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GRAPHS OF TORUS-VALUED HARMONIC MAPS, WITH APPLICATION TO A VARIATIONAL MODEL FOR DISLOCATIONS

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ABSTRACT. This paper deals with a variational problem for dislocations in which the curl of the deformation tensor is constrained by a concentrated measure in a set of lines, called the dislocation density, while the energy density involves the deformation tensor and its gradient, specifically, the curl and the divergence in two distinct terms. To solve this constrained variational problem in finite elasticity, the notion of integral 1-current is used in the spirit of previous work by the same authors. No assumptions on the lines are made except the classical requirement to be closed loops or end at the crystal boundary. Since the displacement field is by essence multiple valued, it is chosen to work with torus-valued maps. Moreover, graphs of harmonics maps are at the heart of such a problem, and therefore our theory is grounded in an analysis of their properties with a view to dislocation modelling. Our main result shows that dislocation density and displacement graph boundary are bound notions. Generalizations of distributional determinants and cofactors appropriate for our purposes are also discussed. Indeed, it is shown that they are Radon measures whose singular parts are expressed in terms of the displacement graph boundary.

CONTENTS

1. Introduction

1.1. **Physical motivation and variational setting.** For Ω an open set of $\mathbb{R}^3$, a typical variational problem for dislocations in the single crystal Ω consists in minimizing an energy of the form

$$\mathcal{W}_{\mathcal{L}}(F, \nabla F), \tag{1.1}$$

over a class of deformation tensors F belonging to L^p whose minors of all orders also belong to L^p, and over a class of dislocation lines C (the associated current is denoted by $\mathcal{L}$), where F happens to be singular. The exponent p must be strictly lower than 2 as a consequence of the constraint

$$\text{Curl } F = -\Lambda_{\mathcal{L}}^T = -b \otimes \mathcal{L}, \tag{1.2}$$

where $\Lambda_{\mathcal{L}}$ is the dislocation density[1], i.e., a measure concentrated on a set of 1-dimensional curves $C \subset \bar{\Omega}$, which is the tensor product of the tangent vector and the Burgers vector of the dislocations, the latter being a constant on each isolated loop of C. The apriori constraint on the geometry of C is the requirement to be a closed curve or end at the crystal boundary.

Without dependence on $\mathcal{L}$, this problem has first been introduced and solved in [12], whereas in [14] the joint minimization in $\mathcal{L}$ and F was carried on for an energy of the form

$$\mathcal{W}_{\mathcal{L}}(F, \text{ Curl } F), \tag{1.3}$$

where the concept of currents was used to model both the line and the deformation tensor in proper mathematical terms. Moreover, to avoid peculiar phenomena such as line densification at optimality (since one wishes to remain at the mesoscopic scale, that is, avoid space-filling effets), in [14] an assumption on $\mathcal{L}$ was made (that is, on the curve C), i.e., to require that the dislocation be contained in a 1-dimensional continuum. To this aim, a term involving the measure of the continuum as related to the energetical cost of dislocation loop nucleation was added to (1.3). Then, in a companion paper [15], the configurational force exerted on the dislocation at optimality was derived, while the functional setting was further described.

In the present work, our initial motivation was to get rid of this nucleation term in the energy. As a matter of fact, it turns out that the variational problem can be solved for an energy of the form

$$\mathcal{W}_{\mathcal{L}}(F, \text{ Curl } F, \text{ div } F), \tag{1.4}$$

the condition on the deformation tensor divergence being in some sense the counterpart of the "continuum" assumption for the line. For a physical justification, note that div F is related to the action of external forces. Note also that written as a perturbation term, that is, $\mathcal{W}^{\epsilon}_{\mathcal{L}}(F, \text{ Curl } F, \text{ div } F) := \mathcal{W}_{\mathcal{L}}(F, \text{ Curl } F) + \epsilon \,\text{div } F$, this model has been considered in the framework of homogenization in linear and nonlinear elasticity in [5]. Another assumption made is that the fairly general $\mathcal{L}$ (in terms of its geometry) is associated to a single Burgers vector b. It is therefore called a dislocation cluster, or network. Note that in practice single crystals are associated with only a few number of Burgers vectors, due to crystallographic constraints.

[1]Symbol T denotes the transpose of a second-rank tensor. This convention is chosen to fit with the second-author references on this topic [17, 18].

A complete discussion of the aforementioned problems, containing in particular the main results of this paper and other yet unpublished results, together with the basic approach to dislocations by means of the notions of currents, Cartesian maps, and torus-valued map, can be found in [13].

1.2. **Scope and structure of the work.** While the aforementioned problem is important in dislocation modelling and wasd indeed its motivation, in this paper it is presented as the application of a series of results on currents and harmonic maps, which are of interest independently of dislocations and sould be considered as the heart of the present work.

In Section 2, we recall the basis for currents and the theory of graphs of torus-valued maps, and prove some important preliminary results, whose counterparts in the Euclidean-valued case are classical. Then, the crucial pointwise and distributional properties of torus-valued harmonic maps are derived in Section 3. Indeed it is possible to construct a real valued displacement field u, which has a constant jump b on a fixed (though arbitrary) surface S enclosed by C. If we quotient the target space $\mathbb{R}^3$ by $\mathbb{Z}^3$, the jump on the (arbitrary) surface is neglected. In other words, the displacement as a Euclidean-valued field is multiple valued since it depends on the arbitrary choice of the surface S^i enclosed by C, whereas working with torus-valued maps means that we have the equivalence $(u_{S^1}, u_{S^2}, ..., u_{S^i}) \sim u_S$. Obviously a smooth enough map has no jump, thus can be said as torus-valued as well and summed to the displacement. Torus-valued maps are therefore generalizations of Euclidean maps appropriate to mathematically model dislocations. As a consequence, the n-forms depending on such maps and from which currents for our purposes are defined by duality, must be periodic, and therefore torus-valued currents is a broader class than Cartesian currents. Note that such displacement field multivaluedness is constantly used by Physicists, for a long time (see, eg., [9]), but without the necessary mathematical rigour, which we believe is worth introducing to seek a profound understanding of the intrinsic difficulties of dislocation modelling.

The main results are presented in two forms, the first in Section 4 holding for torus-valued maps, whereas its counterpart for Euclidean maps is given in the Appendix A.2. With the language of currents and in dislocation terms, our main result basically states that the dislocation density is the boundary of the graph of a displacement field u which is defined with values in the three dimensional torus, viz.,

$$\partial \mathcal{G}_u(\omega) = \mathcal{L} \otimes b(\varphi^\omega),$$

where the n-form φ^ω is a test function depending on ω. In particular, $\partial \mathcal{G}_u$ is proven to be representable by a concentrated Radon measure.

In Section 5, we discuss generalization of the distributional determinant and cofactors of the deformation gradient, which are appropriate to model dislocations. The issue is that the torus-valued displacement is proved to have a SBV representative for which a generalized notion of distributional cofactor and determinant is needed. As a consequence, we propose two solutions. The first as presented in Sections 5.2 and 5.3 is obtained by immerging the torus-valued displacement in $(S^1)^3 \subset \mathbb{R}^6$, whereas the second, given in Appendix B and based on the main result second form, shows concentrated terms on the jump surface S.

Specifically, it is first shown that

$$\langle \text{Cof } (Du)_{ij}, f \rangle = \partial \mathcal{G}_u(\omega_{ij}) + \int_\Omega f \,\text{cof } (Du)_{ij} dx,$$

and

$$\langle \text{ Det } (Du), f\rangle = -\partial_u \mathcal{G}(\omega) + \int_\Omega \text{ det } (Du) f dx,$$

for particular choices of ω, respectively, as found in Section 5. Then, in the $(S^1)^3$ immersion, it is proven that the distributional cofactors and determinants are Radon measures whose absolutely continuous part are the pointwise cofactors and determinants, respectively, whereas in the second form, the concentrated part, that is, the associated current boundary is given an explicit expression.

Eventually, a variational problem is presented and solved in Section 6. Note also that in Section 6.4 another minimum problem is proposed as based on the aforementioned $(S^1)^3$-immersion of the displacment, whereas in Section A.1 it is shown how the main result second form might be used in models involving an energy term on the jump surface, whose complete analysis, though, is left for future work. Concluding remarks are drawn in Section 7.

2. Preliminary results

2.1. **Generalities about currents and graphs.** For all integers $n > 0$ and $k \geq 0$ with $k \leq n$, we denote by $\Lambda_k\mathbb{R}^n$ the space of k-vectors and by $\Lambda^k\mathbb{R}^n$ the space of k covectors. Let α be a multi-index, i.e., an ordered (increasing) subset of $\{1, 2, \dots, n\}$. We denote by $|\alpha|$ the cardinality (or length) of α, and we denote by $\bar\alpha$ the complementary set of α, i.e., the multi-index given by the ordered set $\{1, 2, \dots, n\} \setminus \alpha$.

For a $n \times n$ matrix A with real entries and for α and β multi-indices such that $|\alpha| + |\beta| = n$, $M^\beta_{\bar\alpha}(A)$ will denote the determinant of the submatrix of A given by erasing the i-th columns and the j-th rows, for all $i \in \alpha$ and $j \in \bar\beta$. Moreover, symbol $M(A)$ will denote the n-vector in $\Lambda_n\mathbb{R}^{2n}$ given by

$$M(A) := \sum_{|\alpha|+|\beta|=n} \sigma(\alpha, \bar\alpha) M^\beta_{\bar\alpha}(A) e_\alpha \wedge \varepsilon_\beta,$$

where $\{e_i, \varepsilon_i\}_{i\leq n}$ is the Euclidean basis of $\mathbb{R}^{2n}$, and $\sigma(\alpha, \bar\alpha)$ is the sign of the permutation $(\alpha, \bar\alpha)$. Accordingly, it holds

$$|M(A)| := (1 + \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|>0}} |M^\beta_{\bar\alpha}(A)|^2)^{1/2}.$$

For a matrix $A \in \mathbb{R}^{3\times3}$, the symbols adj A and det A denote the adjunct, i.e., the transpose of the matrix of the cofactors of A, and the determinant of A, respectively. Explicitely,

$$M^i_j(A) = A_{ij}, \quad M^{\bar i}_{\bar j}(A) = (\text{ cof } A)_{ij} = (\text{ adj } A)_{ji} \quad M^{\{1,2,3\}}_{\{1,2,3\}}(A) = \text{ det } A,$$

for $i, j = 1, 2, 3$. Moreover,

$$|M(A)| = \big(1 + \sum_{i,j} A^2_{ij} + \sum_{i,j} \text{cof}(A)^2_{ij} + \det(A)^2\big)^{1/2}. \tag{2.1}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, then $\mathcal{D}^k(\Omega)$ denotes the topological vector space of compactly supported smooth k-forms on Ω, that is the topological vector space of compactly supported and smooth maps on Ω with values in $\Lambda^k\mathbb{R}^n$. The dual space of $\mathcal{D}^k(\Omega)$, denoted by $\mathcal{D}_k(\Omega)$, is said the space of k-currents on Ω. A weak convergence in $\mathcal{D}_k(\Omega)$ is defined by saying that $\mathcal{T}_h \rightharpoonup \mathcal{T}$ as currents if for all $\omega \in \mathcal{D}^k(\Omega)$ we have

$\mathcal{T}_h(\omega) \to \mathcal{T}(\omega)$. For all $\mathcal{T} \in \mathcal{D}_k(\Omega)$ the mass of $\mathcal{T}$ is the number $M(\mathcal{T}) \in [0, +\infty]$ defined as+

$$M(\mathcal{T}) := \sup_{\omega \in \mathcal{D}^k(\Omega),\ |\omega| \leq 1} \mathcal{T}(\omega).$$

If $M(\mathcal{T}) < +\infty$ then $\mathcal{T}$ turns out to be a Borel measure in $\mathcal{M}_b(\Omega, \Lambda_k \mathbb{R}^n)$, and its mass coincides with $M(\mathcal{T})$. Moreover the mass is lower semicontinuous with respect to the weak topology in $\mathcal{D}_k(\Omega)$. Indeed if $\limsup_{h\to\infty} M(\mathcal{T}_h) < +\infty$ and $\mathcal{T}_h \rightharpoonup \mathcal{T}$ then we also find that $\mathcal{T}$ is a Borel measure and $\mathcal{T}_h \rightharpoonup \mathcal{T}$ weakly in $\mathcal{M}_b(\Omega, \Lambda^k \mathbb{R}^n)$, so that the lower-semicontinuity of the mass follows from the lower-semicontinuity of the mass in $M_b(\Omega, \Lambda^k \mathbb{R}^n)$. We also define the quantity

$$N(\mathcal{T}) := M(\mathcal{T}) + M(\partial \mathcal{T}),$$

for every $\mathcal{T} \in \mathcal{D}_k(\Omega)$.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $F : U \to V$ be a smooth map. Then the *push-forward* of a current $\mathcal{T} \in \mathcal{D}_k(U)$ through F is defined as

$$F_\sharp \mathcal{T}(\omega) := \mathcal{T}(\zeta F^\sharp \omega) \qquad \text{for } \omega \in \mathcal{D}^k(V),$$

where $F^\sharp \omega$ is the standard pull-back of ω and ζ is any C^∞ function that is equal to 1 on $\mathrm{spt}\mathcal{T} \cap \mathrm{spt} F^\sharp \omega$. It turns out that $F_\sharp \mathcal{T} \in \mathcal{D}_k(V)$ does not depend on ζ and satisfies

$$\partial F_\sharp \mathcal{T} = F_\sharp \partial \mathcal{T}. \tag{2.2}$$

Let $0 \leq k \leq n$ and let $S \subset \mathbb{R}^n$ be $\mathcal{H}^k$-rectifiable set with approximate tangent space $T_x S$. If $\tau : S \to \Lambda_k(\mathbb{R}^n)$ and $\theta : S \to \mathbb{R}$ are $\mathcal{H}^k$-integrable functions with $\tau(x) \in T_x S$ a simple unit k-vector for $\mathcal{H}^k$-a.e. $x \in S$, then we can define the current $\mathcal{T}$ as

$$\mathcal{T}(\omega) = \int_S \langle \omega(x), \tau(x) \rangle \theta(x) d\mathcal{H}^k(x) \qquad \text{for } \omega \in \mathcal{D}^k(\Omega). \tag{2.3}$$

Every current for which there exists S, τ, and θ as before is said to be *rectifiable.* If also its boundary $\partial\mathcal{T}$ is rectifiable, then we adopt the following notation

$$\mathcal{T} \equiv \{S, \tau, \theta\}. \tag{2.4}$$

The current $\mathcal{T} \in \mathcal{D}_k(\Omega)$ is *rectifiable with integer multiplicity* if it is rectifiable, has rectifiable boundary, and the function θ in (2.3) is integer valued. A integer-multiplicity current $\mathcal{T}$ such that $N(\mathcal{T}) < \infty$ is said *integral current.*

An integer-multiplicity current $T \in \mathcal{D}_M(\mathbb{R}^n)$ is said *indecomposable* if there exists no integral current R such that $R \neq 0 \neq T - R$ and

$$N(T) = N(R) + N(T - R).$$

The following theorem provides the decomposition of every integral current and the structure of integer-multiplicity indecomposable 1-current (see [4, Section 4.2.25]).

Theorem 2.1. *For every integer multiplicity current T there exists a sequence of indecomposable integral currents T_i such that*

$$T = \sum_i T_i \qquad \text{and} \qquad N(T) = \sum_i N(T_i).$$

Suppose T is an indecomposable integer multiplicity 1-current on $\mathbb{R}^n$. Then there exists a Lipschitz function $: \mathbb{R} \to \mathbb{R}^n$ *with* $\mathrm{Lip}(f) \leq 1$ *such that*

$$f \llcorner [0, |T|] \text{ is injective and } T = f_\sharp [0, M(T)].$$

Moreover $\partial T = 0$ if and only if $f(0) = f(M(T))$.

Let us consider the space $\Omega\times\mathbb{R}^3$. We will use the Euclidean coordinates $x=(x_1,x_2,x_3)$ for $x\in\Omega$ and $y=(y_1,y_2,y_3)$ for $y\in\mathbb{R}^3$. Every 3-form $\omega\in\mathcal{D}^3(\Omega\times\mathbb{R}^3)$ can be decomposed as $\omega=\sum\omega_{\alpha\beta}dx_\alpha\wedge dy_\beta$, with $\omega_{\alpha\beta}\in C_c^\infty(\Omega\times\mathbb{R}^3)$, and where the sum is computed over all multi-indices α and β such that $|\alpha|+|\beta|=3$.

For $1\le p<+\infty$ we define

$$\mathcal{A}_p(\Omega):=\{u\in L^p(\Omega,\mathbb{R}^3): u \text{ is approx. differentiable a.e. and } M^\beta_{\bar\alpha}(Du)\in L^p(\Omega) \text{ for all } |\alpha|+|\beta|=3\}.$$

A weak convergence is defined on $\mathcal{A}_p(\Omega)$. We say that the sequence $u_h\in\mathcal{A}_p(\Omega)$ converges to $u\in\mathcal{A}_p(\Omega)$ weakly in $\mathcal{A}_p(\Omega)$ if $u_h\rightharpoonup u$ weakly in $L^p(\Omega,\mathbb{R}^3)$ and $M^\beta_{\bar\alpha}(Du_h)\rightharpoonup M^\beta_{\bar\alpha}(Du)$ weakly in $L^p(\Omega)$ for all multi-indices α and β with $|\alpha|+|\beta|=3$. If $u\in\mathcal{A}_p(\Omega)$ then $\mathcal{G}_u$, the current carried by the graph of u, is well defined as follows:

$$\mathcal{G}_u:=(Id\times u)_\sharp[\![\Omega]\!], \tag{2.5}$$

where $Id\times u:\mathbb{R}^3\to\mathbb{R}^3\times\mathbb{R}^3$ is given by $(Id\times u)(x)=(x,u(x))$, viz.,

$$\mathcal{G}_u(\omega)=\int_\Omega\sigma(\alpha,\bar\alpha)\omega_{\alpha\beta}(x,u(x))M^\beta_{\bar\alpha}(Du(x))dx, \tag{2.6}$$

for all $\omega=\omega_{\alpha\beta}dx_\alpha\wedge dy_\beta\in\mathcal{D}^3(\Omega\times\mathbb{R}^3)$. Moreover if $u\in\mathcal{A}_p(\Omega)$ then $\mathcal{G}_u$ turns out to be a 3-integer-multiplicity current in $\Omega\times\mathbb{R}^3$.

Lemma 2.2. *Let u_ϵ, $u\in\mathcal{A}_p(\Omega)$ be such that $u_\epsilon\rightharpoonup u$ weakly in $\mathcal{A}_p(\Omega)$, then $\mathcal{G}_{u_\epsilon}\rightharpoonup\mathcal{G}_u$ as currents.*

Proof. This is a straightforward consequence of formula (2.6). □

The boundary of the current $\mathcal{G}_u(\omega)$ is, by definition, the 2-current in $\Omega\times\mathbb{R}^3$ given by $\partial\mathcal{G}_u(\omega):=\mathcal{G}_u(d\omega)$ for all 2-forms $\omega\in\mathcal{D}^2(\Omega\times\mathbb{R}^3)$. As soon as $u\llcorner_U$, the restriction of u to a smooth open subset U of Ω, is assumed smooth in U, then $\mathcal{G}_{u\llcorner_U}$ is a surface with rectifiable boundary given by

$$\partial\mathcal{G}_{u\llcorner_U}=\mathcal{G}_{u\llcorner_{\partial U}}:=(Id\times u)_\sharp[\![\partial U]\!].$$

We can also employ Stokes formula to find an explicit formula for $\mathcal{G}_{u\llcorner_{\partial U}}$, i.e.,

$$\mathcal{G}_{u\llcorner_U}(d\omega)=\partial\mathcal{G}_{u\llcorner_U}(\omega)=\int_{\partial U}\sigma(\alpha,\bar\alpha)\omega_{\alpha\beta}(u,u(x))M^\beta_{\bar\alpha}(Du\llcorner_{\partial U})(x)d\mathcal{H}^2(x), \tag{2.7}$$

for all 2-form $\omega=\omega_{\alpha\beta}dx_\alpha\wedge dy_\beta\in\mathcal{D}^2(\Omega\times\mathbb{R}^3)$. This can be seen as follows: by definition of push-forward of a current, $\mathcal{G}_{u\llcorner_{\partial U}}(\omega)=[\![\partial U]\!]((Id\times u)^\sharp(\omega))=\int_{\partial U}\langle(Id\times u)^\sharp\omega,\tau^1\wedge\tau^2\rangle d\mathcal{H}^2=\int_{\partial U}\langle\omega,\frac{\partial\Psi}{\partial\tau^1}\wedge\frac{\partial\Psi}{\partial\tau^2}\rangle d\mathcal{H}^2$ with $\Psi=Id\times u$ and where $\tau^1\wedge\tau^2$ is a volume form for ∂U. Now by Stokes Theorem this is equal to (recall that one can interchange the exterior differentiation and pull-back operations) $\int_U\langle d(Id\times u)^\sharp\omega,e^1\wedge e^2\wedge e^3\rangle d\mathcal{H}^3=\int_U\langle(Id\times u)^\sharp d\omega,e^1\wedge e^2\wedge e^3\rangle d\mathcal{H}^3=\mathcal{G}_{u\llcorner_U}(d\omega)=\partial\mathcal{G}_{u\llcorner_U}(\omega)$ (see [6] p.272 or [10] for details).

