ANISOTROPICALLY DIFFUSED AND DAMPED NAVIER-STOKES EQUATIONS

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Abstract. The incompressible Navier-Stokes equations with anisotropic diffusion and anisotropic damping is considered in this work. For the associated initial-boundary value problem, we prove the existence of weak solutions and we establish an energy inequality satisfied by these solutions. We prove also under what conditions the solutions of this problem extinct in a finite time.

1. Introduction

In this work is considered the initial-boundary value problem posed by the incompressible Navier-Stokes equations modified by the consideration of an anisotropic diffusion and by the presence of an anisotropic damping in the momentum equation. The governing equations are:

$$\operatorname{div}\mathbf{u}=0 \quad \text{in} \quad Q_T:=\Omega\times[0,T]; \tag{1.1}$$

$$\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}+\sum_{i=1}^{N}\kappa_i|u_i|^{\sigma_i-2}u_i\,\mathbf{e}_i=\mathbf{f}-\nabla p+\sum_{i=1}^{N}D_i\left(|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right) \quad \text{in} \quad Q_T; \tag{1.2}$$

$$\mathbf{u}=\mathbf{u}_0 \quad \text{in} \quad \Omega \quad \text{for} \quad t=0; \tag{1.3}$$

$$\mathbf{u}=\mathbf{0} \quad \text{on} \quad \Gamma_T:=\partial\Omega\times[0,T]; \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N\geq 2$, $\partial\Omega$ is the compact boundary of Ω, and $T>0$. Here, the vector function $\mathbf{u}=(u_1,\cdots,u_N)$ and the scalar function p are the unknowns of the problem. The vector function $\mathbf{f}=(f_1,\cdots,f_N)$ is a given problem data, $D_i\mathbf{u}=(\partial_i u_1,\ldots,\partial_i u_N)$ and $\partial_i u_j=\frac{\partial u_j}{\partial x_i}$, and $\{\mathbf{e}_1,\ldots,\mathbf{e}_N\}$ is the canonical basis of $\mathbb{R}^N$. The exponents σ_i and q_i are assumed to be constant with possible distinct values ranging in $(1,\infty)$, and κ_i are non-negative real constants.

The isotropic version of the problem (1.1)-(1.4), *i.e.* when in (1.2) $\sigma_i=\sigma$ and $q_i=q$ for all $i\in\{1,\ldots,N\}$, was considered in the works [11, 12], where it was studied the existence of weak solutions and its asymptotic behavior. In recent years, damped Navier-Stokes equations as the ones studied in [11, 12] found some physical

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justification in the study of porous media flows, where they are known as Brinkman-Forchheimer equations (see *e.g.* [5, 8]). Damped Navier-Stokes equations appear also as a regularization technique to prove existence results for generalized Navier-Stokes equations that govern flows of non-Newtonian fluids (see *e.g.* [6]). The main feature of damped Navier-Stokes equations is that, due to the damping term, one obtains better existence results and some qualitative properties as the extinction in a finite time can be extended to all the range of the power-law exponent. The case of the non-damped problem (1.1)-(1.4), *i.e.* when in (1.2) $\kappa_i = 0$ for all $i \in \{1, \dots, N\}$, was previously considered in [2, 3], where it was proved the existence of weak solutions and it were established some properties of finite time extinction and large time behavior of the solutions. With respect to Navier-Stokes equations with anisotropic diffusion, its main attribute is that, with no need of a damping term, the existence results and many qualitative properties of the solutions can be improved (with respect to its isotropic versions) in almost all directions, or for particular choices of all the diffusion coefficients, with the possibility to achieve optimal results, at least in some directions. In this work, we consider the case in which the generalized Navier-Stokes equations are modified, not only by considering an extra damping term in the momentum equation, but also by assuming that this damping, as well the diffusion, might be fully anisotropic.

This article is organized as follows. In Section 2 we introduce the main concepts of the anisotropic function spaces we are going to work with and we define the notion of weak solution to the problem (1.1)-(1.4). The existence of weak solutions to our problem is established in Section 3 and there we also establish an energy inequality satisfied by these solutions. Properties of extinction in a finite time are studied in Section 4 under different conditions on the diffusion or on the damping terms. The notation used throughout this article and the main notions of the considered (isotropic) function spaces are largely standard in the literature of Partial Differential Equations (see *e.g.* [1, 9]).

2. Weak formulation

Let us define the vectors σ and $\mathbf{q}$ in $\mathbb{R}^N$, whose components are the exponents of the anisotropic terms of damping and of diffusion considered in (1.2), by $\sigma := (\sigma_1, \cdots, \sigma_N)$ and $\mathbf{q} := (q_1, \cdots, q_N)$, where $1 < \sigma_i,\ q_i < \infty$ for all $i \in \{1, \dots, N\}$, and let us set

$$\sigma^+ := \max_{i\in\{1,\dots,N\}} \sigma_i\,,\ \sigma^- := \min_{i\in\{1,\dots,N\}} \sigma_i\,, \quad q^+ := \max_{i\in\{1,\dots,N\}} q_i\,,\ q^- := \min_{i\in\{1,\dots,N\}} q_i,\,.$$

In order to emphasize that σ and $\mathbf{q}$ are multicomponent, throughout the text we will use the notations $\overrightarrow{\sigma}$ and $\overrightarrow{\mathbf{q}}$. We define the anisotropic Banach spaces

$$\mathbf{L}^{\overrightarrow{\mathbf{q}}}(\Omega) = \bigcap_{i=1}^{N} \mathbf{L}^{q_i}(\Omega)\,, \quad \|\mathbf{v}\|_{\mathbf{L}^{\overrightarrow{\mathbf{q}}}(\Omega)} := \sum_{i=1}^{N} \|\mathbf{v}\|_{\mathbf{L}^{q_i}(\Omega)}\,,$$

