On the Free Boundary in Heterogeneous Problems of Obstacle-Type with Two Phases

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Abstract

We consider some properties of the solutions of free boundary problems of obstacle-type with two phases for a class of heterogeneous quasilinear elliptic operators, including the p-Laplacian operator with $1 < p < \infty$. Under a natural non-degeneracy assumption on the interface, when the level set of the change of phase has null Lebesgue measure, we prove a continuous dependence result for the characteristic functions of each phase and we establish sharp estimates on the variation of its Lebesgue measure with respect to the L^1-variation of the data, in a rather general framework. For elliptic quasilinear equations which heterogeneities have appropriate integrable derivatives, we show that the characteristic functions of both phases are of bounded variation for general data with bounded variation. This extends recent results for the obstacle problem and is a first result on the regularity of the free boundary of the heterogeneous two phases problem, which is therefore an interface locally of class C^1 up to a possible singular set of null perimeter.

Dedicated to Nina N. Uraltseva, on the occasion of her 80th birthday

1 Introduction

We consider stationary free boundary problems with two phases in the form

$$Au + \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} = f \qquad \text{a.e. in} \quad \Omega \tag{1.1}$$

associated with a quasi-linear elliptic operator of p-Laplacian type

$$Au = -\operatorname{div}\big(a(x, \nabla u)\big), \tag{1.2}$$

where Ω is a bounded open connected subset of $\mathbb{R}^n$, $n \geq 2$, and the vector field $a(x,\eta)$: $\Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous in $x \in \Omega$ and continuous differentiable in $\eta \in \mathbb{R}^n \backslash \{0\}$. Here $f = f(x)$ and $\lambda_\pm = \lambda_\pm(x) \geq 0$ are given bounded functions and $\chi_{\{u>0\}}, \chi_{\{u<0\}}$ denote the characteristic functions of each phase

$$\{u > 0\} = \{x \in \Omega : \, u(x) > 0\} \quad \text{and} \quad \{u < 0\} = \{x \in \Omega : \, u(x) < 0\}.$$

In the special linear homogeneous case of the Laplacian $A = -\Delta$ with $f \equiv 0$ and constant $\lambda_\pm > 0$, the whole free boundary

$$\Phi = \Omega \cap \big(\partial\{u > 0\} \cup \partial\{u < 0\}\big) \tag{1.3}$$

has finite $(n-1)$-dimensional Hausdorff measure ($\mathcal{H}^{n-1}(\Phi) < \infty$), as it was shown by Weiss [25], and is locally a union of two C^1-surfaces in the neighborhood of each "branch point" (see [17]). In the recent monograph [16] the characterization of free boundary points and the analysis of their properties show that the regular and singular one-phase points $\partial\{u > 0\} \setminus \partial\{u < 0\}$ and $\partial\{u < 0\} \setminus \partial\{u > 0\}$ are locally as in the obstacle problem, while the two-phase free boundary points $\partial\{u > 0\} \cap \partial\{u < 0\}$ may be branch points, if $|\nabla u| = 0$, or, by the implicit function theorem, open portions of $C^{1,\alpha}$ graphs where $|\nabla u| > 0$, which have been shown to be in fact locally real-analytic.

However these results cannot be expected to hold always in the case of non-homogeneous coefficients and more general operators of the type (1.2). In the general case, the problem (1.1) was introduced and treated as a variational inequality by Duvaut and Lions in the framework of temperature control problems regulated by interior heat injection (see [9], Chap. 2). In fact, (1.1) can be regarded as a model for the control of the interface in the steady-state two-phase Stefan problem:

$$\Phi_0 = \{u = 0\} = \{x \in \Omega : \, u(x) = 0\}, \tag{1.4}$$

i.e., of the level set of the melting stationary temperature u separating the liquid phase $\{u > 0\}$ from the solid phase $\{u < 0\}$ (see, for instance, [19] and its references).

In general, we may have $\Phi_0 \supsetneq \Phi$ and (1.1) should be regarded as a quasilinear partial differential equation with discontinuous nonlinearities

$$Au \in F(x,u) \quad \text{a.e. in} \ \ \Omega, \tag{1.5}$$

where $F(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a monotone decreasing function in u for a.e $x \in \Omega$ (see [7] or [20]).

In Section 2, we use a natural regularization of the Heaviside function to solve (1.5) with the additional mixed boundary conditions, which allows us to characterize the equation (1.5) with the right hand side $f - \zeta(u) \in F(u)$ with a bounded function $\zeta \in \partial J(u)$ related to the characteristic functions of $\{u > 0\}$ and $\{u < 0\}$. Indeed ζ is given as the element of the subdifferential in u of the convex functional

$$J(v) = \int_\Omega \Big[\lambda_+(x)\, v^+(x) + \lambda_-(x)\, v^-(x)\Big]\, dx, \tag{1.6}$$

where $v^+ = \max(v, 0)$ and $v^- = (-v)^+$.

Although the classical theory of monotone operators [14] yields existence, uniqueness and global continuous dependence of the solution to our problems, using the L^1-theory we can obtain an additional interesting estimate on the phase variations. This extends the remark of [21], obtained for the one obstacle problem, to the two phases problem in the case of non degenerate interfaces, i.e., essentially when $\Phi = \Phi_0$, for which we provide sufficient conditions on the data $\lambda_\pm$ and f in order to hold that non-degeneracy. These results complement, in Section 3, the general remarks on the stability of both phases in terms of their characteristic functions under a natural non-degeneracy condition.

Finally, in Section 4, we give bounded variation estimates on Au, by extending to the two-phase heterogeneous case earlier results of [4] and [6] for the one obstacle problem. This estimate provides the regularity of the non degenerate free boundary, which outside a possible singular set of null perimeter, is locally of class C^1. These results are new even for the homogeneous p-Laplacian operator ($1 < p < \infty$) and for linear second order partial differential operators with Lipschitz coefficients, and are valid for general bounded data $f, \lambda_\pm$ with bounded variation, with their sum $\lambda_+ + \lambda_-$ positive, continuous and with integrable derivatives.