The class of Cartesian maps is the subspace of $\mathcal{A}_p(\Omega,\mathbb{R}^3)$ defined as

$$\text{Cart}^1(\Omega,\mathbb{R}^3):=\{u\in\mathcal{A}_1(\Omega,\mathbb{R}^3):\partial\mathcal{G}_u=0\}. \tag{2.8}$$

If $u\in W^{1,p}(\Omega,\mathbb{R}^3)$ with $p\ge 3$, then it is easy to see that $u\in\text{Cart}^1(\Omega,\mathbb{R}^3)$. See [7, Section 3.2.2] for details.

Theorem 2.3. *Let u_k be a sequence in $\mathcal{A}_p(\Omega,\mathbb{R}^3)$ such that $u_k \to u$ strongly in $L^p(\Omega,\mathbb{T}^3)$ and suppose that there exist functions $v^\alpha_\beta \in L^p(\Omega)$ such that $M^\beta_{\bar\alpha}(Du_k) \rightharpoonup v^\beta_\alpha$ for all multi-indices α and β with $|\alpha|+|\beta|=3$. If*

$$M(\partial\mathcal{G}_{u_k}) < C < +\infty \tag{2.9}$$

for all $k>0$, then $u \in \mathcal{A}_p(\Omega,\mathbb{R}^3)$ and $v^\beta_\alpha = M^\beta_{\bar\alpha}(Du)$.

This is proved in Theorem 2 of [7, Section 3.3.2]. Since Cartesian maps obviously satisfies condition (2.9), this Theorem applies to this case. Moreover if $u_k \in \mathrm{Cart}^1(\Omega,\mathbb{R}^3)$ then we also have $u \in \mathrm{Cart}^1(\Omega,\mathbb{R}^3)$.

2.2. **Torus-valued maps and currents.** We introduce the torus $\mathbb{T} \cong \mathbb{R}/\sim$, where $\sim$ denotes the equivalent relation given by $a \sim b$ iff $a-b \in 2\pi\mathbb{Z}$. We also denote $\mathbb{T}\times\mathbb{T}\times\cdots\times\mathbb{T}$ by $\mathbb{T}^n$. Now we will consider graphs of maps $u:\Omega\to\mathbb{T}^n$. These turn out to be n-rectifiable currents in $\Omega\times\mathbb{T}^n$. Note that the space of n-forms in $\Omega\times\mathbb{T}^n$, i.e. $\mathcal{D}^n(\Omega\times\mathbb{T}^3)$, is exactly the space of n-forms in $\Omega\times\mathbb{R}^n$ that have coefficients which are smooth and 2π-periodic (with all their derivatives) in the last three variables (actually, they do not have compact support). As a consequence, if $\mathcal{T}$ is a n-current in $\mathcal{D}_n(\Omega\times\mathbb{R}^n)$ that has compact support in $\bar\Omega\times\mathbb{R}^n$, then the following current $T(\mathcal{T}) \in \mathcal{D}_n(\Omega\times\mathbb{T}^n)$ is well defined as

$$T(\mathcal{T}) := \mathcal{T}\llcorner_{\mathcal{D}^n(\Omega\times\mathbb{T}^3)}. \tag{2.10}$$

Moreover, since in general smooth functions in $\Omega\times\mathbb{R}^n$ are not periodic in the last three variables, it turns out that $M(T(\mathcal{T})) \le M(\mathcal{T})$.

Let $u \in \mathcal{A}_p(\Omega,\mathbb{R}^n)$, then we define $T(u):\Omega\to\mathbb{T}^n$ by means of the standard projection $\pi_T:\mathbb{R}\to\mathbb{T}$, i.e. $T(u_i) := \pi_T(u_i)$ for $i=1,2,3$. It is easily seen that, $\mathbb{T}$ being locally isomorphic to $\mathbb{R}$, $T(u)$ is almost everywhere approximatively differentiable with the same approximate derivatives of u. As a consequence $\mathcal{G}_{T(u)}$ is a n-rectifiable current in $\Omega\times\mathbb{T}^n$. It is also easy to see that in such a case $\mathcal{G}_{T(u)} = T(\mathcal{G}_u)$. This fundamental identity follows from the fact that the approximate differentials of u and $T(u)$ coincide almost everywhere, and from (2.6), where we use that if ω is 2π-periodic in the second variable, then $\omega(x,u(x)) = \omega(x,T(u(x)))$. We introduce the space $\mathcal{A}_p(\Omega,\mathbb{T}^n)$ as follows:

Definition 2.4.

$$\begin{aligned}\mathcal{A}_p(\Omega,\mathbb{T}^n) := \{u \in L^1(\Omega,\mathbb{T}^n) : u \text{ is approx. differentiable a.e. on } \Omega, \text{ and}\\ M^\beta_{\bar\alpha}(Du) \in L^p(\Omega) \text{ for all } |\alpha|+|\beta|=n\},\end{aligned} \tag{2.11}$$

where $L^1(\Omega,\mathbb{T}^n)$ means the space of measurable functions $u:\Omega\to\mathbb{T}^n$.

With this definition, we see that for all $u \in \mathcal{A}_p(\Omega,\mathbb{T}^n)$ the graph $\mathcal{G}_u$ is well defined as a n-rectifiable current. A consequence of the fact that the mass of a current does not increase when we compose with T is that, if there exists $\bar u \in \mathcal{A}_p(\Omega,\mathbb{R}^n)$ such that $T(\bar u) = u$ and $\mathcal{G}_{\bar u}$ is an integral current, then $\mathcal{G}_u$ is an integral current. Note that it might happen that such $\bar u$ does exist with $\partial\mathcal{G}_{\bar u}$ unbounded, while $M(\partial\mathcal{G}_u) < \infty$.

Theorem 2.3, being a consequence of the compactness theorem for integral currents, straightforwardly applies also to the case of maps with values in $\mathbb{T}^3$.

Lemma 2.2 readly applies to the case of maps with value in $\mathbb{T}^3$.

Lemma 2.5. *Let $u_\epsilon,\ u \in \mathcal{A}_p(\Omega,\mathbb{T}^3)$ be such that $u_\epsilon \rightharpoonup u$ weakly in $\mathcal{A}_p(\Omega,\mathbb{T}^3)$, then $\mathcal{G}_{u_\epsilon} \rightharpoonup \mathcal{G}_u$ as currents.*

Proof. This is again a consequence of formula (2.6) and the fact that currents in $\mathcal{D}_3(\Omega,\mathbb{R}^3)$ belong also to $\mathcal{D}_3(\Omega,\mathbb{T}^3)$. □

3. Harmonic maps with prescribed jump on a surface

3.1. A preliminar result.

Lemma 3.1. *Let C be a Lipschitz closed curve in $\mathbb{R}^3$ and S a bounded Lipschitz surface with boundary C and unit normal N. Let $b \in \mathbb{R}$. The solution of*

$$\begin{cases} \Delta u = 0 & in \quad \mathbb{R}^3 \setminus S \\ [u] := u^+ - u^- = b & on \quad S \\ [\partial_N u] := \partial_N u^+ - \partial_N u^- = 0 & on \quad S \end{cases} \tag{3.1}$$

is given by (up to a harmonic map in $\mathbb{R}^3$)

$$u(x) = -b \int_S \partial_N \Gamma(x' - x) d\mathcal{H}^2(x'), \tag{3.2}$$

for $x \in \mathbb{R}^3 \setminus S$, where Γ is the solution in $\mathbb{R}^3$ of $\Delta\Gamma = \delta_0$.

Proof. Let $S \subset \hat{\Omega}$ be a smooth surface of discontinuity bounded by C. Let $S^- \neq S$ be another smooth surface bounded by C and laying below S. Let V be the volume comprised between S and S^- and $S_V := S \cup S^-$ with outer unit normal N be such that $\partial V := S_V$. Supposing that u is smooth enough, we have the identities in V

$$\int_V \partial'_k(\partial'_l u(x')\Gamma(x' - x))dx' = \int_{S_V} \partial'_l u(x')\Gamma(x' - x)N_k(x')d\mathcal{H}^2(x')$$

and

$$\int_V \partial'_l(u(x')\partial'_k\Gamma(x' - x))dx' = \int_{S_V} u(x')\partial'_k\Gamma(x' - x)N_l(x')d\mathcal{H}^2(x').$$

Thus by substraction it holds

$$\begin{aligned} &\int_V \partial'_k\partial'_l u(x')\Gamma(x' - x))dx' - \int_V u(x')\partial'_k\partial'_l\Gamma(x' - x))dx' \\ = &\int_{S_V} (\partial'_l u(x'))^- \Gamma(x' - x)N_k(x')d\mathcal{H}^2(x') - \int_{S_V} u_i^-(x')\partial'_k\Gamma(x' - x)N_l(x')d\mathcal{H}^2(x'). \end{aligned}$$

Moreover, the same identities in $\mathbb{R}^3 \setminus \bar{V}$ yield

$$\begin{aligned} &\int_{\mathbb{R}^3\setminus\bar{V}} \partial'_k\partial'_l u(x')\Gamma(x' - x))dx' - \int_{\mathbb{R}^3\setminus\bar{V}} u(x')\partial'_k\partial'_l\Gamma(x' - x))dx' \\ = &-\int_{S_V} (\partial'_l u(x'))^+ \Gamma(x' - x)N_k(x')d\mathcal{H}^2(x') + \int_{S_V} u_i^+(x')\partial'_k\Gamma(x' - x)N_l(x')d\mathcal{H}^2(x'). \end{aligned}$$

and hence, by summing,

$$\begin{aligned} &\int_{\mathbb{R}^3\setminus S_V} \partial'_k\partial'_l u(x')\Gamma(x' - x))dx' - \int_{\mathbb{R}^3\setminus S_V} u(x')\partial'_k\partial'_l\Gamma(x' - x))dx' \\ = &-\int_{S_V} [\partial'_l u(x')]\Gamma(x' - x)N_k(x')d\mathcal{H}^2(x') + \int_{S_V} [u(x')]\partial'_k\Gamma(x' - x)N_l(x')d\mathcal{H}^2(x'). \end{aligned}$$

Contracting with δ_{kl} yields

$$\begin{aligned} &\int_{\mathbb{R}^3\setminus S_V} \Delta u(x')\Gamma(x' - x))dx' - \int_{\mathbb{R}^3\setminus S_V} u(x')\Delta\Gamma(x' - x))dx' \\ = &-\int_{S_V} [\partial'_N u(x')]\Gamma(x' - x)d\mathcal{H}^2(x') + \int_{S_V} [u(x')]\partial'_N\Gamma(x' - x)d\mathcal{H}^2(x'), \end{aligned} \tag{3.3}$$

that is, for $x \in \mathbb{R}^3 \setminus S_V$,

$$\int_{\mathbb{R}^3\setminus S_V} \Delta' u(x')\Gamma(x'-x))dx' - u(x) = - \int_{S_V} [\partial'_N u(x')]\Gamma(x'-x)d\mathcal{H}^2(x') + \int_{S_V} [u(x')]\partial'_N \Gamma(x'-x)d\mathcal{H}^2(x'). \tag{3.4}$$

Taking the particular

$$u = -b\int_S \partial_N \Gamma(y-\cdot)d\mathcal{H}^2(y),$$

u is seen to be harmonic in $\mathbb{R}^3 \setminus S$, $\Delta u(x) = 0$ for $x \in \mathbb{R}^3 \setminus S$, and hence, by (3.4) and for $x \in \mathbb{R}^3 \setminus S_V$,

$$u(x) = \int_{S_V} [\partial'_N u(x')]\Gamma(x'-x)d\mathcal{H}^2(x') - \int_{S_V} [u(x')]\partial'_N \Gamma(x'-x)d\mathcal{H}^2(x'). \tag{3.5}$$

Consider now any smooth tensor test function φ with compact support in place of the tensor Γ. By (3.3), it holds

$$\int_{\mathbb{R}^3\setminus S_V} u(x')\Delta\varphi(x')dx' = \int_{\mathbb{R}^3} u(x')\Delta\varphi(x')dx'$$
$$= \int_{S_V} [\partial'_N u(x')]\varphi(x')d\mathcal{H}^2(x') - \int_{S_V} [u(x')]\partial'_N \varphi(x')d\mathcal{H}^2(x'). \tag{3.6}$$

Define the distribution $\gamma[b]$ concentrated on S as

$$\langle \gamma[b], \varphi\rangle := -\int_S b\partial_N \varphi(y)d\mathcal{H}^2(y).$$

By definition, $u(x) = -b\int_S \partial_N \Gamma(x-y)d\mathcal{H}^2(y) = -\langle \gamma[b], \Gamma(x-\cdot)\rangle$. Observe that

$$\Delta u = -\gamma[b] \tag{3.7}$$

holds in the distribution sense, since for any smooth test function with compact support φ, by definition of the convolution between distributions [16], it holds

$$\begin{aligned}\langle \Delta u, \varphi\rangle = \langle u, \Delta\varphi\rangle &= -\langle\langle \gamma[b], \Gamma(x-\cdot)\rangle, \Delta\varphi(x)\rangle = -\langle \gamma[b], \langle \Delta\Gamma(x-\cdot), \varphi(x)\rangle\rangle \\ &= -\langle \gamma[b], \varphi\rangle. \end{aligned} \tag{3.8}$$

Substracting (3.8) from (3.6) yields

$$\begin{aligned} 0 &= \int_{S_V} [\partial'_N u(x')]\varphi(x')d\mathcal{H}^2(x') - \int_S \left([u(x')] - b\right)\partial'_N \varphi(x')d\mathcal{H}^2(x') \\ &- \int_{S^-} [u(x')]\partial'_N \varphi(x')d\mathcal{H}^2(x'), \end{aligned} \tag{3.9}$$

which since it holds for any test function φ, yields (3.1) by (3.7), achieving the proof. □

Remark that taking an arbitrary $\partial_N \varphi$ on S^- while $\partial_N \varphi = \varphi = 0$ on S in (3.9) yields the continuity of u on S^-. By (3.2), it holds

$$\partial_i u(x) = -b\int_S \left(\frac{N_i}{|x-x'|^3} - 3\frac{N\cdot(x-x')(x_i - x'_i)}{|x-x'|^5}\right) d\mathcal{H}^2(x'). \tag{3.10}$$

More results on this topic can be found in [3].

3.2. **Pointwise properties of gradients of harmonic maps.**

Lemma 3.2. *Let C and S be as in Lemma 3.1 and u be the explicit solution of* (3.1) *given by* (3.2)*. If C is smooth, there exists a constant $c > 0$ depending on the curvature of C, such that it holds*

$$|\partial_i u(x)| \leq cbl + \frac{cb}{d(x,C)}. \tag{3.11}$$

Proof. It is not difficult to prove [17] that if C is smooth, it admits a non-self-intersecting tubular neighborhood.

Step 1. Let us first prove that the value of the derivative $\partial_i u(x)$ does not depend on the surface S appearing in (3.1). Let indeed S' be another smooth surface that does not contain the point x and has C as boundary. For simplicity let us suppose it is disjoint from S. Let u' be the solution of (3.1) with S' replacing S and let A be the open set enclosed by S and S'. By formula (3.2), $(u-u')(x) = c+b\int_{\partial A}\partial'_N\Gamma(x'-x)d\mathcal{H}^2(x') = c+b\chi_A(x)$, the second equality being a consequence of the Divergence theorem. In particular we see that $u-u'$ is constant in a neighborhood of x, so that $\partial_i u(x) = \partial_i u'(x)$. By approximation, we can also extends this to the case of Lipschitz surface S', and then to every rectifiable current S' with $\partial S' = C$ and whose support is outside a neighborhood of x.

Step 2. Let $d = d(x,C)$ be the distance form x to C, let κ be the maximum curvature of C, and let us denote by $R := \kappa^{-1}$ the radius of curvature. Let $B_R(x)$ be a ball with radius R and center x, let P be the point in C such that $d = d(x,P)$, let O be the point on the line Px, on the x side, such that $d(O,P) = R$, and let $B_R(O)$ be a ball with radius R and center O (see figure 1). Let $\pi_R : \mathbb{R}^3 \to \partial B_R(x)$ be the orthogonal projection onto the sphere $\partial B_R(x)$ and let C_R be the image of C throughout π_R. Let us consider the Lipschitz homotopy $\Phi : [0,1]\times[0,l] \to \Omega$ such that $\Phi(0,[0,l]) = C$, $\Phi(1,[0,l]) = C_R$, and $\Phi(\cdot,t)$ is affine for all $t \in [0,l]$. Then $E := \Phi_\sharp[\![[0,1]\times[0,l]]\!]$ is a rectifiable current with boundary $C \cup C_R$. Let D be an integral current on $\partial B_R(x)$ with boundary $-C_R$. The rectifiable current $S' := E + D$ has boundary C, so we can consider the map u' solution of (3.1) with S replaced by S'.

Step 3. We claim that we can choose D in such a way that its total mass $M(D)$ is bounded by lR. Indeed, since C_R in an integral closed 1-current, it decomposes as $C_R = \sum_i C_R^i$, with $\sum_i l_i = M(C_R) \leq l + 2\pi R^2$, where l_i is the length of C_R^i and l is the total length of C. By the isoperimetric inequality on $\partial B_R(x)$, it holds

$$l_i^2 \geq 4\pi A_i - \frac{A_i^2}{R^2},$$

with A_i the minimal area enclosed by C_R^i, so that $A_i \leq 2\pi R^2$. It follows by simple computations that $A_i \leq 2\pi R^2 - R\sqrt{4\pi^2R^2 - l_i^2}$ and since $\sqrt{4\pi^2R^2 - l_i^2} \geq 2\pi R - l_i$ if $l_i \leq 2\pi R$, one deduces by mere substitution that $A_i \leq Rl_i \wedge 2\pi R^2$. As a consequence, $\sum_i A_i \leq lR$.

Step 4. Now, to compute $|\partial_i u(x)|$ we will use formula (3.10), integrating over $E + D$. Inegration over D can be estimated as follows:

$$\begin{aligned} |\partial_i u(x)| &= |b\int_D \left(\frac{N_i}{|x-x'|^3} - 3\frac{N\cdot(x-x')(x_i-x'_i)}{|x-x'|^5}\right)d\mathcal{H}^2(x')| \\ &\leq \frac{4b}{R^3}\int_D d\mathcal{H}^2(x') \leq \frac{4bl}{R^2}, \end{aligned} \tag{3.12}$$

[2]The presence of the term $2\pi R$ is justified as follows: π_R is 1-Lipschitz outside B_R, while it is not Lipschitz inside it. So we must estimate the inner part of C separatly. Since the maximum curvature of C is assigned, it is seen that the part of C inside B_R is bounded and its projection on B_R does not exceed $2\pi R$.

