Cycle spaces of flag domains and $\mathcal{D}$-modules

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This is a discussion about the following reference [FHW]:

G. Fels, A. Huckleberry and J. A. Wolf, *Cycle spaces of flag domains*. Progress in Math. 245, Birkhäuser Boston 2006.

Idea: to give an interpretation and refinement of some results therein, in a more efficient framework.

Matsuki correspondence

- G a semisimple algebraic group over $\mathbb{C}$ (e.g. $G = SL(n, \mathbb{C})$)
- G_0 a s.s. real form of G (e.g. $G_0 = SL(n, \mathbb{R})$ or $G_0 = \underset{p+q=n}{SU(p, q)}$)
- K_0 a maximal compact subgroup of G_0
  [such a K_0 exists unique up to conjugation since G_0 is semisimple]
  (e.g. $G_0 = SO(n, \mathbb{R})$ or $G_0 = SU(p) \times SU(q)$)
- K the complexified of K_0 in G
  (e.g. $K = SO(n, \mathbb{C})$ or $K = SL(p, \mathbb{C}) \times SL(q, \mathbb{C})$)

Let P be a parabolic subgroup of G (e.g. $P = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$)

$X = G/P$ generalized flag mfd (e.g. $X =$ (partial) flags in $\mathbb{C}^n$)

Then both G_0 (real) and K (complex) act on X... any relation?

Matsuki correspondence

The G_0-orbits and the K-orbits in $X = G/P$ are finitely many, and there is a **one-to-one correspondence** between them.
For a G_0-orbit D, the corresponding K-orbit C_D is the only one such that $D \cap C_D$ is nonempty and compact (i.e., is a K_0-orbit).

This correspondence:

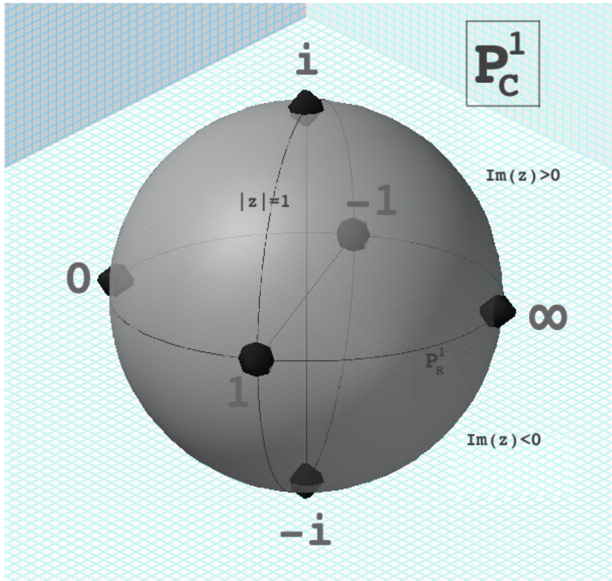
- reverses inclusions (i.e. $D_1 \subset \overline{D_2} \implies \overline{C_{D_1}} \supset C_{D_2}$);
- extends to sheaves (i.e. $\mathbf{D}^b_{K^{\mathrm{an}}}(\mathbb{C}_{X^{\mathrm{an}}}) \simeq \mathbf{D}^b_{G_0}(\mathbb{C}_{X^{\mathrm{an}}})$, [M-U-V]);
- particular cases:
  - if D is open then $\exists^!\ C_D$ compact contained in D;
  - if D is closed (i.e. compact) then $\exists^!\ C_D$ open containing D.

The basic example: $G = SL(2, \mathbb{C})$

$$G = SL(2,\mathbb{C}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$$

G acts on $X = \mathrm{P}^1_{\mathbb{C}}$ by $gz = \frac{az+b}{cz+d}$

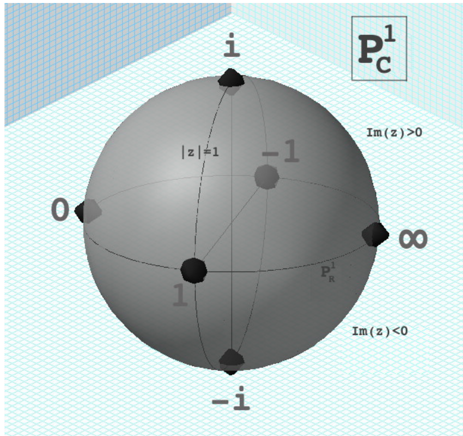
$$G_0 = SL(2,\mathbb{R}) = \left\{ g \in G : \ a, b, c, d \in \mathbb{R} \right\}$$
$$\simeq SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$$