$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) = \bigcap_{i=1}^{N} \mathbf{W}^{1,q_i}_i(\Omega)\,, \quad \|\mathbf{v}\|_{\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega)} := \|\mathbf{v}\|_{\mathbf{L}^1(\Omega)} + \sum_{i=1}^{N} \|D_i\mathbf{v}\|_{\mathbf{L}^{q_i}(\Omega)}\,,$$

where $\mathbf{L}^{q_i}(\Omega)$ is the usual Lebesgue space and $\mathbf{W}^{1,q_i}_i(\Omega)$ denotes the unidirectional Sobolev space $\{\mathbf{v} \in \mathbf{W}^{1,1}(\Omega) : D_i\mathbf{v} \in \mathbf{L}^{q_i}(\Omega)\}$ with norm defined by $\|\mathbf{v}\|_{\mathbf{W}^{1,q_i}_i(\Omega)} := \|\mathbf{v}\|_{\mathbf{L}^1(\Omega)} + \|D_i\mathbf{v}\|_{\mathbf{L}^{q_i}(\Omega)}$. An important limitation of the anisotropic Sobolev space $\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega)$, is that, for bounded domains Ω, the validity of Sobolev imbeddings is restricted to rectangular domains (see *e.g.* [7]). In fact, for rectangular domains Ω, the following imbedding is continuous (*cf.* [13, Theorem 1])

$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow \mathbf{L}^{s}(\Omega) \quad \text{for any } s: \begin{cases} 1 \leq s \leq \overline{q}^{\,*}, & \sum_{i=1}^{N} \frac{1}{q_i} > 1 \\ 1 \leq s < \infty, & \sum_{i=1}^{N} \frac{1}{q_i} \leq 1 \end{cases}, \tag{2.1}$$

where $\overline{q}^{\,*} := \frac{N\overline{q}}{N-\overline{q}}$ and $\overline{q} := \left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{q_i}\right)^{-1}$. As a particular case of (2.1), it can be derived the following inequality

$$\|\mathbf{u}\|_{\mathbf{L}^{s}(\Omega)} \leq C\left(\prod_{i=1}^{N}\|D_i\mathbf{u}\|_{\mathbf{L}^{q_i}(\Omega)}\right)^{\frac{1}{N}} \qquad \forall\ \mathbf{u}\in\mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) \tag{2.2}$$

for $s \leq \overline{q}^{\,*}$ if $\sum_{i=1}^{N}\frac{1}{q_i} > 1$, or $s\in[1,\infty)$ otherwise (see [14, Theorem 1.2]). Moreover, the imbedding (2.1) is compact (*cf.* [13, Theorem 2]), and we denote this fact by

$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^{s}(\Omega) \quad \text{for any } q: 1\leq s<\overline{q}^{\,*}. \tag{2.3}$$

In some situations it is possible to remove the restrictions on the shape's domain and to enlarge the interval of s for the validity of (2.1)-(2.3). Let us see this fact by defining

$$\mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) := \text{ closure of } \mathbf{C}_0^{\infty}(\Omega) \text{ in the anisotropic norm of } \mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega).$$

In this case, we have (*cf.* [4, Theorem 1]), in the interesting case of $\sum_{i=1}^{N}\frac{1}{q_i} > 1$, that (2.1)-(2.3) hold with the critical exponent $\overline{q}^{\,*}$ replaced by

$$q_a^* := \max\left\{\overline{q}^{\,*}, q^+\right\}. \tag{2.4}$$

Observe that for $N=2$, $\overline{q}^{\,*} > q^+$ and therefore $q_a^* = \overline{q}^{\,*}$. But, if $N>2$, it may well happen that $q^+ > \overline{q}^{\,*}$. See [2, Remark 2.1]) for more details and useful examples.

In order to introduce the notions of weak solutions we shall consider in this work, let us recall the well-known function spaces of Mathematical Fluid Mechanics. Given q such that $1<q<\infty$, we set $\mathcal{V} := \{\mathbf{v}\in\mathbf{C}_0^{\infty}(\Omega) : \operatorname{div}\mathbf{v}=0\}$, $\mathbf{H} :=$ closure of $\mathcal{V}$ in the norm $\|\cdot\|_{\mathbf{L}^2(\Omega)}$, and $\mathbf{V}_q :=$closure of $\mathcal{V}$ in the norm $\|\cdot\|_{\mathbf{W}^{1,q}(\Omega)}$. Let us now define the anisotropic analogue of $\mathbf{V}_q$ by

$$\mathbf{V}_{\overrightarrow{\mathbf{q}}} := \text{closure of } \mathcal{V} \text{ in the anisotropic norm of } \mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega).$$

The anisotropic Bochner space considered here is defined by

$$\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}}) := \bigcap_{i=1}^{N}\mathrm{L}^{q_i}(0,T;\mathbf{V}_{q_i}),$$