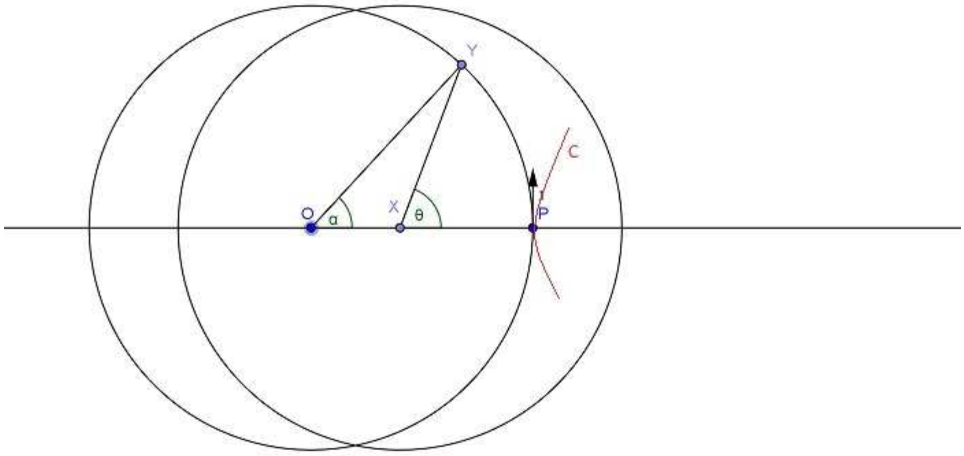

FIGURE 1. Geometry

since $M(D) \le \frac{lR}{\pi}$. Let us now compute the integration over E. Let E^+ (E^-) be the part of E outside $B_R(x)$ (inside, respectively). The integration can be done by polar coordinate (s,ρ) centered at x, where s can be seen as an arc parameter on C_R. Observing that $N \perp (x-x')$, the integral of (3.10) over E^+ is bounded by

$$\int_{C_R} ds \int_R^{+\infty} \frac{b}{\rho^3} ds \le \frac{lb}{2R^2}. \tag{3.13}$$

More delicate is the computation of the integral over E^- (see Figure 1). The the regularity hypotheses on C, the curve C, passing throught P with tangent τ_P, cannot go inside the ball $B_R(O)$. Let C^- be the part of C inside $B_R(x)$. The integral reads

$$\int_{C^-} d\theta \int_{d(x,y(\theta))}^R \frac{b}{\rho^2} d\rho.$$

It is not hard to see that, since the curvature is bounded by R^{-1}, the maximum of this quantity is attained when the curve C^- has constant curvature R^{-1} and moves on the plane tangent to OP and τ, so that its trajectoty is an arc of a circle of radius R on the sphere $\partial B_R(O)$ (with endpoints on $\partial B_R(x)$). We take θ as the angle at x between the point $y(\theta) \in C^-$ and P, and let α be the angle at O between P and $y(\theta)$. Then

$$d(x,y(\theta))^2 = (R\cos\alpha - R + d)^2 + R^2 \sin\alpha^2$$

and

$$\tan\theta = \frac{R\sin\alpha}{R\cos\alpha - R + d},$$

so the integral above becomes

$$2b\int_0^{\arccos\frac{R-d}{2R}} \Big(\frac{1}{((R\cos\alpha - R+d)^2 + R^2\sin\alpha^2)^{\frac{1}{2}}} - \frac{1}{R}\Big)\frac{R\cos\alpha(R\cos\alpha - R+d) + R^2\sin\alpha^2}{(R\cos\alpha - R+d)^2 + R^2\sin\alpha^2} d\alpha, \tag{3.14}$$

and after the change of variable $t = \cos\alpha$,

$$\frac{2b}{R}\int_0^{\frac{R-d}{2R}} \frac{(1-((\frac{R-d}{R})^2-2\frac{R-d}{R}t+1)^{\frac{1}{2}})}{((\frac{R-d}{R})^2-2\frac{R-d}{R}t+1)^{\frac{3}{2}}}\frac{1-\frac{R-d}{R}t}{(1-t^2)^{\frac{1}{2}}}dt.$$

By the estimates

$$\frac{1-\frac{R-d}{R}t}{((\frac{R-d}{R})^2-2\frac{R-d}{R}t+1)^{\frac{3}{2}}} \leq \frac{1-\frac{R-d}{R}t}{((\frac{R-d}{R})^2t^2-2\frac{R-d}{R}t+1)^{\frac{3}{2}}} = \frac{1}{(1-\frac{R-d}{R}t)^2},$$

and

$$1-((\frac{R-d}{R})^2-2\frac{R-d}{R}t+1)^{\frac{1}{2}} \leq 1-((\frac{R-d}{R})^2t^2-2\frac{R-d}{R}t+1)^{\frac{1}{2}} = \frac{R-d}{R}t,$$

valid since $0 \leq t \leq \frac{R-d}{R} < 1$, simple computation leads to estimate the integral (3.14) by

$$2(R-d)(1-\frac{R-d}{R})\frac{bR^{-1}}{d} < \frac{2bR^{-1}}{d}.$$

Summing all the bounds obtained we finally get

$$|\partial_i u(x)| \leq \frac{2b\kappa}{d} + \frac{9}{2}bl\kappa^2, \tag{3.15}$$

from which the thesis follows. □

Remark 3.3. Let C_R^i be a simple loop in $B_R(x)$, let $P \notin C_R^i$ be a point on $\partial B_R(x)$. We construct an homotopy $\Psi^P : [0,1]\times[0,2\pi] \to \partial B_R(x)$ that satisfies $\Psi^P(0,\cdot) \equiv P$ and $\Psi^P(1,[0,2\pi]) = C_R^i$, and we can consider the current $\Psi^P_\sharp[\![0,1]\times[0,2\pi]\!]$. Then we can set $A^i := \Psi^P_\sharp[\![0,1]\times[0,2\pi]\!]$, where P is chosen in such a way that $\Psi^P_\sharp[\![0,1]\times[0,2\pi]\!]$ has minimal mass.

Remark 3.4. In Lemma 3.2 we also proved that the integral in (3.2) does not depend on the particular surface S, but only on its boundary C.

Corollary 3.5. *Let C be the union of $N > 0$ smooth closed curves C_k, let S be the union of the corresponding surfaces S_k with boundary C_k respectively, and let u be the solution to* (3.1) *given by* (3.2). *Then* (3.11) *holds true.*

Proof. Actually the same proof as of Lemma 3.2 applies. □

Lemma 3.6. *Let S and C be as above and smooth. Let u be the solution to*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3\setminus S \\ u^+ - u^- = 1 & \text{on } S \\ \partial_N u^+ - \partial_N u^- = 0 & \text{on } S. \end{cases} \tag{3.16}$$

Then, if U is a tubular neighborhood of C, for all $(\rho,\theta,\tau) \in U$ with $\theta \neq 0$, it holds

(i) *there exists the limit* $\lim_{\epsilon\to 0^+} u(\epsilon\rho,\theta,\tau) = u(\hat\tau) = u^+(\tau) := \frac{\theta}{2\pi} + c$, *where c is a fixed arbitrary constant, and $\hat\tau = (0,0,\tau)$.*

(ii) $\lim_{\epsilon\to 0^+} |\partial_\tau u(\epsilon\rho,\theta,\tau)| < c < +\infty$ *for some constant $c > 0$ that depends only on the curve C.*

(iii) $\lim_{\epsilon\to 0^+} |\partial_\rho u(\epsilon\rho,\theta,\tau)| < c < +\infty$ *for some constant $c > 0$ that depends only on the curve C.*

Proof. With no loss of generality we can suppose that the curve C which represents the boundary of S passes through the origin of an Euclidean coordinate system where it is tangent to the z-axis. Moreover we choose the coordinates x_1 and x_2 in such a way that $x_1 = \rho\cos\theta$ and $x_2 = \rho\sin\theta$, so that it follows that the point $(\epsilon\rho,\theta,z)$ coincides with $(\epsilon x_1,\epsilon x_2,z)$. For simplicity we take $z = 0$ and denote $x = (x_1,x_2,0)$, while S is orthogonal to the x_2-axis in 0. From Lemma 3.1 we have the following explicit formula (for simplicity u will not be renamed after a change of variables):

$$u(\epsilon\rho,\theta,0)=u(\epsilon x_1,\epsilon x_2,0)=-\int_S \partial_N\Gamma(x'-\epsilon x,y'-\epsilon y,z')d\mathcal{H}^2(x',y',z'),$$

with the change of variables $(\epsilon x_1'',\epsilon x_2'',\epsilon z'')=(x_1',x_2',z')$ we obtain

$$u(\epsilon x_1,\epsilon x_2,0)=-\int_{\frac{1}{\epsilon}S}\partial_N\Gamma(x_1''-x_1,x_2''-x_2,z'')d\mathcal{H}^2(x_1'',x_2'',z''),$$

where we have used the explicit expression of Γ, with ∂_N being the partial derivative in the new variable. Letting ϵ go to zero we obtain

$$\lim_{\epsilon\to0^+}u(\epsilon\rho,\theta,z)=-\int_{\Pi_0}\partial_N\Gamma(x''-x)d\mathcal{H}^2(x''),$$

where Π_0 is the half-plane $\{z=x_2=0,x_1>0\}$ and we have used the shorter notation $x''=(x_1'',x_2'',z'')$. Thanks to Lemma 3.1, we see that the right-hand side coincides with $u(x_1,x_2,0)$, where u is the solution of (3.16) with $S=\Pi_0$. But it is well known that such solutions are given by, in cylindrical coordinates, $u(\rho,\theta,z)=\frac{\theta}{2\pi}+c$ for arbitrary constants c. In particular we have $\lim_{\epsilon\to0^+}u(\epsilon\rho,\theta,z)=\frac{\theta}{2\pi}+c$.

To prove statement (ii) we use the explicit expression (3.10), which reads, after the change of variables $x'=\epsilon x''$ (here, again with abuse of notations, $\partial_z u=\partial_\tau u$ at $z=0$),

$$\partial_z u(\epsilon x_1,\epsilon x_2,0)=-\frac{1}{\epsilon}\int_{\frac{1}{\epsilon}S}\left(\frac{N_z}{|x-x''|^3}-3\frac{N\cdot(x-x'')(z-z'')}{|x-x''|^5}\right)d\mathcal{H}^2(x''). \quad (3.17)$$

We fix $R>0$ and consider the ball B_ϵ with center $(\epsilon x_1,\epsilon x_2,0)$ and radius R. We then write the last integral as

$$-\frac{1}{\epsilon}\int_{\frac{1}{\epsilon}S\cap B_\epsilon}\left(\frac{N_z}{|x-x''|^3}-3\frac{N\cdot(x-x'')(z-z'')}{|x-x''|^5}\right)d\mathcal{H}^2(x'')$$
$$-\frac{1}{\epsilon}\int_{\frac{1}{\epsilon}S\cap B_\epsilon^c}\left(\frac{N_z}{|x-x''|^3}-3\frac{N\cdot(x-x'')(z-z'')}{|x-x''|^5}\right)d\mathcal{H}^2(x''),$$

and thanks to Remark 3.4, up to choose R small enough, we can assume that the surface S is everywhere orthogonal to the vector $(\epsilon x-x')$ in B_ϵ, that is, to $(x-x'')$ in $\frac{1}{\epsilon}B_\epsilon$, so that the integral above becomes

$$-\frac{1}{\epsilon}\int_{\frac{1}{\epsilon}S\cap B_\epsilon}\frac{N_z}{|x-x''|^3}d\mathcal{H}^2(x'')$$
$$-\frac{1}{\epsilon}\int_{\frac{1}{\epsilon}S\cap B_\epsilon^c}\left(\frac{N_z}{|x-x''|^3}-3\frac{N\cdot(x-x'')(z-z'')}{|x-x''|^5}\right)d\mathcal{H}^2(x''). \quad (3.18)$$

Let us now estimate the second term in (3.18). In B_ϵ^c it holds $|\epsilon x-x'|>R$, that is, $|x-x''|>\epsilon^{-1}R$, so it is easy to see that this term can be estimated by

$$\frac{\mathcal{H}^2(S)}{R^3}\le\gamma\frac{|C|^2}{R^3},$$

where $|C|$ is the length of C and $\gamma>0$ is the constant of the isoperimetric inequality.

It remains to estimate the first term. Let us consider the plane Π passing through 0 and tangent to the versor $\bar{z}$ and $\overline{x-0}$. Let Π^+ be the half-plane in Π bounded by the axis $\hat{z}$ and not containing the point x. Thanks to the smoothness of C, we can assume that there exists a smooth one-to-one map $\Phi:\Pi^+\cap B_\epsilon\to S\cap B_\epsilon$, so that also $N\circ\Phi:\Pi^+\ni\hat{x}'\mapsto N(x')$ is smooth, and then in $B_\epsilon\cap\Pi^+$ we can use the Taylor expansion of $N \circ \Phi$ at 0. Going back to the variable $x' = \epsilon x''$ (and $\hat{x}' := \epsilon \hat{x}''$), we find that the first term in (3.18) reads

$$-\int_{S\cap B_\epsilon} \frac{N_z(x')}{|\epsilon x - x'|^3} d\mathcal{H}^2(x'') = -\int_{S\cap B_\epsilon} \frac{\nabla^2 N_z(0)\hat{x}'\cdot\hat{x}'}{|\epsilon x - x'|^3} + \frac{r_N(|\hat{x}'|^2)}{|\epsilon x - x'|^3} d\mathcal{H}^2(x').$$

The Taylor expansion of Φ at 0 provides $x' = \hat{x}' + \nabla^2\Phi(0)\hat{x}'\cdot\hat{x}' + r_\Phi(\hat{x}')$ and if R is small enough we can assume that $|\nabla^2\Phi(0)\hat{x}'\cdot\hat{x}' + r_\Phi(\hat{x}')| < \frac{1}{2}|\hat{x}'|$. Note that, since C is smooth, we can find such a $R > 0$ satisfying the last inequality globally, i.e., R is independent of the point x. In particular we find $|\epsilon x - x'| > |\epsilon x - \hat{x}'| - |\nabla^2\Phi(0)\hat{x}'\cdot\hat{x}' + r_\Phi(\hat{x}')| > |\hat{x}'| - \frac{1}{2}|\hat{x}'| = \frac{1}{2}|\hat{x}'|$ for all $\epsilon > 0$, so that the integral is bounded by

$$\int_{\Pi^+\cap B_\epsilon} \frac{|\nabla^2 N_z(0)\hat{x}'\cdot\hat{x}'|}{|\hat{x}'|^3} + \frac{r_N(|\hat{x}'|^2)}{|\hat{x}'|^3} d\mathcal{H}^2(x'), \tag{3.19}$$

and taking into account that $R > 0$ can be small as we want, we assume that $|r_N(\hat{x}')| < |\nabla^2 N_z(0)\hat{x}'\cdot\hat{x}'|$, whereby the last integral can be estimated by

$$C_0 \int_{\Pi^+\cap B_\epsilon} \frac{1}{|\hat{x}'|} d\mathcal{H}^2(\hat{x}'),$$

where the constant C_0 is independent of R and x, and whose limit as $\epsilon \to 0$ reads by the monotone convergence theorem

$$C_0 \int_{\Pi^+\cap B(0,R)} \frac{1}{|\hat{x}'|} d\mathcal{H}^2(\hat{x}'),$$

which is uniformly bounded. Now, since the value of R is independent of the point x but only depends on the geometry of the curve C, we achieved the proof.

Statement (iii) can be proved taking into account that choosing R small enough a formula similar to (3.18) holds, and then arguing as for statement (ii). □

Remark 3.7. Let us point out that Lemma 3.6 still holds true if we do not assume that C is connected. Indeed if C is the union of a finite family of smooth closed curves, the surface S will be the union of a finite family of smooth surfaces and the arguments used in the proof of Lemma 3.6 still work.

Remark 3.8. The curve regularity required in Lemma 3.6 is $W^{3,\infty}$, because of estimates such as (3.19). Note also that C_0 depends on the curve curvature.

Lemma 3.9. *Let $b \in 2\pi\mathbb{Z}$. Then the solution u of* (3.1) *belongs to $C^\infty(\Omega \setminus C, \mathbb{T})$ and it is harmonic in $\Omega \setminus C$.*

Proof. As we have proved in Corollary 3.2 if we choose a surface S' with boundary C disjoint from S, and denote by u' the corresponding solution of (3.1), then $u - u' = b\chi_A$, with A the open set with boundary $S \cup S'$. Since $b \in 2\pi\mathbb{Z}$ we see that $u = u'$ as a map into $\mathbb{T}$. Moreover if $x \notin S$ then u is smooth at x, so in particular, up to change the surface S, we obtain that it belongs to $C^\infty(\Omega \setminus C, \mathbb{T})$ and u is harmonic at x for all $x \notin C$. □

3.3. **Distributional properties of gradients of harmonic maps.**

Lemma 3.10. *Let C be a closed Lipschitz curve in Ω and let $b \in 2\pi\mathbb{Z}^3$. Then for any Lipschitz surface S with boundary C, every solution u to* (3.1) *belongs to $BV^p(\Omega, \mathbb{R}^3)$ with $1 \le p \le \frac{3}{2}$, satisfies* $\mathrm{Div}\,\nabla u = 0$ *and* $-\mathrm{Curl}\,\nabla u = b \otimes \mathcal{L}$ *as distributions, with ∇u the part of the gradient of u that is absolutely continuous with respect to the Lebesgue measure. Moreover if the curve C is smooth, then $\nabla u \in L^p(\Omega, \mathbb{R}^{3\times 3})$ for all $1 \le p < 2$.*

Proof. Let u be a solution to (3.1) and let us first assume C be smooth. By (3.11), $\nabla u \in L^p(\Omega, \mathbb{R}^{3\times 3})$ for $p < 2$. It has been shown that u is smooth outside S where it has a jump of amplitude b. In particular u belongs to $SBV(\Omega, \mathbb{R}^3)$ and its distributional derivative is given by

$$\langle Du, \varphi \rangle := -\langle u, \text{div } \varphi \rangle = S(\varphi) + \langle \nabla u, \varphi \rangle, \tag{3.20}$$

for all $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^{3\times 3})$, where S denotes the distribution $S(\varphi) = -\int_S N_j b_i \varphi_{ij} d\mathcal{H}^2$.

Let us prove that $-\text{ Curl } \nabla u = \mathcal{L} \otimes b$. To this aim let us take $\psi \in \mathcal{D}(\Omega, \mathbb{R}^{3\times 3})$ and write

$$\begin{aligned} -\langle \text{ Curl } \nabla u, \psi \rangle &:= -\langle \nabla u, \text{ Curl } \psi \rangle = -\langle Du, \text{ Curl } \psi \rangle + S(\text{ Curl } \psi) \\ &= \int_C \tau_j b_i \psi_{ij} d\mathcal{H}^1 = b \otimes \mathcal{L}(\psi), \end{aligned} \tag{3.21}$$

where the second equality follows from (3.20) with $\varphi = \text{ Curl } \psi$, and the third one by Stokes theorem.

We now prove that $\text{ Div } \nabla u = 0$. Again, we take $\psi \in \mathcal{D}(\Omega, \mathbb{R}^3)$ and write

$$-\langle \text{ Div } \nabla u, \psi \rangle := \langle \nabla u, \nabla \psi \rangle = \langle Du, \nabla \psi \rangle - S(\nabla \psi), \tag{3.22}$$

and using the explicit formula (3.2) for u we obtain

$$\begin{aligned} \langle Du, \nabla \psi \rangle &= b_k \langle D_i \int_S \partial'_N \Gamma'_k (x' - \cdot) d\mathcal{H}^2(x'), D_i \psi \rangle \\ &= -\int_S b_k \langle \Delta \Gamma'_k (x' - \cdot), D_j \psi N_j \rangle d\mathcal{H}^2(x') \\ &= -b_k \int_S \partial'_N \psi_k (x') d\mathcal{H}^2(x') = S(\nabla \psi), \end{aligned}$$

where $\Gamma'(x')(x) := \Gamma(x - x')$ for $x \in \mathbb{R}^3$, so that plugging the last identity in (3.22) we obtain $\text{ Div } \nabla u = 0$.

Let us now treat the general case. We proceed by approximation, so let C_n be a sequence of curves converging uniformly and in the sense of currents to C, let S_n be surfaces converging in the same sense to S, and let u_n be the corresponding solutions. Let V_n be the solution to the system (6.12) below, with $\mu := -b \otimes \mathcal{L}_n$, so that for this solution we entail $\|V_n\|_{L^p(\Omega)} \leq |b||\mathcal{L}_n|$ for $1 \leq p \leq \frac{3}{2}$. So far we have proved that ∇u_n differ from $-V_n^T$ by the gradient of an harmonic map whose boundary datum at $\partial\Omega$ are bounded since $d(C_n, \partial\Omega) > \delta > 0$ (it can be seen computing $\nabla u_n N$ by formula (3.10)). In particular we find that there is a constant $C > 0$ such that $\|\nabla u_n\|_{L^p(\Omega)} \leq C|b||\mathcal{L}_n|$ for $1 \leq p \leq \frac{3}{2}$. Now it is easy to see that $u_n \rightharpoonup u$ weakly* in $BV^p(\Omega, \mathbb{R}^3)$, and the conclusion easily follows. □

Remark 3.11. In order to prove that $\text{ Div } \nabla u = 0$, we might also argue as follows. Let $\hat{S} \supset S$ such that $\hat{S}$ separates Ω in two parts Ω^- and Ω^+. Then for every test function $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$ it holds

$$\begin{aligned} &\int_\Omega \nabla u \nabla \varphi dx = \int_{\Omega^+} \nabla u \nabla \varphi dx + \int_{\Omega^-} \nabla u \nabla \varphi dx = \\ &-\int_{\Omega^+} \text{ Div } \nabla u \varphi dx - \int_{\Omega^-} \text{ Div } \nabla u \varphi dx + \int_{\hat{S}^+} \partial_N u^+ \varphi dx - \int_{\hat{S}^-} \partial_N u^- \varphi dx = 0. \end{aligned}$$

Remark 3.12. The statement of Lemma 3.10 readly applies to the case of C being a finite union of Lipschitz curves.

