$

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Moreover, assume that $(\mathscr{M}, F)$ is a good f-elliptic pair with f-proper support. Then, $\underline{f}_!(\mathscr{M} \otimes F)$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y^{\mathrm{op}})$ and there is a duality isomorphism:

$$\underline{f}_!(\underline{\mathrm{D}}_X(\mathscr{M}) \otimes \mathrm{D}'F) \xrightarrow{\sim} \underline{\mathrm{D}}_Y[f_!(\mathscr{M} \otimes F)].$$

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Theorem: [P. Schapira, J.P. Schneiders]

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Assume also that:

- $\mathscr{M}$ is a good $\mathscr{D}_X$-module
- $\mathrm{supp}(\mathscr{M}) \cap \mathrm{supp}(F)$ is compact

Then, the complexes

$$\mathrm{R}\Gamma(X, \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(F, \mathscr{O}_X)))$$
$$\mathrm{R}\Gamma(X, \Omega_X[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathscr{M} \otimes_{\mathbb{C}_X} F))$$

belong to $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C})$ and are dual one to each other.

Elliptic pairs over $\mathbb{C}^\hbar$

Definitions:

(a) The $\hbar$-characteristic variety of $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ with respect to f is defined by $\mathrm{char}_{\hbar,f}(\mathscr{M}) := \mathrm{char}_f(\mathrm{gr}_\hbar(\mathscr{M}))$.

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(b) One says that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar})$ is a good $\mathscr{D}_X^{\hbar}$-module if $\mathrm{gr}_{\hbar}(\mathscr{M})$ is a good $\mathscr{D}_X$-module.

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(c) An f-elliptic pair $(\mathscr{M}, F)$ over $\mathbb{C}^\hbar$ is the data of an object $\mathscr{M}$ of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^\hbar_X)$ and an object F of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}^\hbar_X)$, such that

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Proposition:

Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^\hbar_X)$ and let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}^\hbar_X)$. Then $(\mathscr{M}, F)$ is an elliptic pair over $\mathbb{C}^\hbar$ if and only if $(\mathrm{gr}_\hbar(\mathscr{M}), \mathrm{gr}_\hbar(F))$ is an elliptic pair over $\mathbb{C}$.

Regularity for elliptic pairs over $\mathbb{C}^\hbar$

Lemma:

Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$. Then:

(a) $\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}$ is cohomologically complete.

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Theorem: [D.R.]

Let $(\mathscr{M}, F)$ be an f-elliptic pair over $\mathbb{C}^\hbar$. Then the canonical morphism below is an isomorphism:

$$F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} (\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}) \xrightarrow{r} \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(\mathrm{D}'F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}).$$

Proof:

Recall that for $G, F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$ the canonical morphism $\mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(G, \mathbb{C}_X^\hbar) \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} F \to \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(G, F)$ is an isomorphism if $\mathrm{SS}(F) \cap \mathrm{SS}(G) \subset T^*_X X$.

Regularity for elliptic pairs over $\mathbb{C}^\hbar$

Lemma:

Let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$. Then, there is a natural bifunctorial morphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$:

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Theorem: [D.R.]

Consider an elliptic pair of the form $(\mathscr{M}, F^\hbar)$ over $\mathbb{C}^\hbar$ such that $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$. Then there is a commutative diagram of isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$:

$$\begin{array}{ccc}
\mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{D}'_{\mathbb{C}^\hbar} F^\hbar \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \mathscr{O}^\hbar_X) & \xrightarrow{\bar{\alpha}} & \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, (\mathrm{D}'_{\mathbb{C}} F \otimes \mathscr{O}_X)^{\mathrm{R}\hbar}) \\
\downarrow \bar{r_1} & & \downarrow \bar{r_2} \\
\mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_X}(F^\hbar, \mathscr{O}^\hbar_X)) & \xrightarrow{\sim} & \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om(F, \mathscr{O}_X)^{\mathrm{R}\hbar}).
\end{array}$$

Finiteness for elliptic pairs over $\mathbb{C}^{\hbar}$

Theorem: [D.R.]

Let $(\mathscr{M}, F)$ be a good f-elliptic pair over $\mathbb{C}^{\hbar}$ with f-proper support. Then, $\underline{f}^{\hbar}_{*}(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^{\hbar}_X} F)$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}^{\hbar}_Y)$.

