Existence and linearized stability of solitary waves for a quasilinear Benney system

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Abstract

We prove the existence of solitary wave solutions to the quasilinear Benney system

$$\begin{cases} iu_t + u_{xx} = a|u|^p u + uv \\ v_t + f(v)_x = (|u|^2)_x, \end{cases}$$

where $f(v) = -\gamma v^3$, $-1 < p < +\infty$ and $a, \gamma > 0$. We establish, in particular, the existence of travelling waves with speed arbitrary large if $p < 0$ and arbitrary close to 0 if $p > \frac{2}{3}$. We also show the existence of standing waves in the case $-1 < p \leq \frac{2}{3}$, with compact support if $-1 < p < 0$. Finally, we obtain, under certain conditions, the linearized stability of such solutions.

Keywords: Long wave - short wave interactions; Solitary waves; Dispersive equations; Hyperbolic systems; Linearized stability.

1 Introduction

In the seminal works [9], [10], D.J. Benney introduced a number of universal models describing the interaction between short and long waves propagating along a direction (Ox) in a dispersive media. One of these models is the system

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{\partial x^2} = m_1|u|^2 u + m_2 uv \\ \\ \dfrac{\partial v}{\partial t} + m_3\dfrac{\partial v}{\partial x} = m_4\dfrac{\partial}{\partial x}(|u|^2), \quad x \in \mathbb{R},\, t \geq 0. \end{cases} \tag{1}$$

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Here, m_j are real constants, $u = u_{(y)} + iu_{(z)}$ represents, in complex notation, the transverse components $(u_{(y)}, u_{(z)})$ of the short wave, and v the density perturbation induced by the long wave.
This model has been successfully applied to several physical contexts, such as the study of the formation and annihilation of solitons resulting from the interaction between Langmuir and ion sound waves in a magnetized plasma, in the case where the perturbation propagates with a speed close to that of sound ([28], [39]), or the interaction between Alfvén and magneto-acoustic waves in a cold plasma subjected to a strong external magnetic field ([13], [35]). In water waves theory, applications of this model include the interaction between gravity-capillary waves in a two-layer fluid, when the group velocity of the surface waves coincides with the phase velocity of the internal waves (see [21],[22],[36]. See also [34] for an alternative derivation of Benney's equations from the Zakharov formulation of surface gravity waves). Other examples, such as long-wave short-wave interaction in bubbly liquids ([1]) or optical-microwave interactions in nonlinear mediums ([14]) can be given.

The mathematical study of system (1), namely the well-posedness of the associated Cauchy Problem or the existence and stability of solitary waves, has been extensively conducted over the years by many authors (see for instance [6], [12], [26], [32], [37],[38] and references therein).

As pointed out in [9], this system is an adequate model in the case where the amplitude of the long wave is considerably smaller than the amplitude of the short wave. When both amplitudes are of the same order, the effect of long waves becomes considerably weaker, and, in this context, (1) should be replaced by a system of the form

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{\partial x^2} = |u|^2 u + uv \\ \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x} f(v) = \dfrac{\partial}{\partial x}(|u|^2), \end{cases} \tag{2}$$

where f is a nonlinear polynomial. Contrarely to the linear case $f(v) = mv$, only recently some attention has been given to the mathematical study of these more general systems. In [2], the case of the Schrödinger-Burgers' system ($f(v) = mv^2$) was adressed in the half-line. The existence and linear stability of shockwave solutions to (2) was proved in [4]. By combining methods from dispersive equations and systems of hyperbolic conservation laws, in [15], [19], the authors studied the existence of global weak solutions and local strong solutions for the corresponding Cauchy problem in the energy space, in the case where $f(v) = av^2 - bv^3$, $a \in \mathbb{R}$, $b > 0$ (see also [3], [5], [16], [17], [18] and [20] for related results concerning similar systems).

Also recently, Bégout and Díaz ([7], [8]) considered nonlinear Schrödinger equations with an "absorbing" singular potential of the form $|u|^p$, $p < 0$ such as the homogenous equation

$$iu_t + \Delta u = \alpha |u|^p u, \qquad -1 < p < 0.$$

Nonlinear Schrödinger equations with singular potentials arise in a large variety of contexts (see e.g. [31],[27]). The authors proved in particular that under some circumstances such equations admit standing wave solutions of the form $u(x,t) = \phi(x)e^{i\beta t}$ with compact support, under the fundamental condition $-1 < p < 0$. Such localization of solutions is well-known not to exist for ordinary Schrödinger equations and seem to be a special feature of singular potentials of this type.

With these motivations, in the present work, we are concerned with the existence and behaviour of solitary waves for quasilinear Benney systems of the type

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{\partial x^2} = m_1 |u|^p u + uv \\ \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x} f(v) = m_4 \dfrac{\partial}{\partial x}(|u|^2), \end{cases} \tag{3}$$

where $f(v) = m_2 v^3$ and $-1 < p < +\infty$.