4. Main result: graph boundary of $\mathbb{T}^3$-valued harmonic maps

We introduce the following notation. For all $b \in \mathbb{R}^3$ we define the 1-current $\vec{b} \in \mathcal{D}_1(\mathbb{T}^3)$ as

$$\vec{b}(\omega) := -\frac{1}{2\pi}\int_0^{2\pi} \langle \omega(\frac{b_1\theta}{2\pi}, \frac{b_2\theta}{2\pi}, \frac{b_3\theta}{2\pi}), b\rangle d\theta, \tag{4.1}$$

for any 1-form $\omega \in \mathcal{D}^1(\mathbb{T}^3)$. It is easy to see that $M(\vec{b}) = |b|$. The fact that we are on the torus, i.e., ω is 2π-periodic on $\mathbb{R}^3$, implies that $\vec{b}$ is a closed current whenever $b \in 2\pi\mathbb{Z}^3$. Moreover it is convenient to define, for all $b \in \mathbb{R}^3$ and all $r \in \mathbb{R}^3$, the 1-current $\vec{b}_r \in \mathcal{D}_1(\mathbb{T}^3)$ as

$$\vec{b}_r(\omega) := -\frac{1}{2\pi}\int_0^{2\pi} \langle \omega(r_1 + \frac{b_1\theta}{2\pi}, r_2 + \frac{b_2\theta}{2\pi}, r_3 + \frac{b_3\theta}{2\pi}), b\rangle d\theta, \tag{4.2}$$

for any 1-form $\omega \in \mathcal{D}^1(\mathbb{T}^3)$. Also in this case $M(\vec{b}_r) = 2\pi|b|$ for all $r \in \mathbb{R}^3$. Note that, if $b \in 2\pi\mathbb{Z}^3$ and there is a real number δ such that $b = \delta r$, then the currents $\vec{b} = \vec{b}_r$ thanks to the periodicity of the forms in $\mathcal{D}^1(\mathbb{T}^3)$.

We denote by $\mathcal{L} \wedge \vec{b}$ the 2-current in $\Omega \times \mathbb{T}^3$ defined as

$$\mathcal{L} \wedge \vec{b}(\omega) = -\frac{1}{2\pi}\int_C \int_0^{2\pi} \langle \omega(x, \frac{b\theta}{2\pi}), \bar{\tau} \wedge \bar{b}\rangle d\theta d\mathcal{H}^1(x), \tag{4.3}$$

for any 2-form $\omega \in \mathcal{D}^2(\Omega \times \mathbb{T}^3)$ (we mean $\bar{\tau} = (\vec{\tau}, 0) \in \mathbb{R}^3 \times \mathbb{R}^3$, the tangent vector to C in $\Omega \times \mathbb{R}^3$, and $\bar{b} = (0, b) \in \mathbb{R}^3 \times \mathbb{R}^3$, with $\vec{\tau}$ the tangent vector to C in Ω).

Let C be a closed loop of class C^1. There is a cylindrical neighborhood D_R of C, with cylindrical coordinates $(\rho, \theta, z) \in [0, R] \times [0, 2\pi] \times [0, l]/\sim$, where $\sim$ means that the coordinate $\theta = 0$ (and $z = 0$) is identified with $\theta = 2\pi$ (resp. $z = l$). The neighborhood D_R is also parametrized by the coordinates (x, y, z) setting $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Let S be a smooth surface with boundary C and such that $S \cap U$ coincides with the set $\{\theta = 0\}$.

In the sequel we will use the notation $\Phi := Id \times u : \Omega \to \Omega \times \mathbb{T}^3$.

Theorem 4.1. *Let S be a smooth surface in Ω whose boundary C is a smooth and closed curve in Ω. Let $b = (b_1, b_2, b_3) \in 2\pi\mathbb{Z}^3$ and let $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ be the map with u_i given by (3.2) with $b = b_i$. Then $\mathcal{G}_u$ is an integral current in $\mathcal{D}_3(\Omega \times \mathbb{T}^3)$ and its boundary is given by*

$$\partial\mathcal{G}_u(\omega) = \mathcal{L} \wedge \vec{b}(\omega), \tag{4.4}$$

for all $\omega \in \mathcal{D}^2(\Omega \times \mathbb{T}^3)$.

Before proving Theorem 4.1 we state the following preliminary fact:

Lemma 4.2. *Let u be as in Theorem 4.1. Then $u \in \mathcal{A}_p(\Omega, \mathbb{T}^3)$ for all $1 \le p < 2$.*

Proof. Lemma 3.9 shows that u is well-defined in $\mathbb{T}^3$. In order to prove that it belongs to $\mathcal{A}_p(\Omega, \mathbb{T}^3)$ we need to show that all its minors $M^\beta_{\bar{\alpha}}(Du)$ belong to $L^p(\Omega)$. Thanks to Lemma 3.2 it is easy to see that every 1×1-minor belongs to $L^p(\Omega)$. Moreover from Lemma 3.1 we have that u_1, u_2, and u_3 differ from a multiplicative constant, so that the rows of the matrix Du are linearly dependent. In particular all the minors greater than 1×1 vanish, and the thesis follows. □

Proof of Theorem 4.1. Let u_ϵ be the restriction of the map u to $\Omega_\epsilon := \Omega \setminus \bar{D}_\epsilon$, $u_\epsilon := u \llcorner_{\Omega_\epsilon}$, where $D_\epsilon := \{(\rho, \theta, z) \in [0, R] \times [0, 2\pi] \times [0, l]/\sim : \rho < \epsilon\}$. The graph $\mathcal{G}_{u_\epsilon}$ is the restriction of the graph $\mathcal{G}_u$ to the open set $\Omega_\epsilon \times \mathbb{T}^3$. Formula (2.6) and the Dominated Convergence Theorem readily implies that $\mathcal{G}_{u_\epsilon} \rightharpoonup \mathcal{G}_u$ as currents. As a consequence we find

$$\partial\mathcal{G}_{u_\epsilon} \rightharpoonup \partial\mathcal{G}_u.$$

In order to compute explicitly the boundary of $\mathcal{G}_u$ we write $\partial\mathcal{G}_{u_\epsilon}(\omega) = \mathcal{G}_{u_\epsilon}(d\omega)$, for $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^3)$. Lemma 3.9 implies that u is smooth outside a neighborhood of C, so that we can apply the Stokes Theorem and find

$$\partial\mathcal{G}_{u_\epsilon}(\omega) = \int_{\partial D_\epsilon} \langle \omega \circ \Phi, \frac{\partial\Phi}{\partial\sigma} \wedge \frac{\partial\Phi}{\partial\tau}\rangle d\mathcal{H}^2(x),$$

where (σ, τ) is an orthogonal coordinate system in the tangent space to ∂D_ϵ. The gradient of Φ reads

$$(D\Phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}. \tag{4.5}$$

Let $\partial D_\epsilon \cong [0, 2\pi] \times [0, l]/\sim$ for all $(\theta, \tau) \in \partial D_\epsilon$. In the coordinate system $(\rho, \sigma, \tau, y_1, y_2, y_3)$ it holds

$$(D(\Phi_{\llcorner \partial D_\epsilon})) = (\frac{\partial\Phi}{\partial\sigma}, \frac{\partial\Phi}{\partial\tau}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \frac{\partial u_1}{\partial\sigma} & \frac{\partial u_1}{\partial\tau} \\ \frac{\partial u_2}{\partial\sigma} & \frac{\partial u_2}{\partial\tau} \\ \frac{\partial u_3}{\partial\sigma} & \frac{\partial u_3}{\partial\tau} \end{pmatrix}. \tag{4.6}$$

If $\omega = \omega_{ij} dz_i \wedge dz_j$, with $1 \le i < j \le 6$, where we have defined $z_1 = \rho$, $z_2 = \sigma$, $z_3 = \tau$, and $z_{k+3} = y_k$ for $k = 1, 2, 3$, we can write

$$\int_{\partial D_\epsilon} \langle \omega \circ \Phi, \frac{\partial\Phi}{\partial\sigma} \wedge \frac{\partial\Phi}{\partial\tau}\rangle d\mathcal{H}^2(x) = \int_{\partial D_\epsilon} \sigma(i, \bar{i})\omega_{ij}(x, u(x))\tilde{M}_i^j(D(\Phi_{\llcorner \partial D_\epsilon}(x)))d\mathcal{H}^2(x), \tag{4.7}$$

with $\tilde{M}_i^j(D(\Phi_{\llcorner \partial D_\epsilon}(x)))$ being the minor of $D(\Phi_{\llcorner \partial D_\epsilon})$ given by the i-th and j-th rows. From (4.6) we see that the (2×2)-minors of $D(\Phi_{\llcorner \partial D_\epsilon})$ which are nonzero are the only ones involving either the second or third row. So (4.7) reads

$$\begin{aligned} &\int_{\partial D_\epsilon} \left(\omega_{23}(x, u(x)) - \sum_{k=4}^{6}(\omega_{2k}(x, u(x))\frac{\partial u_{k-3}}{\partial\tau}(x) + \omega_{3k}(x, u(x))\frac{\partial u_{k-3}}{\partial\sigma}(x))\right) d\mathcal{H}^2(x) = \\ &\int_{\partial D_\epsilon} \omega_{23}(x, u(x))d\mathcal{H}^2(x) - \sum_{k=4}^{6}\int_0^{2\pi}\int_0^l \epsilon\tilde{\omega}_{2k}(\epsilon, \theta, \tau, u(\epsilon, \theta, \tau))\frac{\partial u_{k-3}}{\partial\tau}(\epsilon, \theta, \tau)d\tau d\theta \\ &- \sum_{k=4}^{6}\int_0^l\int_0^{2\pi} \epsilon\tilde{\omega}_{3k}(\epsilon, \theta, \tau, u(\epsilon, \theta, \tau))\frac{\partial u_{k-3}}{\partial\sigma}(\epsilon, \theta, \tau))d\theta d\tau, \end{aligned} \tag{4.8}$$

where $\tilde{\omega} := \omega \det \Psi$, with $\Psi : [0, \epsilon] \times [0, 2\pi] \times [0, l] \to D_\epsilon$ is the map of change of variables. Note that by the assumption of smoothness of C, we have that Ψ is smooth and $\det \Psi = 1$ on C. Now the first term of the right-hand side of (4.8) vanishes as $\epsilon \to 0$ since ω is bounded and $\mathcal{H}^2(\partial D_\epsilon) \to 0$. Integrating by parts the second term and using Lemma 3.6 we obtain

$$\sum_{k=4}^{6}\int_0^{2\pi}\epsilon\int_0^l\frac{\partial\tilde\omega_{2k}}{\partial\tau}(\epsilon,\theta,\tau,u(\epsilon,\theta,\tau))u_{k-3}(\epsilon,\theta,\tau)d\tau d\theta=$$
$$=\sum_{k=4}^{6}\int_0^{2\pi}\epsilon\int_0^l\frac{\partial\tilde\omega_{2k}}{\partial\tau}(\epsilon,\theta,\tau,u(\epsilon,\theta,\tau))(\frac{\theta b_{k-3}}{2\pi}+O(1))d\tau d\theta$$
$$+\sum_{k=4}^{6}\int_0^{2\pi}\epsilon\int_0^l\sum_{h=1}^{3}\frac{\partial\tilde\omega_{2k}}{\partial x_{3+h}}(\epsilon,\theta,\tau,u(\epsilon,\theta,\tau))\frac{\partial u_h}{\partial\tau}(\epsilon,\theta,\tau)(\frac{\theta b_{k-3}}{2\pi}+O(1))d\tau d\theta,\quad(4.9)$$

where $|O(1)|\le C$ as $\epsilon\to0$, so that its absolute value can be estimated by $\epsilon\|\frac{\partial\tilde\omega_{2k}}{\partial x_i}\|_\infty C$ thanks to Lemma 3.6, whereby this term vanishes as well as $\epsilon\to0$.

As for the third term of (4.8), we first set

$$R(\epsilon):=-\sum_{k=4}^{6}\int_0^{2\pi}\int_0^l\epsilon\Delta\tilde\omega_{3k}(\epsilon,\theta,\tau,u(\epsilon,\theta,\tau))\frac{\partial u_{k-3}}{\partial\sigma}(\epsilon,\theta,\tau))d\tau d\theta,$$

with, recalling that $\lim_{\epsilon\to0^+}u(\epsilon\rho,\theta,\tau)=u(\hat\tau)=u^+(\tau):=\frac{\theta}{2\pi}+c$,

$$\Delta\tilde\omega_{3k}(\epsilon,\theta,\tau,u(\epsilon,\theta,\tau)):=\tilde\omega_{3k}(\epsilon,\theta,\tau,u(\epsilon,\theta,\tau))-\tilde\omega_{3k}(\hat\tau,u^+(\tau)),$$

where $\hat\tau:=(0,0,\tau)$. Since $\frac{\partial}{\partial\sigma}=\frac{1}{\epsilon}\frac{\partial}{\partial\theta}$, we obtain

$$-\sum_{k=4}^{6}\int_0^l\int_0^{2\pi}\tilde\omega_{3k}(\hat\tau,u^+(\tau))\frac{\partial u_{k-3}}{\partial\theta}(\epsilon,\theta,\tau))d\theta d\tau+R(\epsilon)=$$
$$=-\sum_{k=4}^{6}\int_0^l\tilde\omega_{3k}(\hat\tau,\frac{b_1\theta}{2\pi}+c,\frac{b_2\theta}{2\pi}+c,\frac{b_3\theta}{2\pi}+c)u_{k-3}(\epsilon,\theta,\tau)\Big|_0^{\theta=2\pi}d\tau$$
$$+\sum_{k=4}^{6}\int_0^l\int_0^{2\pi}\frac{d}{d\theta}\tilde\omega_{3k}(\hat\tau,\frac{b_1\theta}{2\pi}+c,\frac{b_2\theta}{2\pi}+c,\frac{b_3\theta}{2\pi}+c)u_{k-3}(\epsilon,\theta,\tau))d\theta d\tau+R(\epsilon).\quad(4.10)$$

Using Lemma 3.2, for some constant $\gamma>0$, we have

$$|R(\epsilon)|\le\sum_{k=4}^{6}\int_0^{2\pi}\int_0^l\epsilon\|\Delta\omega_{3k}(x,u(x))\|_{L^\infty(\partial D_\epsilon)}|\frac{\partial u_{k-3}}{\partial\sigma}(\epsilon,\tilde\theta,\tau)|d\tau d\theta$$
$$\le\gamma\int_0^{2\pi}\|\Delta\omega_{3k}(x,u(x))\|_{L^\infty(\partial D_\epsilon)}d\theta\to0,$$

as $\epsilon\to0$, by (3.11) and being $\tilde\omega_{3k}(\epsilon,\theta,\tau,u(\epsilon,\theta,\tau))$ uniformly continuous at $\epsilon=0$, again thanks to Lemma 3.6 and the fact that C is compact. So that letting $\epsilon\to0$ in (4.10), using Lemma 3.6, integrating by parts again, and taking into account the periodicity of ω, (4.10) becomes

$$-\sum_{k=4}^{6}\int_0^l\int_0^{2\pi}\omega_{3k}(\hat\tau,\frac{b_1\theta}{2\pi}+c,\frac{b_2\theta}{2\pi}+c,\frac{b_3\theta}{2\pi+c})\frac{b_{k-3}}{2\pi}d\theta d\tau$$
$$=-\frac{1}{2\pi}\int_0^l\int_0^{2\pi}\langle\omega(\hat\tau,\frac{b_1\theta}{2\pi},\frac{b_2\theta}{2\pi},\frac{b_3\theta}{2\pi}),\bar\tau\wedge\bar b\rangle d\theta d\tau$$
$$=\mathcal{L}\wedge\vec b(\omega),\quad(4.11)$$

since in the local basis $\bar\tau=(\vec\tau,0)=(0,0,1,0,0,0)$. The proof is completed. □

Remark 4.3. For all $\omega \in \mathcal{D}^2(\Omega \times \mathbb{T}^3)$ let us write $\omega = \sum_{i=0}^2 \omega^i$ where we have set

$$\omega^i := \sum_{\alpha:|\alpha|=i} \omega_{\alpha\beta} dx_\alpha \wedge dy_\beta. \tag{4.12}$$

In other words ω^i represents the components of ω whose coefficient $dx_\alpha \wedge dy_\beta$ has a i-dimensional horizontal component (dx_α) and a $2-i$ dimensional vertical component (dy_β). Let $\Phi_\theta(x) := (x, \frac{b_1\theta}{2\pi}, \frac{b_2\theta}{2\pi}, \frac{b_3\theta}{2\pi})$ and define the tensor test function φ^ω associated to any form $\omega \in \mathcal{D}^2(\Omega \times \mathbb{T}^3)$ componentwise as

$$\varphi^\omega_{\alpha\beta} := \int_0^{2\pi} \langle \omega^1 \circ \Phi_\theta, e_\alpha \wedge \varepsilon_\beta \rangle d\theta, \tag{4.13}$$

where $\alpha, \beta = 1, 2, 3$.

Then, our main result can be restated as

$$\partial\mathcal{G}_u(\omega) = \Lambda(\varphi^\omega) := \mathcal{L} \otimes b(\varphi^\omega), \tag{4.14}$$

for all $\omega \in \mathcal{D}^3(\Omega \times \mathbb{T}^3)$, where φ^ω is defined by (4.13).

Corollary 4.4. *Let S be a Lipschitz surface in Ω whose boundary C is a Lipschitz and closed curve in Ω. Let $b = (b_1, b_2, b_3) \in 2\pi\mathbb{Z}^3$ and let $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ be a map with u_i satisfying* (3.2) *with $b = b_i$. Then $\mathcal{G}_u$ is an integral current in $\mathcal{D}_3(\Omega \times \mathbb{T}^3)$ and* (4.4) *holds.*

Proof. We procede by approximation. Let $\{C_k\}_{k>0}$ be a sequence of smooth closed curves approximating C (uniformly and in the sense of 1-currents) and let $\{S_k\}_{k>0}$ be smooth surfaces with boundary $\{C_k\}_{k>0}$ and converging (uniformly and in the sense of currents) to S. Let u_k be maps as in Theorem 4.1 with C replaced by C_k and S replaced by S_k. Thanks to the uniform convergence of S_k to S and using formula (3.2) we see that u_k converges pointwise to u, and then strongly in $L^p(\Omega, \mathbb{T}^3)$. Since C_k are converging uniformly to C whose length is finite, the lengths of C_k are uniformly bounded so the same argument employed in Lemma 3.10 gives a uniform bound in $L^p(\Omega)$, with $p < \frac{3}{2}$, for the 1×1 minors of Du_k, while the higher-order minors are all null. Therefore there are maps $v^\beta_{\bar\alpha} \in L^p(\Omega)$ such that, up to a subsequence, $M^\beta_{\bar\alpha}(Du_k) \rightharpoonup v^\beta_{\bar\alpha}$ weakly in $L^p(\Omega)$. Finally, the lengths of C_k being uniformly bounded, Theorem 4.1 provides an uniform bound on the masses of $\partial\mathcal{G}_{u_k}$. Now Theorem 2.3 applies and implies that $u \in \mathcal{A}_p(\Omega, \mathbb{T}^3)$. In particular we have that $u_k \rightharpoonup u$ weakly in $\mathcal{A}_p(\Omega, \mathbb{T}^3)$, thus Lemma 2.2 implies that $\partial\mathcal{G}_{u_k} \rightharpoonup \partial\mathcal{G}_u$ as currents, and the fact that for u_k the explicit form (4.4) holds true implies that it holds also at the limit, concluding the proof. □

Theorem 4.5. *Let S be the union of $N > 0$ Lipschitz surfaces S_k in Ω whose boundary is C, the union of the corresponding boundaries C_k, that are closed curves in Ω. Let $b = (b_1, b_2, b_3) \in 2\pi\mathbb{Z}^3$ and let $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ be a map with u_i satisfying* (3.2) *with $b = b_i$. Then $\mathcal{G}_u$ is an integral current in $\mathcal{D}_3(\Omega \times \mathbb{T}^3)$ and* (4.4) *holds.*

Proof. Let us first suppose that S_k and C_k are smooth and that the curves C_k are mutually disjoint. Then we will obtain the general result by approximation by mean of Theorem 2.3, arguing as in the proof of Corollary 4.4. Since N is finite, we see that C is compact and there is a tubular neighborhood around C. We can then argue as in the proof of Theorem 4.1, obtaining a formula similar to (4.10). Here $R(\epsilon)$ in (4.10) still vanishes thanks to Corollary 3.5, and $\frac{\partial u_{k-3}}{\partial\sigma}(\epsilon, \theta, \tau)$ in (4.9) still tends to 0. The thesis follows. □

Theorem 4.6. *Let S, C, b and u as in Theorem 4.5, and let $v \in C^1(\bar\Omega, \mathbb{R}^3)$. Then $\mathcal{G}_{u+v}$ is the integral current in $\mathcal{D}_3(\Omega \times \mathbb{T}^3)$ given by*

$$\partial \mathcal{G}_{u+v}(\omega) = \mathcal{L} \wedge \vec{b}(\omega) + \mathcal{C}_{u+v}(\omega), \tag{4.15}$$

for all $\omega \in \mathcal{D}^2(\Omega \times \mathbb{T}^3)$, with $\mathcal{C}_{u+v}$ defined as

$$\mathcal{C}_{u+v}(\omega) = -\frac{1}{2\pi} \int_C \int_0^{2\pi} \langle \omega(x, \frac{b\theta}{2\pi} + v(x)), \frac{\partial \bar v}{\partial \tau} \wedge \bar b \rangle d\theta d\mathcal{H}^1(x), \tag{4.16}$$

for all $\omega \in \mathcal{D}^2(\Omega \times \mathbb{T}^3)$ and with $\bar v = (0, v) \in C^1(\bar\Omega, \mathbb{R}^3 \times \mathbb{R}^3)$. In particular, it holds

$$M(\partial \mathcal{G}_{u+v}) \leq C(1 + \|Dv\|_{L^\infty(\Omega)}) |\mathcal{L} \otimes b|(\Omega). \tag{4.17}$$

Proof. As in Theorem 4.1, we first prove the result for a smooth loop C and then we obtain the general case arguing as in Theorem 4.5. Let us check that $u + v \in \mathcal{A}_p(\Omega, \mathbb{T}^3)$. To this aim let us prove that $\operatorname{adj}(Du + Dv)$ and $\det(Du + Dv)$ are summable functions. Since the rows of Du are linearly dependent and recalling the identity $\det A = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}$, it follows that

$$\det(Du + Dv) = \det \begin{pmatrix} Dv_1 \\ Dv_2 \\ Dv_3 \end{pmatrix} + \det \begin{pmatrix} Du_1 \\ Dv_2 \\ Dv_3 \end{pmatrix} + \det \begin{pmatrix} Dv_1 \\ Du_2 \\ Dv_3 \end{pmatrix} + \det \begin{pmatrix} Dv_1 \\ Dv_2 \\ Du_3 \end{pmatrix}.$$

Since $Dv_i \in C^0(\bar\Omega, \mathbb{R}^3)$, in particular it is bounded, so that all the determinants belong to $L^p(\Omega, \mathbb{R}^3)$ thanks to (3.11). A similar arguments applies for $\operatorname{adj}(Du + Dv)$.