whose norm is defined by $\|\mathbf{v}\|_{L^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})} := \|\mathbf{v}\|_{\mathbf{L}^1(Q_T)} + \sum_{i=1}^{N}\|D_i\mathbf{v}\|_{\mathbf{L}^{q_i}(Q_T)}$, where $\mathbf{V}_{q_i} :=$closure of $\mathcal{V}$ in the norm of $\mathbf{W}_i^{1,q_i}(\Omega)$ and $\mathrm{L}^{q_i}(0,T;\mathbf{V}_{q_i})$ is the unidirectional Bochner space, with the norm defined by $\|\mathbf{v}\|_{L^{q_i}(0,T;\mathbf{V}_{q_i})} := \|\mathbf{v}\|_{\mathbf{L}^1(Q_T)} + \|D_i\mathbf{v}\|_{\mathbf{L}^{q_i}(Q_T)}$. Note that, for a bounded domain Ω and for a finite T, the continuous imbedding $\mathrm{L}^{q^+}(0,T;\mathbf{V}_{q^+}) \hookrightarrow \mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}}) \hookrightarrow \mathrm{L}^{q^-}(0,T;\mathbf{V}_{q^-})$ holds. Therefore, as a closed subspace of $\mathrm{L}^{q^-}(0,T;\mathbf{V}_{q^-})$, the anisotropic space $\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$ is separable and reflexive. The dual spaces of $\mathrm{L}^{q_i}(0,T;\mathbf{V}_{q_i})$ and $\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$ are denoted by $\mathrm{L}^{q_i'}(0,T;\mathbf{V}_{q_i}')$ and $\mathrm{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}}')$, respectively, where $\mathbf{V}_{q_i}'$ and $\mathbf{V}_{\overrightarrow{\mathbf{q}}}'$ stay for the dual spaces of $\mathbf{V}_{q_i}$ and $\mathbf{V}_{\overrightarrow{\mathbf{q}}}$.

Definition 2.1. Let $N \geq 2$ and $\overrightarrow{\mathbf{q}} = (q_1, \dots, q_N)$, $\overrightarrow{\sigma} = (\sigma_1, \dots, \sigma_N)$, with $1 < q_i,\ \sigma_i < \infty$ for all $i \in \{1, \dots, N\}$. Assume that $\mathbf{f} \in \mathbf{L}^{\overrightarrow{\mathbf{q}}\,'}(0,T;\mathbf{V}'_{\overrightarrow{\mathbf{q}}})$ and

$$q_a^* \geq 2\,. \tag{2.5}$$

A vector field $\mathbf{u}$ is a weak solution to the problem (1.1)-(1.4), if:

(1) $\mathbf{u} \in \mathrm{L}^\infty(0,T;\mathbf{H}) \cap \mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$, with $\mathbf{e}_i \cdot \mathbf{u} \in \mathrm{L}^{\sigma_i}(\Omega)$ for all $i \in \{1, \dots, N\}$;
(2) $\mathbf{u}(0) = \mathbf{u}_0$;
(3) For every $\mathbf{v} \in \mathbf{V}_{\overrightarrow{\mathbf{q}}} \cap \mathbf{L}^\theta(\Omega)$, with $\mathbf{e}_i \cdot \mathbf{v} \in \mathrm{L}^{\sigma_i}(\Omega)$ for all $i \in \{1, \dots, N\}$,

(2.6)
$$\frac{d}{d\,t}\int_\Omega \mathbf{u}(t)\cdot\mathbf{v}\,d\mathbf{x} + \sum_{i=1}^N \int_\Omega |D_i\mathbf{u}(t)|^{q_i-2} D_i\mathbf{u}(t)\cdot D_i\mathbf{v}\,d\mathbf{x} +$$
$$\sum_{i=1}^N \kappa_i \int_\Omega |u_i(t)|^{\sigma_i-2} u_i(t)\,\mathbf{e}_i\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega [(\mathbf{u}(t)\cdot\nabla)\,\mathbf{u}(t)]\cdot\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f}(t)\cdot\mathbf{v}\,d\mathbf{x}$$

for all $t \in [0,T]$, where $\frac{1}{q_a^*} + \frac{1}{q^-} + \frac{1}{\theta} = 1$.

Observe that if $\sigma^+ \leq q^- + 2 - \frac{2q^-}{q_a^*}$, then the continuous imbedding $\mathrm{L}^\infty(0,T;\mathbf{H}) \cap \mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}}) \hookrightarrow \mathrm{L}^{\sigma^+}(Q_T)$ holds and therefore in Definition 2.1 we only need to require that $\mathbf{u} \in \mathrm{L}^\infty(0,T;\mathbf{H}) \cap \mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$. Note also that $\mathbf{v} \in \mathbf{L}^\theta(\Omega)$ is necessary to control the boundedness of the convective integral term when $\mathbf{u}$ merely belongs to $\mathrm{L}^\infty(0,T;\mathbf{H}) \cap \mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$. But, if $\theta \leq q_a^*$ or if $\theta \leq \sigma^-$ we only need to require that $\mathbf{v} \in \mathbf{V}_{\overrightarrow{\mathbf{q}}}$, with $\mathbf{e}_i \cdot \mathbf{v} \in \mathrm{L}^{\sigma_i}(\Omega)$ for all $i \in \{1, \dots, N\}$, since, in this case, the imbeddings $\mathbf{V}_{\overrightarrow{\mathbf{q}}} \hookrightarrow \mathbf{L}^\theta(\Omega)$ and $\mathbf{L}^{\sigma_i}(\Omega) \hookrightarrow \mathbf{L}^\theta(\Omega)$, for $i = 1, \dots, N$, hold. On the other hand, the assumption (2.5) is needed to control the first integral when $\mathbf{u}$ and $\mathbf{v}$ solely belong to the mentioned spaces.

3. Existence

First we observe that, according to what is customary in Mathematical Fluid Mechanics, the determination of the pressure p is not a problem. In fact, after we determine $\mathbf{u}$, we can recover p by applying de Rham's theorem (see *e.g.* [9]). Therefore, with regard to the existence, in this work we shall be concerned only with the existence of the unknown field $\mathbf{u}$.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume that*

$$\mathbf{u}_0 \in \mathbf{H}\,, \qquad \mathbf{f} \in \mathbf{L}^{\overrightarrow{\mathbf{q}}'}(Q_T), \quad \textit{where } \overrightarrow{\mathbf{q}}' = (q_1', \dots, q_N'). \tag{3.1}$$

If $q_a^* \geq q_*$, where $q_* := \frac{2q^-(q^--1)}{(q^-+1)(q^--2)}$ if $2 < q^- < 3$ and $q_* := \frac{2q^-}{q^--1}$ if $q^- \geq 3$, then there exists, at least, a weak solution to the problem (1.1)-(1.4), in the sense of Definition 2.1, for any $\overrightarrow{\sigma}$ with $\sigma_i > 1$ for all $i \in \{1, \dots, N\}$.