The rest of this paper is organized as follows:

In Sections 2 and 3 we establish the existence of a two-parameter family of solitary-wave solutions to (3) of the form

$$(u(x,t), v(x,t)) = (e^{iwt} e^{i\frac{c}{2}(x-ct)} \phi(x-ct), \psi(x-ct)), \tag{4}$$

where ϕ and $-\psi$ are non-negative radially decreasing functions vanishing at infinity. This result relies on the derivation of sharp estimates for the Lagrange multiplier associated to a variational minimization problem. These estimates allow us also to exhibit solitary waves with positive speed c arbitrary large in the case $-1 < p < 0$ and arbitrary close to 0 for $p > \frac{2}{3}$.

When $0 \leq p \leq \frac{2}{3}$, we prove, in Section 4, the existence of standing-wave solutions ($c = 0$) of the form

$$(u(x,t), v(x,t)) = (e^{iwt}\phi(x), \psi(x))$$

by applying a result due to Berestycki and Lions ([11]). We also establish the existence of standing waves with compact support. The condition for the existence of such localized solutions is $-1 < p < 0$, related in particular to the convergence of a singular integral of the type $\displaystyle\int_0^a \frac{dx}{x^{1+\frac{p}{2}}}$. Although we use totally different methods, this is, as mentionned above, the exact same condition used in [7], [8] to derive solutions with compact support.

Finally, in Section 5, after establishing the global well-posedness of a non-autonomous system consisting of the linearization of (3) around a solitary wave, we prove, in the spirit of [23], the linearized stability of solitary wave solutions in the case $p > -\frac{2}{3}$, with $c = 0$ if $p < 0$ (and without restrictions on the speed c if $p > 0$).
Our results are synthetized in the following table:

Existence and stability of solitary-wave solutions to (3)

	Regularity	Speed	Further Properties
$-1 < p < 0$	$H^1 \times H^1$	$c > 0$, arbitrarily large	
$-1 < p \leq -\frac{2}{3}$	$(C^2 \cap W^{2,\infty}) \times (C^1 \cap W^{1,\infty})$	$c = 0$	Compactly supported
$-\frac{2}{3} < p \leq \frac{2}{3}$	$(C^3 \cap W^{3,\infty}) \times (C^2 \cap W^{2,\infty})$	$c = 0$	Linearly stable; Compactly supported if $p < 0$
$p > \frac{2}{3}$	$H^\infty \times H^\infty$	$c > 0$, arbitrarily small	Linearly stable

2 Existence of Solitary waves for $-1 < p < 0$

We consider the system

$$\begin{cases} iu_t + u_{xx} = a|u|^p u + uv \\ v_t + f(v)_x = (|u|^2)_x, \end{cases} \tag{5}$$

where $f(v) = -\gamma v^3$, $-1 < p < 0$, $\gamma > 0$ and $a > 0$.

We look for solutions of the form

$$(u(x,t), v(x,t)) = (e^{iwt} e^{i\frac{c}{2}(x-ct)} \phi(x-ct), \psi(x-ct)),$$

with ϕ and ψ real-valued and vanishing at infinity. We obtain the system

$$\left\{ \begin{array}{rcl} -\phi'' + c^*\phi & = & -\phi\psi - a|\phi|^p\phi \\ c\psi & = & -\phi^2 + f(\psi), \end{array} \right. \tag{6}$$

where $c^* = w - \frac{c^2}{4}$.

By showing the existence of solutions to (6), we will prove the following theorem, describing a two-parameter family of soutions to (5):

Theorem 2.1 *Let* $\dfrac{1}{3} < \alpha \leq 1$.

There exists $\mu_0 = \mu(\alpha) > 0$ *such that for all* $\mu > \mu_0$, *the system (5) has non-trivial solutions of the form*

$$\left\{ \begin{array}{l} u(x,t) = e^{iwt} e^{i\frac{c}{2}(x-ct)} \phi_{\mu,\alpha}(x-ct), \\ \\ v(x,t) = \psi_{\mu,\alpha}(x-ct) \end{array} \right.$$

where $\phi_{\mu,\alpha}$ *and* $-\psi_{\mu,\alpha}$ *are non-negative radially decreasing* H^1 *functions such that*

$$\|\phi_{\mu,\alpha}\|_{H^1}^2 + \|\psi_{\mu,\alpha}\|_2^2 \geq \mu^{\frac{3}{4}(1-\alpha)}.$$

Furthermore,

$$c = c(\mu, \alpha) \approx_{\mu \to +\infty} \mu^{\frac{1}{2}(1+\alpha)}.$$

The minimization problem

For $u \in H^1(\mathbb{R}) \cap L^{p+2}(\mathbb{R})$, $-1 < p < 0$, and $v \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, let

$$\tau(u,v) = \frac{2a}{p+2} \int |u|^{p+2} + \int vu^2 + \frac{\gamma}{4} \int v^4.$$