To compute the boundary of $\mathcal{G}_{u+v}$ we proceed as in the proof of Corollary 4.4 (and Theorem 4.1), resulting in (4.7). This formula, setting $w := u + v$, takes the form

$$\begin{aligned}
&\int_{\partial D_\epsilon} \omega_{23}(x, w(x)) - \sum_{k=1}^3 (\omega_{2k}(x, w(x)) \frac{\partial u_k}{\partial \tau}(x) + \omega_{3k}(x, w(x)) \frac{\partial u_k}{\partial \sigma}(x)) d\mathcal{H}^2(x) \\
&- \int_{\partial D_\epsilon} \sum_{k=1}^3 (\omega_{2k}(x, w(x)) \frac{\partial v_k}{\partial \tau}(x) + \omega_{3k}(x, w(x)) \frac{\partial v_k}{\partial \sigma}(x)) d\mathcal{H}^2(x) + \\
&+ \sum_{4 \leq i < j \leq 6} \int_{\partial D_\epsilon} \omega_{ij}(x, w(x)) \tilde M_i^j (D(\mathrm{Id} \times v)_{\llcorner \partial D_\epsilon}(x)) d\mathcal{H}^2(x) \\
&+ \sum_{4 \leq i \neq j \leq 6} \int_{\partial D_\epsilon} \omega_{ij}(x, w(x)) (\frac{\partial u_{i-3}}{\partial \tau} \frac{\partial v_{j-3}}{\partial \sigma} - \frac{\partial u_{i-3}}{\partial \sigma} \frac{\partial v_{j-3}}{\partial \tau}) d\mathcal{H}^2(x),
\end{aligned} \tag{4.18}$$

where in the last term we have used the fact that $\omega_{ij} = -\omega_{ji}$. The first row, as seen by (4.8), tends to (4.11), the second and the third ones vanish as $\epsilon \to 0$ since v is smooth and its partial derivatives are bounded. The terms of the last row containing $\frac{\partial u_{i-3}}{\partial \tau}$ and $\frac{\partial u_{j-3}}{\partial \tau}$ vanish again thanks to Lemma 3.6 and the smoothness of C. It remains to study the term

$$\begin{aligned}
&- \sum_{4 \leq i \neq j \leq 6} \int_{\partial D_\epsilon} \omega_{ij}(x, w(x)) \frac{\partial u_{i-3}}{\partial \sigma} \frac{\partial v_{j-3}}{\partial \tau} d\mathcal{H}^2(x) = R^w(\epsilon) \\
&- \sum_{4 \leq i \neq j \leq 6} \int_0^l \int_0^{2\pi} \epsilon \tilde\omega_{ij}(\hat\tau, w(\hat\tau)) \frac{\partial v_{j-3}}{\partial \tau}(\hat\tau) \frac{\partial u_{i-3}}{\partial \sigma}(\epsilon, \theta, \tau) d\theta d\tau,
\end{aligned} \tag{4.19}$$

with $\hat\tau := (0, 0, \tau)$ and

$$R^w(\epsilon) = - \sum_{4 \leq i \neq j \leq 6} \int_0^l \int_0^{2\pi} \epsilon \Delta(\omega_{ij} \frac{\partial v_{j-3}}{\partial \tau})(\epsilon, \theta, \tau) \frac{\partial u_{i-3}}{\partial \sigma}(\epsilon, \theta, \tau) d\theta d\tau,$$

and

$$\Delta(\omega_{ij}\frac{\partial v_{j-3}}{\partial\tau})(\epsilon,\theta,\tau) :=\tilde{\omega}_{ij}(\epsilon,\theta,\tau,w(\epsilon,\theta,\tau))\frac{\partial v_{j-3}}{\partial\tau}(\epsilon,\theta,\tau) - \tilde{\omega}_{ij}(\hat{\tau},w(\hat{\tau}))\frac{\partial v_{j-3}}{\partial\tau}(\hat{\tau}).$$

Arguing as for $R(\epsilon)$ in (4.10) we see that $R^w(\epsilon)$ is negligible as $\epsilon\to 0$, while arguing as in (4.10) and taking into account that we see that $w(\hat{\tau}) = \frac{b\theta}{2\pi} + c + v(\hat{\tau})$, the expression (4.19) tends to

$$\begin{aligned}
&-\sum_{4\le i\ne j\le 6}\int_0^l\int_0^{2\pi}\omega_{ij}(\hat{\tau},\frac{b\theta}{2\pi}+c+v(\hat{\tau}))\frac{\partial v_{j-3}}{\partial\tau}(\hat{\tau})\frac{b_{i-3}}{2\pi}d\theta d\tau\\
&=-\sum_{4\le i\ne j\le 6}\int_0^l\int_0^{2\pi}\omega_{ij}(\hat{\tau},\frac{b\theta}{2\pi}+v(\hat{\tau}))\frac{\partial v_{j-3}}{\partial\tau}(\hat{\tau})\frac{b_{i-3}}{2\pi}d\theta d\tau\\
&=-\sum_{4\le i< j\le 6}\int_0^l\int_0^{2\pi}\omega_{ij}(\hat{\tau},\frac{b\theta}{2\pi}+v(\hat{\tau}))(\frac{\partial v_{j-3}}{\partial\tau}(\hat{\tau})\frac{b_{i-3}}{2\pi}-\frac{\partial v_{i-3}}{\partial\tau}(\hat{\tau})\frac{b_{j-3}}{2\pi})d\theta d\tau\\
&=\mathcal{C}_{u+v}(\omega),
\end{aligned}$$

where we have considered the periodicity of ω. The bound (4.17) now readily follows. □

5. Generalized Distributional Cofactors and Determinants

Let $u\in W^{1,p}(\Omega;\mathbb{R}^3)$, and suppose $u_iDu_j\in L^1(\Omega,\mathbb{R}^3)$ for all $i\ne j$, we define the *distributional cofactor* of Du, the distribution $\text{Cof } Du$ writing componentwise

$$(\text{ Cof } Du)_{ij} := D_{j+1}(u_{i+1}Du_{(i+2)(j+2)}) - D_{j+2}(u_{i+1}Du_{(i+2)(j+1)}), \tag{5.1}$$

with indices $i,j\in\{1,2,3\}$ (taken mod 3 when summed and with the derivatives intended in the sense of distributions), i.e.,

$$\begin{aligned}
\langle(\text{ Cof } Du)_{ij},\varphi\rangle := &-\int_\Omega D_{j+1}\varphi(u_{i+1}Du_{(i+2)(j+2)})dx\\
&+\int_\Omega D_{j+2}\varphi(u_{i+1}Du_{(i+2)(j+1)})dx, \quad \forall\varphi\in C_c^\infty(\Omega).
\end{aligned}$$

Moreover, $\text{Adj}Du$ is the *distributional adjunct* of Du, that is the transpose matrix of the distributional cofactor $\text{Cof}Du$. In general it is not true that the pointwise and distributional adjuncts coincide. Suppose $u_k(\text{adj}Du)^k\in L^1(\Omega,\mathbb{R}^3)$ for $k=1,2,3$, with $(\text{adj}Du)^k := (\text{adj}(Du)_{1k},\text{adj}(Du)_{2k},\text{adj}(Du)_{3k})$ being the k-column of $\text{adj}Du$. The *distributional determinant* of Du is the distribution $\text{Det } Du$ given taking the distributional divergence of $\frac{1}{3}(\sum_{k=1}^3(-1)^{k+1}u_k(\text{adj}Du)^k)$, i.e.,

$$\langle\text{Det}Du,\varphi\rangle := -\frac{1}{3}\int_\Omega(\sum_{k=1}^3(-1)^{k+1}u_k(\text{adj}Du)^k)\cdot D\varphi dx, \qquad \forall\varphi\in C_c^\infty(\Omega,\mathbb{R}^3).$$

As for the adjunct, in general $\text{Det } Du$ and $\det Du$ differ.

5.1. Generalized distributional cofactor and determinant in $\mathbb{R}^3$.

Let us choose the 2-form $\omega_{ij} = f(x_1,x_2,x_3)y_{i+1}dx_j\wedge dy_{i+2}$ (again taken mod 3). Compute the external derivative

$$\begin{aligned}
d\omega_{ij} =&(\partial_{j+1}f)y_{i+1}dx_{j+1}\wedge dx_j\wedge dy_{i+2} + (\partial_{j+2}f)y_{i+1}dx_{j+2}\wedge dx_j\wedge dy_{i+2}\\
&- f dx_j\wedge dy_{i+1}\wedge dy_{i+2},
\end{aligned}$$

so that,

$$\begin{aligned}\partial\mathcal{G}_u(\omega_{ij}) =& -\int_\Omega (\partial_{j+1} f u_{i+1}\partial_{j+2}u_{i+2} - \partial_{j+2} f u_{i+1}\partial_{j+1}u_{i+2})dx \\ & -\int_\Omega f \operatorname{cof}\,(Du)_{ij}dx \\ &= \langle \operatorname{Cof}\,(Du)_{ij}, f\rangle - \int_\Omega f \operatorname{cof}\,(Du)_{ij}dx. \end{aligned} \tag{5.2}$$

In particular we find that the distributional cofactor can be written as

$$\langle \operatorname{Cof}\,(Du)_{ij}, f\rangle = \partial\mathcal{G}_u(\omega_{ij}) + \int_\Omega f \operatorname{cof}\,(Du)_{ij}dx, \tag{5.3}$$

for all $f \in C_c^\infty(\Omega)$, where ω_{ij} is the form $fy_{i+1}dx_j \wedge dy_{i+2}$.

Let $\omega_0 := \frac{1}{3}(y_1dy_2 \wedge dy_3 - y_2dy_1 \wedge dy_3 + y_3dy_1 \wedge dy_2)$, taking the 2-form $\omega := f(x_1, x_2, x_3)\omega_0$, with external derivative

$$d\omega = \sum_{i=1}^{3} \partial_i f\ \omega_0 + f dy_1 \wedge dy_2 \wedge dy_3,$$

we find

$$\partial\mathcal{G}_u(\omega) = \frac{1}{3}\int_\Omega \sum_{i=1}^{3} \partial_i f\ (\sum_{k=1}^{3}(-1)^{k+1}u_k(\mathrm{adj}Du)_i^k)dx + \int_\Omega \det\,(Du)f dx,$$

from which, according to the definition of distributional determinant, it follows

$$\langle \operatorname{Det}\,(Du), f\rangle = -\partial_u\mathcal{G}(\omega) + \int_\Omega \det\,(Du)f dx, \tag{5.4}$$

for all $f \in C_c^\infty(\Omega)$, where ω is the form $f\omega_0$.

Thus, according to (4.14), Det (Du) is a Radon measure with det (Du) as absolutely continuous part and $-\mathcal{L} \otimes b(\varphi^\omega)$ as singular concentrated part. The same remark holds to the distributional cofactors.

5.2. **Generalized distributional cofactor and determinant in $\mathbb{R}^6$.** Formulae (5.3) and (5.4) show how we can write the distributional cofactor and determinant in terms of the graph of a map $u \in W^{1,p}(\Omega;\mathbb{R}^3)$ with $u_iDu_j \in L^1(\Omega,\mathbb{R}^3)$ for all $i \neq j$. However, we must point out that the deformations w we consider in dislocation problems (actually, maps with a jump on a surface as in Theorem 3.1, see also the map w of Theorem A.1) does not satisfy these latter assumptions, but are functions in $SBV(\Omega,\mathbb{R}^3)$. Therefore (5.3) and (5.4) cannot be directly applied, unless considering terms concentrated on a a surface S, which as we have seen is multiply defined.

Going back to our main result, it turns out that the standard volume form $dy_1 \wedge dy_2 \wedge dy_3$ is not an exact form in $\mathbb{T}^3$, and therefore cannot be written as the external derivative of a form ω_0. Since all previous developments have been made provided duality with periodic differential forms (in the last three components, i.e., in the displacement u, or w), it is convenient to watch at the torus $\mathbb{T}^3$ as $(S^1)^3 \subset \mathbb{R}^6$ where we use the Euclidean coordinates $(z_1, w_1, z_2, w_2, z_3, w_3)$, $(z_i, w_i) \in S^1 \subset \mathbb{R}^2$. Such identification is given by the map

$$H : \mathbb{T}^3 \ni (y_1, y_2, y_3) \mapsto (\cos\tilde{y}_1, \sin\tilde{y}_1, \cos\tilde{y}_2, \sin\tilde{y}_2, \cos\tilde{y}_3, \sin\tilde{y}_3) \in \mathbb{R}^6, \tag{5.5}$$

where $y_i \in \mathbb{T}$ is naturally identified by a unique number $\tilde{y}_i \in [0, 2\pi)$. In other words, $\tilde{y}_i$ represents the angle at the origin with the abscissa axis. Not to weight up the notation $\tilde{y}_i$ will be still noted by y_i. At a fixed point $P \in S^1$ with polar coordinates $P = (1,\theta)$, the covector dy, tangent to S^1, is written in Euclidean coordinates as $dy = -\sin\theta dz + \cos\theta dw$. Viceversa, a covector $adz + bdw$ is written in polar coordinates $(a\cos\theta + b\sin\theta)d\rho - (a\sin\theta + b\cos\theta)dy$, thus obtained as the image throught the map

$$Q = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{5.6}$$

Thus, any covector $\omega = \gamma_1 dz_1 + \eta_1 dw_1 + \gamma_2 dz_2 + \eta_2 dw_2 + \gamma_3 dz_3 + \eta_3 dw_3$ has coordinates in the basis $\{d\rho_1, d\theta_1, d\rho_2, d\theta_2, d\rho_3, d\theta_3\}$ given by $Q^3(\omega)$, with $Q^3 := (Q\ Q\ Q)$. Let Π denote the projection onto the tangent space to $(S^1)^3$, i.e., $\Pi : (\rho_1, \theta_1, \rho_2, \theta_2, \rho_3, \theta_3) \mapsto (0, \theta_1, 0, \theta_2, 0, \theta_3)$, and define $S := \Pi \circ Q^3$. Let $w = u + v$ be a map satisfying the hypotheses of Theorem 4.6. Then, denoting by $\hat{w} : \Omega \to (S^1)^3$ the map obtained from w identifying $\mathbb{T}^3$ with $(S^1)^3$, i.e., $\hat{w} = H(w)$, we can write the graph of $\hat{w}$ in $\Omega \times \mathbb{R}^6$ as

$$\mathcal{G}_{\hat{w}}(\omega) = \mathcal{G}_w(S(\omega)), \tag{5.7}$$

for any 3-form $\omega \in \mathcal{D}^3(\Omega \times \mathbb{R}^6)$. The same being true for its boundary, formula (4.16) provides, denoting by $h = h(\theta, x) := \frac{b\theta}{2\pi} + v(x)$

$$\begin{aligned}
&\partial\mathcal{G}_{\hat{w}}(\omega) = \\
&-\sum_{|\beta|=1} \frac{1}{2\pi} \int_C \int_0^{2\pi} \big(-\sin(h_\beta)\omega^z_{3\beta}(x, H(h)) + \cos(h_\beta)\omega^w_{3\beta}(x, H(h))\big) d\theta d\mathcal{H}^1(x) \\
&-\sum_{|\beta|=2} \int_C \int_0^{2\pi} \sin(h_{\beta_1})\sin(h_{\beta_2})\omega^{zz}_{\beta_1\beta_2}(x, H(h))\Big(\frac{\partial v_{\beta_2}}{\partial\tau}\frac{b_{\beta_1}}{2\pi} - \frac{\partial v_{\beta_1}}{\partial\tau}\frac{b_{\beta_2}}{2\pi}\Big) d\theta d\mathcal{H}^1(x) \\
&+\sum_{|\beta|=2} \int_C \int_0^{2\pi} \sin(h_{\beta_1})\cos(h_{\beta_2})\omega^{zw}_{\beta_1\beta_2}(x, H(h))\Big(\frac{\partial v_{\beta_2}}{\partial\tau}\frac{b_{\beta_1}}{2\pi} - \frac{\partial v_{\beta_1}}{\partial\tau}\frac{b_{\beta_2}}{2\pi}\Big) d\theta d\mathcal{H}^1(x) \\
&-\sum_{|\beta|=2} \int_C \int_0^{2\pi} \cos(h_{\beta_1})\cos(h_{\beta_2})\omega^{ww}_{\beta_1\beta_2}(x, H(h))\Big(\frac{\partial v_{\beta_2}}{\partial\tau}\frac{b_{\beta_1}}{2\pi} - \frac{\partial v_{\beta_1}}{\partial\tau}\frac{b_{\beta_2}}{2\pi}\Big) d\theta d\mathcal{H}^1(x),
\end{aligned} \tag{5.8}$$

where we have used the representation for a 2-form $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^6)$ as

$$\begin{aligned}
\omega = &\sum_{|\alpha|=2} \omega_{(\alpha_1,\alpha_2)} dx_{\alpha_1} \wedge dx_{\alpha_2} \\
&+ \sum_{|\alpha|=1, |\beta|=1} \big(\omega^z_{\alpha\beta} dx_\alpha \wedge dz_\beta + \omega^w_{\alpha\beta} dx_\alpha \wedge dw_\beta\big) \\
&+ \sum_{|\beta|=2} \big(\omega^{zz}_{(\beta_1,\beta_2)} dz_{\beta_1} \wedge dz_{\beta_2} + \omega^{zw}_{(\beta_1,\beta_2)} dz_{\beta_1} \wedge dw_{\beta_2} + \omega^{ww}_{(\beta_1,\beta_2)} dw_{\beta_1} \wedge dw_{\beta_2}\big).
\end{aligned} \tag{5.9}$$

The coordinates (x_1, x_2, x_3) are still chosen in such a way that x_3 is the tangent component to the curve C.