Proof. The proof of Theorem 3.1 will be split into several steps.

Step 1: Existence of approximative solutions. For the smallest integer $s > 1 + \frac{N}{2}$, we define $\mathbf{V}^s :=$ closure of $\mathcal{V}$ in $\mathbf{W}^{s,2}(\Omega)$. Let $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ be a set of non-trivial solutions $\mathbf{v}_j$ of the following spectral problem associated to the eigenvalues $\lambda_j > 0$:

$$\sum_{|\gamma|=s} \int_\Omega \mathrm{D}^\gamma \mathbf{v}_j \cdot \mathrm{D}^\gamma \varphi\,d\mathbf{x} = \lambda_j \int_\Omega \mathbf{v}_j \cdot \varphi\,d\mathbf{x} \quad \forall\, \varphi \in \mathbf{V}^s\,.$$

The family $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ is orthogonal in $\mathbf{V}^s$ and can be chosen as being orthonormal in $\mathbf{H}$ (see *e.g.* [10, p. 290]). Given $m\in\mathbb{N}$, let us also consider the correspondingly m-dimensional space, say $\mathbf{V}^m$, spanned by $\mathbf{v}_1,\ldots,\mathbf{v}_m$. For each $m\in\mathbb{N}$, we search for an approximative solution $\mathbf{u}^m(t)$ of (2.6) in the form $\mathbf{u}^m(t)=\sum_{k=1}^m c_k^m(t)\mathbf{v}_k$, where $\mathbf{v}_k\in\mathbf{V}^m$. This function is found by solving the following system of m nonlinear ordinary differential equations, with respect to the m unknowns $c_1^m,\ldots,c_m^m$, obtained from (2.6):

$$\begin{aligned}&\int_\Omega \frac{\partial\,\mathbf{u}^m(t)}{\partial t}\cdot\mathbf{v}_k\,d\mathbf{x}+\sum_{i=1}^N\int_\Omega |D_i\mathbf{u}^m(t)|^{q_i-2}D_i\mathbf{u}^m(t)\cdot D_i\mathbf{v}_k\,d\mathbf{x}\\ &+\sum_{i=1}^N\int_\Omega |u_i^m(t)|^{\sigma_i-2}u_i^m(t)\,\mathbf{e}_i\cdot\mathbf{v}_k\,d\mathbf{x}+\int_\Omega\left[\left(\mathbf{u}^m(t)\cdot\nabla\right)\mathbf{u}^m(t)\right]\cdot\mathbf{v}_k\,d\mathbf{x}\\ &=\int_\Omega \mathbf{f}(t)\cdot\mathbf{v}_k\,d\,\mathbf{x},\qquad \text{for } k=1,\ldots,m;\end{aligned}\tag{3.2}$$

supplemented with the initial conditions

$$\mathbf{u}^m(0)=\mathbf{u}_0^m\,,\quad \text{and assume that}\quad \mathbf{u}_0^m\to\mathbf{u}_0\,,\quad \text{as } m\to\infty, \text{ in } \mathbf{H}.\tag{3.3}$$

Due to the orthonormality of $\{\mathbf{v}_k\}_{k=1}^N$ in $\mathbf{H}$, the system (3.2)-(3.3) can be rewritten as a system of ODEs (see *e.g.* [10, p. 207]) for which, due to Peano's theorem, there exist $t^m\in(0,T)$ and $\mathbf{c}^m(t)\equiv(c_1^m(t),\ldots,c_N^m(t))$ such that $\mathbf{c}^m(t)$ is a solution thereof in the interval $[0,t^m]$. Then, we multiply (3.2) by c_k^m, we add up from $k=1$ until $k=m$ and we integrate the resulting equation between 0 and t, with $t\in(0,t^m)$. In the obtained equation, we take the essential supreme in $[0,T]$, we use the assumptions (3.1) and (3.3), and we apply Cauchy's inequality with a suitable ϵ. After all, we get

$$\|\mathbf{u}^m\|_{\mathrm{L}^\infty(0,T;\mathbf{H})}+\sum_{i=1}^N\left(\|D_i\mathbf{u}^m\|^{q_i}_{\mathbf{L}^{q_i}(Q_T)}+\|u_i^m\|^{\sigma_i}_{\mathrm{L}^{\sigma_i}(Q_T)}\right)\leq C,\tag{3.4}$$

where C is an independent of m positive constant. Thus, from the Theory of the ODEs, we can take $t^m=T$.

Step 2: Convergence of the approximative solutions. Due to (3.4) and by means of separability and reflexivity, there exists $\mathbf{u}$ and a subsequence (still denoted by) $\mathbf{u}^m$ such that

$$\mathbf{u}^m\longrightarrow\mathbf{u}\quad\text{weakly-}*\text{ in } \mathrm{L}^\infty(0,T;\mathbf{H}),\quad \text{as } m\to\infty,\tag{3.5}$$

$$\mathbf{u}^m\longrightarrow\mathbf{u}\quad\text{weakly in } \mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}}),\quad \text{as } m\to\infty\,,\tag{3.6}$$

$$u_i^m\longrightarrow u_i\quad\text{weakly in } \mathrm{L}^{\sigma_i}(Q_T),\quad \text{as } m\to\infty\,,\quad i=1,\ldots,N.\tag{3.7}$$