2 Approximation and continuous dependence of the solution

In this paper we suppose the standard structural assumptions, with $1 < p < \infty$, for the operator A given by (1.2), with $a(x, 0) = 0$ and

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial \eta_j}(x, \eta)\, \xi_i \xi_j \geq \gamma_0 \big(\kappa + |\eta|^2\big)^{\frac{p-2}{2}} |\xi|^2, \tag{2.1}$$

$$\sum_{i,j=1}^{n} \left| \frac{\partial a_i}{\partial \eta_j}(x, \eta) \right| \leq \gamma_1 \big(\kappa + |\eta|^2\big)^{\frac{p-2}{2}}, \tag{2.2}$$

$$\big|a(x_1, \eta) - a(x_2, \eta)\big| \leq \gamma_2 |x_1 - x_2| \big(\kappa + |\eta|^2\big)^{\frac{p-1}{2}}, \tag{2.3}$$

for some $\kappa \in [0, 1]$ and some positive constants γ_0, γ_1, γ_2, for $x \in \Omega$, $\eta \in \mathbb{R}^n \backslash \{0\}$ and for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. It is well known that this implies the existence of $\gamma > 0$, such that ([23], Lemma 1):

$$\sum_{i=1}^{n} \big(a_i(x, \eta) - a_i(x, \xi)\big)(\eta_i - \xi_i) \geq \gamma \begin{cases} \big(\kappa + |\eta| + |\xi|\big)^{p-2} |\eta - \xi|^2 & \text{if } \ p \leq 2, \\ |\eta - \xi|^p & \text{if } \ p \geq 2. \end{cases} \tag{2.4}$$

Therefore, in particular, we cover heterogeneous quasilinear operators of the p-Laplacian type when $\kappa = 0$,

$$Au = -\operatorname{div}\Big(M(x)\big(\kappa + |\nabla u|^2\big)^{\frac{p-2}{2}} \nabla u\Big),$$

where $M(x)$ may be a Lipschitz continuous positive definite matrix uniformly in $x \in \bar{\Omega}$ and it may also include, for $p = 2$, linear second order operators with variable coefficients $M_{ij}(x)$ in divergence form.

We suppose the Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where the two regular components are, such that, $\mathcal{H}^{n-1}(\Gamma_D) > 0$, with $\mathcal{H}^{n-1}$ denoting the $(n-1)$-dimensional Hausdorff measure.

For the equation (1.1) we consider a mixed Dirichlet and Neumann boundary conditions

$$u = h \quad \text{on} \quad \Gamma_D \qquad \text{and} \qquad \frac{\partial u}{\partial \nu_A} = a(\nabla u) \cdot \vec{n} = g \quad \text{on} \quad \Gamma_N. \tag{2.5}$$

We assume

$$f \in L^\infty(\Omega), \quad g \in L^\infty(\Gamma_N) \quad \text{and} \quad h \in W^{1,p}(\Omega), \tag{2.6}$$

$$\lambda_\pm \in L^\infty(\Omega), \quad \lambda_\pm \geq 0 \quad \text{and} \quad \lambda_+(x) + \lambda_-(x) > 0 \qquad \text{a.e.} \quad x \in \Omega, \tag{2.7}$$

and we introduce the functional framework

$$V_h = \Big\{ v \in W^{1,p}(\Omega) : \ v = h \quad \text{on} \quad \Gamma_D \Big\}, \tag{2.8}$$

$$\langle Au, v \rangle = \int_\Omega a(x, \nabla u) \cdot \nabla v \quad \text{and} \quad \langle L, v \rangle = \int_\Omega fv + \int_{\Gamma_N} gv, \qquad v \in V_0, \tag{2.9}$$

Following [9], our problem may now be formulated as the elliptic variational inequality

$$u \in V_h : \quad J(v) - J(u) \geq \langle L - Au, v - u \rangle, \qquad v \in V_h. \tag{2.10}$$

The standard theory of monotone operators allow us to state that our problem is well posed in the following sense (see, for instance [14] or [22]).

Proposition 2.1. *Under the assumptions* (2.6)–(2.7) *there exists a unique solution to* (2.10) *and there exists $\zeta \in V_0'$, such that,*

$$\zeta = L - Au \in \partial J(u). \tag{2.11}$$

Moreover for a sequence of data $h_\eta \to h$ in $W^{1,p}(\Omega)$, $f_\eta \to f$, and $\lambda_{\eta\pm} \to \lambda_\pm$ in $L^{p'}(\Omega)$, $g_\eta \to g$ in $L^{p'}(\Gamma_N)$, the corresponding solutions u_η to (2.10) *satisfy $u_\eta \to u$ in $W^{1,p}(\Omega)$ as $\eta \to \infty$.*

Remark 2.1. If in (2.5) the Neumann condition is replaced by a two-phase boundary condition of the type

$$\frac{\partial u}{\partial \nu_A} + \mu_+ \chi_{\{u>0\}} - \mu_- \chi_{\{u<0\}} = g \qquad \text{on} \quad \Gamma_N.$$

then (2.10) will also be the the variational formulation of this problem if we replace J by J_N defined by

$$J_N(v) = \int_\Omega (\lambda_+ v^+ + \lambda_- v^-) + \int_{\Gamma_N} (\mu_+ v^+ + \mu_- v^-).$$

Although we may expect, from (2.11) and from our departure equation (1.1), that

$$\zeta(x) = \lambda_+(x)\chi_{\{u>0\}}(x) - \lambda_-(x)\chi_{\{u<0\}}(x) \qquad \text{a.e. in} \quad x \in \Omega, \tag{2.12}$$

this characterization may not always hold, in particular when $\mathcal{L}^n(\Phi_0) > 0$, i.e., if the mushy region $\{u = 0\}$ has positive Lebesgue measure.

A first characterization towards (2.12) may be obtained by approximating the solution u of (2.10) by solutions u_ε of the regularized equation

$$Au_\varepsilon + \lambda_+ H_\varepsilon(u_\varepsilon) - \lambda_- H_\varepsilon(-u_\varepsilon) = f \qquad \text{in} \quad \Omega \tag{2.13}$$

with the same mixed boundary conditions (2.5). Here $\varepsilon > 0$ and H_ε is the Lipschitz approximation of the Heaviside function

$$H_\varepsilon(t) = 0, \quad t \le 0, \qquad H_\varepsilon(t) = \frac{t}{\varepsilon}, \quad 0 \le t \le \varepsilon, \qquad H_\varepsilon(t) = 1, \quad t \ge \varepsilon. \tag{2.14}$$

Theorem 2.1. *The unique solution u_ε to* (2.13),(2.5) *is uniformly bounded in $W^{1,p}(\Omega) \cap C^{1,\alpha}(\Omega)$ for some α, $0 < \alpha < 1$, and such that, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u \qquad in \quad W^{1,p}(\Omega) \cap C^{1,\alpha'}(\Omega)$$

for all $\alpha' < \alpha$, where u is the solution of (2.10)*. Moreover,*

$$\|\nabla(u - u_\varepsilon)\|^p_{L^p(\Omega)} \le \varepsilon\, C_\gamma \qquad if \quad p \ge 2, \tag{2.15}$$

$$\|\nabla(u - u_\varepsilon)\|^2_{L^2(\Omega')} \le \varepsilon\, C' \qquad if \quad 1 < p < 2, \tag{2.16}$$

where $C_\gamma = \frac{1}{\gamma}(\|\lambda_+\|_{L^1(\Omega)} + \|\lambda_-\|_{L^1(\Omega)})$ and $C' > 0$ depends on C_γ, on $\Omega' \subset\subset \Omega$, and on an upper bound Λ'_α of $\|\nabla u_\varepsilon\|_{C^{0,\alpha}(\overline{\Omega'})}$, but not on ε.