Also, for $d, \mu > 0$, let

$$\begin{aligned} X_{\mu,d} = \{(u,v) \in H^1(\mathbb{R}) \cap L^{p+2}(\mathbb{R}) \times (L^2(\mathbb{R}) \cap L^4(\mathbb{R})) \ : \\ N_d(u,v) = \|u\|_2^2 + \|u'\|_2^2 + d\|v\|_2^2 = \mu\} \end{aligned}$$

and

$$\mathcal{I}(\mu, d) = inf\{\tau(u, v) \,:\, (u, v) \in X_{\mu,d}\}. \tag{7}$$

If (u, v) is a minimizer, then there exists a Lagrange multiplier λ such that $\nabla\tau = \lambda\nabla N_d$, that is

$$\left\{ \begin{array}{lcl} 2a|u|^p u + 2vu & = & \lambda(-2u'' + 2u) \\ u^2 + \gamma v^3 & = & 2\lambda dv \end{array} \right. \tag{8}$$

and

$$\left\{ \begin{array}{lcl} \lambda u'' - \lambda u & = & -uv - a|u|^p u \\ -2d\lambda v & = & -u^2 + f(v). \end{array} \right. \tag{9}$$

If $\lambda < 0$, the change of variable $x' = x\sqrt{-\lambda}$ leads to a solution

$$(\phi(x), \psi(x)) = \left(u\left(\sqrt{-\lambda}x\right), v\left(\sqrt{-\lambda}x\right)\right) \tag{10}$$

of system (6) for

$$c* = -\lambda \quad \text{ and } \quad c = -2\lambda d. \tag{11}$$

Proposition 2.2 *For $\mu, d > 0$, $\mathcal{I}(\mu, d) > -\infty$.*

Proof:
We only have to notice that for $(u, v) \in X_{\mu,d}$,

$$\tau(u, v) \geq -\int |v|u^2 \geq -\|v\|_2\|u\|_4^2 \geq -C\|v\|_2\|u'\|_2^{\frac{1}{2}}\|u\|_2^{\frac{3}{2}} \geq -C\frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}},$$

by the Gagliardo-Nirenberg inequality $(C > 0)$. ■

Proposition 2.3 *For $\mu, d > 0$,*

$$\mathcal{I}(\mu, d) \leq -\frac{3}{8\sqrt{\pi}}\frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}} + C\left(\mu^{1+\frac{p}{2}} + \gamma\frac{\mu^2}{d^2}\right),$$

where C is a positive constant.
In particular, for $\frac{1}{3} < \alpha \leq 1$, $d = \mu^\alpha$ and μ large enough, $\mathcal{I}(\mu, d) < 0$.

Proof:
For $B > 0$, we consider the following functions

$$u(x) = \frac{B}{1 + x^2}$$

and

$$v(x) = -\frac{1}{\sqrt{d}} u(x).$$

A simple computation shows that

$$\|u\|_2^2 + \|u'\|_2^2 + d\|v\|_2^2 = B^2\pi,$$

hence, by taking $B = \sqrt{\frac{\mu}{\pi}}$, $(u, v) \in X_{\mu,d}$.

Furthermore,

$$\int vu^2 = -\frac{B^3}{\sqrt{d}} \int \left(\frac{1}{1+x^2}\right)^3 = -\frac{\frac{3\pi}{8}}{\pi^{\frac{3}{2}}} \frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}} = -\frac{3}{8\sqrt{\pi}} \frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}},$$

hence

$$\tau(u, v) = \frac{2a}{p+2} \int |u|^{p+2} + \int vu^2 + \frac{\gamma}{4} \int v^4 \leq -\frac{3}{8\sqrt{\pi}} \frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}} + C\left(\mu^{1+\frac{p}{2}} + \gamma\frac{\mu^2}{d^2}\right),$$

where $C > 0$. ■

Proposition 2.4 *Let* $\mu, d > 0$ *and* $(u, v) \in X_{\mu,d}$.
There exists $\tilde{u}$ *non-negative and* $\tilde{v}$ *non-positive,* $\tilde{u}$ *and* $|\tilde{v}|$ *radially decreasing, such that* $\tau(\tilde{u}, \tilde{v}) \leq \tau(u, v)$ *and* $(\tilde{u}, \tilde{v}) \in X_{\mu,d}$.