On the other hand, it can be proved that for any $i\in\{1,\ldots,N\}$

$$\|D_i\left(|D_i\mathbf{u}^m|^{q_i-2}D_i\mathbf{u}^m\right)\|_{\mathrm{L}^{q_i'}(0,T;\mathbf{V}'_{q_i})}\leq C\,.$$

Then, for each $i\in\{1,\ldots,N\}$, there exists $\mathbf{S}_i\in\mathrm{L}^{q_i'}(0,T;\mathbf{V}'_{q_i})$ such that

$$D_i\left(|D_i\mathbf{u}^m|^{q_i-2}D_i\mathbf{u}^m\right)\longrightarrow\mathbf{S}_i\quad\text{weakly in } \mathrm{L}^{q_i'}(0,T;\mathbf{V}'_{q_i}),\quad \text{as } m\to\infty.\tag{3.8}$$

Next, using the orthogonal projection operator $P^m:\mathbf{H}\to\mathbf{V}^m$ in the spirit of [9, p. 76] and owing to (3.2), (3.4) and $(3.1)_2$, it can be proved that

$$\frac{\partial\mathbf{u}^m}{\partial t}\in\mathrm{L}^\infty(0,T;\mathbf{V}'_s)\cap\mathrm{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}'_s)\cap\left(\cup_{i=1}^N\mathbf{L}^{\sigma_i'}(Q_T)\right)\,,$$

where $\mathbf{V}_s'$ is the dual of the space $\mathbf{V}_s$ defined in Step 1 and $\mathrm{L}^{\vec{\mathbf{q}}'}(0,T;\mathbf{V}_s')$ is the dual space of $\mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_s)$. In particular, for finite T, $\mathrm{L}^{\infty}(0,T;\mathbf{V}_s') \hookrightarrow \mathrm{L}^{\vec{\mathbf{q}}'}(0,T;\mathbf{V}_s') \hookrightarrow \mathrm{L}^{(q^+)'}(0,T;\mathbf{V}_s')$ and $\mathbf{L}^{\sigma_i'}(Q_T) \hookrightarrow \mathbf{L}^{\sigma_i'}(0,T;\mathbf{V}_s') \hookrightarrow \mathbf{L}^{(\sigma^+)'}(0,T;\mathbf{V}_s')$ for any $i \in \{1,\dots,N\}$. As a consequence

$$\frac{\partial \mathbf{u}^m}{\partial t} \in \mathrm{L}^{\iota'}(0,T;\mathbf{V}_s'), \qquad \iota' := \min\{(q^+)',(\sigma^+)'\}\,. \tag{3.9}$$

By means of reflexivity, we have

$$\frac{\partial \mathbf{u}^m}{\partial t} \longrightarrow \frac{\partial \mathbf{u}}{\partial t} \quad \text{weakly in } \mathrm{L}^{\iota'}(0,T;\mathbf{V}_s')\,. \tag{3.10}$$

On the other hand, for a finite T, there holds $\mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}}) \hookrightarrow \mathrm{L}^{q^-}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$ and, due to (3.4),

$$\mathbf{u}^m \in \mathrm{L}^{q^-}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})\,. \tag{3.11}$$

Now, we observe that by the choice of s, the continuous imbedding

$$\mathbf{H} \hookrightarrow \mathbf{V}_s' \tag{3.12}$$

holds and, by (2.3) with q_a^*, the anisotropic compactness imbedding

$$\mathbf{V}_{\vec{\mathbf{q}}} \hookrightarrow\hookrightarrow \mathbf{H}\,, \quad \text{with } 2 < q_a^*, \tag{3.13}$$

also holds. Then, due to (3.9)-(3.13), we can apply Aubin-Lions compactness Lemma (see [9, p. 58]) to deduce that

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{strongly in } \mathrm{L}^{q^-}(0,T;\mathbf{H}), \quad \text{as } m \to \infty. \tag{3.14}$$

Using parabolic interpolation, we obtain from (3.4) and (3.14) that

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{strongly in } \mathrm{L}^{s}(0,T;\mathbf{H}), \text{ as } m \to \infty, \text{ for any } s \geq 1. \tag{3.15}$$

Consequently, for any $i \in \{1,\dots,N\}$, we also have

$$u_i^m \longrightarrow u_i \quad \text{strongly in } \mathrm{L}^{s}(Q_T), \text{ as } m \to \infty, \text{ for any } s \geq 1. \tag{3.16}$$

Step 3: Passing to the limit. Fixing k, we pass the equation (3.2) to the limit $m \to \infty$ by using (3.6), (3.8), (3.10), (3.15) and (3.16), and yet observing the definition of the space $\mathbf{V}^s$ given in Step 1, we obtain for all $t \in [0,T]$

$$\begin{aligned} &\int_\Omega \frac{\partial\, \mathbf{u}(t)}{\partial t} \cdot \mathbf{v}_k \, d\mathbf{x} + \sum_{i=1}^N \int_\Omega \mathbf{S}_i(t) \cdot \mathbf{v}_k \, d\mathbf{x} + \sum_{i=1}^N \int_\Omega |u_i(t)|^{\sigma_i - 2} u_i(t)\, \mathbf{e}_i \cdot \mathbf{v}_k \, d\mathbf{x} \\ &+ \int_\Omega [(\mathbf{u}(t)\cdot\nabla)\,\mathbf{u}(t)] \cdot \mathbf{v}_k \, d\mathbf{x} = \int_\Omega \mathbf{f}(t) \cdot \mathbf{v}_k \, d\mathbf{x} \qquad \forall\ \mathbf{v}_k \in \mathbf{V}^m\,. \end{aligned} \tag{3.17}$$