In order to simplify notation let us set $(z_1, w_1, z_2, w_2, z_3, w_3) = (\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22}, \zeta_{31}, \zeta_{32})$. In the spirit of (5.4), for every 3-multiindex $\gamma \in \{11, 12, 21, 22, 31, 32\}$ we define the 2-form $\omega^0_\gamma \in \mathcal{D}^2(\mathbb{R}^6)$ by

$$\omega^0_\gamma := \frac{1}{3}(\zeta_{\gamma_1} d\zeta_{\gamma_2} \wedge d\zeta_{\gamma_3} - \zeta_{\gamma_2} d\zeta_{\gamma_1} \wedge d\zeta_{\gamma_3} + \zeta_{\gamma_3} d\zeta_{\gamma_1} \wedge d\zeta_{\gamma_2}), \tag{5.10}$$

and therefore we can define the distributional determinant of $\hat{w}$ as the family of twenty distributions

$$\langle \operatorname{Det}_\gamma(D\hat{w}), f\rangle = -\partial\mathcal{G}_{\hat{w}}(\omega) + \int_\Omega M^{\{1,2,3\}}_\gamma(D\hat{w}) f dx, \tag{5.11}$$

for all $f \in C_c^\infty(\Omega)$, where ω is the form $f\omega_\gamma^0 \in \mathcal{D}^2(\Omega \times \mathbb{R}^6)$. Let us point out that, after the change of variables (5.5), $D\hat{w}$ reads

$$D\hat{w} = \begin{pmatrix} -\sin w_1 \frac{\partial w_1}{\partial x_1} & -\sin w_1 \frac{\partial w_1}{\partial x_2} & -\sin w_1 \frac{\partial w_1}{\partial x_3} \\ \cos w_1 \frac{\partial w_1}{\partial x_1} & \cos w_1 \frac{\partial w_1}{\partial x_2} & \cos w_1 \frac{\partial w_1}{\partial x_3} \\ -\sin w_2 \frac{\partial w_2}{\partial x_1} & -\sin w_2 \frac{\partial w_2}{\partial x_2} & -\sin w_2 \frac{\partial w_2}{\partial x_3} \\ \cos w_2 \frac{\partial w_2}{\partial x_1} & \cos w_2 \frac{\partial w_2}{\partial x_2} & \cos w_2 \frac{\partial w_2}{\partial x_3} \\ -\sin w_3 \frac{\partial w_3}{\partial x_1} & -\sin w_3 \frac{\partial w_3}{\partial x_2} & -\sin w_3 \frac{\partial w_3}{\partial x_3} \\ \cos w_3 \frac{\partial w_3}{\partial x_1} & \cos w_3 \frac{\partial w_3}{\partial x_2} & \cos w_3 \frac{\partial w_3}{\partial x_3} \end{pmatrix}, \tag{5.12}$$

which is a 6×3 matrix, and the distributions above correspond to all the twenty weak 3×3-determinants. Note that $\hat{w}_i D\hat{w}_i \in L^1(\Omega)$ and hence the distributional cofactor and determinant are well defined.

Substituting in (5.11) the expression (5.8) we obtain

$$\begin{aligned} &\langle \operatorname{Det}_\gamma(D\hat{w}), f\rangle = \int_\Omega M_\gamma^{\{1,2,3\}}(D\hat{w}) f(x)dx \\ &- \frac{\epsilon_{ijk}}{3} \int_C \int_0^{2\pi} f(x) \sin(h_j)\sin(h_k)\cos(h_i)(\theta, x) \frac{\partial v_k}{\partial \tau}(x) \frac{b_j}{2\pi} d\theta d\mathcal{H}^1(x), \end{aligned} \tag{5.13}$$

if $\gamma = (11, 21, 31)$, and, for $\gamma = (11, 22, 31)$,

$$\begin{aligned} &\langle \operatorname{Det}_\gamma(D\hat{w}), f\rangle = \int_\Omega M_\gamma^{\{1,2,3\}}(D\hat{w}) f(x)dx \\ &+ \sum_{\sigma \in A_3} \frac{(-1)^{\sigma_2}}{3} \int_C \int_0^{2\pi} f \sin(h_{\sigma_2})\cos(h_{\sigma_3})\cos(h_{\sigma_1})\Big(\frac{\partial v_{\sigma_3}}{\partial \tau}\frac{b_{\sigma_2}}{2\pi} - \frac{\partial v_{\sigma_2}}{\partial \tau}\frac{b_{\sigma_3}}{2\pi}\Big) d\theta d\mathcal{H}^1(x), \end{aligned} \tag{5.14}$$

where the sum is computed on the cyclic permutations of $\{1, 2, 3\}$. For the other values of γ there are similar expressions, but we observe that $\operatorname{Det}_\gamma \neq 0$ only if $\gamma_1 \in \{11, 12\}$, $\gamma_2 \in \{21, 22\}$, $\gamma_3 \in \{31, 32\}$. In particular the presence of sin or cos depends on the second digit of γ_i, actually sin for digit 1 and cos for digit 2. Since the function f does not depend on the angle variable θ, it turns out that the singular part (the integral on C) of expressions (5.13) and (5.14) are null, since integration of sin and cos on the interval $[0, 2\pi]$ is always zero (except in the case $b_1 = b_2 = b_3 = 0$, but then the second factor in the integrals vanishes identically). Note also that, denoting by $\operatorname{Det}_\gamma^a$ the absolutely continuous part of $\operatorname{Det}_\gamma$, one has $\operatorname{Det}_\gamma^a = \operatorname{Det}_\gamma = M_\gamma^{\{1,2,3\}}$ and there holds

$$\Big(\sum_\gamma \operatorname{Det}_\gamma(D\hat{w})^2\Big)^{\frac{1}{2}} = \Big(\sum_\gamma M_\gamma^{\{1,2,3\}}(D\hat{w})^2\Big)^{\frac{1}{2}} = |\det(Dw)|, \tag{5.15}$$

where w is the map in (5.7). Thus the vector $(\operatorname{Det}_\gamma)_\gamma$ is in $L^2(\Omega)$.

We can also arguing similarly for the distributional cofactor. Specifically for all $i \in \{1, 2, 3\}$ and all couples $\{\beta_1, \beta_2\} = \beta \subset \{11, 12, 21, 22, 31, 32\}$ we define the distributional cofactor

$$\langle \operatorname{Cof}^i_\beta(D\hat{w}), f\rangle = -\partial\mathcal{G}_{\hat{w}}(\omega) + \int_\Omega M_{\bar{\beta}}^{\bar{i}}(D\hat{w}) f dx, \tag{5.16}$$

for all $f \in C_c^\infty(\Omega)$, where ω is the form $f\zeta_{\beta_1} dx_i \wedge d\zeta_{\beta_2}$. Also in this case the result is nonzero only if $\beta_1^1 \neq \beta_2^1$, where $\beta_1 = \beta_1^1\beta_1^2 \in \{11, 12, 21, 22, 31, 32\}$ and $\beta_2 = \beta_2^1\beta_2^2 \in \{11, 12, 21, 22, 31, 32\}$. Moreover, as for the determinant, it turns out that the concentrated parts of the cofactors vanish, while the absolutely continuous parts satisfy the identity

$$\Big(\sum_{j=1,2,k=1,2} \text{Cof }_{\{\beta_1^1 j,\beta_2^1 k\}}(D\hat{w})^2\Big)^{\frac{1}{2}} = |\,\text{cof }(Dw)_{i\beta^1}| = |M^i_{\bar{\beta}^1}(Dw)|, \tag{5.17}$$

where again $\beta_h = \beta_h^1\beta_h^2 \in \{11,12,21,22,31,32\}$ for $h = 1,2$, $\beta^1 = \{\beta_1^1,\beta_2^1\} \subset \{1,2,3\}$, and w being the map in (5.7).

5.3. **The concentrated part of the Determinant.** There exist other definitions of distributional determinant in literature. For instance, consider the map $\hat{w} : \Omega \to S^1$, assume that $\hat{w} \in W^{1,1}(\Omega, S^1)$ and set

$$j(\hat{w}) := \sum_j (\hat{w}_1 \frac{\partial \hat{w}_2}{\partial x_j} - \hat{w}_2 \frac{\partial \hat{w}_1}{\partial x_j}) dx_j.$$

Then the following distributional Jacobian of $\hat{w}$ is well defined:

$$[J\hat{w}] := \frac{1}{3} dj(\hat{w}), \tag{5.18}$$

(see, e.g., [8], [1] and references therein, see also [14] were a derivation is given with some detail). It is worth quoting the fact ([8, example 5, Section 2]) that, if $\hat{w}$ has winding number k around a fixed curve C (i.e., if any small loop around C such that its image by $\hat{w}$ has homotopy class k in S^1), it holds

$$[J\hat{w}] = 2k\pi\mathcal{L},$$

where $\mathcal{L}$ is current supported by the dislocation set C (seen as a measure). Actually, this can be seen from the fact that the Curl is identified with the external derivative, and then the last formula coincides with our condition on the curl. It has been shown in [14, Theorem 6.1] that the deformation tensor might be written componentwise as $F_{ij} = \hat{w}_1^i \partial_j \hat{w}_2^i - \hat{w}_2^i \partial_j \hat{w}_1^i$ for a certain $\hat{w} \in W^{1,1}(\Omega, (S^1)^3)$ in such a way that $j^i(\hat{w}) := (\hat{w}_1^i D\hat{w}_2^i - \hat{w}_2^i D\hat{w}_1^i) \cdot dx$ and $\frac{1}{3} dj^i(\hat{w}) = b_i\mathcal{L}$ where the ith component of the Burgers vector $b_i \in 2\pi\mathbb{Z}$ is the winding number of $\hat{w}^i$ around C multiplied by 2π. Hence, putting together the three components of our displacement field $\hat{w} : \Omega \to \mathbb{T}^3 = (S^1)^3$, we find

$$[J\hat{w}] = b \otimes \mathcal{L}. \tag{5.19}$$

6. A variational model for dislocations at the continuum scale

6.1. **Preliminaries on dislocations at the continuum scale.** A dislocation loop is a simple closed curve C in Ω which has an associated Burgers vector $b \in \mathbb{Z}^3$. The deformation gradient F around C satisfies the condition $-\text{Curl } F = \Lambda^T_{\mathcal{L}^b} := b \otimes \vec{\tau}\mathcal{H}^1 \llcorner_C$, where $\vec{\tau}$ is an oriented tangent vector to C. For any $b \in 2\pi\mathbb{Z}^3$ we call a b-dislocation current a 1-integer multiplicity current $\mathcal{L}^b$ that produces a curl of the deformation gradient given by the *density* $\Lambda_{\mathcal{L}^b}$, hence satisfying

$$\langle \Lambda_{\mathcal{L}^b}, w\rangle = \mathcal{L}^b((wb)^*), \tag{6.1}$$

for every $w \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$, where in the right-hand side $\omega := (wb)^*$ is the covector writing $(wb)^* := w_{kj} b_j dx_k$ (with sums on the repeated indices). Moreover, $\Lambda_{\mathcal{L}^b}$ is a Radon measure as soon as $M(\mathcal{L}^b)$ is finite. In the sequel we will use the following shortcut notation:

$$\Lambda_{\mathcal{L}^b} = \mathcal{L}^b \otimes b = \vec{\tau}^b \otimes b\theta^b\ \mathcal{H}^1 \llcorner L^b. \tag{6.2}$$

Definition 6.1 (Regular dislocation)**.** A *regular dislocation* is a sequence of b-dislocation currents $\mathcal{L} := \{\mathcal{L}^b\}_{b\in\mathcal{B}}$. We associate to each dislocation a *dislocation current*, still denoted by $\mathcal{L}$, and the associated *dislocation density* $\Lambda_{\mathcal{L}}$,

$$\mathcal{L} := \sum_{b\in\mathcal{B}} \mathcal{L}^b, \qquad \Lambda_{\mathcal{L}} := \sum_{b\in\mathcal{B}} \Lambda_{\mathcal{L}^b}. \tag{6.3}$$

With this definition it is possible to model the dislocations with every possible Burgers vector. However, it is possible to split the current $\mathcal{L}$ in the canonical basis of $\mathbb{R}^3$,

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3,$$

in such a way that $\mathcal{L}_i$ has e_i as associated Burgers vector and satisfies

$$\Lambda_{\mathcal{L}} = \Lambda_{\mathcal{L}_1} + \Lambda_{\mathcal{L}_2} + \Lambda_{\mathcal{L}_3} = \sum_{i=1}^{3} \mathcal{L}_i \otimes e_i.$$

Moreover, as proved in [14] one has

$$|\mathcal{L}_i|_\Omega \le C\|\Lambda_{\mathcal{L}}\|_{\mathcal{M}(\Omega)}, \tag{6.4}$$

for some constant C independent of i and Ω.

6.2. **Functional properties.** Let $1 \le p < \infty$ and introduce the vector space of tensor-valued fields

$$\mathcal{BC}^p(\Omega, \mathbb{R}^{3\times 3}) := \{F \in L^p(\Omega, \mathbb{R}^{3\times 3}) \text{ s.t. } \operatorname{Curl} F \in \mathcal{M}_{\rm div}(\bar{\Omega}, \mathbb{R}^{3\times 3})\}, \tag{6.5}$$

which endowed with norm

$$\|F\|_{\mathcal{BC}^p} := \|F\|_p + |\operatorname{Curl} F|(\bar{\Omega}), \tag{6.6}$$

is a Banach space. Let us define

$$L^p_{\rm div}(\Omega, \mathbb{R}^{3\times 3}) \quad := \quad \{F \in L^p(\Omega) \quad \text{s.t.} \quad \operatorname{div} F = 0\},$$

and the space

$$\tilde{\mathcal{V}}^p(\Omega) := \left\{V \in L^p_{\rm div}(\Omega, \mathbb{R}^{3\times 3}) \text{ s.t. } \operatorname{Curl} V \in L^p(\Omega, \mathbb{R}^{3\times 3}), VN = 0 \text{ on } \partial\Omega\right\}. \tag{6.7}$$

In order to discuss minimization problems, the open set Ω is given, and another open set $\hat{\Omega}$ is prescribed such that $\Omega \subset\subset \hat{\Omega}$. We will also assume that both Ω and $\hat{\Omega}$ are simply connected.

Let $1 < p < 2$ and let $F \in \mathcal{BC}^p(\hat{\Omega}, \mathbb{R}^{3\times 3})$ be such that $-\operatorname{Curl} F = b \otimes \mathcal{L}$, with $b \in 2\pi\mathbb{Z}^3$ and $\mathcal{L}$ a 1-integer multiplicity current which is closed and with compact support in $\hat{\Omega}$. The Helmholtz decomposition in $L^p(\hat{\Omega}, \mathbb{R}^3)$ provides $v \in W^{1,p}(\hat{\Omega}, \mathbb{R}^3)$ and $G \in \tilde{\mathcal{V}}^p(\hat{\Omega})$ such that (see [15] for details and further references)

$$F = Dv + \operatorname{Curl} G. \tag{6.8}$$

If we set $V := \operatorname{Curl} G$, then of course $\operatorname{div} V = 0$, while since $-\operatorname{Curl} F = b \otimes \mathcal{L}$, we also have $-\operatorname{Curl} V = b \otimes \mathcal{L}$. Thanks to the decomposition theorem for 1-integer multiplicity currents (Theorem 2.1) we find a sequence of Lipschitz maps

$$f_k : S^1 \to \hat{\Omega} \quad \text{such that} \quad \mathcal{L} = \sum_{k>0} f_{k\sharp}[\![S^1]\!]. \tag{6.9}$$

Let us denote by C_k the closed Lipschitz curves $f_k(S^1)$.

Theorem 6.2. *Let $b \in 2\pi\mathbb{Z}^3$ be fixed, $1 < p < 2$, $\mathcal{L}$ be a closed integral current with compact support in $\hat{\Omega}$, and let $V \in \tilde{\mathcal{V}}^p(\hat{\Omega})$ be such that $-\operatorname{Curl} V = b \otimes \mathcal{L}$ in $\hat{\Omega}$. Then there exists a map $\tilde{u} \in \mathcal{A}_p(\hat{\Omega}, \mathbb{T}^3)$ such that $\nabla\tilde{u} = V$ almost everywhere in $\hat{\Omega}$, and*

$$M(\partial\mathcal{G}_{\tilde{u}}) \le C|\mathcal{L} \otimes b|(\hat{\Omega})(1 + |\mathcal{L} \otimes b|(\hat{\Omega})), \tag{6.10}$$

with $C > 0$ a constant depending only on $\hat{\Omega}$. Moreover $\tilde{u} = u - v$ with $v \in C^1(\bar{\hat{\Omega}}, \mathbb{R}^3)$, $u \in \mathcal{A}_p(\hat{\Omega}, \mathbb{T}^3)$, and

$$\partial \mathcal{G}_u(\omega) = \mathcal{L} \wedge \vec{b}(\omega), \tag{6.11}$$

for all $\omega \in \mathcal{D}^3(\hat{\Omega} \times \mathbb{T}^3)$.

Proof. Let us first assume $p \leq \frac{3}{2}$. To prove the Theorem we will use the fact that the following system

$$\begin{cases} -\operatorname{Curl} U &= \mu & \text{in} & \hat{\Omega} \\ \operatorname{div} U &= 0 & \text{in} & \hat{\Omega} \\ UN &= 0 & \text{on} & \partial\hat{\Omega}, \end{cases} , \tag{6.12}$$

has a unique solution that also satisfies $\|U\|_{L^p} \leq C|\mu|(\hat{\Omega})$, with $C = C(\hat{\Omega})$. This is proved in [15]. Another key fact is the following: if $\hat{\Omega}$ is a bounded open set with smooth boundary, $g \in C^0(\partial\hat{\Omega}, \mathbb{R}^3)$ with $\int_{\partial\hat{\Omega}} gNd\mathcal{H}^2 = 0$, and $v \in C^1(\hat{\Omega}, \mathbb{R}^3)$ is the zero-mean-value solution to

$$\begin{cases} \Delta v &= 0 & \text{in} & \hat{\Omega} \\ \partial_N v &= g & \text{on} & \partial\hat{\Omega}, \end{cases} , \tag{6.13}$$

then $\|v\|_{C^1} \leq C\|g\|_{C^0}$, with $C = C(\hat{\Omega})$.

We use the decomposition (6.9) for $\mathcal{L}$ and we first suppose that the maps f_k are smooth. The general case will follow using an approximation argument and proceding as in the proof of Theorem 4.5. If C_k is a smooth closed curve, we can choose a smooth surface S_k with boundary C_k. Then we set $S := \cup_k S_k$ and $C := \cup_k C_k$, we seek a solution u of (3.1) with these S and C. Let us also set $\hat{S}_n := \cup_{k=0}^n S_k$ and $\hat{C}_n := \cup_{k=0}^n C_k$. For $i = 1, 2, 3$, let u_i^n be the solution of (3.1) with $\hat{S}_n$, $\hat{C}_n$, and b_i. Lemma 3.10 and Remark 3.12 show that the distributional divergence of ∇u^n is zero, while the curl is given by $-b \otimes \sum_{k=0}^n f_{k\sharp}[\![S^1]\!]$. Up to subtracting a constant to u^n, we also suppose it has zero mean value.

By hypotheses it holds $\inf_k d(C_k, \partial\hat{\Omega}) > 0$, and then u^n are of class C^∞ on $\partial\hat{\Omega}$, and their $\mathcal{C}^h$ norms are uniformly bounded with respect to n for all $h > 0$ (taking into account that the set $C = \cup_k C_k$ has finite length, and then $S = \cup_k S_k$ has finite $\mathcal{H}^2$measure). Let v^n be the solution to (6.13) with $g := \partial_N u^n$. From the estimates of this solution we find $\|v^n\|_{C^1} \leq C_1\|\partial_N u^n\|_{C^0} < C_2$, for some constant C_2 independent of n. Setting $\tilde{u}^n := u^n - v^n$, we see that $\nabla\tilde{u}^n$ solves system (6.12) with $\mu = \mu^n := b \otimes \sum_{k=0}^n f_{k\sharp}[\![S^1]\!]$, so that we also have $\|\nabla\tilde{u}^n\|_p \leq |\mu^n|(\hat{\Omega}) < C_3$, with C_3 independent of n. In particular we get $\|u^n\|_{W^{1,p}} \leq \|v^n\|_{W^{1,p}} + \|\tilde{u}^n\|_{W^{1,p}} \leq C$, for a constant $C > 0$ independent of n. Therefore $u^n \rightharpoonup u$ weakly in $W^{1,p}(\hat{\Omega}, \mathbb{R}^3)$, for some $u \in W^{1,p}(\hat{\Omega}, \mathbb{R}^3)$. Similarly $\tilde{u}^n \rightharpoonup \tilde{u}$ and $v^n \rightharpoonup v$ weakly in $W^{1,p}(\hat{\Omega}, \mathbb{R}^3)$, with $u = \tilde{u} + v$. Since the rows of ∇u^n are equal up to a multiplicative factor, we also get that all the minors of u^n are uniformly bounded in L^p. Then, by Theorem 2.3 and Lemma 4.2, u^n weakly converge in $\mathcal{A}_p(\hat{\Omega}, \mathbb{T}^3)$ to u. Moreover Theorem 4.5 implies that for every $n > 0$ equation (4.4) holds for u^n, with $\mathcal{L}$ replaced by $\sum_{k=0}^n f_{k\sharp}[\![S^1]\!]$. Now, Lemma 2.5 implies that $\mathcal{G}_u$ is an integral current whose boundary satisfies

$$\partial\mathcal{G}_u(\omega) = \sum_{k=1}^{\infty} f_{k\sharp}[\![S^1]\!] \wedge \vec{b}(\omega), \tag{6.14}$$

for all $\omega \in \mathcal{D}^3(\hat{\Omega} \times \mathbb{T}^3)$. To conclude the proof it suffices to observe that the maps $\tilde{u}^n$ are smooth in a neighborhood of $\partial\hat{\Omega}$ with $\partial_N\tilde{u}^n$ vanishing, and hence $\partial_N\tilde{u}$ also vanishes, in such a way that $\nabla\tilde{u}$ satisfies (6.12) with $\mu := b \otimes \mathcal{L}$. By the smoothness properties of v^n, it is also true that v satisfies (6.13) with a bounded and smooth $g = \partial_N u$, so it is smooth in $\hat{\Omega}$ and Lemma 4.6 implies (4.17). We now compute g

by using formula (3.10) and $d(C_k, \partial\hat{\Omega}) > 0$, so that the same argument employed in Lemma 3.2 (actually estimates (3.12) and (3.13)) shows that there is a constant $C_4 > 0$ such that $\|g\|_{C^1} \leq C_4 |b \otimes \mathcal{L}|(\hat{\Omega})$, so that the inequality $\|v\|_{C^1} \leq C\|g\|_{C^0}$ together with (4.17) gives (6.10). The thesis is proved when $p \leq \frac{3}{2}$. Assume $V \in L^p(\hat{\Omega}, \mathbb{R}^{3\times3})$ with $\frac{3}{2} < p < 2$. In particular $V \in L^p(\hat{\Omega}, \mathbb{R}^{3\times3})$ with $p \leq \frac{3}{2}$, so the previous argument shows that $V = \nabla\bar{u} = \nabla u - \nabla v$ with $\bar{u}$ satisfying (6.10), u satisfying (6.11), and v of class C^1. In particular $\nabla u \in L^p(\hat{\Omega}, \mathbb{R}^{3\times3})$ with $\frac{3}{2} < p < 2$, and the thesis follows.