Proof. We remark that

$$-\lambda_-(x) \le \zeta_\varepsilon(x) = \lambda_+ H_\varepsilon(u_\varepsilon) - \lambda_- H_\varepsilon(-u_\varepsilon) \le \lambda_+(x) \qquad \text{a.e.} \quad x \in \Omega, \tag{2.17}$$

and therefore $Au_\varepsilon \in L^\infty(\Omega)$ uniformly in $\varepsilon > 0$. Hence, by the general $C^{1,\alpha}$ local regularity under the assumptions (2.1)–(2.3) for any $\Omega' \subset\subset \Omega$ we have (see [12], [8] or [23])

$$\|u_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega'})} \le \Lambda'_\alpha \qquad \text{independent of} \quad \varepsilon > 0.$$

Since by compactness, there is $u \in V_h$ such that, as $\varepsilon \to 0$ and all $0 < \alpha' < \alpha$

$$u_\varepsilon \rightharpoonup u \qquad \text{in} \quad W^{1,p}(\Omega)\text{-weak} \;\; \text{and in} \;\; C^{1,\alpha'}(\overline{\Omega'}),$$

$$H_\varepsilon(u_\varepsilon) \rightharpoonup \chi_+ \qquad \text{and} \qquad H_\varepsilon(-u_\varepsilon) \rightharpoonup \chi_- \quad \text{in} \;\; L^\infty(\Omega)\text{-weak*},$$

for some functions χ_+ and χ_- satisfying

$$0 \le \chi_+(x) \le 1 \;\; \text{and} \quad 0 \le \chi_-(x) \le 1 \quad \text{a.e.} \quad x \in \Omega.$$

By the uniform convergence of u_ε and the definition (2.14), in the open subsets $\{u>0\}$ and $\{u<0\}$ we have, respectively, $\chi_+\equiv 1$, $\chi_-\equiv 0$ and $\chi_+\equiv 0$ and $\chi_-\equiv 1$, since this holds at interior points of those subsets for $H_\varepsilon(u_\varepsilon)$ and $H_\varepsilon(-u_\varepsilon)$ for ε sufficiently small. Hence,

$$\zeta_\varepsilon \rightharpoonup \lambda_+\chi_+ - \lambda_-\chi_- \qquad \text{in} \quad L^\infty(\Omega)\text{-weak*}.$$

Then, as in Chap. 1 of [9], passing to the limit in the variational formulation of (2.13),(2.5)

$$\int_\Omega a(\nabla u_\varepsilon)\cdot\nabla w + \int_\Omega \zeta_\varepsilon w = \int_\Omega f w + \int_{\Gamma_N} g w, \qquad \forall\, w\in V_0, \tag{2.18}$$

we may show that u is the unique solution to (2.10). Then, by (2.10) and (2.11) we may conclude that

$$\zeta = \lambda_+\chi_+ - \lambda_-\chi_- \qquad \text{a.e. in} \quad \Omega, \tag{2.19}$$

by the definition of subdifferential, and we have

$$\int_\Omega a(\nabla u)\cdot\nabla w + \int_\Omega \zeta w = \int_\Omega f w + \int_{\Gamma_N} g w, \qquad \forall\, w\in V_0. \tag{2.20}$$

Consequently, we obtain

$$\int_\Omega \big[a(\nabla u) - a(\nabla u_\varepsilon)\big]\cdot\nabla(u-u_\varepsilon) = \int_\Omega (\zeta_\varepsilon - \zeta)\,(u-u_\varepsilon). \tag{2.21}$$

In order to estimate the right hand side, we start with the term corresponding to the positive phase:

$$\begin{aligned}
&\int_\Omega \lambda_+\big[H_\varepsilon(u_\varepsilon)-\chi_+\big]\,(u-u_\varepsilon) \le \\
&\qquad \le \int_{\{u>0\}} \lambda_+\big[H_\varepsilon(u_\varepsilon)-1\big]\,(u-u_\varepsilon) + \int_{\{u=0\}} \lambda_+\big[H_\varepsilon(u_\varepsilon)-\chi_+\big]\,(-u_\varepsilon)\\
&\qquad \le \int_{\{u>0\}} \lambda_+\big[H_\varepsilon(u_\varepsilon)-1\big]\,(-u_\varepsilon)\\
&\qquad\quad + \int_{\{u=0\}\cap\{u_\varepsilon<0\}} \lambda_+\chi_+u_\varepsilon + \int_{\{u=0\}\cap\{u_\varepsilon>0\}} \lambda_+\big[H_\varepsilon(u_\varepsilon)-1\big]\,(-u_\varepsilon)\\
&\qquad \le \varepsilon\|\lambda_+\|_{L^1(\Omega)}
\end{aligned}$$

by the property $[1-H_\varepsilon(t)]t\le\varepsilon$, for $t\in\mathbb{R}$, and as we have seen

$$0\le \chi_{\{u>0\}} \le \chi_+ \le 1-\chi_{\{u<0\}} \qquad \text{a.e. in} \quad \Omega. \tag{2.22}$$

Repeating the symmetric argument for the negative phase, we conclude

$$\int_\Omega (\zeta_\varepsilon-\zeta)\,(u-u_\varepsilon) \le \varepsilon\big(\|\lambda_+\|_{L^1(\Omega)} + \|\lambda_-\|_{L^1(\Omega)}\big)$$

and (2.15) follows immediately from (2.21) and (2.4) with $p\ge 2$. For $1<p<2$ we apply (2.4) in $\Omega'\subset\subset\Omega$ and, using the a priori bound of the gradients in Ω', we easily conclude (2.16) with $C' = (2\Lambda'_\alpha)^{2-p}(\|\lambda_+\|_{L^1(\Omega)} + \|\lambda_-\|_{L^1(\Omega)})/\gamma$. The strong convergence in $W^{1,p}(\Omega)$ is now an easy consequence of those estimates. □

Remark 2.2. By the $C^{1,\alpha}$ estimates up to the boundary (see [13]), if the boundary is of class $C^{1,\gamma}$, for the Dirichlet problem ($\Gamma_N = \emptyset$) if $h \in C^{1,\gamma}(\partial\Omega)$, $0 < \gamma < 1$, then the solution $u \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < \gamma$, and, as a consequence the estimate (2.16) holds in the whole Ω. Analogously, for the Neumann problem ($\Gamma_D = \emptyset$), under certain compatibility conditions on the data (see [1]), we may obtain also $C^{1,\alpha}$-regularity up to the boundary for $g \in C^{0,\gamma}(\partial\Omega)$, $0 < \gamma < 1$.

Remark 2.3. As a consequence of the proof of Theorem 2.1 the solution u to (2.10) also solves (1.1) in a weak form

$$Au + \lambda_+\chi_+ - \lambda_-\chi_- = f \quad \text{a.e. in} \quad \Omega \tag{2.23}$$

with the boundary conditions (2.5) with bounded functions χ_+ and χ_- satisfying (2.22) and $0 \leq \chi_{\{u<0\}} \leq \chi_- \leq 1 - \chi_{\{u>0\}}$ a.e. in Ω.