Proof:
Let $u_* = |u|^*$ and $v_* = -|v|^*$, where f^* denotes the Schwarz symmetrization of f.
On one hand,

$$\tau(|u|, -|v|) = \frac{2a}{p+2} \int |u|^{p+2} - \int |v|u^2 + \frac{\gamma}{4} \int v^4 \leq \tau(u, v).$$

Furthermore, since for $r \geq 1$, $\int (f^*)^r = \int f^r$ for every positive function f in $L^r(\mathbb{R})$ and $\int |u|^2|v| \leq \int (|u|^*)^2|v|^*$,

$$\tau(u_*, v_*) \leq \tau(u, v).$$

By the Polya-Szego inequality, $\int ((u_*)')^2 \leq \int (u')^2$, hence

$$N_d(u_*, v_*) \leq N_d(u, v) = \mu.$$

If $N_d(u_*, v_*) = \mu$, we put $(\tilde{u}, \tilde{v}) = (u_*, v_*)$.
If $N_d(u_*, v_*) < \mu$ we set, for $k > 0$,

$$\tilde{u}(x) = k^{\frac{1}{4p}} u_* \left(\frac{x}{k^{\frac{p+2}{-4p}}} \right) \text{ and } \tilde{v}(x) = k^{\frac{1}{4}} v_*(kx). \tag{12}$$

Since $\displaystyle\int |\tilde{u}|^2 = k^{-\frac{1}{4}} \int u_*^2$ and $\displaystyle\int |\tilde{v}|^2 = k^{-\frac{1}{2}} \int v_*^2$ and at least one of these quantities is different from 0, there exists $0 < k < 1$ such that

$$N_d(\tilde{u}, \tilde{v}) = \mu.$$

Furthermore,

$$\int \tilde{v}^4 = \int v_*^4,$$

$$\int \tilde{u}^{p+2} = \int u_*^{p+2}$$

and

$$\int \tilde{u}^2 \tilde{v} = k^{\frac{1}{2p} + \frac{1}{4}} \int u_*^2 \left(\frac{x}{k^{\frac{p+2}{-4p}}} \right) v_*(kx) < k^{\frac{1}{2p} + \frac{1}{4}} \int u_*^2 \left(\frac{x}{k^{\frac{p+2}{-4p}}} \right) v_* \left(\frac{x}{k^{\frac{p+2}{-4p}}} \right)$$

since

$$|kx| < \left| \frac{x}{k^{\frac{p+2}{-4p}}} \right| \tag{13}$$

for $x \neq 0$ and $-v_*$ is non-negative and radially decreasing. Finally,

$$\int \tilde{u}^2 \tilde{v} \leq \left(k^{\frac{1}{2p} + \frac{1}{4} - \frac{p+2}{4p}} \right) \int u_*^2 v_* = \int u_*^2 v_*$$

and $\tau(\tilde{u}, \tilde{v}) < \tau(u_*, v_*) \leq \tau(u, v)$, which completes the proof. ■

Proposition 2.5 *Let $\mu, d > 0$. There exists a solution (u, v) for the minimization problem (7), with u and $-v$ non-negative and radially decreasing.*

Proof:
Let (u_n, v_n) a minimizing sequence in $(H^1_{rd}(\mathbb{R}) \cap L^{p+2}(\mathbb{R})) \times (L^2_{rd}(\mathbb{R}) \cap L^4(\mathbb{R}))$. By the compacteness of the injection $H^1_{rd}(\mathbb{R}) \hookrightarrow L^r(\mathbb{R})$, $r > 2$, there exists a subsequence still denoted u_n such that

- $u_n \to u$ in $L^4(\mathbb{R})$;

- $u_n \rightharpoonup u$ in $H^1(\mathbb{R})$ weak;
- $u_n \to u$ almost everywhere (in particular, u is radial decreasing).

Also, since $\|v_n\|_2^2 \leq \dfrac{\mu}{d}$ is bounded, we can extract a subsequence still denoted v_n such that $v_n \rightharpoonup v$ in $L^2(\mathbb{R})$ weak.

Hence, since $u_n^2 \to u^2$ in L^2 strong and $v_n \rightharpoonup v$ in L^2 weak,

$$\int v_n u_n^2 \to \int u^2 v.$$

The sequence

$$\frac{\gamma}{4}\int v_n^4 = \tau(u_n, v_n) - \int v_n u_n^2 - \frac{2a}{p+2}\int |u|^{p+2}$$

is thus bounded, and we can extract a subsequence still denoted v_n such that $v_n \rightharpoonup v$ in L^4 weak.

Since $\displaystyle\int v^4 \leq \liminf \int v_n^4$ and $\displaystyle\int |u|^{p+2} \leq \liminf \int |u_n|^{p+2}$,

$$\tau(u, v) \leq \liminf \tau(u_n, v_n) = \mathcal{I}(u, v).$$

Now, if $\|u\|_2^2 + \|u'\|_2^2 + d\|v\|_2^2 < \mu$, the construction made in the proof of Proposition 2.4 shows that there exists $(\tilde{u}, \tilde{v}) \in X_{\mu,d}$ such that

$$\tau(\tilde{u}, \tilde{v}) < \tau(u, v).$$

Finally,

$$\mathcal{I}(\mu, d) \leq \tau(\tilde{u}, \tilde{v}) < \tau(u, v) \leq \liminf \tau(u_n, v_n) = \mathcal{I}(\mu, d),$$

which is absurd, hence $(\tilde{u}, \tilde{v}) \in X_{\mu,d}$ is a minimizer.