□

Remark 6.3. By definition of the u^k, we have observed that for all k the three components u^k_i, $i = 1, 2, 3$, differ by a multiplicative factor. In particular we have seen that their gradients ∇u^k_i (i.e., the rows of the matrix ∇u^k) are linearly dependent. As a consequence the same is true for the gradients ∇u_i. Thus, the three components of the harmonic function v have as boundary data $\partial_N u_i$ three linearly dependent vector fields. This implies, by the uniqueness of solution of elliptic equations, that also ∇v_i are linearly dependent and hence that the final matrix $V = \nabla u = \nabla\tilde{u} - \nabla v$ has linearly dependent rows. Therefore, its pointwise adjunct and determinant are constantly zero.

6.3. **The minimum problem.** Let us recall that $\Omega \subset\subset \hat{\Omega}$, with Ω and $\hat{\Omega}$ simply connected and smooth. We deal with an energy $\mathcal{W}$ with the form

$$\mathcal{W}(F) := \mathcal{W}_e(F, \operatorname{div} F) + \mathcal{W}_{\text{defect}}(\operatorname{Curl} F), \tag{6.15}$$

where we assume the following properties on $\mathcal{W}_e$ and $\mathcal{W}_{\text{defect}}$:

(i) The following coerciveness condition holds: there exists positive constants α_0, α_1, β_0, β_1, and $p, q \geq 1$ such that

$$\mathcal{W}_e(F, \operatorname{div} F) \geq \beta_1(\|F\|^p_{L^p} + \|\operatorname{adj}F\|^p_{L^p} + \|\det F\|^p_{L^p} + \|\operatorname{div} F\|^q_{L^q}) - \beta_0,$$
$$\mathcal{W}_{\text{defect}}(\Lambda) \geq \alpha_1|\Lambda|(\Omega) - \alpha_0.$$

(ii) $\mathcal{W}_{\text{defect}}$ is a function on $\mathcal{M}_b(\Omega, \mathbb{R}^{3\times3})$ which is lower semicontinuous with respect to the weak* convergence.

(iii) $\mathcal{W}_e$ is a function of $M(F)$ (i.e., of F, adj F, and det F) and div F, and it is lower semicontinuous in $M(F)$ with respect to the weak convergence in L^p, and lower semicontinuous in div F with respect to the weak convergence in L^q.

Remark 6.4. In expression (6.15) of the energy, a particular form of gradient elasticity is chosen, in which the rotational part of the strain derivatives are incorporated in the defect contribution of the energy, whereas the strain divergence appears in its elastic part.

Remark 6.5. Hypotheses (i), (ii), and (iii) are readily satisfied in the following standard situations: one might assume $\mathcal{W}_e$ to be polyconvex in $\mathcal{M}(F)$ and div F, with growth conditions as in (i). As for the defect part of the energy we might assume

$$\mathcal{W}_{\text{defect}}(\Lambda) = \int_C \psi(\theta b, \tau) d\mathcal{H}^1(\tau),$$

where $\Lambda = b \otimes \mathcal{L} = b \otimes \theta\tau\mathcal{H}^1 \llcorner C$ is the density of a dislocation with Burgers vector $b = \beta_i e_i, \beta_i \in \mathbb{Z}$ ($b \neq 0$) and supported on C. Under suitable hypotheses on ψ this is seen to satisfy (iii) (see [2]).

We fix a boundary condition α for the dislocation and a map $\hat{F} \in L^p(\hat{\Omega}, \mathbb{R}^{3\times3})$ with $-\operatorname{Curl} \hat{F} = \Lambda^T_\alpha$ on $\hat{\Omega}$, and the additional property that div $\hat{F} \in L^q(\hat{\Omega}, \mathbb{R}^3)$ for some $q > 1$. Let $b \in 2\pi\mathbb{Z}^3$ a fixed Burgers vector, let $p, q > 1$, then we define the class of admissible functions as

$$\mathcal{F}_b^{p,q} := \{F \in L^p(\hat\Omega, \mathbb{R}^{3\times3}) : -\operatorname{div} F \in L^q(\hat\Omega, \mathbb{R}^3),\ -\operatorname{Curl} F = b \otimes \mathcal{L}$$
$$\text{for some closed integral 1-current } \mathcal{L}, \text{ and } F = \hat F \text{ on } \hat\Omega \setminus \Omega\}. \tag{6.16}$$

The existence of a minimizer of $\mathcal{W}$ in $\mathcal{F}_b^{p,q}$ is provided by the following:

Theorem 6.6. *Let $1 < p < 2$ and $q > 3$. If $\mathcal{W}$ satisfies (i), (ii), and (iii), then there exists a minimizer $F \in \mathcal{F}_b^{p,q}$ of $\mathcal{W}$.*

Proof. We will apply the direct method. Let $\{F_k\}_{k>0}$ be a minimizing sequence. From the coerciveness (i) we see that there exist $F \in L^p(\hat\Omega, \mathbb{R}^{3\times3})$, $A \in L^p(\hat\Omega, \mathbb{R}^{3\times3})$, $D \in L^p(\hat\Omega)$, and $R \in L^p(\hat\Omega, \mathbb{R}^3)$ such that

$$F_k \rightharpoonup F \quad \text{weakly in } L^p(\hat\Omega, \mathbb{R}^{3\times3}), \tag{6.17a}$$
$$\operatorname{adj} F_k \rightharpoonup A \quad \text{weakly in } L^p(\hat\Omega, \mathbb{R}^{3\times3}), \tag{6.17b}$$
$$\det F_k \rightharpoonup D \quad \text{weakly in } L^p(\hat\Omega), \tag{6.17c}$$
$$\operatorname{div} F_k \rightharpoonup R \quad \text{weakly in } L^q(\hat\Omega, \mathbb{R}^3). \tag{6.17d}$$

Moreover we find a measure $\Lambda \in \mathcal{M}_b(\hat\Omega, \mathbb{R}^{3\times3})$ with

$$\Lambda_k \rightharpoonup \Lambda \quad \text{weakly* in } \mathcal{M}_b(\hat\Omega, \mathbb{R}^{3\times3}), \tag{6.18}$$

where we have set $\Lambda_k = \Lambda_{\mathcal{L}_k} = -(\operatorname{Curl} F_k)^T$. As [14, Lemma 7.5] shows, there exists a regular dislocation current $\mathcal{L}$ such that $(\mathcal{L}_k)_i \rightharpoonup \mathcal{L}_i$ in $\mathcal{D}_1(\hat\Omega)$ and $\Lambda = \Lambda_{\mathcal{L}} = \sum_{i=1}^3 \mathcal{L}_i \otimes e_i$. In order to prove the Theorem we have to show that $\operatorname{div} F = R$, $-\operatorname{Curl} F = \Lambda_{\mathcal{L}}$, $A = \operatorname{adj} F$, and $D = \det F$.

The Helmholtz decomposition gives

$$F_k = Dw_k + \operatorname{Curl} G_k, \tag{6.19}$$

with $w_k \in W^{1,p}(\hat\Omega, \mathbb{R}^3)$ satisfying $-\Delta w_k = -\operatorname{div} F_k$ with $\partial_N w_k = F_k N = \hat F N$ on $\partial\hat\Omega$[3], and $G_k \in \tilde{\mathcal{V}}^p(\hat\Omega)$. Since $\operatorname{div} F_k \in L^q(\hat\Omega, \mathbb{R}^3)$, with $q > 3$, by the regularity theory of elliptic problems and the Sobolev embedding Theorem, we find that $w_k \in C^1(\hat\Omega, \mathbb{R}^{3\times3})$ and that the L^∞ norm of their gradients are bounded by a constant,

$$\|Dw_k\|_\infty < C. \tag{6.20}$$

Moreover we have, up to a subsequence, that

$$w_k \rightharpoonup w \quad \text{weakly in } W^{1,q}(\hat\Omega, \mathbb{R}^3), \tag{6.21}$$

for some $w \in W^{1,q}(\hat\Omega, \mathbb{R}^3)$.

Let us set $V_k := \operatorname{Curl} G_k$. Now $-\operatorname{Curl} V_k = \Lambda_{\mathcal{L}_k}$, and Theorem 6.2 provides functions $u_k \in \mathcal{A}_p(\hat\Omega, \mathbb{T}^3)$ and $v_k \in C^1(\hat\Omega, \mathbb{T}^3)$ such that $\nabla u_k - \nabla v_k = V_k$ satisfying

$$\partial \mathcal{G}_{u_k}(\omega) = \mathcal{L}_k \wedge \vec b(\omega), \tag{6.22}$$

for all $\omega \in \mathcal{D}^3(\hat\Omega \times \mathbb{T}^3)$, and

$$\|Dv_k\|_\infty \le C|b \otimes \mathcal{L}_k|(\hat\Omega). \tag{6.23}$$

Thanks to (6.17a), (6.21), and (6.23), we can assume that there exist $u \in W^{1,p}(\hat\Omega, \mathbb{T}^3)$ and $v \in W^{1,p}(\hat\Omega, \mathbb{R}^{3\times3})$ such that $u_k \to u$ and $v_k \to v$ strongly in $L^p(\hat\Omega, \mathbb{R}^3)$,

$$\nabla u_k \rightharpoonup \nabla u \quad \text{weakly in } L^p(\hat\Omega, \mathbb{R}^{3\times3}). \tag{6.24}$$

[3]Note that $F_k N$ and $\hat F N$ have a distributional meaning, the divergences of F_k and $\hat F$ being in L^p, see [15].

and

$$\nabla v_k \rightharpoonup \nabla v \quad \text{weakly in } L^p(\hat\Omega, \mathbb{R}^{3\times 3}). \tag{6.25}$$

Thanks to estimates (6.20) and (6.23), Lemma 4.6 applies providing

$$M(\partial \mathcal{G}_{w_k+u_k-v_k}) \le C(1+|\Lambda_{\mathcal{L}_k}|(\hat\Omega))|\Lambda_{\mathcal{L}_k}|(\hat\Omega) < C. \tag{6.26}$$

This allows us to apply Theorem 2.3, obtaining

$$\text{adj}(D\psi_k) \rightharpoonup \text{adj}(D\psi) \quad \text{weakly in } L^p(\hat\Omega, \mathbb{R}^{3\times 3}), \tag{6.27}$$

$$\det(D\psi_k) \rightharpoonup \det(D\psi) \quad \text{weakly in } L^p(\hat\Omega), \tag{6.28}$$

with $\psi_k := w_k + u_k - v_k$ and $\psi := w + u - v$. As a consequence of (6.17a), convergences (6.27) and (6.28) read $\text{adj}F_k \rightharpoonup \text{adj}F$ weakly in $L^p(\hat\Omega, \mathbb{R}^{3\times 3})$ and $\det F_k \rightharpoonup \det F$ weakly in $L^p(\hat\Omega, \mathbb{R}^3)$. Therefore $A = \text{adj}F$ by (6.17b), and $\det F = D$ from (6.17c). Moreover, for every test function $\varphi \in C^\infty_c(\hat\Omega, \mathbb{R}^3)$ we have

$$\langle F, \nabla\varphi\rangle = \langle \nabla w, \nabla \varphi\rangle = \lim_{k\to\infty} \langle \nabla w_k, \nabla\varphi\rangle = \lim_{k\to\infty} \langle \text{ div } F_k, \nabla\varphi\rangle,$$

and from (6.17d) it follows $R = \text{ div } F$. Finally we write

$$\langle \text{ Curl } F_k, \varphi\rangle = \langle F_k, \text{ Curl } \varphi\rangle \to \langle F, \text{ Curl } \varphi\rangle = \langle \text{ Curl } F, \varphi\rangle,$$

for all $\varphi \in \mathcal{D}(\hat\Omega, \mathbb{R}^{3\times 3})$, and by (6.18) we conclude $-\text{ Curl } F = \Lambda^T_{\mathcal{L}}$ achieving the proof. □

6.4. **A displacement-based minimum problem.** In view of the definition of the map H in (5.5), we can associate to any F a function $\hat w \in W^{1,1}(\Omega, (S^1)^3)$ in such a way that $\hat w_1 D\hat w_2 - \hat w_2 D\hat w_1 = F$ (here $\hat w_1 = \hat w_{\gamma_1}$ and $\hat w_2 = \hat w_{\gamma_2}$ with the triples γ_i being $\gamma_1 = (1,3,5)$ and $\gamma_2 = (2,4,6)$ respectively, consistently with (5.5) and (5.12)). Thus we recover the existence of a minimizer for the following problem:

$$\inf \tilde{\mathcal{W}}(\hat w), \tag{6.29}$$

where the infimum is computed on the class of all functions $\hat w \in W^{1,1}(\Omega, (S^1)^3)$ satisfying some fixed boundary datum, and under the following assumptions on the energy.

(a) $\mathcal{W}_e$ is a function of all the minors $M^\beta_{\bar\alpha}(D\hat w)$ and of $\text{div } (\hat w_1 D\hat w_2 - \hat w_2 D\hat w_1) = \text{div } F$, with some growth and lower semicontinuity conditions like in (i) and (iii).

(b) $\mathcal{W}_{\text{defect}}(\hat w) := \int_\Omega J[\hat w]$.

Moreover, thanks to (5.15), (5.17), we see that the dependence of $\mathcal{W}_e$ on the 3×3 determinants of $D\hat w$ or cofactors can be weakened to the dependence on their distributional counterparts.

Specifically, we set $\text{Cof } D\hat w = (\text{ Cof }_\beta D\hat w)_\beta$ be the vector of all distributional cofactors (and thus β varies among the 2-multiindex with entries in $\{11, 12, 21, 22, 31, 32\}$) and similarly $\text{Det } D\hat w = (\text{ Det }_\gamma D\hat w)_\gamma$, with γ be a 3-multiindex. Thanks to (5.15) and (5.17) it turns out that $\|\text{ Det } D\hat w\|_p = \|\det Dw\|_p$ and $\|\text{ Cof } D\hat w\|_p = \|\text{ cof } Dw\|_p$, so that $\text{Det } D\hat w,\ \text{Cof } D\hat w \in L^p$. Hence we might assume

(a') $\mathcal{W}_e(D\hat w) = \int_\Omega g(D\hat w, \text{ Cof } D\hat w, \text{ Det } D\hat w, \text{ Div } F)$, with g be a convex function with the coerciveness property:

$$g(D\hat w, \text{ Cof } D\hat w, \text{ Det } D\hat w, \text{ Div } F) \ge \|D\hat w\|^p_p + \|\text{ Cof } D\hat w\|^p_p + \|\text{ Det } D\hat w\|^p_p + \|\text{ Div } F\|^q_q.$$

Moreover the dependence on $\text{div } F$ might also be generalized to a dependence on all the terms $(\text{ div } \hat w_i D\hat w_j)_{ij}$, provided the growth condition in (a') is satisfied. Under hypotheses (a') and (b) the minimum problem above can be reconducted to the one in the previous section, and thus we entail existence of minimizers $\hat w \in W^{1,1}(\Omega, (S^1)^3)$.

7. Concluding remarks

This paper is a direct follow up and a generalization of the first work of the authors on dislocations [14]. Whereas in the latter work, the aim was to provide an existence results for the most general continuum dislocation, in the present work our aim was to avoid the necessity of a control on the curve in the energy. As a consequence a more general result on the boundedness of the graph of the displacement field is used, i.e., with no need to appeal to Cartesian maps as in [12,14]. To proceed in this way, our first task has been to study harmonic maps with jump on the surface, which in the application happens to be the surface enclosed by the dislocation. Our second task has been to introduce torus-valued maps, to take into account the intrinsic displacement field multivaluedness. Our main result is an explicit expression of the boundary of the graph of the displacement, which shows to be written in function of the dislocation density. This expression was used in the last section to prove an existence result for one single dislocation loop, which can show complex geometric structures, but which must be associated to a single Burgers vector. To achieve this aim, it has been necessary to introduce an energy control of the deformation divergence. An intermediate work of the authors [15] was also required to elaborate complete proofs.

The plan for future works is to exploit the explicit expressions of the distributional determinants and adjuncts to prove an existence result where our hope is to get rid of, or at least to elucidate, the divergence term. Let us nonetheless observe that such a term is physically justified, as found in higher-gradient-elasticity models. It turns out to be related to the boundedness of the applied volume force.

Appendix A. Expression of the boundary graph for $\mathbb{R}^3$-valued displacement

A.1. **Motivation.** Recall that the dislocated solid is separated into two elastic half-spaces joined by atomic-level forces across their common interface, known as the glide plane.

So far, the energy was made of two terms, the elastic part and the defect part which was assumed as concentrated in C. With a view to time evolution of dislocation, an important physical term should also be added on the glide plane S, which is prescribed in single crystals at moderate temperature ranges, the so-called "misfit" energy taking into account nonlinear atomic interactions, viz.,

$$\mathcal{W}_{\rm mis} := \int_S W_{\rm mis}(\delta(x))dS(x)$$

where $0 < \delta(x) < |b|$ and $W_{\rm mis}$ is the interplanar potential energy density. This problem was originately introduced and solved by Nabarro [11], and received considerable attention nowadays, since it permits a reasonable alternative to atomic models.

In this paper the main results are presented without referring to the surface S (in the final formulae), since torus-valued maps are considered. Nonetheless, the formulae in the Euclidean setting might be of interest with a view to the introduction of misfit energy terms as explained above. The derivation of such formulae is rather technical, though they do not require new notions with respect to those exposed in the paper core. This is the reason why they are given in this appendix.

A.2. **Main result second form.** Let us introduce the following notation. For all $b \in \mathbb{R}^3$ and all $r \in \mathbb{R}^3$ we define the 1-current $\vec{b}_r \in \mathcal{D}_1(\mathbb{R}^3)$ as

$$\vec{b}_r(\omega) := -\frac{1}{2\pi}\int_0^{2\pi} \langle \omega(r_1 + \frac{b_1\theta}{2\pi}, r_2 + \frac{b_2\theta}{2\pi}, r_3 + \frac{b_3\theta}{2\pi}), b\rangle d\theta, \tag{A.1}$$

for any 1-form $\omega \in \mathcal{D}^1(\mathbb{R}^3)$. Moreover $M(\vec{b}_r) = 2\pi|b|$ for all $r \in \mathbb{R}^3$.

Let τ be an arc length parameter along the curve C. As in Lemma 3.6 we will denote by $u(\tau)$ the value of the limit as $\epsilon \to 0$ of $u(\epsilon, \theta = 0, \tau)$, with $\{\epsilon, \theta, \tau\}$ a system of cylindrical coordinates around C taken in such a way that $\theta = 0$ corresponds to the set of points in the surface S, so that the lower and upper traces of u on S can be denoted by $u(\epsilon, 0, \tau)$ and $u(\epsilon, 2\pi, \tau)$, respectively (at least in a small neighborhood of C). Since the jump of u is the constant b on S we can assume $u(\epsilon, 2\pi, \tau) = b + u(\epsilon, 0, \tau)$. With this convention $u(\tau) := \lim_{\epsilon\to 0} u(\epsilon, 0, \tau)$.

We have denoted by $\mathcal{L} \wedge \vec{b}_{u(\tau)}$ the 2-current in $\Omega \times \mathbb{R}^3$ defined as

$$\mathcal{L} \wedge \vec{b}_u(\omega) = -\frac{1}{2\pi}\int_C \int_0^{2\pi} \langle \omega(x, u(\tau) + \frac{b\theta}{2\pi}), \bar{\tau} \wedge \bar{b}\rangle d\theta d\mathcal{H}^1(x), \tag{A.2}$$

for any 2-form $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^3)$.