Since $\mathbf{V}^s = \bigcup_{m=1}^\infty \mathbf{V}^m$, this equation holds for all $\mathbf{v} \in \mathbf{V}^s$. By a continuity argument, the equation (3.17) holds true for all $\mathbf{v} \in \mathbf{V}_{\vec{\mathbf{q}}}$ as long as the integrals there remain bounded for $\mathbf{u} \in \mathrm{L}^{\infty}(0,T;\mathbf{H}) \cap \mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$ and with $u_i \in \mathrm{L}^{\sigma_i}(Q_T)$ for all $i \in \{1,\dots,N\}$. The only difficulty here is to show the boundedness of the convective integral term. For this, we use Hölder's inequality, to show that for all $t \in [0,T]$

$$\int_\Omega [(\mathbf{u}(t)\cdot\nabla)\,\mathbf{u}(t)] \cdot \mathbf{v} \, d\mathbf{x} = -\int_\Omega \mathbf{u}(t) \otimes \mathbf{u}(t) : \nabla \mathbf{v} \, d\mathbf{x} \leq \|\mathbf{u}(t)\|^2_{\mathbf{L}^{2(q^-)'}(\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^{q^-}(\Omega)}\,.$$

The boundedness follows by using the imbedding $\mathbf{V}_{\vec{\mathbf{q}}} \hookrightarrow \mathbf{V}_{q^-}$ and the anisotropic Sobolev imbedding (2.1) which holds for

$$q_a^* \geq \frac{2q^-}{q^- - 1}\,. \tag{3.18}$$

Step 4: Use of the monotonicity. Since the diffusion term is the sum of N possible different diffusion terms which are strictly monotonous, the anisotropic diffusion term is strictly monotonous:

$$\sum_{i=1}^{N} \langle |D_i\xi|^{q_i-2}D_i\xi - |D_i\eta|^{q_i-2}D_i\eta, D_i\xi - D_i\eta\rangle_{\mathbf{L}^{q_i'}(Q_T)\times\mathbf{L}^{q_i}(Q_T)} > 0$$

for every $\xi,\ \eta \in \mathbf{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$, with $\xi \neq \eta$. Consequently, appealing to this strict monotonicity property, we can construct a suitable test function $\mathbf{v} \in \mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$ to use the Minty trick in the spirit of Lions [9, pp. 212-215]) and to identify

$$\mathbf{S}_i = D_i\left(|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right) \quad \forall\, i \in \{1,\ldots,N\}.$$

For the application of the this reasoning, one needs to show that

$$\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\,d\mathbf{x} = -\int_\Omega \mathbf{u}\otimes\mathbf{u} : \nabla\,\mathbf{v}\,d\mathbf{x} \ \in \mathrm{L}^1(0,T) \tag{3.19}$$

for $\mathbf{u} \in \mathrm{L}^\infty(0,T;\mathbf{H}) \cap \mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$ and $\mathbf{v} \in \mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$. In [3] it is proved that (3.19) holds provided that

$$q_a^* \geq \frac{2q^-(q^- - 1)}{(q^- + 1)(q^- - 2)} \quad \text{and} \quad q^- > 2\,. \tag{3.20}$$

Finally, we observe that (3.18) implies (3.13) and the requirements of (3.18) and (3.20) follow from the assumption that $q_a^* \geq q_*$. $\square$

In the next result is established the energy inequality satisfied by the solutions to the problem (1.1)-(1.4).

Theorem 3.2. *Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.4) in the sense of Definition 2.1. Assume that $(3.1)_1$ holds and $\mathbf{f} = -\mathbf{div}\,\mathbf{F}$, with $\mathbf{F} \in \mathbf{L}^{\vec{\mathbf{q}}'}(Q_T)$. Then there exist two independent of t constants C_1 and C_2 such that, for any $t \in [0,T]$,*

$$\frac{d}{dt}\|\mathbf{u}(t)\|_{\mathbf{H}}^2 + C_1\sum_{i=1}^{N}\left(\|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i} + \|u_i(t)\|_{\mathrm{L}^{\sigma_i}(\Omega)}^{\sigma_i}\right) \leq C_2\sum_{i=1}^{N}\int_\Omega |\mathbf{F}(t)|^{q_i'}\,d\mathbf{x}\,. \tag{3.21}$$

Note that the assumption $\mathbf{f} = -\mathbf{div}\,\mathbf{F}$ is made only to simplify the exposition.

Proof. We proceed similarly as in the proof of Theorem 3.1, but by taking $\mathbf{f} = -\mathbf{div}\,\mathbf{F}$ in the equation (3.2). We multiply this equation by c_k^m, we add up from $k = 1$ until $k = m$ and we integrate the resulting equation between t_0 and t_1, with $t_0 < t_1$ and $t_0,\ t_1 \in [0,T)$. After all, we obtain the following energy equality

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}^m(t)\|_{\mathbf{H}}^2\Big|_{t=t_0}^{t=t_1} + \sum_{i=1}^{N}\int_{t_0}^{t_1}\left(\|D_i\mathbf{u}^m(t)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i} + \|u_i^m(t)\|_{\mathrm{L}^{\sigma_i}(\Omega)}^{\sigma_i}\right)dt \\
&= \int_{t_0}^{t_1}\int_\Omega \mathbf{F} : \nabla\mathbf{u}^m\,d\mathbf{x}dt\,.
\end{aligned}$$