Note that, since $\mathcal{I}(\mu, d) = \tau(u, v)$, we have in fact that $\int |u|^{p+2} = \liminf \int |u_n|^{p+2}$, $\int v^4 = \liminf \int v_n^4$, $\int v^2 = \liminf \int v_n^2$ and $\int u^2 + u'^2 = \liminf \int u_n^2 + u_n'^2$, hence $u_n \to u$ in $L^{p+2}(\mathbb{R}) \cap H^1(\mathbb{R})$ strong and $v_n \to v$ in $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ strong.
By choosing a new subsequence, $v_n \to v$ almost everywhere, hence $-v$ is non-negative and radially decreasing. ■

If $(u,v) \in X_{\mu,d}$ is a solution to the minimization problem, $u, -v \geq 0$, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\left\{ \begin{array}{rcl} \lambda u'' - \lambda u & = & -uv - a|u|^p u \\ -2d\lambda v & = & -u^2 + f(v). \end{array} \right. \tag{14}$$

The next result states the asymptotic behaviour of λ:

Proposition 2.6 *Let $\frac{1}{3} < \alpha \leq 1$ and $d = \mu^\alpha$. There exists positive constants M_1, M_2 such that for μ large enough,*

$$M_1 \left(\frac{\mu}{d}\right)^{\frac{1}{2}} \leq -\lambda \leq M_2 \left(\frac{\mu}{d}\right)^{\frac{1}{2}}.$$

Proof:
Multiplying the equations in (14) respectively by u and v and integrating by parts,

$$2\lambda\mu = 3\int u^2 v + 2a\int u^{p+2} + \gamma\int v^4. \tag{15}$$

In particular,

$$\begin{aligned} -2\lambda\mu &= -3\int u^2 v - 2a\int u^{p+2} - \gamma\int v^4 \\ &\leq 3\left(\int u^4\right)^{\frac{1}{2}} \left(\int v^2\right)^{\frac{1}{2}} \leq 3C\mu\sqrt{\frac{\mu}{d}}, \end{aligned} \tag{16}$$

$C > 0$, by the Gagliardo-Nirenberg inequality.
This proves the second inequality by choosing $M_2 = \frac{3C}{2}$.

Now, since

$$2a\int u^{p+2} = (p+2)\tau(u,v) - (p+2)\int u^2 v - (p+2)\frac{\gamma}{4}\int v^4,$$

we obtain by (15) that

$$2\lambda\mu = (1-p)\int u^2 v + (p+2)\tau(u,v) + \frac{\gamma}{4}(2-p)\int v^4. \tag{17}$$

Since

$$\frac{\gamma}{4}\int v^4 = \tau(u,v) - \frac{2a}{p+2}\int u^{p+2} - \int u^2 v \leq \tau(u,v) - \int u^2 v, \tag{18}$$

$$2\lambda\mu \leq 4\tau(u,v) - \int u^2 v \leq 4\tau(u,v) + \left(\int v^2\right)^{\frac{1}{2}} \left(\int u^4\right)^{\frac{1}{2}}$$
$$\leq 4\tau(u,v) + C_0^{\frac{1}{2}} \frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}},$$
where C_0 is the smaller constant for the Gagliardo-Nirenberg inequality $\|u\|_4^4 \leq C_0 \|u'\|_2 \|u\|_2^3$.

By Proposition 2.3,

$$2\lambda\mu \leq \left(C_0^{\frac{1}{2}} - \frac{3}{2\sqrt{\pi}}\right) \frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}} + 4C \left(\mu^{1+\frac{p}{2}} + \gamma \frac{\mu^2}{d^2}\right), \tag{19}$$

where $C > 0$. Also, one can choose $C_0 = \frac{1}{\sqrt{3}}$. Indeed, it is known that the sharp constant in the Gagliardo-Nirenberg inequality is given by $C_0 = \frac{4}{\sqrt{3}\|Q\|_2^2}$, where $Q(x) = \sqrt{2}\,\mathrm{sech}(x)$ is the positive radial solution of $Q'' + Q^3 = Q$: $\|Q\|_2^2 = 4$ (see for instance [24], [25]).

Now, taking $d = \mu^\alpha$ and putting $\epsilon = \dfrac{3}{2\sqrt{\pi}} - \dfrac{1}{3^{\frac{1}{4}}} > 0$,

$$c := -2\lambda\mu \geq \epsilon \frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}} - C'\left(\mu^{1+\frac{p}{2}} + \frac{\gamma}{2}\frac{\mu^2}{d^2}\right) = \epsilon\mu^{\frac{3}{2}-\frac{\alpha}{2}} - C'\mu^{1+\frac{p}{2}} - C'\frac{\gamma}{2}\mu^{2(1-\alpha)}$$
$$\geq \frac{\epsilon}{2}\mu^{\frac{3}{2}-\frac{\alpha}{2}}$$
for μ large enough, since for $1 \geq \alpha > \frac{1}{3}$, we have $1 + \frac{p}{2} < \frac{1}{2}(3-\alpha)$ and $2(1-\alpha) < \frac{1}{2}(3-\alpha)$.
The proof is now complete by taking $M_1 = \dfrac{\epsilon}{4}$. ∎