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We get that $\mathscr{N} := \underline{f}^\hbar_!(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} F)$ is cohomologically complete as a consequence of the regularity theorem.

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Applying Schapira-Schneiders finiteness result to the elliptic pair $(\mathrm{gr}_{\hbar}(\mathscr{M}), \mathrm{gr}_{\hbar}(F))$ we conclude that $\mathrm{gr}_{\hbar}(\mathscr{N})$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{Y})$.
Hence, $\mathscr{N}$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^{\hbar}_{Y})$.

Duality for elliptic pairs over $\mathbb{C}^\hbar$

From now on we shall consider the case $f = a_X$.
Denote by ω_X the dualizing sheaf on X: $\omega_X \simeq a_X^! \mathbb{C}^\hbar \simeq \mathbb{C}^\hbar[2d_X]$.
Since there is a morphism of functors $a_{X!} a_X^! \to id$, we get a canonical morphism: $\mathrm{R}\Gamma_c(X; \omega_X) \to \mathbb{C}^\hbar$.

Given a complex $I \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar)$, denote by I^* the dual
$I^* := \mathrm{RHom}_{\mathbb{C}^\hbar}(I, \mathbb{C}^\hbar)$.

Lemma:

For each $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$, there is a canonical morphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar)$:

$$\varphi : \mathrm{R}\Gamma(X, \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \Omega_X^\hbar[d_X])) \to (\mathrm{R}\Gamma_c(X; \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{O}_X^\hbar))^*.$$

Duality for elliptic pairs over $\mathbb{C}^\hbar$

Proof:

The morphism φ is defined by the sequence of morphisms below:

$$\begin{aligned}
& \mathrm{R}\Gamma(X;\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M},\Omega_X^\hbar[d_X])) \\
\to\ & \mathrm{R}\Gamma(X;\mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{O}_X^\hbar,\Omega_X^\hbar[d_X]\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{O}_X^\hbar) \\
\to\ & \mathrm{R}\Gamma(X;\mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{O}_X^\hbar,\mathbb{C}_X^\hbar[2d_X]) \\
\simeq\ & \mathrm{R}\Gamma(X;\mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{O}_X^\hbar,\omega_X) \\
\to\ & \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar}(\mathrm{R}\Gamma_c(X;\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{O}_X;\mathrm{R}\Gamma(X;\omega_X))) \\
\to\ & \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar}(\mathrm{R}\Gamma_c(X;\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{O}_X;\mathbb{C}^\hbar)) \\
=\ & (\mathrm{R}\Gamma(X;\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{O}_X))^*.
\end{aligned}$$

Duality for elliptic pairs over $\mathbb{C}^\hbar$

Theorem: [D.R.]

Let $(\mathscr{M}, F)$ be a good elliptic pair over $\mathbb{C}^\hbar$ with compact support.
Then, there is an isomorphism in $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$:

$$\varphi : \mathrm{R}\Gamma(X; \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} \mathscr{M}, \Omega_X^\hbar[d_X])) \xrightarrow{\sim} (\mathrm{R}\Gamma(X; F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{O}_X^\hbar))^*.$$

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Proof:

The morphism is obtained by applying the previous lemma to the $\mathscr{D}^\hbar_X$-module $F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \mathscr{M}$. By the finiteness theorem both sides of φ are objects of $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$. Since $\mathrm{gr}_\hbar : \mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar) \to \mathsf{D}^{\mathrm{b}}_f(\mathbb{C})$ is a conservative functor, it is enough to prove that $\mathrm{gr}_\hbar(\varphi)$ is an isomorphism. But, by construction, $\mathrm{gr}_\hbar(\varphi)$ is an isomorphism by the duality theorem of Schapira and Schneiders.

Example 1:

Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{C}^\hbar$. Then $\mathrm{SS}(\mathbb{K}_X) = T^*_X X$ and $(\mathscr{M}, \mathbb{K}_X)$ is an elliptic pair over $\mathbb{C}^\hbar$, for every $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^\hbar_X)$.

