CAUCHY ELASTICITY WITH DISLOCATIONS IN THE SMALL STRAIN ASSUMPTION

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Abstract. In this letter it is shown how the singularities created by dislocations in an elastic body must not preclude from a linear approach. Cauchy elasticity is considered and hence no variational approach is needed and the displacement field only appears in a second step, as an interpretation of the first of the two model variables. The second model variable obeys nonclassical PDEs relying on the incompatibility operator.

Keywords: Elasticity, dislocations, incompatibility operator, Beltrami decomposition, Helmholtz decomposition.

1. Introduction

Let Ω be a simply-connected smooth and bounded subset of $\mathbb{R}^3$, where $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with Γ_0 and Γ_1 mutually disjoint. Let the dislocation density $\Lambda \in \mathcal{M}(\Omega, \mathbb{M}^3)$ be given as a concentrated Radon measure. It is well known that in the presence of dislocations, the Cauchy stress is given by the symmetric $\sigma \in L^p(\Omega, \mathbb{M}^3)$ with $1 \leq p < 2$. The quadratic energy being unbounded, two approaches are found in the literature. The widely followed procedure is to hide the line singularity in a core of finite diameter and proceed as in the Hilbertian case away from this tube. Alternatively, to consider finite strain (Green) elasticity as based on minimization principles as in [9]. The first approach drawback is to avoid the intrinsic concentration properties of dislocations at the mesoscopic scale, whereas the second requires a heavy artilery which is unnecessary away from the core. In the present letter, we suggest a simple approach based on linear PDEs, where the classical Lamé system of elasticity is solved, though with a variable u which is not primarily the displacement field, rather originating from a strain decomposition which also provides a dislocation-dependent field F. This latter field solves an incompatibility-based PDE, which is here discussed.

Being f, g and U the body and surface forces, and the prescribed boundary load, the following result will be proved: there exists $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq 3/2$ such that

$$\left\{ \begin{array}{rcll} -\operatorname{div}\left(A\nabla^S u\right) & = & f + \mathcal{F} & \text{in} \quad \Omega \\ \left(A\nabla^S u\right) N & = & g + \mathcal{G} & \text{on} \quad \Gamma_1 \\ u & = & U & \text{on} \quad \Gamma_0, \end{array} \right. , \tag{1.1}$$

where $\mathcal{F}$ and $\mathcal{G}$ are dislocation-induced body and surface forces in $W^{-1,p}(\Omega)$ and $W^{-1/p,p}(\Gamma_1)$, respectively, with $1 \leq p \leq 3/2$, and vanishing as soon as $\Lambda = 0$.

The u is interpreted as the displacement field. Note that $\nabla^S u$ has less regularity (i.e., is in $L^p(\Omega)$ with $1 \leq p \leq 3/2$) than σ if σ is taken in $L^q(\Omega)$ with $3/2 < q < 2$.

2. Preliminary results

The divergence and curl of a tensor T is defined componentwise as $(\operatorname{div} T)_i := \partial_j T_{ij}$ and $(\operatorname{Curl} T)_{ij} := \epsilon_{jkl}\partial_k T_{il}$, respectively. The incompatibility of a tensor E is defined componentwise as follows: $(\operatorname{inc} E)_{ij} := (\operatorname{Curl}(\operatorname{Curl} E)^{\mathrm{T}})_{ij} =$

$\epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l E_{mn}$. Let $\mathbb{M}^3$ denote the space of square 3-matrices, and $\mathbb{S}^3$ of symmetric 3-matrices. Introduce

$$\begin{aligned} L^p_{\rm div}(\Omega,\mathbb{M}^3) &:= \{F\in L^p(\Omega,\mathbb{M}^3)\quad \text{s.t.}\quad {\rm div}F=0\},\\ \mathcal{V}^p(\Omega) &:= \left\{V\in L^p_{\rm div}(\Omega,\mathbb{M}^3) \text{ s.t. Curl } V\in L^p(\Omega,\mathbb{M}^3), V\times N=0 \text{ on } \partial\Omega\right\}\\ \tilde{\mathcal{V}}^p(\Omega) &:= \left\{V\in L^p_{\rm div}(\Omega,\mathbb{M}^3) \text{ s.t. Curl } V\in L^p(\Omega,\mathbb{M}^3), VN=0 \text{ on } \partial\Omega\right\}. \end{aligned}$$