End of the proof of Theorem 2.1:

In particular, from Proposition 2.6, $\lambda < 0$. By the change of variables (10), we obtain from a minimizer $(u,v) \in X_{\mu,d}$ a solution $(\phi_{\mu,\alpha}, \psi_{\mu,\alpha})$ of system (6).
Note that

$$\mu = \|u\|_2^2 + \|u'\|_2^2 + d\|v\|_2^2 = \left\|\phi_{\mu,\alpha}\left(\frac{\cdot}{\sqrt{-\lambda}}\right)\right\|_2^2 + \left\|\phi'_{\mu,\alpha}\left(\frac{\cdot}{\sqrt{-\lambda}}\right)\right\|_2^2 + d\left\|\psi_{\mu,\alpha}\left(\frac{\cdot}{\sqrt{-\lambda}}\right)\right\|_2^2$$

and
$$\mu = \sqrt{-\lambda}\|\phi_{\mu,\alpha}\|_2^2 + \frac{1}{\sqrt{-\lambda}}\|\phi'_{\mu,\alpha}\|_2^2 + d\sqrt{-\lambda}\|\psi_{\mu,\alpha}\|_2^2.$$

Hence,

$$\mu \approx \mu^{\frac{1}{4}(1-\alpha)}\|\phi_{\mu,\alpha}\|_2^2 + \mu^{\frac{1}{4}(\alpha-1)}\|\phi'_{\mu,\alpha}\|_2^2 + \mu^{\frac{1}{4}(1+3\alpha)}\|\psi_{\mu,\alpha}\|_2^2 \tag{20}$$
$$\leq \mu^{\frac{1}{4}(1+3\alpha)}\left(\|\phi_{\mu,\alpha}\|_{H^1}^2 + \|\psi_{\mu,\alpha}\|_2^2\right)$$

and

$$\|\phi_{\mu,\alpha}\|_{H^1}^2 + \|\psi_{\mu,\alpha}\|_2^2 \geq C\mu^{\frac{3}{4}(1-\alpha)}, \quad C > 0.$$

■

3 Existence of Solitary waves for $p > \frac{2}{3}$

In the case of $p > \frac{2}{3}$, we prove the following result:

Theorem 3.1 *Let* $1 - p < \alpha < \dfrac{1}{3}$.

There exists $\mu_0 = \mu(\alpha) > 0$ *such that for all* $0 < \mu < \mu_0$*, the system (5) has non-trivial solutions of the form*

$$\begin{cases} u(x,t) = e^{iwt}e^{i\frac{c}{2}(x-ct)}\phi_{\mu,\alpha}(x-ct), \\ \\ v(x,t) = \psi_{\mu,\alpha}(x-ct) \end{cases}$$

where $\phi_{\mu,\alpha}$ *and* $-\psi_{\mu,\alpha}$ *are non-negative radially decreasing smooth functions such that, for* $0 < \alpha < \dfrac{1}{3}$,

$$\|\phi_{\mu,\alpha}\|_{H^1}^2 + \|\psi_{\mu,\alpha}\|_2^2 \leq C\mu^{\frac{1}{4}(1-3\alpha)}, \quad C > 0. \tag{21}$$

Furthermore,

$$c = c(\mu,\alpha) \approx_{\mu\to 0^+} \mu^{\frac{1}{2}(1+\alpha)}. \tag{22}$$

Proof:
We begin by noticing that Propositions 2.2 and 2.3 hold for $p > \dfrac{2}{3}$. Furthermore, estimate (13) holds for $p < 0$ and for $p > \dfrac{2}{3}$.
Hence, the conclusions in Propositions 2.4 and 2.5 can be drawn also in this case.
Finally, estimate (16)

$$-\lambda\mu \leq \frac{3C}{2}\frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}}$$

remains valid for all p, and, for $2-p \geq 0$, estimate (19)

$$2\lambda\mu \leq \left(C_0^{\frac{1}{2}} - \frac{3}{2\sqrt{\pi}}\right)\frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}} + 4C\left(\mu^{1+\frac{p}{2}} + \gamma\frac{\mu^2}{d^2}\right),$$

can be derived in the exact same way as in the case $p < 0$. On the other hand, if $p > 2$, we get from (17) and (18) that

$$2\lambda d = 4\tau(u,v) - \int u^2 v + 2a\frac{p-2}{p+2}\int u^{p+2}.$$

By Proposition 2.3,

$$2\lambda\mu \leq \left(C_0^{\frac{1}{2}} - \frac{3}{2\sqrt{\pi}}\right)\frac{\mu^{\frac{3}{2}}}{d^{\frac{1}{2}}} + 4C\left(\mu^{1+\frac{p}{2}} + \gamma\frac{\mu^2}{d^2}\right) + 2a\frac{p-2}{p+2}\int u^{p+2}.$$