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In particular, as a corollary of finiteness and duality theorems, we conclude that if $\mathscr{M}$ is a good $\mathscr{D}^\hbar_X$-module with compact support, then the complex of global solutions $\mathrm{R}\Gamma(X, \mathrm{Sol}_\hbar(\mathscr{M}))$ has finitely generated cohomology over $\mathbb{C}^\hbar$ and it is isomorphic to the dual of the complex $\mathrm{R}\Gamma(X, \Omega_X[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})$.

Example 2:

In particular if we consider a coherent $\mathscr{O}_X^\hbar$-module $\mathscr{F}$, then $(\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X^\hbar} \mathscr{D}_X^\hbar, \mathbb{C}_X^\hbar)$ is an elliptic pair over $\mathbb{C}^\hbar$.

Example 2:

In particular if we consider a coherent $\mathscr{O}_X^\hbar$-module $\mathscr{F}$, then $(\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X^\hbar} \mathscr{D}_X^\hbar, \mathbb{C}_X^\hbar)$ is an elliptic pair over $\mathbb{C}^\hbar$.

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Since $\mathrm{char}_\hbar(\mathscr{O}_X^\hbar) = T_X^*X$, $(\mathscr{O}_X^\hbar, F)$ is an elliptic pair over $\mathbb{C}^\hbar$, for every $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X^\hbar)$.

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Example 4:

Let X be the complexification of a real analytic manifold M.

(a) We say that $\mathscr{M}$ is an elliptic $\mathscr{D}_X^\hbar$-module if $(\mathscr{M}, \mathbb{C}_M^\hbar)$ is an elliptic pair over $\mathbb{C}^\hbar$.

(b) We say that $P \in \mathscr{D}_X^\hbar$ is an elliptic operator if $\mathscr{D}_X^\hbar / \mathscr{D}_X^\hbar P$ is an elliptic $\mathscr{D}_X^\hbar$-module.

Real analytic functions and hyperfunctions with $\hbar$-parameter

Let X be the complexification of a real analytic manifold M. The classical sheaves of real analytic functions and hyperfunctions on X are respectively defined by:

$$\mathscr{A}_M := \mathbb{C}_M \otimes_{\mathbb{C}_X} \mathscr{O}_X,$$
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In the $\hbar$-framework, we can consider the sheaves:

$$\mathscr{A}_{M,\hbar} := \mathbb{C}^\hbar_M \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \mathscr{O}^\hbar_X, \quad \mathscr{A}^\hbar_M := (\mathbb{C}_M \otimes_{\mathbb{C}_X} \mathscr{O}_X)^\hbar;$$
$$\mathscr{B}_{M,\hbar} := \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_X}(\mathrm{D}'_{\mathbb{C}^\hbar}\mathbb{C}^\hbar_M, \mathscr{O}^\hbar_X), \quad \mathscr{B}^\hbar_M := \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(\mathrm{D}'_{\mathbb{C}}\mathbb{C}_M, \mathscr{O}_X)^\hbar.$$

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Proposition:

There is a monomorphism $i : \mathscr{A}_{M,\hbar} \hookrightarrow \mathscr{A}^{\hbar}_M$ and an isomorphism $\mathscr{B}_{M,\hbar} \simeq \mathscr{B}^{\hbar}_M$.

Real analytic functions and hyperfunctions with $\hbar$-parameter

Corollary: [D.R.]

Let $\mathscr{M}$ be an elliptic $\mathscr{D}_X^{\hbar}$-module on M.

(a) There is a commutative diagram of isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$:

$$\begin{array}{ccc} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar}}(\mathscr{M},\mathscr{A}_{M,\hbar}) & \xrightarrow{\sim} & \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar}}(\mathscr{M},\mathscr{A}_M^{\hbar}) \\ & \searrow^{\sim} & \downarrow{\scriptstyle\sim} \\ & & \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar}}(\mathscr{M},\mathscr{B}_M^{\hbar}). \end{array}$$

(b) If M is compact and $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^{\hbar})$, then $\mathrm{R}\Gamma(M;\mathrm{R}\mathscr{H}om_{\mathscr{D}_M^{\hbar}}(\mathscr{M},\mathscr{B}_{M,\hbar}))$ belongs to $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^{\hbar})$ and is isomorphic to the dual of $\mathrm{R}\Gamma(M;\Omega_X^{\hbar}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^{\hbar}}\mathscr{M})$.

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