Theorem 1 (Helmholtz-Weyl-Hodge-Yanagisawa [5])**.** *Let $1<p<\infty$ and let Ω be a bounded, simply-connected and smooth open set in $\mathbb{R}^3$. For every $F\in L^p(\Omega,\mathbb{M}^3)$, there exists $u_0\in W^{1,p}_0(\Omega,\mathbb{R}^3)$ and a solenoidal $V\in\tilde{\mathcal{V}}^p(\Omega)$, such that $F=Du_0+$ Curl V. Alternatively, there exists $u\in W^{1,p}(\Omega,\mathbb{R}^3)$ and a solenoidal $V_0\in\mathcal{V}^p(\Omega)$, such that $F=Du+$ Curl V_0. Moreover the decompositions are unique.*

In particular, if ${\rm div}F=0$ then $F=$ Curl V for some $V\in\tilde{\mathcal{V}}^p(\Omega)$.

Theorem 2 (Beltrami decomposition [3])**.** *Let $\Omega\subseteq\mathbb{R}^3$ be a simply-connected domain with smooth boundary, let $p\in(1,+\infty)$ be a real number and let $e\in L^p(\Omega,\mathbb{S}^3)$ be a symmetric tensor. Then, there exist a vector field $u\in W^{1,p}(\Omega,\mathbb{R}^3)$ and a tensor $F\in L^p(\Omega,\mathbb{S}^3)$ with* Curl $F\in L^p(\Omega,\mathbb{S}^3)$, ${\rm inc}F\in L^p(\Omega,\mathbb{S}^3)$, ${\rm div}F=0$ *and $FN=0$ on $\partial\Omega$ such that*

$$e=\nabla^S u+{\rm inc}F. \tag{2.1}$$

The following result is given without proof, which is classical.

Lemma 1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded open set with boundary of class C^1 and let $F\in L^p(\Omega,\mathbb{M}^3)$ be such that ${\rm div}F\in L^p(\Omega,\mathbb{R}^3)$. Then there exists $FN\in W^{-1/p,p}(\partial\Omega):=\left(W^{1/p,p'}(\partial\Omega)\right)'$ such that*

$$\langle FN,\gamma(\Phi)\rangle := \langle\, {\rm div}F,\Phi\rangle+\langle F,D\Phi\rangle \tag{2.2}$$

for all $\Phi\in W^{1,p'}(\Omega,\mathbb{R}^3)$, with $\gamma(\Phi)\in W^{1/p,p'}(\partial\Omega,\mathbb{R}^3)$ the boundary trace of Φ, and where $\langle\cdot\rangle$ always mean the duality product in appropriate spaces.

Lemma 2. *Let Ω be a domain and $u_0\in W^{1/p',p}(\partial\Omega,\mathbb{R}^3), g_0\in W^{-1/p,p}(\partial\Omega,\mathbb{R}^3)$, and let v be the solution of the system*

$$\begin{cases} \mathcal{L}_{0,1}(v):=\nabla{\rm div}v+\Delta v=0 & \text{in } \Omega,\\ v=u_0 & \text{on } \Gamma_0\subset\partial\Omega,\\ (\nabla^S v)N=g_0 & \text{on } \Gamma_1\subset\partial\Omega. \end{cases} \tag{2.3}$$

Then there exists a $G\in L^p(\Omega,\mathbb{S}^3)$ with Curl $G\in L^p(\Omega,\mathbb{M}^3)$, ${\rm div}G=0$ *in Ω and $GN=0$ on $\partial\Omega$, and such that*

$$\nabla^s v={\rm inc}G. \tag{2.4}$$

Proof. Let us sketch the proof whose details can be found in [3]. From (2.3) we see that $\nabla^s v$ is divergence free. By Theorem 3.2, there exists a divergence-free B such that $\nabla^s v=$ Curl B solution of[1]

$$\left\{\begin{array}{ccc} \text{Curl Curl } B &=& \text{Curl } \nabla^S v \quad \text{in} \quad \Omega\\ BN &=& 0 \quad \text{on} \quad \partial\Omega\\ B\times N &=& (\nabla^S v)\times N \quad \text{on} \quad \partial\Omega \end{array}\right., \tag{2.5}$$