Using the Gagliardo-Nirenberg inequality $\|u\|_{p+2} \leq C\|u\|^{\frac{p}{2p+4}}\|u'\|^{\frac{p+4}{2p+4}}$, we obtain $\int u^{p+2} \leq C\mu^{1+\frac{p}{2}}$, hence, in all cases,

$$c = -2\lambda d \geq \epsilon\mu^{\frac{3}{2}-\frac{\alpha}{2}} - C_1\mu^{1+\frac{p}{2}} - C_2\frac{\gamma}{2}\mu^{2(1-\alpha)},$$

where ϵ, C_1, and C_2 are positive constants and $d = \mu^\alpha$.

Taking $1-p < \alpha < \frac{1}{3}$, $1+\frac{p}{2} > \frac{3}{2} - \frac{\alpha}{2}$ and $2(1-\alpha) > \frac{3}{2} - \frac{\alpha}{2}$, there exists $\mu_0 > 0$ such that for all $0 < \mu < \mu_0$,

$$c \geq \frac{\epsilon}{2}\mu^{\frac{3}{2}-\frac{\alpha}{2}},$$

which, with estimate (16), completes the proof of (22).
Finally, estimate (21) follows from (20).

Remark 3.2 *In whats concerns the regularity of ϕ and ψ, note that the monotony of ϕ and ψ garantee, via Lebesgue's Theorem, that ϕ' and ψ' exist almost everywhere. Differentiating the second equation in (6) then yields*

$$\psi'(c + 3\gamma\psi^2) = -2\phi\phi'.$$

Since $\phi \in H^1(\mathbb{R})$,

$$\int(\psi')^2 \leq \frac{4}{c^2}\int \phi^2\phi'^2 \leq \frac{4}{c^2}\|\phi\|_\infty^2\|\phi'\|_2^2 < +\infty$$

and $\psi \in H^1(\mathbb{R})$.
Now, in the case where $p \geq 0$, the first equation in (6) shows that $\phi'' \in L^2(\mathbb{R})$, that is, $\phi \in H^2(\mathbb{R})$. And again, by differentiating the second equation,

$$\psi''(c + 3\gamma\psi^2) = -2(\phi')^2 - 2\phi\phi'' - 6\gamma\psi(\psi')^2,$$

and we easily get that in fact $\psi \in H^2(\mathbb{R})$. A bootstrap argument then shows that in this case $\phi, \psi \in H^\infty(\mathbb{R})$.

Remark 3.3 *For $p \geq 0$ and $c \geq 0$, let (ϕ, ψ) be $C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ solutions of (6) with $\phi \geq 0$ and $\psi \leq 0$. Then $\phi^p \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$.*
Indeed, in a neighbourhood of a point x such that $\phi(x) = 0$, from the second equation in (6), $\psi \sim \phi^2$ if $c > 0$ ($\psi \sim \phi^{\frac{2}{3}}$ if $c = 0$). Hence, we derive from the first equation in (6) that $\phi'' \sim \phi$. Noticing that ϕ is non-negative and non-increasing, $\phi(x) = 0$ implies that $\phi(y) = 0$ if $y > x$ and, in particular, $\phi'(x) = 0$. Writing $\phi'(y) = \int_x^y \phi''(t)dt$ then shows that $\phi' \sim \phi$. Finally, $\phi^{p-1}\phi'$, $\phi^{p-2}(\phi')^2$ and $\phi^{p-1}\phi''$ vanish at x, which gives the desired result.

4 Existence of standing waves for $-1 < p \leq \frac{2}{3}$

In this section we show the existence of smooth non-trivial standing wave solutions to (3). More precisely:

Proposition 4.1 *Let $-1 < p \leq \dfrac{2}{3}$ and $\gamma, a, \omega > 0$, with $\gamma^{-\frac{1}{3}} > a$ if $p = \frac{2}{3}$. Then (3) admits non trivial solutions of the form*

$$(u(x,t), v(x,t)) = (e^{iwt}\phi(x), \psi(x)),$$

where $\phi \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ and $-\psi = \left(\frac{\phi^2}{\gamma}\right)^{\frac{1}{3}} \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ are non-negative, radially decreasing functions.
Moreover:

- *if $p > -\frac{2}{3}$, $\phi \in C^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})$ and $\psi \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$;*
- *if $-1 < p < 0$, ϕ and ψ are compactly supported.*

Proof:
Let us consider the system (6) with $c = 0$ and $\phi \geq 0$:

$$\begin{cases} -\phi'' + w\phi &= -\phi\psi - a\phi^{p+1} \\ \phi^2 &= -\gamma\psi^3. \end{cases} \tag{23}$$