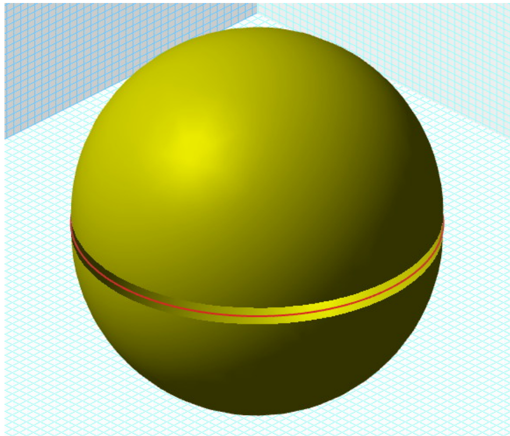


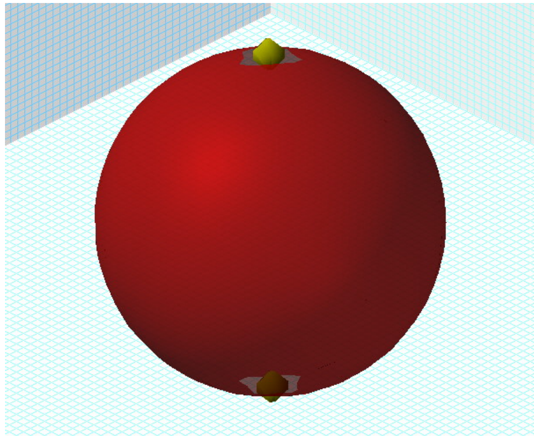
What is Matsuki correspondence in this case?

(1) Think at $G_0 = SL(2,\mathbb{R})$: then the three G_0-orbits on X are

$$D_0 = \mathrm{P}^1_{\mathbb{R}} \ \text{(closed)} \qquad \text{and} \qquad D_{\mp} = \{\operatorname{Im} z \lessgtr 0\} \ \text{(open)}.$$







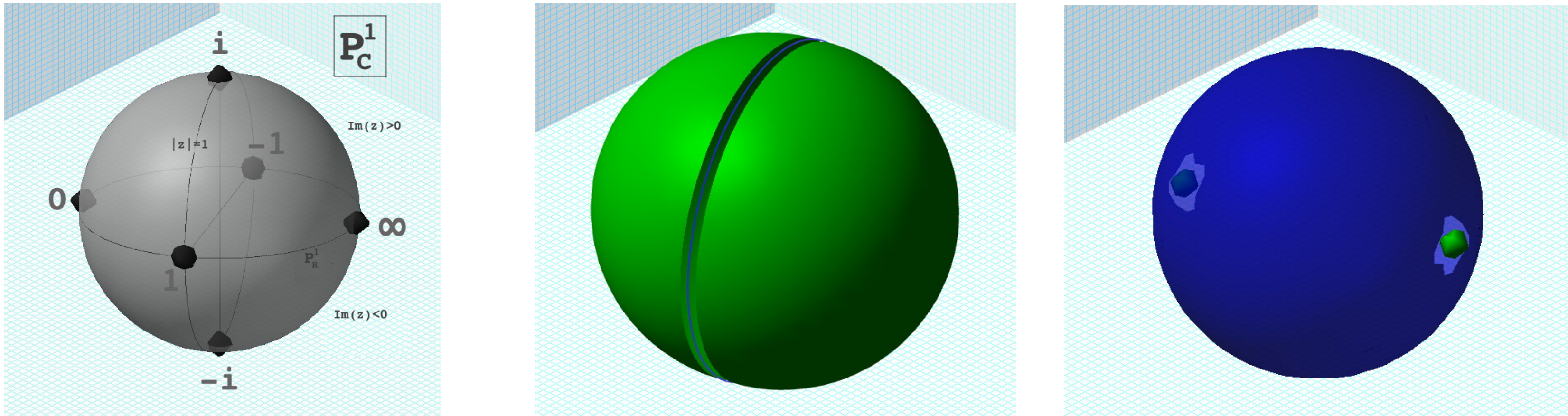
Here $K_0 = SO(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\} \simeq \mathbb{S}^1$,
and $K = SO(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}$, so the K-orbits on X are

$$C_1 = \{i\}, \qquad C_2 = \{-i\} \qquad \text{and} \qquad C_3 = X \setminus \{i, -i\}.$$

Hence $C_1 = C_{D_+}$, $C_2 = C_{D_-}$ and $C_3 = C_{D_0}$.