Moreover let the zero-mean h and H be solution of

$$\left\{\begin{array}{ccc} {\rm div}h &=& H \quad \text{in} \quad \Omega\\ h &=& 0 \quad \text{on} \quad \partial\Omega \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ccc} \Delta H &=& 0 \quad \text{in} \quad \Omega\\ \partial_N H &=& -\partial_i B_{ij}N_j \quad \text{on} \quad \partial\Omega \end{array}\right., \tag{2.6}$$

[1]This strong form is formal since the boundary value of $\nabla^S v$ must not be defined. The solution is intendent in a weak sense as shown in [5]. It suffices that $\nabla^S v$ be p-integrable in Ω.

respectively. Defining $B' := B + \nabla h$, it is then shown that $B' = (\text{ Curl } G)^T$ with the divergence-free G solution to

$$\begin{cases} \text{Curl Curl } G &= \text{Curl } (B')^T \quad \text{in} \quad \Omega \\ GN &= 0 \quad \text{on} \quad \partial\Omega \\ G \times N &= (B')^T \times N \quad \text{on} \quad \partial\Omega \end{cases}, \tag{2.7}$$

(well posedness for (2.7) and existence are studied in [5]) achieving the proof by observing the the symmetry of G follows by (2.4) since $\epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l G_{mn} = \epsilon_{jlm}\epsilon_{ikn}\partial_l\partial_k G_{mn} = \epsilon_{jlm}\epsilon_{ikn}\partial_k\partial_l G_{mn}$ and hence the skewsymmetric part of G equals to a symmetric gradient, meaning that it vanishes. □

3. Governing PDEs

Define the elastic strain as $\epsilon^e := A^{-1}\sigma$, where A is the assumed constant elasticity tensor, i.e., $A = \mu\mathbb{I}_4 + \lambda\mathbb{I}_2 \otimes \mathbb{I}_2$. Conservation of momentum (or Equilibrium) reads

$$\begin{cases} -\text{div}\sigma &= f \quad \text{in} \quad \Omega \\ \sigma N &= g \quad \text{on} \quad \Gamma_1 \\ u &= U \quad \text{on} \quad \Gamma_0, \end{cases} \tag{3.1}$$

with $f \in W^{-1,p}(\Omega)$, $U \in W^{1/p',p}(\Gamma_0)$ and $g \in W^{-1/p,p}(\Gamma_1)$ with $1 \leq p < 2$.

The problem reads: find ϵ, u and F such that

- (3.1) holds for $\sigma = \hat{\sigma}(\epsilon, u, F) = \tilde{\sigma}(\epsilon, \Lambda)$
- $\Lambda = 0 \Rightarrow F = 0$ and $\epsilon = \epsilon^e = \nabla^S u$.

Note that ϵ is found up to a symmetric, solenoidal and traceless tensor $\bar{\epsilon}$ satisfying $(A\bar{\epsilon})N = 0$ on Γ_1. For any such $\bar{\epsilon}$, we have by Theorem 2 the decomposition

$$\epsilon := \epsilon^e + \bar{\epsilon} = \nabla^S \tilde{u} + \text{inc}\tilde{F}, \tag{3.2}$$

where $\tilde{u}$ satisfies

$$\begin{cases} \mathcal{L}_{0,1}(\tilde{u}) &= \text{div}\epsilon \quad \Omega \\ (\nabla^S \tilde{u})N &= (\epsilon - \text{inc}\tilde{F})N \quad \text{on} \quad \partial\Omega, \end{cases}, \tag{3.3}$$

while the solenoidal $\tilde{F}$ is solution of

$$\begin{cases} \text{inc}\,\text{inc}\tilde{F} &= \text{inc}\epsilon \quad \text{in} \quad \Omega \\ \tilde{F}N &= 0 \quad \text{on} \quad \partial\Omega \\ \left(\partial_N \tilde{F} \times N\right)^t \times N &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}. \tag{3.4}$$

Moreover, for any such $\bar{\epsilon}$, Equilibrium is rewritten as

$$\begin{cases} -\text{div}(A\nabla^S \tilde{u}) &= \tilde{f} := f + \text{div}(A\,\text{inc}\tilde{F}) \quad \Omega \\ (A\nabla^S \tilde{u})N &= \tilde{g} := g - (A\,\text{inc}\tilde{F})N \quad \text{on} \quad \partial\Omega \end{cases}. \tag{3.5}$$