Then by (3.3) and (3.5)-(3.7), and by a classical property of weak limits, we obtain

$$\frac{1}{2}\|\mathbf{u}(t)\|_{\mathbf{H}}^2\Big|_{t=t_0}^{t=t_1}+\sum_{i=1}^{N}\int_{t_0}^{t_1}\left(\|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i}+\|u_i(t)\|_{\mathrm{L}^{\sigma_i}(\Omega)}^{\sigma_i}\right)dt\leq\int_{t_0}^{t_1}\int_{\Omega}\mathbf{F}:\nabla\mathbf{u}\,d\mathbf{x}dt\,,$$

for all $t_0,\ t_1\in[0,T]$, with $t_0<t_1$. Thus we can write for every $t,\ t+\Delta t\in[0,T]$, with $\triangle\,t>0$,

$$\begin{aligned}\frac{1}{|\Delta t|}\ \|\mathbf{u}(s)\|_{\mathbf{H}}^2\Big|_{s=t}^{s=t+\Delta t}\leq&-\frac{2}{|\Delta t|}\sum_{i=1}^{N}\int_{t}^{t+\Delta t}\left(\|D_i\mathbf{u}(s)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i}+\|u_i(s)\|_{\mathrm{L}^{\sigma_i}(\Omega)}^{\sigma_i}\right)ds\\&+\frac{2}{|\Delta t|}\int_{t}^{t+\Delta t}\int_{\Omega}\mathbf{F}:\nabla\mathbf{u}\,d\mathbf{x}ds\,.\end{aligned}$$

Since $\mathbf{u}\in\mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$, with $u_i\in\mathrm{L}^{\sigma_i}(\Omega)$ for all $i\in\{1,\dots,N\}$, and $\mathbf{F}\in\mathbf{L}^{\vec{\mathbf{q}}'}(Q_T)$, then $\|D_i\mathbf{u}\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i}\,,\ \|u_i\|_{\mathrm{L}^{\sigma_i}(\Omega)}^{\sigma_i}\in\mathrm{L}^1[0,T]$ for any $i\in\{1,\dots,N\}$ and $\int_\Omega|\mathbf{F}:\nabla\mathbf{u}|d\mathbf{x}\in\mathrm{L}^1[0,T]$ as well. In consequence, every term on the right-hand side of the previous inequality has a limit, for all $t\in[0,T]$, as $\Delta t\to 0$. This in turn yields the existence of a limit of the left-hand side of this inequality, for all $t\in[0,T]$, as $\Delta t\to 0$. Whence we can write for all $t\in[0,T]$

$$\frac{d}{dt}\|\mathbf{u}(t)\|_{\mathbf{H}}^2+2\sum_{i=1}^{N}\left(\|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i}+\|u_i(t)\|_{\mathrm{L}^{\sigma_i}(\Omega)}^{\sigma_i}\right)\leq 2\int_\Omega|\mathbf{F}(t):\nabla\mathbf{u}(t)|d\mathbf{x}\,.$$

Then making use of Young's inequality, we obtain (3.21). □

4. Extinction in a finite time

In this section, we assume that the time interval $[0,T]$ is so large that the vanishing moments belong to the interior of $[0,T]$. To begin with, let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.4) and let us denote the kinetics energy associated with this problem by

$$e(t):=\frac{1}{2}\int_\Omega|\mathbf{u}(t)|^2\,d\mathbf{x}\,. \tag{4.1}$$

In this section, we are interested in solutions to the problem (1.1)-(1.4) with a finite energy e. Therefore, we may assume that exists a positive constant M such that

$$e(t)<M\quad\Rightarrow\quad E(t):=\frac{e(t)}{M}<1\quad\text{for all } t\in[0,T]\,. \tag{4.2}$$

Theorem 4.1. *Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.4) in the sense of Definition 2.1 and such that all the conditions of Theorem 3.2 are fulfilled. In addition, assume that (2.5) and (4.2) hold, and that*

$$\sum_{i=1}^{N}\frac{1}{q_i}>\frac{N}{2}\,. \tag{4.3}$$

(1) *If $\mathbf{F}=\mathbf{0}$ a.e. in Q_T, then there exists $t^*>0$ such that $\mathbf{u}=\mathbf{0}$ a.e. in Ω and for all $t\geq t^*$.*

(2) *Let $\mathbf{F} \neq \mathbf{0}$ and assume that exist positive constants $\delta_1, \dots, \delta_N$, $\gamma_1, \dots, \gamma_N$ and γ, and positive times $t_{\mathbf{F}}^1, \dots, t_{\mathbf{F}}^N$ and $t_{\mathbf{F}}$ such that, for any $i \in \{1, \dots, N\}$ and for all $t \in [0, T]$,*

$$\int_\Omega |\mathbf{F}(t)|^{q_i'}\, d\mathbf{x} \leq \delta_i \left(1 - \frac{t}{t_{\mathbf{F}}^i}\right)_+^{\gamma_i} \quad \text{and} \quad \frac{\gamma_i}{\gamma} \geq \frac{t_{\mathbf{F}}^i}{t_{\mathbf{F}}}, \tag{4.4}$$

where $(\cdot)_+ = \max\{0, \cdot\}$, $\gamma := (1-\mu)^{-1}$, $t_{\mathbf{F}} := [C_1(1-\mu)]^{-1}$ and μ is defined in (4.6) below. Then there exists a constant $\delta_0 > 0$ such that $\mathbf{u} = \mathbf{0}$ a.e. in Ω and for all $t \geq t_{\mathbf{F}}$, provided that $0 < \max_{i\in\{1,\dots,N\}} \delta_i \leq \delta_0$.