(2) Think at $G_0 = SU(1,1)$: then the three G_0-orbits on X are

$$D_{\text{unit}} = \{|z| = 1\} \text{ (closed)} \qquad \text{and} \qquad D_{\lessgtr 1} = \{|z| \lessgtr 1\} \text{ (open)}.$$



Here $K_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : |a| = 1 \right\} \simeq \mathbb{S}^1$, and
and $K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \neq 0 \right\} \simeq \mathbb{C}^\times$, so the K-orbits on X are

$$C_1' = \{0\}, \qquad C_2' = \{\infty\} \qquad \text{and} \qquad C_3' = X \setminus \{0, \infty\}.$$

Hence $C_1' = C_{D_{<1}}$, $C_2' = C_{D_{>1}}$ and $C_3' = C_{D_{\text{unit}}}$.

Cycle spaces: definition and Stein property

- Assume G_0 acts *not transitively* on $X = G/P$
  [exclude G_0 is compact, plus two noncompact special cases]
- D a *open* G_0-orbit (a *flag domain*);
  C_D the corresponding compact K-orbit in D (the *base cycle* of D)
- $q = \dim_{\mathbb{C}} C_D$; $J = \mathrm{stab}_G(C_D) = \{g : gC_D = C_D \text{ as subset of } X\}$
- $Y \ (= M_{X,D}) \ = \{gC_D : g \in G\} \simeq G/J$;
  $M_D = \{gC_D : gC_D \subset D\}^\circ$ (the *cycle space* of D) is open in Y

Now $K \subseteq J \subsetneq G$ (action non transitive!) $\implies$ only two possibilities:

- The generic case: $J = K$ (in fact, generically K is maximal)
- The special cases: $J = KS^{\mp}$ (a parabolic subgroup)

When G_0 is “of hermitian type” and G_0/K_0 is a “bounded symmetric domain” then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}^+ \oplus \mathfrak{s}^-$ with $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{s}^\pm$ nilpotent

($\Rightarrow \mathfrak{k}$ is not maximal!)

In this case there could be flag domains either of *holomorphic type* (special, $J = KS^\mp$) or of *nonholomorphic type* (generic, $J = K$).

Example. $G = SL(n, \mathbb{C})$, $G_0 = SU(p, q)$: then $K_0 = SU(p) \times SU(q)$ and $K = SL(p, \mathbb{C}) \times SL(q, \mathbb{C})$ (note that K is not maximal, but the reductive part of opposite parabolic subgroups $KS^\pm$).

In any case:

to the flag domain $D \subset X$ is attached the complex manifold M_D.

The (real) geometry of D could be complicated; but M_D...

Theorem (FHW)

M_D is a Stein manifold (i.e. the GAGA-analytic analogue of an affine variety).

Examples. The first low-dimensional cases...

- $G = SL(2,\mathbb{C})$, $G_0 = SU(1,1)$: here both $D_{\lessgtr 1} \subset X = \mathrm{P}^1_{\mathbb{C}}$ are of holomorphic type (i.e. J parabolic); $Y \ (= M_{X,D_{\lessgtr 1}}) \ = \mathrm{P}^1_{\mathbb{C}}$, $M_{D_{\lessgtr 1}} = D_{\lessgtr 1}$.
- $G = SL(3,\mathbb{C})$, $G_0 = SU(1,2)$ (hermitian form: $H = (+ \ - \ -)$). Let $X = \mathrm{P}^2_{\mathbb{C}}$.
  $K_0 = \{1\} \times SU(2)$, $K = \{1\} \times SL(2,\mathbb{C})$. Set $y_0 = \langle e_1 \rangle$, $y_0^* = \langle e_2, e_3 \rangle$.
  G_0-orbits on X: $D_{1,2} = \{z \in X : H|_z \gtrless 0\}$ (open), $D_3 = \{z : H|_z \equiv 0\}$ (closed)
  K-orbits on X: $C_1 = \{y_0\}$, $C_2 = \{z \in X : z \subset y_0^*\} \simeq \mathrm{P}^1_{\mathbb{C}}$, $C_3 =$ all the rest
  Stabilizers of C_1 and C_2 are parabolic (all these domains are of hermitian type)
  Here $M_{D_1} = D_1$ (open ball), $M_{D_2} = \{$2-planes on which $H > 0\}$ (also open ball)
- $G = SL(3,\mathbb{C})$, $G_0 = SL(3,\mathbb{R})$: the first generic case. Let $X = \mathrm{P}^2_{\mathbb{C}}$.
  Think $V^3_{\mathbb{R}} \hookrightarrow V^3_{\mathbb{C}}$, $K_0 = SO(V^3_{\mathbb{R}}) \simeq SO(3,\mathbb{R})$, $K = SO(V^3_{\mathbb{C}}) \simeq SO(3,\mathbb{C})$
  G_0-orbits on X: $D_1 = \mathrm{P}^2_{\mathbb{C}} \setminus \mathrm{P}^2_{\mathbb{R}}$ (open), $D_2 = \mathrm{P}^2_{\mathbb{R}}$ (closed)
  K-orbits on X: $C_1 = \{z \in X : z_1^2 + z_2^2 + z_3^2 = 0\}$, $C_2 = X \setminus C_1$
  Stabilizer of C_1 (isotropic lines): $J \ = \ \{g : {}^t zz = 0 \Rightarrow {}^t(gz)(gz) = 0\}$
  $= \ \{g : {}^t zz = 0 \Rightarrow {}^t z({}^t gg)z = \mathbf{0}\} \ = \ \{g : {}^t gg = \mathbf{1}\} = K$, as expected