Remark 2.4. When $\kappa > 0$ in (2.1)–(2.2) we can guarantee that the solution u has integrable second derivatives, namely (see Prop. 1 of [23], for instance)

$$u \in H^2_{\text{loc}}(\Omega) \quad \text{if} \ \ p \geq 2 \qquad \text{and} \qquad u \in W^{2,p}_{\text{loc}}(\Omega) \quad \text{if} \ \ p \leq 2.$$

3 Stability of the Phases under Non-Degeneracy

In general we cannot preclude a "thick interface" Φ_0 and by (2.22) and Remark 2.3 it is clear that

$$\{0 < \chi_+ < 1\} \cup \{0 < \chi_- < 1\} \subset \{u = 0\} = \Phi_0,$$

where these possible non-empty subsets are defined up to null sets for the Lebesgue measure $\mathcal{L}^n$. Hence a natural non-degeneracy condition is

$$\mathcal{L}^n(\Phi_0) = \text{meas}\{u = 0\} = 0 \tag{3.1}$$

which is obviously equivalent to say

$$\chi_+ = \chi_{\{u>0\}} \ \text{ and } \ \chi_- = \chi_{\{u<0\}} \qquad \text{a.e. in} \quad \Omega. \tag{3.2}$$

For the class of differential operators with the property

$$Av = Aw \qquad \text{a.e. in} \quad \{v = w\}, \tag{3.3}$$

it is simple to provide a sufficient condition on the external force f in order to generate the non-degeneracy condition (3.1):

$$f(x) > \lambda_+(x) \ \text{ or } \ f(x) < -\lambda_-(x), \qquad \text{a.e.} \quad x \in \Omega. \tag{3.4}$$

Indeed, from (2.23) and (2.19), if u is a solution to the two-phase problem (2.11), with an operator satisfying the property (3.3), with $\text{meas}\{u = 0\} > 0$, we have $Au = 0$ a.e. in $\{u = 0\}$ and therefore

$$-\lambda_- \leq \lambda_+\chi_+ - \lambda_-\chi_- = f \leq \lambda_+ \qquad \text{a.e. in} \quad \{u = 0\},$$

which contradicts (3.4).

Therefore we have shown the following interesting result.

Proposition 3.1. *Under the assumptions* (3.3) *and* (3.4) *the interface* Φ_0 *has zero Lebesgue measure, i.e.* (3.1) *holds.*

Remark 3.1. The property (3.3) is associated with the local regularity of the solutions v and w. For operators in the class (2.1)–(2.3) with $\kappa > 0$, for bounded data f, the standard regularity of nonlinear operators yields the integrability of the second order derivatives (see [12], [15] or [23], for instance), since the vector field $a(x, \eta)$ is Lipschitz continuous in x, as we have recalled in Remark 2.4. Similarly, for $\kappa = 0$ and $p \leq 2$, in particular, for linear operators and the singular heterogeneous p-Laplacian operator, the local regularity still implies the condition (3.3). However, for $\kappa = 0$ and $p > 2$, i.e., for heterogeneous p-Laplacians of degenerate type, the integrability of the second order derivatives fails, but we still have (3.3) by a well known theorem of Stampacchia and knowing that we may prove that $a(\cdot, \nabla u(\cdot)) \in (W^{1,p/(p-1)}_{\rm loc}(\Omega))^n$, by the usual methods of [12], [15] or [23].

Another immediate consequence of (3.1) and (3.2) is

$$\chi_{\{u>0\}} = 1 - \chi_{\{u<0\}} \qquad \text{a.e.} \quad \Omega, \tag{3.5}$$

and the strong approximation of the characteristic functions of both phases follows easily from (2.13):

$$H_\varepsilon(u_\varepsilon) \to \chi_{\{u>0\}} \quad \text{and} \quad H_\varepsilon(-u_\varepsilon) \to \chi_{\{u<0\}} \qquad \text{in} \quad L^q(\Omega), \quad \forall\, q < \infty.$$

Indeed, we already know these convergences hold weakly, and it is enough to conclude for $q \geq 2$ the convergence of the L^q-norms. This follows from

$$\begin{aligned} \int_\Omega \chi_{\{u>0\}} = \lim_{\varepsilon\to 0} \int_\Omega H_\varepsilon(u_\varepsilon) \geq \overline{\lim_{\varepsilon\to 0}} \int_\Omega H^q_\varepsilon(u_\varepsilon) \geq \underline{\lim_{\varepsilon\to 0}} \int_\Omega H^q_\varepsilon(u_\varepsilon) \geq \\ \geq \int_\Omega \chi^q_{\{u>0\}} = \int_\Omega \chi_{\{u>0\}}, \end{aligned} \tag{3.6}$$

and similarly for $H_\varepsilon(-u_\varepsilon)$ towards $\chi_{\{u<0\}}$.

As in the one obstacle problem [18] and [21], these arguments give also a kind of stability of the two phases in terms of their characteristic functions.

Theorem 3.1. *Let* $(u_\eta, \chi_{\eta+}, \chi_{\eta-})$ *be a solution to* (2.23),(2.5) *corresponding to data* f_η, g_η, h_η, $\lambda_{\eta\pm}$ *converging, as in Proposition 2.1, to a solution* u *to the limit problem* (1.1)–(2.5), *under the condition* (3.2)*. Then we have*

$$\chi_{\eta+} \to \chi_{\{u>0\}} \quad \textit{and} \quad \chi_{\eta-} \to \chi_{\{u<0\}} \qquad \textit{strongly in} \quad L^q(\Omega), \quad \forall\, q < \infty, \tag{3.7}$$

$$\chi_{\{u_\eta>0\}} \to \chi_{\{u>0\}} \quad \textit{and} \quad \chi_{\{u_\eta<0\}} \to \chi_{\{u<0\}} \qquad \textit{strongly in} \quad L^q(\Omega), \quad \forall\, q < \infty. \tag{3.8}$$

Proof. Let, for some subsequence,

$$\chi_{\eta+} \rightharpoonup \chi^*_+ \ \text{ and } \ \chi_{\eta-} \rightharpoonup \chi^*_- \qquad \text{in} \quad L^\infty(\Omega)\text{-weakly*}.$$

Since we know that $\chi_{\eta+} = 0$ in $\{u_\eta < 0\}$, we conclude

$$0 \le \chi_+^* \le 1 - \chi_{\{u<0\}} \qquad \text{a.e. in} \quad \Omega,$$

from the convergence

$$0 = \int_\Omega u_\eta^- \chi_{\eta+} \quad \to \quad \int_\Omega u^- \chi_+^* = 0.$$

Since also $\chi_{\eta+} = 1$ in $\{u_\eta > 0\}$, taking an arbitrary measurable set $\mathcal{O} \subset \Omega$, we have

$$\int_{\mathcal{O}} u_\eta^+ \chi_{\eta+} = \int_{\mathcal{O}} u_\eta^+ \to \int_O u^+.$$