By Lemma 2, let us rewrite (3.2) as

$$\epsilon = \nabla^S \tilde{u} + \nabla^S v - \text{inc}G + \text{inc}\tilde{F}, \tag{3.6}$$

with the solenoidal G solution of (2.7) and v solution of

$$\begin{cases} \mathcal{L}_{0,1}(v) = 0 & \text{in } \Omega, \\ v = U - \tilde{u} & \text{on } \Gamma_0 \\ (\nabla^S v)N = 0 & \text{on } \Gamma_1. \end{cases} \tag{3.7}$$

Now, by (3.6) we define $u := \tilde{u} + v$ and the symmetric $F := \tilde{F} - G$ and hence

$$\epsilon = A^{-1}\sigma + \bar{\epsilon} = \nabla^S u + \text{inc}F, \tag{3.8}$$

with, recalling $A=\mu\mathbb{I}_4+\lambda\mathbb{I}_2\otimes\mathbb{I}_2$ and the solenoideal property of $\mathrm{inc}\, F$, Equilibrium rewritten as

$$\left\{\begin{array}{ccc} -\operatorname{div}(A\nabla^S u) & = & \tilde{f}:=f+\lambda\nabla\,\mathrm{tr}(\,\mathrm{inc}\,F) \quad \text{in} \quad \Omega \\ (A\nabla^S u)N & = & \tilde{g}:=g-\lambda\,\mathrm{tr}(\,\mathrm{inc}\,F)N \quad \text{on} \quad \Gamma_1 \\ u & = & U \quad \text{on} \quad \Gamma_0 \end{array}\right. , \tag{3.9}$$

where $\mathrm{tr}(\,\mathrm{inc}\,F)=\Delta\,\mathrm{tr}F$.

Now, it has been proved in [10] that there exists a $\bar{\epsilon}$ such that $\mathrm{inc}(\epsilon^e+\bar{\epsilon})=\mathrm{Curl}\,(\Lambda-\frac{I}{2}\,\mathrm{tr}\Lambda)$. Therefore, (3.4) reads

$$\left\{\begin{array}{ccc} \mathrm{inc}\,\mathrm{inc}\tilde{F} & = & \mathrm{Curl}\,(\Lambda-\frac{I}{2}\,\mathrm{tr}\Lambda) \quad \text{in} \quad \Omega \\ \tilde{F}N & = & 0 \quad \text{on} \quad \partial\Omega \\ \left(\partial_N\tilde{F}\times N\right)^t\times N & = & 0 \quad \text{on} \quad \partial\Omega \end{array}\right. . \tag{3.10}$$

Note that it also holds

$$\left\{\begin{array}{ccc} \mathrm{inc}\,\mathrm{inc}F & = & \mathrm{Curl}\,(\Lambda-\frac{I}{2}\,\mathrm{tr}\Lambda) \quad \text{in} \quad \Omega \\ FN & = & 0 \quad \text{on} \quad \partial\Omega \end{array}\right. . \tag{3.11}$$

4. Solution and regularity of the PDEs

Note that by Morrey embedding $W^{2,p'}(\Omega)\curvearrowright C^1(\Omega)$ if $2\geq 1+3/p'$, that is, if $1\leq p\leq 3/2$ with $1/p+1/p'=1$. Therefore, the strain incompatibility belongs to

$$\eta=\mathrm{Curl}\,(\Lambda-\frac{I}{2}\,\mathrm{tr}\Lambda)\in W^{-2,p}(\Omega),$$

with $1\leq p\leq 3/2$, since Λ is a Radon measure. Thus, if Λ is known, we need to first solve (3.10) to get $\tilde{F}$, then (3.3) to get $\tilde{u}$, from which (3.7) is found. Eventually $F=\tilde{F}-G$ is found, provided the solutions of (2.5), (2.6) and (2.7) are known. For (2.5) and (2.7) we refer to [5], where Lemma 1 gives a sense to $\partial_i B_{ij}N_j$ on $\partial\Omega$. Finally, (3.9) is solved to get u, which is interpreted as a infinitesimal displacement field in a linear elastic body with mesoscopic dislocations.

4.1. Well-posedness and existence for (3.10).

The main concern about (3.10) is to prove that the boundary conditions are necessary and sufficient, that is, allow one to have a unique solution $\tilde{F}$.