Proof. We first observe that, since (2.5) holds, we can use the anisotropic inequality (2.2) with $s = 2$. Then making use of the notations (4.1)-(4.2) and using the algebraic inequality $A^a \times B^b \leq (A+B)^{a+b}$, which is valid for any real numbers A, B, a, b, with A, $B \geq 0$ and a, $b > 0$, we get

$$\begin{aligned} E(t) \leq & \frac{C}{2M}\left(\prod_{i=1}^N \|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}\right)^{\frac{2}{N}} \leq \frac{C}{2M}\left(\sum_{i=1}^N \|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i}\right)^{\frac{2}{N}\sum_{i=1}^N \frac{1}{q_i}} \\ \leq & \frac{C}{2M}\left(\sum_{i=1}^N \left(\|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i} + \|u_i(t)\|_{\mathrm{L}^{\sigma_i}(\Omega)}^{\sigma_i}\right)\right)^{\frac{2}{N}\sum_{i=1}^N \frac{1}{q_i}} . \end{aligned} \tag{4.5}$$

Then plugging (4.5) into the energy inequality (3.21), we get

$$\frac{d}{dt}E(t) + C_1 E(t)^\mu \leq C_2 \sum_{i=1}^N \int_\Omega |\mathbf{F}(t)|^{q_i'}\, d\mathbf{x}, \quad \mu := \left(\frac{2}{N}\sum_{i=1}^N \frac{1}{q_i}\right)^{-1} \tag{4.6}$$

for all $t \in [0, T]$. Due to assumption (4.3), we have $0 < \mu < 1$. Thus we are precisely in the conditions of [3, Theorem 5.1] and the proof follows in the same way. □

Observe that for the validity of Theorem 4.1, the damping term is needless. In this particular case and if we consider the diffusion to be isotropic, *i.e.* when, in the equation (1.2), $\kappa_i = 0$ and $q_i = q$ for all $i \in \{1, \dots, N\}$, then, from (4.3), we recover the condition $q < 2$ that characterizes shear-thinning fluids for which it is well-known the phenomenon of extinction in a finite time (see *e.g.* [1, 11]). Note also that in the important case of $\overline{q}^* \geq q^+$, the assumptions (2.5) and (4.3) show us that the property of finite time extinction holds provided that $\frac{N}{2} < \sum_{i=1}^N \frac{1}{q_i} \leq 1 + \frac{N}{2}$. The following result gives us the conditions for the validity of this property when $1 < \sum_{i=1}^N \frac{1}{q_i} \leq \frac{N}{2}$.

Theorem 4.2. *Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.4) in the sense of Definition 2.1 and such that all the conditions of Theorem 3.2 are fulfilled. In addition, assume that (2.5) and (4.2) hold, and that*

$$\sum_{i=1}^N \frac{1}{\sigma_i} > \frac{N}{2}. \tag{4.7}$$

Then the assertions of Theorem 4.1 also hold, but with μ now defined in (4.9) below.

Proof. The same reasoning used to prove (4.5), but now preceded by interpolation between q_a^* and σ^-, yields

$$\begin{aligned} E(t) \leq & \frac{1}{2M}\|\mathbf{u}(t)\|^{2\theta}_{\mathbf{L}^{q_a^*}(\Omega)}\|\mathbf{u}(t)\|^{2(1-\theta)}_{\mathbf{L}^{\sigma^-}(\Omega)}, \qquad \theta := \left(\frac{1}{\sigma^-}-\frac{1}{2}\right)\left(\frac{1}{\sigma^-}-\frac{1}{q_a^*}\right)^{-1} \\ \leq & \frac{C}{2M}\left(\prod_{i=1}^{N}\|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}\right)^{\frac{2\theta}{N}}\left(\prod_{i=1}^{N}\|u_i(t)\|_{\mathrm{L}^{\sigma_i}(\Omega)}\right)^{\frac{2(1-\theta)}{N}} \\ \leq & \frac{C}{2M}\left(\sum_{i=1}^{N}\left(\|D_i\mathbf{u}(t)\|^{q_i}_{\mathbf{L}^{q_i}(\Omega)}+\|u_i(t)\|^{\sigma_i}_{\mathrm{L}^{\sigma_i}(\Omega)}\right)\right)^{\frac{2\theta}{N}\sum_{i=1}^{N}\frac{1}{q_i}+\frac{2(1-\theta)}{N}\sum_{i=1}^{N}\frac{1}{\sigma_i}}. \end{aligned} \tag{4.8}$$

Observe that $\theta \in (0,1)$ provided that $q_a^* > 2 > \sigma^-$ or that $\sigma^- > q_a^* > 2$. In this case, we plug (4.8) into the energy inequality (3.21) to get the differential inequality (4.6), but with the exponent of nonlinearity defined by

$$\mu := \left(\frac{2\theta}{N}\sum_{i=1}^{N}\frac{1}{q_i}+\frac{2(1-\theta)}{N}\sum_{i=1}^{N}\frac{1}{\sigma_i}\right)^{-1}. \tag{4.9}$$

Since $\theta \in (0,1)$, we have by assumption (4.7), that $\theta\sum_{i=1}^{N}\frac{1}{q_i}+(1-\theta)\sum_{i=1}^{N}\frac{1}{\sigma_i} > \frac{N}{2}$. As a consequence we also have $0 < \mu < 1$. Again we are in the conditions of [3, Theorem 5.1] and the proof follows in the same way. □

If in the proof of Theorem 4.2, we let be $\theta = 1$ or $\theta = 0$, then $q_a^* = 2$ or $\sigma^- = 2$, respectively, and we fall in the case of Theorem 4.1 for which the extinction in a finite time only holds for $\sum_{i=1}^{N}\frac{1}{q_i} > \frac{N}{2}$. From Theorems 4.1 and 4.2, we can see that condition (4.7) is crucial for the extinction in a finite time property when $1 < \sum_{i=1}^{N}\frac{1}{q_i} \leq \frac{N}{2}$. In this situation, we get $\sum_{i=1}^{N}\frac{1}{q_i} < \sum_{i=1}^{N}\frac{1}{\sigma_i}$, which, in the isotropic case, *i.e.* when $\sigma_i = \sigma$ and $q_i = q$ for all $i \in \{1,\dots,N\}$, reduces to the well-known condition of fast diffusion $\sigma < q$ (see [1]).

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