On the other hand, since $u_\eta^+ \to u^+$ in $L^q(\Omega)$, $\forall\, q < +\infty$,

$$\int_{\mathcal{O}} u_\eta^+ \chi_{\eta+} \to \int_{\mathcal{O}} u^+ \chi_+^*,$$

we obtain $\int_{\mathcal{O}} u^+ \chi_+^* = \int_{\mathcal{O}} u^+$ for all $\mathcal{O} \subset \Omega$. Therefore, $\chi_+^* = 1$ a.e. in $\{u > 0\}$, and so

$$0 \le \chi_{\{u>0\}} \le \chi_+^*.$$

By the assumption (3.2) we conclude

$$\chi_+^* = \chi_{\{u>0\}} = 1 - \chi_{\{u<0\}},$$

and, symmetrically, also

$$\chi_-^* = \chi_{\{u<0\}} = 1 - \chi_{\{u>0\}}.$$

The strong convergences follow as in (3.6) and the same arguments apply by replacing in the above proof $\chi_{\eta+}$ by $\chi_{\{u_\eta>0\}}$ and $\chi_{\eta-}$ by $\chi_{\{u_\eta<0\}}$, concluding the results. □

Remark 3.2. We stress that the non-degeneracy condition (3.2) is required only for the limit problem and not for the approximating problems. Therefore the positive and negative phases have a kind of weak stability in Lebesgue measure if the interface has null measure.

In fact, using the L^1-contraction property of m-accretive operators in Banach spaces (see [3]) and extending the stability property of [5] applied to the one obstacle problem in [18] and [21], we can prove the following estimate.

Theorem 3.2. *Let* (u, ζ) *and* $(\widehat{u}, \widehat{\zeta})$ *denote the solutions of* (2.11) *corresponding to data* (f, g) *and* $(\widehat{f}, \widehat{g})$*, i.e., of* (2.23),(2.5) *with the same* h *and* $\lambda_\pm$ *as in Proposition 2.1. Then*

$$\|\zeta - \widehat{\zeta}\|_{L^1(\Omega)} \le \|f - \widehat{f}\|_{L^1(\Omega)} + \|g - \widehat{g}\|_{L^1(\Gamma_N)}. \tag{3.9}$$

If, in addition, both solutions satisfy the non-degeneracy condition (3.1) *and*

$$\lambda_+(x) + \lambda_-(x) \ge \mu > 0 \qquad a.e. \quad x \in \Omega, \tag{3.10}$$

then

$$\text{meas}\big(\{u > 0\} \div \{\widehat{u} > 0\}\big) \le \frac{1}{\mu}\big(\|f - \widehat{f}\|_{L^1(\Omega)} + \|g - \widehat{g}\|_{L^1(\Gamma_N)}\big) \tag{3.11}$$

where $\div$ *denotes the symmetric difference of sets* $B \div D = (B \backslash D) \cup (D \backslash B)$.

Proof. Although the estimate (3.9) is well-known and a consequence of the general theory of m-accretive operators in $L^1(\Omega)$ (see, for instance, [26] for a Dirichlet problem, or [1] for a nonlinear Neumann problem), for completeness, we sketch a simple proof for our mixed problem (see also [5] or [21]).

Multiply the difference of the equations (2.23) for u and $\widehat{u}$

$$\zeta - \widehat{\zeta} = f - \widehat{f} - (Au - A\widehat{u}) \qquad \text{a.e. in} \quad \Omega$$

by the measurable function

$$s(x) = \begin{cases} -1 & \text{on} \ \{u < \widehat{u}\} \cup \{\zeta < \widehat{\zeta}\}, \\ 0 & \text{on} \ \{u = \widehat{u}\} \cap \{\zeta = \widehat{\zeta}\}, \\ 1 & \text{on} \ \{u > \widehat{u}\} \cup \{\zeta > \widehat{\zeta}\}, \end{cases}$$

which satisfies $s \in \sigma(u - \widehat{u})$, where σ denotes the maximal monotone graph of the sign function ($\sigma = \partial r$, $r(t) = |t|$).

Integrating by parts we obtain

$$\|\zeta - \widehat{\zeta}\|_{L^1(\Omega)} = \int_\Omega (\zeta - \widehat{\zeta}) s \leq \int_\Omega (f - \widehat{f}) s + \int_{\Gamma_N} |g - \widehat{g}| \leq \|f - \widehat{f}\|_{L^1(\Omega)} + \|g - \widehat{g}\|_{L^1(\Gamma_N)},$$

since we have

$$-\int_\Omega (Au - A\widehat{u}) s \leq \int_{\Gamma_N} |g - \widehat{g}|, \qquad \forall\, s \in \sigma(u - \widehat{u}).$$

This inequality follows by using the remark

$$(Au - A\widehat{u}) s = (Au - A\widehat{u}) \operatorname{sign}(u - \widehat{u}) \qquad \text{a.e.} \quad x \in \Omega,$$

for any $s \in \sigma(u - \widehat{u})$, by (3.3) and considering a smooth approximation $\sigma_\varepsilon(t) \to_{\varepsilon \to 0} \operatorname{sign}(t)$ with the boundary conditions (2.5) for u and $\widehat{u}$ in the integration by parts.

Finally, under the condition (3.1) we have

$$\zeta = \lambda_+ \chi_+ - \lambda_- \chi_-$$

with

$$\chi_+ = \chi_{\{u>0\}} = 1 - \chi_{\{u<0\}} = 1 - \chi_-$$

with similar definitions for $\widehat{\zeta}$ with $\widehat{\chi}_+ = \chi_{\{\widehat{u}>0\}}$ and $\widehat{\chi}_- = \chi_{\{\widehat{u}<0\}} = 1 - \widehat{\chi}_+$.

Taking into account (3.10), we immediately obtain (3.11) from the estimate (3.9) and the following inequality a.e. in Ω:

$$|\zeta - \widehat{\zeta}| = \big|\lambda_+(\chi_+ - \widehat{\chi}_+) - \lambda_-(\chi_- - \widehat{\chi}_-)\big| = \big|(\lambda_+ + \lambda_-)\,(\chi_+ - \widehat{\chi}_+)\big| \geq \mu|\chi_+ - \widehat{\chi}_+|.$$

□

Remark 3.3. The estimate (3.11) can be used to estimate the free boundary stability whenever $\Phi_0 = \{u = 0\}$ and $\widehat{\Phi}_0 = \{\widehat{u} = 0\}$ are non-degenerate and both u and $\widehat{u}$ have a monotonicity property, for instance, if $u_{x_n} = \frac{\partial u}{\partial x_n} \geq 0$ and $\widehat{u}_{x_n} \geq 0$. Hence in a cylinder subdomain $D = \omega \times (-L, L) \subset \Omega$ containing both Φ_0 and $\widehat{\Phi}_0$ we may define the upper semi-continuous functions in $x' \in \omega$

$$\varphi(x') = \inf\big\{x_n :\ u(x', x_n) > 0\big\} \qquad \text{and} \qquad \widehat{\varphi}(x') = \inf\big\{x_n :\ \widehat{u}(x', x_n) > 0\big\},$$

and from (3.11) we may conclude

$$\|\varphi - \widehat{\varphi}\|_{L^1(\omega)} = \int_D \big|\chi_{\{u>0\}} - \chi_{\{\widehat{u}>0\}}\big| \leq \frac{1}{\mu}\big(\|f - \widehat{f}\|_{L^1(\Omega)} + \|g - \widehat{g}\|_{L^1(\Gamma_N)}\big) \equiv \delta. \qquad (3.12)$$