For solenoidal and symmetric E, the following identity holds:

$$\begin{aligned}(\mathrm{inc}(\mathrm{inc}E))_{ij} &= \varepsilon_{ikl}\varepsilon_{jmn}\varepsilon_{lpq}\varepsilon_{nrs}\partial_k\partial_m\partial_p\partial_r E_{qs} \\ &= \partial_i\partial_j\partial_q\partial_s E_{qs}-\partial_k^2\partial_j\partial_s E_{is}-\partial_k^2\partial_r\partial_j E_{ij}+\partial_p^2\partial_r^2 E_{ij}=\Delta^2 E_{ij}.\end{aligned}$$

Thus, we will show that (3.10) is well posed as written as

$$\left\{\begin{array}{ccc} \Delta^2 E & = & \eta\in W^{-2,p}(\Omega) \quad \text{in} \quad \Omega \\ EN & = & 0 \quad \text{on} \quad \partial\Omega \\ \mathrm{div}E & = & 0 \quad \text{on} \quad \partial\Omega \\ \partial_N\,\mathrm{div}E & = & 0 \quad \text{on} \quad \partial\Omega \\ (\partial_N E\times N)^t\times N & = & 0 \quad \text{on} \quad \partial\Omega \end{array}\right. . \tag{4.1}$$

First, note that taking the divergence of (4.1) yields $\mathrm{div}E=0$ in Ω by uniqueness of the zero-solution to the homogeneous bilaplacian equation.

Let τ be a tangent vector and N the unit normal vector at $x\in\partial\Omega$. It is classically known that this systems admits a unique solution $E\in W^{2,p}(\Omega;\mathbb{M}^3)$ with $1\leq p\leq 3/2$ provided the boundary conditions are complementary in the sense of Agmon, Douglis and Nirenberg [1]. To check this fact one introduce the characteristic polynomials of the operator, Δ^2, viz. $L(\xi):=|\xi|^4\mathbb{I}$ with $\xi\in\mathbb{R}^3$ and $\mathbb{I}\in\mathbb{M}^3$ the identity matrix. The order of the PDE is 6 since the degree of $\det L=12$. The number of independent unknowns is six.

The first three 0-order Dirichlet boundary conditions (4.1)-b- are associated to the symbol

$$B_1(\xi) := \begin{pmatrix} N_1 & N_2 & N_3 \end{pmatrix}, \tag{4.2}$$

The next six (1st and 2nd -order, (4.1)-c- and -d-, respectively) conditions, are given by

$$B_2(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ N\cdot\xi\xi_1 & N\cdot\xi\xi_2 & N\cdot\xi\xi_3 \end{pmatrix}. \tag{4.3}$$

For the last three degrees of freedom, some preliminary calculations are done. Consider a portion of $\partial\Omega$ such that $N_3 \neq 0$. Then the conditions $G \times N = \varphi^1$ implies that $\varphi \cdot N = 0$ and these two conditions together are equivalent to

$$\begin{cases} G_{i2}N_3 - G_{i3}N_2 & = (\varphi^1)_{i1} \\ G_{i3}N_1 - G_{i1}N_3 & = (\varphi^1)_{i2} \\ \varphi^1 \cdot N & = 0 \end{cases}. \tag{4.4}$$

An analogue treatment is made for the complement portions of the boundary (i.e., with $N_2 \neq 0$ or $N_1 \neq 0$). Now, denoting $G := \partial_N E \times N$, we first notice that the condition $GN = 0$ entails that G has two independent colums that we write $G_{i\alpha}$ with $i = 1, 2, 3$ and $\alpha = 1, 2$. Therefore, the conditions $G^t \times N = \varphi^2$ together with $\varphi^2 N = 0$ are written by means of the 4×4 matrix $(G^t \times N)_{\alpha\beta} = \epsilon_{\alpha km}\epsilon_{\beta ln}N_k N_l \partial_N E_{mn}$, which by the assumed symmetry of E, is also symmetric, and hence shows 3 degrees of freedom. Then,

$$(B_3(\xi))_{\alpha\beta mn} := \epsilon_{\alpha km}\epsilon_{\beta ln}N_k N_l (N\cdot\xi). \tag{4.5}$$