Theorem A.1. *Let C be a smooth closed curve in Ω and let S be a smooth surface with $\partial S = C$. Let $b \in \mathbb{R}^3$ and let $u_i \in SBV(\Omega)$ be the solution to (3.1) with $b = b_i$ for $i = 1, 2, 3$. Let $v \in C^1(\bar{\Omega}, \mathbb{R}^3)$ and set $u = (u_1, u_2, u_3)$, and $w := u + v$. Then $\mathcal{G}_u$ and $\mathcal{G}_w$ are integral currents in $\mathcal{D}_3(\Omega \times \mathbb{R}^3)$ and it holds*

$$\begin{aligned}\partial\mathcal{G}_u(\omega) =& \mathcal{L} \wedge \vec{b}_u(\omega) + \int_S (\omega_{12}(x, u(x) + b) - \omega_{12}(x, u(x)))d\mathcal{H}^2(x)\\ &+ \sum_{k=4}^{6}\int_S b_{k-3}(\omega_{1k}(x, u(x) + b) - \omega_{1k}(x, u(x)))\frac{\partial \hat{u}}{\partial \tau_2} d\mathcal{H}^2(x)\\ &- \sum_{k=4}^{6}\int_S b_{k-3}(\omega_{2k}(x, u(x) + b) - \omega_{2k}(x, u(x)))\frac{\partial \hat{u}}{\partial \tau_1} d\mathcal{H}^2,\end{aligned} \tag{A.3}$$

for all $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^3)$, where $\{\tau_1, \tau_2\}$ is an orthogonal basis for S, and $\hat{u}$ is the solution to (3.1) with $b = 1$. In the previous formula u and $u + b$ are the two traces of u on the lower and upper face of S. Moreover

$$\begin{aligned}\partial\mathcal{G}_w(\omega) =& \sum_{4\le i<j\le 6}\int_S (\omega_{ij}(x, w(x) + b) - \omega_{ij}(x, w(x)))(\frac{\partial w_{i-3}}{\partial \tau_1}\frac{\partial w_{j-3}}{\partial \tau_2} - \frac{\partial w_{j-3}}{\partial \tau_1}\frac{\partial w_{i-3}}{\partial \tau_2})d\mathcal{H}^2(x)\\ &+ \int_S (\omega_{12}(x, w(x) + b) - \omega_{12}(x, w(x)))d\mathcal{H}^2(x)\\ &+ \sum_{k=4}^{6}\int_S (\omega_{1k}(x, w(x) + b) - \omega_{1k}(x, w(x)))\frac{\partial w_{k-3}}{\partial \tau_2} d\mathcal{H}^2(x)\\ &- \sum_{k=4}^{6}\int_S (\omega_{2k}(x, w(x) + b) - \omega_{2k}(x, w(x)))\frac{\partial w_{k-3}}{\partial \tau_1} d\mathcal{H}^2(x)\\ &+ \mathcal{L} \wedge \vec{b}_w(\omega) + \mathcal{C}_w(\omega),\end{aligned} \tag{A.4}$$

for all $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^3)$, where $\mathcal{C}_w(\omega)$ is a rectifiable 2-current with support in $C \times \mathbb{R}^3$ given by

$$\mathcal{C}_w(\omega) := -\frac{1}{2\pi}\int_C \int_0^{2\pi} \langle \omega(x, \frac{b\theta}{2\pi} + w(x)), \frac{\partial \bar{v}}{\partial \tau}(x) \wedge \bar{b}\rangle d\theta d\mathcal{H}^1(x), \tag{A.5}$$

with $\bar{v} = (0, v)$ *and for all* $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^3)$*. Moreover*

$$M(\mathcal{C}_w) \leq |\mathcal{L} \otimes b|(\Omega) \|Dv\|_\infty .$$

In particular there exists a constant $c > 0$ *depending only on the geometry of* C *such that*

$$M(\partial \mathcal{G}_u) \leq |\mathcal{L} \otimes b|(\Omega) + c\mathcal{H}^2(S), \tag{A.6}$$

and

$$M(\partial \mathcal{G}_w) \leq (1 + \|Dv\|_\infty)\big(|\mathcal{L} \otimes b|(\Omega) + c\mathcal{H}^2(S)\big) + \mathcal{H}^2(S)\|Dv\|_\infty^2 . \tag{A.7}$$

Proof. We will follow the lines of the proof of Theorem 4.1. We consider a smooth tubular neighborhood D_ϵ of the curve C, and a neighborhood U_δ, of widht 2δ, of the surface S. Then we consider the currents $\partial \mathcal{G}_{u^{\epsilon,\delta}}$ in $\Omega \times \mathbb{R}^3$, with $u^{\epsilon,\delta} := u \llcorner_{\Omega \setminus (D_\epsilon \cup U_\delta)}$ and let first $\delta \to 0$, and then $\epsilon \to 0$.

To simplify the notation, we denote by ∂D_ϵ only the part of the boundary of D_ϵ that does not belong to U_δ, and similarly ∂U_δ is the boundary of U_δ which does not belong to D_ϵ. Therefore, for all $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^3)$ it holds

$$\begin{aligned} \partial \mathcal{G}_{u^{\epsilon,\delta}}(\omega) = & \int_{\partial D_\epsilon} \sigma(i, \bar{i}) \omega_{ij}(x, u(x)) \tilde{M}_i^j (D(\Phi \llcorner_{\partial D_\epsilon}(x))) d\mathcal{H}^2(x) \\ & + \int_{\partial U_\delta} \sigma(i, \bar{i}) \omega_{ij}(x, u(x)) \tilde{M}_i^j (D(\Phi \llcorner_{\partial U_\delta}(x))) d\mathcal{H}^2, \end{aligned} \tag{A.8}$$

where $\Phi := Id \times u$. Let us study the second term of the last expression. Using the coordinates $\{\tau_1, \tau_2\}$ on S and denoting by N the unit normal to S, this can be written as

$$\begin{aligned} & \int_{\partial S} \omega_{12}(x + \delta N, u(x + \delta N)) + \sum_{k=4}^{6} \omega_{1k}(x + \delta N, u(x + \delta N)) \frac{\partial u_{k-3}}{\partial \tau_2}(x + \delta N) d\mathcal{H}^2(x) \\ & - \int_{\partial S} \sum_{k=4}^{6} \omega_{2k}(x + \delta N, u(x + \delta N)) \frac{\partial u_{k-3}}{\partial \tau_1}(x + \delta N) d\mathcal{H}^2(x) \\ & - \int_{\partial S} \omega_{12}(x - \delta N, u(x - \delta N)) - \sum_{k=4}^{6} \omega_{1k}(x - \delta N, u(x - \delta N)) \frac{\partial u_{k-3}}{\partial \tau_2}(x - \delta N) d\mathcal{H}^2(x) \\ & + \int_{\partial S} \sum_{k=4}^{6} \omega_{2k}(x - \delta N, u(x - \delta N)) \frac{\partial u_{k-3}}{\partial \tau_1}(x - \delta N) d\mathcal{H}^2(x). \end{aligned} \tag{A.9}$$

Using the fact that $d(\partial S_\delta, C) \geq \epsilon$, we see that such a current, thanks to (3.11), has uniformly bounded mass for all δ, so that the currents $\mathcal{G}_{u^{\epsilon,\delta}}$ converge to $\mathcal{G}_{u^{\epsilon,\delta=0}}$ in the sense of currents. Letting $\delta \to 0$ in (A.9), taking into account that $u_k = b_k \hat{u}$, we obtain exactly the second, third, and fourth terms in (A.3), with the only exception that we are integrating on $S \setminus D_\epsilon$, viz.,

$$\begin{aligned} \partial \mathcal{G}_{u^{\epsilon,0}}(\omega) = & \int_{\partial D_\epsilon} \sigma(i, \bar{i}) \omega_{ij}(x, u(x)) \tilde{M}_i^j (D(\Phi \llcorner_{\partial D_\epsilon}(x))) d\mathcal{H}^2(x) \\ & + \int_{S \setminus D_\epsilon} (\omega_{12}(x, u(x) + b) - \omega_{12}(x, u(x))) d\mathcal{H}^2(x) \\ & + \sum_{k=4}^{6} \int_{S \setminus D_\epsilon} b_{k-3}(\omega_{1k}(x, u(x) + b) - \omega_{1k}(x, u(x))) \frac{\partial \hat{u}}{\partial \tau_2} d\mathcal{H}^2(x) \\ & - \sum_{k=4}^{6} \int_{S \setminus D_\epsilon} b_{k-3}(\omega_{2k}(x, u(x) + b) - \omega_{2k}(x, u(x))) \frac{\partial \hat{u}}{\partial \tau_1} d\mathcal{H}^2(x). \end{aligned} \tag{A.10}$$

Now, we can explicitly write the first integral as in (4.8), i.e.,

$$\int_{\partial D_\epsilon} \tilde{\omega}_{23}(x,u(x))d\mathcal{H}^2 - \sum_{k=4}^{6}\int_0^{2\pi}\int_0^l \epsilon\tilde{\omega}_{2k}(\epsilon,\theta,\tau,u(\epsilon,\theta,\tau))\frac{\partial u_{k-3}}{\partial\tau}(\epsilon,\theta,\tau)d\tau d\theta$$
$$-\sum_{k=4}^{6}\int_0^l\int_0^{2\pi}\epsilon\tilde{\omega}_{3k}(\epsilon,\theta,\tau,u(\epsilon,\theta,\tau))\frac{\partial u_{k-3}}{\partial\sigma}(\epsilon,\theta,\tau))d\theta d\tau,$$

and following the same computations as in the proof of Theorem 4.1, neglecting the vanishing terms, we obtain the following expression

$$-\sum_{k=4}^{6}\int_0^l \omega_{3k}(\hat{\tau},u^+(\tau))u_{k-3}(\epsilon,\theta,\tau)\Big|_0^{\theta=2\pi}d\tau$$
$$+\sum_{k=4}^{6}\int_0^l\int_0^{2\pi}\frac{d}{d\theta}\omega_{3k}(\hat{\tau},u^+(\tau))u_{k-3}(\epsilon,\theta,\tau))d\theta d\tau, \tag{A.11}$$

with $\hat{\tau}=(0,0,\tau)$. Letting ϵ go to 0 and integrating by parts in θ as in (4.11), we obtain

$$-\sum_{k=4}^{6}\int_0^l\int_0^{2\pi}\omega_{3k}(\hat{\tau},u^+(\tau)+\frac{b}{2\pi}\theta)\frac{b_{k-3}}{2\pi}d\theta d\tau, \tag{A.12}$$

where we have used that $\lim_{\epsilon\to 0}u(\epsilon,\theta,\tau)=u^+(\tau)=u^+(\tau)+\frac{b\theta}{2\pi}$. This last expression is exactly the first term of (A.3). To see that the second, third, and fourth terms of (A.10) tend to the correspondent terms in (A.3), it suffices to observe that the total mass of the currents represented by them are bounded (uniformly with respect to ϵ) thanks to the estimates (ii) and (iii) of Lemma 3.6, so that $\mathcal{G}_{u^{\epsilon,0}}$ converges to $\mathcal{G}_u$ in the sense of currents and we are done.

It remains to prove (A.4). We argue as in the proof of Theorem 4.6. The previous computations applied to the function w gives rise to a formula like (A.10) for the part of the boundary on S plus the term contating the 2×2 determinants of Dw. These terms converge to the first four lines of (A.4). As for the part of the boundary on C we have an approximation as in (4.18), where the first row can be treated as in (A.11). This gives rise to $\mathcal{L}\wedge\vec{b}_w(\omega)$. The other non-vanishing terms are given by the expression (4.19), with the same notation. In constrast we have that ω is not anymore periodic. Then arguing as in (4.19) we find the final expression

$$-\sum_{4\le i\neq j\le 6}\int_0^l\int_0^{2\pi}\omega_{ij}(\hat{\tau},\frac{b\theta}{2\pi}+w(\hat{\tau}))\frac{\partial v_{j-3}}{\partial\tau}(\hat{\tau})\frac{b_{i-3}}{2\pi}d\theta d\tau$$
$$=\mathcal{C}_w(\omega),$$

with $\hat{\tau}=(0,0,\tau)$ as expressed in the local basis. Now $\mathcal{C}_w$ can be estimated as in Theorem 4.6. The estimates (A.6) and (A.7) are straightforward consequences of (A.3) and (A.4). □

Appendix B. Generalized distributional determinant and cofactor

Since the functions considered in Theorem A.1 are not Sobolev but only belong to SBV, it turns out that formulae (5.3) and (5.4), if applied to this case, must be considered as definitions of distributional cofactor and determinants, in order to generalize the classical definitions for Sobolev maps. The main difference relies in the fact that, in the case of the cofactor, the derivatives inside the brackets in the right-hand side of (5.1) are not intended in the sense of distribution, but are here considered as their absolutely continuous part.

Explicitly writing (5.3) and (5.4) for the map w of Theorem A.1, the cofactor reads

$$\begin{aligned}\langle \text{ Cof } (Dw)_{ij}, f\rangle =& \int_\Omega f \text{ cof } (Dw)_{ij} dx - \int_C b_{i+2} f(x)(w_{i+1}(x) + 2b_{i+1})\tau_j d\mathcal{H}^1(x)\\ &+ \int_S b_{i+1}b_{i+2} f(x)(\tau_j^1 \frac{\partial \hat{u}}{\partial \tau^2}(x) - \tau_j^2 \frac{\partial \hat{u}}{\partial \tau^1}(x))d\mathcal{H}^2(x)\\ &+ \int_S b_{i+1} f(x)(\tau_j^1 \frac{\partial v_{i+2}}{\partial \tau^2}(x) - \tau_j^2 \frac{\partial v_{i+2}}{\partial \tau^1}(x))d\mathcal{H}^2(x), \end{aligned} \tag{B.1}$$

with $\{\tau^1, \tau^2\}$ an orthonormal basis of the tangent space to S, and τ the oriented tangent vector to C (indices are considered mod 3). For the sake of clearness, the terms in the second and third lines of (B.1) is computed by substraction of the third and fourth lines of (A.4), since $\omega_{1k}(x, w) = f w_{k+1} dx_1 \wedge dy_{k+2}$, by $i = k + 3$ and $dx_j = \tau_j^1 d\tau^1 + \tau_j^2 d\tau^2$, and recalling that $w = b\hat{u} + v$. Moreover the second term in the RHS of the first line of (B.1) is exactly the first term of the fifth line of (A.4). Note that the last term of (A.4) vanishes for this particular ω as referring to its expression in the last line of (4.18).

Note that (B.1) rewrites in compact form as (recalling the identity $\varepsilon_{ljk}\varepsilon_{lmn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk}$ and since $N_l = \varepsilon_{lmn}\tau_m^1\tau_n^2$)

$$\begin{aligned}\langle \text{ Cof } (Dw)_{ij}, f\rangle =& \int_\Omega f \text{ cof } (Dw)_{ij} dx - \int_C b_{i+2} f(x)(w_{i+1}(x) + 2b_{i+1})\tau_j(x) d\mathcal{H}^1(x)\\ &+ \int_S b_{i+1} f(x)\varepsilon_{ljk}\varepsilon_{lmn}\tau_m^1\tau_n^2 \partial_k w_{i+2}(x) d\mathcal{H}^2(x)\\ =& \int_\Omega f \text{ cof } (Dw)_{ij} dx - \int_C b_{i+2} f(x)(w_{i+1}(x) + 2b_{i+1})\tau_j(x) d\mathcal{H}^1(x)\\ &+ \int_S b_{i+1} f(x)\varepsilon_{lpk}\partial_k \left(w_{i+2}(x)\delta_{jp}\right) N_l(x) d\mathcal{H}^2(x) \end{aligned} \tag{B.2}$$

$$\begin{aligned} =& \int_\Omega f \text{ cof } (Dw)_{ij} dx - 2\int_C b_{i+2} f(x) b_{i+1}\tau_j(x) d\mathcal{H}^1(x)\\ &+ \int_C f(x)\left(b_{i+1}v_{i+2}(x) - b_{i+2}v_{i+1}(x)\right)\tau_j(x) d\mathcal{H}^1(x)\\ &+ \int_S b_{i+1} f(x)\varepsilon_{ljk}\partial_k v_{i+2}(x) N_l(x) d\mathcal{H}^2(x) \end{aligned} \tag{B.3}$$

where S is a level set for $\hat{u}$ and we have plugged $w = b\hat{u} + v$ in. As for the determinant, we have

$$\begin{aligned}\langle \text{ Det } (Dw), f\rangle =& -\frac{1}{3}\int_S b_1 f(x)(\frac{\partial w_2}{\partial \tau^1}(x)\frac{\partial w_3}{\partial \tau^2}(x) - \frac{\partial w_3}{\partial \tau^1}(x)\frac{\partial w_2}{\partial \tau^2}(x))d\mathcal{H}^2(x)\\ &- \frac{1}{3}\int_S b_2 f(x)(\frac{\partial w_3}{\partial \tau^1}(x)\frac{\partial w_1}{\partial \tau^2}(x) - \frac{\partial w_1}{\partial \tau^1}(x)\frac{\partial w_3}{\partial \tau^2}(x))d\mathcal{H}^2(x)\\ &- \frac{1}{3}\int_S b_3 f(x)(\frac{\partial w_1}{\partial \tau^1}(x)\frac{\partial w_2}{\partial \tau^2}(x) - \frac{\partial w_2}{\partial \tau^1}(x)\frac{\partial w_1}{\partial \tau^2}(x))d\mathcal{H}^2(x)\\ &- \frac{1}{3}\int_C f(x)(\frac{b_1}{2} + w(x))(\frac{\partial v_3}{\partial \tau}b_2 - \frac{\partial v_2}{\partial \tau}b_3)d\mathcal{H}^1(x)\\ &- \frac{1}{3}\int_C f(x)(\frac{b_2}{2} + w(x))(\frac{\partial v_1}{\partial \tau}b_3 - \frac{\partial v_3}{\partial \tau}b_1)d\mathcal{H}^1(x)\\ &- \frac{1}{3}\int_C f(x)(\frac{b_3}{2} + w(x))(\frac{\partial v_2}{\partial \tau}b_1 - \frac{\partial v_1}{\partial \tau}b_2)d\mathcal{H}^1(x)\\ &+ \int_\Omega f \det (Dw) dx, \end{aligned} \tag{B.4}$$

which, pluging $w = b\hat{u} + v$, rewrites in compact form as

$$\begin{aligned}\langle \operatorname{Det}(Dw), f\rangle = &-\frac{\varepsilon_{ijk}}{3}\int_S b_i f(x)\frac{\partial v_j}{\partial \tau^1}(x)\frac{\partial v_k}{\partial \tau^2}(x)d\mathcal{H}^2(x)\\ &-\frac{\varepsilon_{ijk}}{3}\int_C f(x)v_i(x)\frac{\partial v_k}{\partial \tau}b_j d\mathcal{H}^1(x)\\ &+\int_\Omega f \det(Dw)dx. \end{aligned} \tag{B.5}$$

Note that the Jacobian of u is always zero (since its components differ in a multiplicative factor), so that for $w = b\hat{u} + c$ (i.e., $v = c$ where c is a constant vector) one has

$$\langle \operatorname{Det}(Dw), f\rangle = \langle \operatorname{Det}(Du), f\rangle = \det(D\hat{u}) = 0 = \langle(\operatorname{cof}(Du))_{ij}, f\rangle. \tag{B.6}$$

Remark B.1. In particular, Theorem A.1, implies that, if the dislocation $\mathcal{L}$ is composed by the images of a countable number of loops $\gamma : S^1 \to \Omega$ (not necessarily injective) that are uniformly bounded in $W^{2,\infty}(S^1)$, then we can choose a representative $w \in SBV(\Omega, \mathbb{R}^3)$ for the displacement, such that its distributional determinant and adjunct are bounded measures whose singular part is supported by the jump set of w (i.e., the surface S) and by the dislocation set C. This can be seen by equations (5.3) and (5.4), and by the fact that the constant $c > 0$ appearing in formulae (A.6) and (A.7) depends on the second derivative (the curvature) of C, since it is choosen as the constant in (ii) and (iii) of Lemma 3.6.

Moreover it is seen that the absolutely continuous parts of the distributional determinant and of the adjunct of w always coincide with the pointwise determinant and adjoint, respectively, independently of the choice of the representative w (in other words, independently of the choice of the surface S). The same is true for the dislocation depending parts of the measures, i.e., the currents $\mathcal{L} \wedge \vec{b}_w(\omega)$ and $\mathcal{C}_w$, which are also independent of the curvature of the dislocation curves. On the other hand, the part of the distributional determinant and adjnct which is concentrated on the interior of S obviously depends on the choice of S. It should also be emphasized that estimate (A.7) does not ensure that the total mass is finite for general dislocation geometry, since the constant $c > 0$ given in this formula does depend on the L^∞ norm of the curvature of C.

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