If, in addition, both u and $\widehat{u}$ are such that, for some $\varepsilon > 0$

$$u_e = \nabla \cdot e \geq 0 \ \text{ and } \ \widehat{u}_e \geq 0, \qquad \forall\, e \in C_\varepsilon = \big\{x \in \mathbb{R}^n :\ x_n > \varepsilon |x'|\big\},$$

i.e., are monotone in some cone with axis e_n and opening $2\arctan(1/\varepsilon)$, then, under the assumption (3.1) the free boundaries may be locally given by Lipschitz graphs

$$\Phi_0 \cap D = \big\{x_n = \varphi(x'),\ x' \in \omega\big\} \quad \text{and} \quad \widehat{\Phi}_0 \cap D = \big\{x_n = \widehat{\varphi}(x'),\ x' \in \omega\big\}$$

with $|\nabla' \varphi| \leq \varepsilon$ and $|\nabla \widehat{\varphi}| \leq \varepsilon$. Then, arguing as in Theorem 6:5.3 in page 200 of [18], from (3.12) and using a Gagliardo–Nirenberg interpolation inequality we may also estimate the Hölder norm

$$\|\varphi - \widehat{\varphi}\|_{C^{0,\alpha}(\omega)} \leq C_\varepsilon\, \delta^{\frac{1-\alpha}{n}}, \qquad \text{for any} \quad 0 \leq \alpha < 1.$$

4 Regularity of the Free Boundary

In order to obtain the local boundedness of the $\mathcal{H}^{n-1}$-measure of the essential non degenerate free boundary Φ_0, we shall require a weak differentiability of the data

$$f \ \text{ and } \ \lambda_\pm \quad \text{are in} \quad BV_{\text{loc}}(\Omega), \qquad (4.1)$$

i.e., are of bounded variation in Ω' for all $\Omega' \subset\subset \Omega$ in the sense that

$$\|\nabla f\|(\Omega') = \sup\left\{\sum_{i=1}^{\infty} \int_{\Omega'} f \operatorname{div} \vec{\varphi};\ \vec{\varphi} \in C_c^\infty(\Omega')^n,\ \|\vec{\varphi}\|_\infty \leq 1\right\} < \infty.$$

We also require, in addition, for some $\kappa \in [0, 1]$:

$$\sum_{i,j=1}^{n} \left|\frac{\partial^2 a_i}{\partial x_i\, \partial x_j}(x, \eta)\right| \leq \gamma_3\big(\kappa + |\eta|^2\big)^{\frac{p-1}{2}}, \qquad (4.2)$$

$$\sum_{i,j,\ell=1}^{n} \left|\frac{\partial^2 a_\ell}{\partial \eta_j\, \partial x_i}(x, \eta)\right| \leq \gamma_4\big(\kappa + |\eta|^2\big)^{\frac{p-2}{2}}, \qquad (4.3)$$

for positive constants γ_3, γ_4, for a.e. $x \in \Omega$ and all $\eta \in \mathbb{R}^n$.

Theorem 4.1. *Let u be the solution of* (2.10) *under the assumptions of Theorem 2.1. If* (4.2)–(4.3) *holds, then*

$$Au \in BV_{\text{loc}}(\Omega). \tag{4.4}$$

Proof. We recall from (2.20) and the proof of Theorem 2.1 that u can be approximated by the solution u_ε of (2.13) where now we replace λ_+, λ_- and f by smooth approximations $\lambda_{\varepsilon+}$, $\lambda_{\varepsilon-}$ and f_ε with their gradients bounded in $L^1_{\text{loc}}(\Omega)$ uniformly in ε, by the assumption (4.1). Hence we have

$$Au_\varepsilon = f_\varepsilon - \lambda_{\varepsilon+}H_\varepsilon(u_\varepsilon) + \lambda_{\varepsilon-}H_\varepsilon(-u_\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} f - \lambda_+\chi_+ + \lambda_-\chi_- = Au \qquad \text{in} \quad L^q(\Omega)\text{-weak}$$

for all $q < \infty$, and it suffices to show that

$$\int_\omega \left|(Au_\varepsilon)_{x_\ell}\right| \leq C_\omega, \qquad \ell = 1, ..., n, \tag{4.5}$$

for an arbitrary $\omega \subset\subset \Omega$ and some constant $C_\omega > 0$ independent of $\varepsilon > 0$.

Consider a cutoff function $\varphi \in C_c^\infty(\Omega)$, such that, $\varphi \equiv 1$ in ω and $0 \leq \varphi \leq 1$ in Ω. Let $\sigma_\delta(t)$, $\delta > 0$, be a smooth approximation to $\text{sign}(t)$, i.e., such that $|\sigma_\delta(t)| \leq 1$, $\sigma'_\delta \geq 0$, $\sigma_\delta(0) = 0$ and $\lim_{\delta\to 0}\sigma_\delta(t) = \text{sign}(t)$. Observe that $\sigma_\delta(u_{\varepsilon x_\ell})\, u_{\varepsilon x_\ell}$ is a non-negative function and

$$\lim_{\delta\to 0} H'_\varepsilon(\pm u_\varepsilon)\, u_{\varepsilon x_\ell}\, \sigma_\delta(u_{\varepsilon x_\ell}) = \left|\big(H_\varepsilon(\pm u_\varepsilon)\big)_{x_\ell}\right| \qquad \text{a.e. in} \quad \omega.$$

From the approximating equation (2.13) we find

$$(Au_\varepsilon)_{x_\ell} = F_\varepsilon - \left[\lambda_{\varepsilon+}H'_\varepsilon(u_\varepsilon) + \lambda_{\varepsilon-}H'_\varepsilon(-u_\varepsilon)\right] u_{\varepsilon x_\ell} \tag{4.6}$$

where $F_\varepsilon = f_{\varepsilon x_\ell} - (\lambda_{\varepsilon+})_{x_\ell}H_\varepsilon(u_\varepsilon) + (\lambda_{\varepsilon-})_{x_\ell}H_\varepsilon(-u_\varepsilon)$ is uniformly bounded in $L^1_{\text{loc}}(\Omega)$. Hence, if we prove that

$$\int_\Omega \varphi\, \sigma_\delta(u_{\varepsilon x_\ell})\, (-Au_\varepsilon)_{x_\ell} \leq C_\varphi, \qquad \ell = 1, ..., n, \tag{4.7}$$

for some constant $C_\varphi > 0$ independent of ε and δ, we conclude (4.5) from the estimate