The boundary condition operator writing as

$$\begin{pmatrix} \delta_{mi}\left(B_1(\tau+\nu N)\right)_n \\ \delta_{mi}\left(B_2(\tau+\nu N)\right)_{\gamma n} \\ \left(B_3(\tau+\nu N)\right)_{\alpha\beta mn} \end{pmatrix} E_{mn}$$

then yields the twelve independent boundary conditions associated to the six independent components of E. Following Agmon, Douglis and Nirenberg, and Gelfand [1,4], the boundary conditions are complementary and thus the PDE well posed and there exists a unique solution if the lines of $|\tau+\nu N|^4 \begin{pmatrix} \left(B_1(\tau+\nu N)\right)_n E_{in} \\ \left(B_2(\tau+\nu N)\right)_{\gamma n} E_{in} \\ \left(B_3(\tau+\nu N)\right)_{\alpha\beta in} E_{in} \end{pmatrix} \in \mathbb{R}^{4\times 3}$, considered as polynomial in the variable $\nu \in \mathbb{C}$, are linearly independent modulo $(\nu - i|\tau|)^6$. It is obvious that the constant B_1 is linearly independent of the -degree one and two- lines of $B_2(\tau+\nu N)$. It is also obvious that the lines of $B_2(\tau+\nu N)$ are independent among themselves. It is obvious that $\delta_{m\alpha}(B_2(\tau + \nu N))_{\beta n} = \delta_{m\alpha}\nu^{\beta-1}(\tau_n + \nu N_n)$ is independent of $(B_3(\xi))_{\alpha\beta mn}$ for $\beta = 2$, while for $\beta = 1$ (for which both are of degree two) observe that for fixed n the former will vanish for $i \neq \alpha$ contrarily to the latter, and vice-versa for $i = \alpha$.

4.2. **Well-posedness and existence for** (3.9)**.** We will not dwell on the linear elasticity system which is well studied and documented. Let us simply mention that the solution may be found by surperposition of the two following cases.

4.2.1. *Nonhomogeneous elasticity system.* The field F being in $W^{2,p}(\Omega)$, one has $\tilde{f} \in W^{-1,p}(\Omega)$ with $1 \leq p \leq 3/2$. Existence and uniqueness of $u \in W^{m+1,p}(\Omega)$ to

$$\begin{cases} -\operatorname{div}\left(A\nabla^S u\right) & = & \tilde{f} & \text{in} & \Omega \\ \left(A\nabla^S u\right)N & = & 0 & \text{on} & \Gamma_1 \\ u & = & 0 & \text{on} & \Gamma_0, \end{cases} \tag{4.6}$$

with $\tilde{f} \in W^{m,p}(\Omega)$ follows from classical arguments as Fredholm alternative and regularity estimates. Note that the our case $m = -1$ is not explicitely found in classical textbooks as [8], but holds with similar arguments as the Fourier transform, following the method of [7] (for $p = 2$), since one has the identification

$$W^{-1,p}(\Omega) = \{w_{|\Omega} : w \in \mathcal{S}'(\mathbb{R}^3) \; s.t. \; \mathcal{F}^{-1}[(1 + \|\xi\|^2)^{-1/2}\mathcal{F}[w]] \in L^p(\mathbb{R}^3)\}.$$

4.2.2. *Homogeneous elasticity system.* The field F being in $W^{2,p}(\Omega)$, one has the divergence-free inc$F \in L^p(\Omega)$ with $1 \leq p \leq 3/2$, and hence by Lemma 1, $(\,\mathrm{inc}F)N \in W^{-1/p,p}(\partial\Omega)$. In particular, $\tilde{g}, (\,\mathrm{tr}\,\mathrm{inc}F)N \in W^{-1/p,p}(\partial\Omega)$. The solution of

$$\left\{ \begin{array}{rcll} -\operatorname{div}\left(A\nabla^S u\right) & = & 0 & \text{in} \quad \Omega \\ \left(A\nabla^S u\right) N & = & \tilde{g} & \text{on} \quad \Gamma_1 \\ u & = & U & \text{on} \quad \Gamma_0, \end{array} \right. \tag{4.7}$$

is obtained by first and second layer potentials and the Fredholm alternative, with methods similar as those of [2, 6] for the Laplacian.

5. Concluding remarks

The present method should in future be complemented by numerical simulations, where in a first stage the defect distribution Λ will be prescribed. The main feature is the simplicity of PDE solving (the two systems are decoupled), since linear elasticity is recovered by providing a dislocation-induced body force depending explicitely on the defect variable F. The second aspect is that it is stress based, whereas displacement only appears in a second step, after PDEs are solved.

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