The idea is **to transfer data from D to its cycle space M_D**.
There is no direct map $D \dashrightarrow M_D$ ... rather, a *double fibration*

$$\begin{array}{ccccc} & & S_D & & \\ & \swarrow^{\mu} & & \searrow^{\nu} & \\ D & & & & M_D \end{array}$$

where $S_D = \{(x, y) \in D \times M_D : x \in y\}$

[FHW]: The map μ has contractible fibers, and ν is proper. Hence...

Theorem (FHW)

Let E be a G_0-homogeneous holomorphic vector bundle on D. Then $\exists$ a G_0-homogeneous holomorphic vector bundle E' on M_D and a natural map $\boxed{\varphi : H^q(D; \mathcal{O}_E) \longrightarrow \Gamma(M_D; \mathcal{O}_{E'})}$ *, injective if E is "negative enough" so that $H^j(C_D; \Omega^k_\mu(E)|_{C_D}) = 0$ for $j < q$, $k > 0$.*

The construction of φ is quite standard (cf. Eastwood-Penrose-Wells).

Two questions arise about the map $\varphi : H^q(D; \mathcal{O}_E) \longrightarrow \Gamma(M_D; \mathcal{O}_{E'})$.

1. For a given E, **how to describe** E' ? And then: How to **describe the image of** φ **by a system of PDEs on** M_D ?
2. The injectivity condition of φ (i.e. E is "negative enough" so that...) appears to depend on E, G_0, D and P, and is quite complicated. Could one **find an easier and less depending condition** ?

Idea: to study this integral transform in the homological language of sheaves and $\mathcal{D}$-modules (formalized by D'Agnolo-Schapira, ~1995).

- Sheaves: encode the geometric aspects ("what goes where").
  $X \xleftarrow{f} S \xrightarrow{g} Y$: F sheaf on Y (e.g. $F = \mathbb{C}_{B|Y}$) $\rightsquigarrow$ $F^\wedge = Rf_! g^{-1} F$ sheaf on X.
- $\mathcal{D}$-modules: the analytic ones ("equations on data & transforms").
  Take a datum $u = (u_1, \ldots, u_n) \in (\mathcal{C}_X^\infty)^n$ s.t. $\sum_{j=1}^n P_{ij} u_j = 0$ for $i = 1, \ldots, m$
  $\rightsquigarrow$ operator $P : (\mathcal{C}_X^\infty)^n \to (\mathcal{C}_X^\infty)^m$ $\rightsquigarrow$ coherent $\mathcal{D}_X$-module $\mathcal{M}_P := \frac{\mathcal{D}_X^n}{\mathcal{D}_X^m \cdot P}$
  $\rightsquigarrow$ transformed (complex of) $\mathcal{D}_Y$-modules $(\mathcal{M}_P)^\wedge := Dg_! Df^* \mathcal{M}_P$
  $\rightsquigarrow$ operator P' on $\mathcal{C}_Y^\infty$, satisfied by $u^\wedge = \int_g f^* u$ (i.e. $P' u^\wedge = 0$).
- Adjunction form.: $R\Gamma(X; Sol(\mathcal{M}, F^\wedge \otimes \mathcal{O}_X)) \xrightarrow{\sim} R\Gamma(Y; Sol(\mathcal{M}^\wedge, F \otimes \mathcal{O}_Y))$ [shift].

How to interpret our transform (final aim: $H^q(D;\mathcal{O}_E) \xrightarrow{\varphi} \Gamma(M_D;\mathcal{O}_{E'})$) in the language of sheaves and $\mathcal{D}$-modules?