$$\begin{aligned}
\int_\Omega \varphi\left[\lambda_{\varepsilon+}\left|\big(H_\varepsilon(u_\varepsilon)\big)_{x_\ell}\right| + \lambda_{\varepsilon-}\left|\big(H_\varepsilon(-u_\varepsilon)\big)_{x_\ell}\right|\right] &= \\
&= \lim_{\delta\to 0}\int_\Omega \varphi\left[\lambda_{\varepsilon+}H'_\varepsilon(u_\varepsilon) + \lambda_{\varepsilon-}H'_\varepsilon(-u_\varepsilon)\right] u_{\varepsilon x_\ell}\, \sigma_\delta(u_{\varepsilon x_\ell}) \\
&= \lim_{\delta\to 0}\int_\Omega \varphi\, \sigma_\delta(u_{\varepsilon x_\ell})\left[F_\varepsilon - (Au_\varepsilon)_{x_\ell}\right] \leq \int_\Omega \varphi|F_\varepsilon| + C_\varphi
\end{aligned}$$

by recalling that $\lambda_{\varepsilon+}$ and $\lambda_{\varepsilon-}$ are nonnegative and uniformly bounded.

Finally, in order to prove the remaining estimate (4.7) we integrate by parts, as in [6],

$$
\begin{aligned}
\int_\Omega \varphi\, \sigma_\delta(u_{\varepsilon x_\ell})\,(Au_\varepsilon)_{x_\ell} \;&= -\int_\Omega \big[a(x,\nabla u_\varepsilon)\big]_{x_\ell}\cdot \nabla\big[\varphi\,\sigma_\delta(u_{\varepsilon x_\ell})\big] \;= \\
&= -\sum_{i=1}^n \int_\Omega \frac{\partial a_i}{\partial x_\ell}(x,\nabla u_\varepsilon)\big(\varphi\,\sigma_\delta(u_{\varepsilon x_\ell})\big)_{x_i} - \sum_{i,j=1}^n \int_\Omega \frac{\partial a_i}{\partial \eta_j}\, u_{\varepsilon x_\ell x_j}\,\varphi_{x_i}\,\sigma_\delta(u_{\varepsilon x_\ell}) \\
&\quad - \sum_{i,j=1}^n \int_\Omega \frac{\partial a_i}{\partial \eta_j}\, u_{\varepsilon x_\ell x_j}\, u_{\varepsilon x_\ell x_i}\,\sigma'_\delta(u_{\varepsilon x_\ell})\,\varphi \\
&= J_1 + J_2 + J_3,
\end{aligned}
\tag{4.8}
$$

and using the structural assumptions we have

$$
\begin{aligned}
J_1 \;&= \sum_{i=1}^n \int_\Omega \left(\frac{\partial^2 a_i}{\partial x_\ell\,\partial x_i} + \sum_{j=1}^n \frac{\partial^2 a_i}{\partial x_\ell\,\partial \eta_j}\, u_{\varepsilon x_j x_i}\right) \varphi\,\sigma_\delta(u_{\varepsilon x_\ell}) \\
&\le \gamma_3 \int_\Omega \big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-1}{2}} \varphi \,+ \gamma_4 \int_\Omega \big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}} |D^2 u_\varepsilon|\,\varphi \;\le\; \frac12\, C_\varphi, \\
J_2 \;&\le \gamma_1 \int_\Omega \big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}} |D^2 u_\varepsilon|\,|\nabla\varphi| \;\le\; \frac12\, C_\varphi, \\
J_3 \;&\le -\gamma_0 \int_\Omega \big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}} |\nabla u_{\varepsilon x_\ell}|^2\,\sigma'_\delta(u_{\varepsilon x_\ell})\,\varphi \;\le\; 0,
\end{aligned}
$$

where the choice of $C_\varphi > 0$ independently of ε is possible since $|\nabla u_\varepsilon| \in L^\infty_{\rm loc}(\Omega)$, which bound is independent of κ and ε, and $u_\varepsilon \in H^2_{\rm loc}(\Omega)$ uniformly in ε when $\kappa > 0$.

Since this last estimate does not hold in general for $\kappa = 0$, we need to estimate J_1 and J_2 independently of $\varepsilon > 0$, with the help of Lemma 4.1 below, with $\kappa = \varepsilon$, where u_ε is now an approximating solution of (2.13),(2.5) with A regularized by A_ε with each $a_\varepsilon(x,\eta)$ satisfying (2.1)–(2.3) and (4.2)–(4.3) with $\kappa = \varepsilon > 0$. □

Lemma 4.1. *Under the assumptions* (4.1)–(4.3) *and* (2.1)–(2.3) *we have the estimate*

$$
\int_{\Omega'} \Big[\big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}} |D^2 u_\varepsilon| \Big]^2 \le C', \qquad \forall\, \Omega' \subset\subset \Omega, \tag{4.9}
$$

where the constant $C' > 0$ depends on $\|\nabla f\|(\Omega'')$, $\|\nabla \lambda_\pm\|(\Omega')$ and $\|\nabla u_\varepsilon\|_{L^\infty(\Omega'')}$ with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, but is independent of $\kappa \in (0,1]$ and $\varepsilon > 0$.

Proof. Let $G = G(t)$ be a smooth odd nondecreasing function and $\varphi \in C^\infty_c(\Omega)$, $\varphi \ge 0$. Since $G(u_{\varepsilon x_\ell})\, u_{\varepsilon x_\ell} \ge 0$, if we multiply (4.6) by $\varphi^2 G(u_{\varepsilon x_\ell})$ and integrate in Ω we obtain, as in (4.8),

$$
\int_\Omega \big[a(x,\nabla u_\varepsilon)\big]_{x_\ell} \cdot \nabla\big(\varphi^2 G(u_{\varepsilon x_\ell})\big) \;\le\; \int_\Omega F_\varepsilon\, \varphi^2\, G(u_{\varepsilon x_\ell}). \tag{4.10}
$$

Setting $t_\varepsilon = (\kappa + |\nabla u_\varepsilon|^2)^{1/2}$ and developing the left hand side

$$\sum_{i=1}^{n} \int_\Omega \frac{\partial a_i}{\partial x_\ell} \big(\varphi^2 G(u_{\varepsilon x_\ell})\big)_{x_i} + \sum_{i,j=1}^{n} \int_\Omega \frac{\partial a_i}{\partial \eta_j} u_{\varepsilon x_\ell x_j} \big(\varphi^2 G'(u_{\varepsilon x_\ell})\, u_{\varepsilon x_\ell x_i} + 2\,\varphi\, \varphi_{x_i}\, G(u_{\varepsilon x_\ell})\big),$$

we use the structural assumptions (2.1)–(2.2) and (4.2)–(4.3) to obtain, from (4.10), for $\ell = 1, ..., n$:

$$\begin{aligned}
\gamma_0 \int_\Omega t_\varepsilon^{p-2} |\nabla u_{\varepsilon x_\ell}|^2 \varphi^2\, G'(u_{\varepsilon x_\ell}) \;&\le \sum_{i,j=1}^{n} \int_\Omega \frac{\partial a_i}{\partial \eta_j} u_{\varepsilon x_\ell x_i}\, u_{\varepsilon x_\ell x_j}\, \varphi^2\, G'(u_{\varepsilon x_\ell}) \;\le \\
&\le \int_\Omega \left[F_\varepsilon - \sum_{i=1}^{n} \left(\frac{\partial^2 a_i}{\partial x_\ell\, \partial x_i} + \sum_{j=1}^{n} \frac{\partial^2 a_i}{\partial x_\ell\, \partial \eta_j}\, u_{\varepsilon x_j x_i} \right) \right] \varphi^2\, G(u_{\varepsilon x_\ell}) \\
&\quad - \sum_{i,j=1}^{n} \int_\Omega \frac{\partial a_i}{\partial \eta_j} u_{\varepsilon x_\ell x_j}\, 2\, \varphi\, \varphi_{x_i}\, G(u_{\varepsilon x_\ell}) \qquad (4.11) \\
&\le C'_\varphi \int_\Omega \Big[\big(|F_\varepsilon| + \gamma_3\, t_\varepsilon^{p-1} \big) \varphi + (\varphi\, \gamma_4 + \gamma_1)\, t_\varepsilon^{p-2} |D^2 u_\varepsilon| \Big] \varphi |G(t_\varepsilon)|
\end{aligned}$$

since $|G(u_{\varepsilon x_\ell})| \le |G(t_\varepsilon)|$.

For $p \ge 2$, we set $G(t) = t$ and from (4.11) we have

$$\gamma_0 \int_\Omega t_\varepsilon^{p-2} |D^2 u_\varepsilon|^2 \varphi^2 \;\le n\, C'_\varphi \int_\Omega \varphi^2 \Big[\big(|F_\varepsilon| + \gamma_3\, M_\varphi^{p-1} \big) + (\varphi\, \gamma_4 + \gamma_1)\, t_\varepsilon^{p-2}\, |D^2 u_\varepsilon|\, \varphi \Big] M_\varphi,$$

where $M_\varphi = \|t_\varepsilon\|_{L^\infty(\operatorname{supp}\varphi)}$ may be chosen independent of ε and κ.

By Cauchy–Schwartz inequality and monotonicity of t^{p-2} we have

$$\int_{\Omega'} \big[t_\varepsilon^{p-2} |D^2 u_\varepsilon| \big]^2 \le M_\varphi^{p-2} \int_\Omega t_\varepsilon^{p-2} |D^2 u_\varepsilon|^2\, \varphi^2 \le M_\varphi^{p-1}\, C'' \left(1 + M_\varphi^{p-1} + \int_\Omega |F_\varepsilon|\, \varphi^2 \right)$$

provided $\varphi \ge 1$ in Ω', which proves (4.9), since $|F_\varepsilon| \le |\nabla f_\varepsilon| + |\nabla \lambda_{+\varepsilon}| + |\nabla \lambda_{-\varepsilon}|$ and f, λ_+ and λ_- are locally of bounded variation.

For $1 < p < 2$, we set $G(t) = (\varepsilon + t^2)^{\frac{p-2}{2}}\, t$ and since we have $G'(t) \ge (p-1)\,(\varepsilon + t^2)^{\frac{p-2}{2}}$ we get from (4.11) with $s_\varepsilon = (\varepsilon + |u_{\varepsilon x_\ell}|^2)^{1/2} \le t_\varepsilon$ and $\ell = 1, ..., n$

$$\gamma_0 (p-1) \int_\Omega s_\varepsilon^{p-2}\, t_\varepsilon^{p-2} |\nabla u_{\varepsilon x_\ell}|^2 \varphi^2 \;\le C'_\varphi \int_\Omega \Big[\big(|F_\varepsilon| + \gamma_3\, t_\varepsilon^{p-1} \big) \varphi + (\varphi\, \gamma_4 + \gamma_1)\, t_\varepsilon^{p-2}\, |D^2 u_\varepsilon| \Big] \varphi\, t_\varepsilon^{p-1}.$$

Again by Cauchy–Schwartz inequality and noting now $s_\varepsilon^{p-2} \ge t_\varepsilon^{p-2}$ we may conclude as before

$$\int_\Omega \big[t_\varepsilon^{p-2} |D^2 u_\varepsilon| \big]^2 \varphi^2 \le C^*\, M_\varphi^{p-1} \left(M_\varphi^{p-1} + \int_\Omega |F_\varepsilon|\, \varphi^2 \right).$$

□

As a consequence of equation (1.1), in the non-degenerate interface case, using (3.5) we may write

$$\chi_{\{u>0\}} = \frac{f - Au + \lambda_-}{\lambda_+ + \lambda_-} \qquad \text{a.e. in} \quad \omega \tag{4.12}$$

where we have introduced the subset

$$\omega = \big\{x \in \Omega : \ (\lambda_+ + \lambda_-)(x) > 0\big\}. \tag{4.13}$$

Assuming now

$$\lambda_+ + \lambda_- \in C(\Omega) \cap W^{1,1}_{\text{loc}}(\Omega) \tag{4.14}$$

as an immediate consequence of Theorem 4.1, from (4.12) it follows that the characteristic functions $\chi_{\{u>0\}}$ and $\chi_{\{u<0\}}$ of both phases are locally of bounded variation in ω and, by a well known theorem of De Giorgi (see [11], for instance), that yields the following regularity of the free boundary.

Theorem 4.2. *Under the structural conditions* (2.1)–(2.3), (4.2)–(4.3) *on the heterogeneous operator A and the assumptions* (2.6)–(2.7)*,* (4.1) *and* (4.14)*, where the interface is nondegenerate, i.e., if $\mathcal{L}^n(\Phi_0 \cap \omega) = 0$, the free boundary is, up to a set of null perimeter (i.e., of $\|\nabla \chi_{\{u>0\}}\|$-measure zero), the union of at most a countable family of C^1-hypersurfaces.*

Remark 4.1. As it is known from measure theory (see [10], for instance), an open set $\mathcal{O} \subset \mathbb{R}^n$ which characteristic function is locally of bounded variation has a boundary $\partial\mathcal{O}$ locally with finite perimeter. Its singular component $\Sigma \subset \partial\mathcal{O}$, that is the subset of points with null upper n-dimensional Lebesgue densities with respect to $\mathcal{O}$ and $\mathbb{R}^n \backslash \mathcal{O}$, has null perimeter (i.e. $\|\nabla\chi_{\mathcal{O}}\|(\Sigma) = 0$) and its essential boundary $\partial_e\mathcal{O} = \partial\mathcal{O}\backslash\Sigma$ has locally finite $(n-1)$-dimensional Hausdorff measure. Although the corresponding regularity for the heterogeneous one obstacle problem has been shown in [6], implying that a similar conclusion holds for the free boundary in the one-phase local situation, no concrete conclusion is known at the free boundary branch points for general operators with non-homogeneous data.

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