1. Choose $X = G/P$, $Y = G/K$ (ok also in the hermitian case), $S = \{(x,y) \in X \times Y : x \in y\}$
2. **Geometry.** A flag domain $D \subset X$ $\rightsquigarrow$ our target is $M_D \subset Y$. Take the sheaf $\mathbb{C}_{M_D|Y}$ on Y $\Longrightarrow$ $(\mathbb{C}_{M_D|Y})^\wedge$ on X is what?

[FHW] (exegesis) : One has $(\mathbb{C}_{M_D|Y})^\wedge \simeq \mathbb{C}_{D|X}{}^{[-2d_{S/X}]}$.

3. **Analysis.** What equations on our data? Nothing! Hence $P = 0$ $\rightsquigarrow$ $\mathcal{M}_P \simeq \mathcal{D}^m$ (locally) $\rightsquigarrow$ Take the $\mathcal{D}_X$-module $\mathcal{M}_\lambda = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(-w_P\lambda)$ (λ a P-dom. wgt) $\Longrightarrow$ $(\mathcal{M}_\lambda)^\wedge$ on Y is what?

We'll talk about the (complex of) $\mathcal{D}_Y$-modules $(\mathcal{M}_\lambda)^\wedge$ in a moment.

4. Putting the above in [D'A-S] adjunction formulas, one gets ◂

$(\Diamond)\quad R\Gamma(D;\mathcal{O}_X(\lambda)) \xrightarrow{\sim} R\Gamma\big(M_D; Sol((\mathcal{M}_\lambda)^\wedge, \mathcal{O}_Y\big)\,[-q]\,.$ ▸ Adj. Form.

5. $(\mathcal{M}_\lambda)^\wedge$ is a quasi-G-equivariant (complex of) $\mathcal{D}_Y$-modules $\rightsquigarrow$ (enough at $y_0 = eK$) corresponds to the $(\mathfrak{g}, K)$-module

$$(\Diamond\Diamond)\qquad R\Gamma\big(X; \mathcal{B}_{C_D|X} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\gamma(\lambda))\big)$$

where γ depends only on P (it is $\gamma(\lambda) = -w_P\lambda + 2(\rho - \rho_P)$)

Can use tools of $(\mathfrak{g}, K)$-modules (e.g. BGG) to compute $(\Diamond\Diamond)$

6. Now: when is $(\mathcal{M}_\lambda)^\wedge$ (hence $(\Diamond\Diamond)$) concentrated in degree 0?
Let $\widetilde{X}$ the full flag manifold, $\pi : \widetilde{X} \to X$ the (smooth) projection
$\mathcal{N}_\lambda = \mathcal{B}_{C_D|X} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\gamma(\lambda))$ is a twisted $\mathcal{D}_{X,\gamma(\lambda)}$-module $\rightsquigarrow$
$D\pi^{-1}\mathcal{N}_\lambda$ also a twisted $\mathcal{D}_{\widetilde{X},\gamma(\lambda)}$-mod, and $R\Gamma(\widetilde{X}; D\pi^{-1}\mathcal{N}_\lambda) \simeq (\Diamond\Diamond)$

7. [Beilinson-Bernstein]: if μ integrally antidom. ($\langle \mu - \rho, \alpha^\vee\rangle \notin \mathbb{Z}_{>0}$) then $R\Gamma(\widetilde{X}; \mathcal{L})$ is in degree 0 for any twisted $\mathcal{D}_{\widetilde{X},\mu}$-module $\mathcal{L}$.
$\implies$ A sufficient condition for $(\mathcal{M}_\lambda)^\wedge$ to be in degree 0 is that

$(\bigstar)\quad \gamma(\lambda) - \rho = \boxed{-w_P\lambda + \rho - 2\rho_P \text{ is integrally antidominant}}$

8. Assuming ($\bigstar$), let $\Box_\lambda : \mathcal{O}(H'_\lambda) \to \mathcal{O}(H''_\lambda)$ be a system of PDEs associated to $(\mathcal{M}_\lambda)^\wedge$. Applying H^q to adjunction formula ($\Diamond$):

$$H^q(D; \mathcal{O}_X(\lambda)) \xrightarrow{\sim} \{v \in \Gamma(M_D; \mathcal{O}(H'_\lambda)) : \Box_\lambda v = 0\} \ .$$

So, what about the two problems of [FHW] ?

1. For a given E, **how to describe** E' and the image of φ by a system of PDEs on M_D ?

   Given λ, computing $\Box_\lambda$, H'_λ and H''_λ is a matter of $(\mathfrak{g}, K)$-modules.

2. The injectivity condition of φ seems to depend on E, G_0, D and P, and is complicated.

   Find an easier and less depending condition ?

   Condition ($\bigstar$) depends only on E and P, and not on G_0 and D.