From the second equation, we obtain $\psi = -\left(\frac{\phi^2}{\gamma}\right)^{\frac{1}{3}}$.
Replacing in the first equation leads to

$$\phi'' = a\phi^{p+1} + \omega\phi - \gamma^{-\frac{1}{3}}\phi^{\frac{5}{3}}. \tag{24}$$

We first analyse the case $-1 < p < 0$.
By multiplying (24) by ϕ' and integrating, we deduce, for a solution verifying $\phi'(\xi) = 0$ in all points ξ such that $\phi(\xi) = 0$, that

$$\phi'^2 = \frac{2a}{p+2}\phi^{p+2} + w\phi^2 - \frac{3}{4}\gamma^{-\frac{1}{3}}\phi^{\frac{8}{3}} := h(\phi). \tag{25}$$

Now, taking $\phi_0 > 0$ such that $h(\phi_0) = 0$ and $h(\phi) \neq 0$ for $\phi \in]0, \phi_0[$, we can derive from (25) the existence of a solution $\phi \in C^2(\mathbb{R})$ to (24) with compact support, non-negative, radially decreasing, such that $\max \phi = \phi(0) = \phi_0$ and $supp(\phi) = [-x_0, x_0]$, $x_0 = \int_0^{\phi_0}(h(\phi))^{-\frac{1}{2}}d\phi$ (note that this integral is finite for $p < 0$).
Moreover, if $-\dfrac{2}{3} < p < 0$, we can easily establish, from (24) and (25), that $\phi \in C^3(\mathbb{R})$ and $\psi \in C^2(\mathbb{R})$, with the same support.

We now turn to the case $0 \leq p \leq \dfrac{2}{3}$, with $\gamma^{-\frac{1}{3}} > a$ if $p = \dfrac{2}{3}$. Equation (24) can be written as

$$-\phi'' = g(\phi) := -a\phi^{p+1} - w\phi + \gamma^{-\frac{1}{3}}\phi^{\frac{5}{3}}. \tag{26}$$

We have $g \in C^1(\mathbb{R})$, $g(0) = 0$ and $g'(0) = -w < 0$. Moreover, putting $F(\phi) = \displaystyle\int_0^{\phi} g(\xi)d\xi$ and $\phi_0 = inf\{\xi > 0 : F(\xi) = 0\}$, $\phi_0 > 0$ and $g(\phi_0) = F'(\phi_0) > 0$. By applying Theorem 5 and Remark 6.3 in [11], there exists a unique solution $\phi \in C^3(\mathbb{R})$ of (26) such that $\phi(0) = \phi_0$, ϕ positive and radially decreasing, and such that

$$\phi(x), |\phi'(x)|, |\phi''(x)| \leq Ce^{-\delta|x|}, \tag{27}$$

where C and δ are positive constants.

We can easily deduce from (24), (25) and (27) that $\psi = -\frac{1}{\gamma}\phi^{\frac{2}{3}} \in C^2(\mathbb{R})$ with

$$|\psi(x)|, |\psi'(x)|, |\psi''(x)| \leq C'e^{-\frac{2\delta}{3}|x|}, \quad C' > 0. \tag{28}$$

5 Linearized Stability for $p > -\frac{2}{3}$

In this section we will consider, for $p > -\dfrac{2}{3}$, special solutions $(\tilde{u}, \tilde{v})$ of system (5), of the form

$$\left\{\begin{array}{l} \tilde{u}(x,t) = e^{iwt}e^{i\frac{c}{2}(x-ct)}\phi(x-ct) \\ \tilde{v}(x,t) = \psi(x-ct), \end{array}\right. \tag{29}$$

satisfying the following conditions:

- $c \geq 0$ and $c = 0$ if $-\dfrac{2}{3} < p < 0$;
- $\phi, \psi \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$;

(30)

- $\phi, -\psi \geq 0$, and $\phi, -\psi$ radially decreasing;
- $\phi^p \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ if $p \geq 0$ (cf. Remark 3.3).

By linearizing the system (5) around $(\tilde{u}, \tilde{v})$ (cf. [4],[23]), identifying the first order terms and, for sake of simplicity, replacing the solution (U, V) by the new dependent variables $u(x,t) = e^{-iwt}e^{i\frac{c^2}{2}t}U(x,t)$ and $v(x,t) = V(x,t)$, we obtain the system

$$\left\{\begin{array}{lcl} iu_t + u_{xx} & = & (w - \frac{c^2}{2})u + \frac{a}{2}\phi^p[(p+2)u + pe^{icx}\overline{u}] + e^{i\frac{c}{2}x}\phi v + \psi u \\ & & \\ v_t - 3\gamma(\psi^2 v)_x & = & 2Re(e^{i\frac{c}{2}x}\phi u)_x, \end{array}\right. \tag{31}$$

which we complete with initial data

$$(u_0, v_0) \in H^2(\mathbb{R}) \times H^1(\mathbb{R}). \tag{32}$$

Since, for $p < 0$, ϕ^p is not, in general, a $C^2 \cap W^{2,\infty}(\mathbb{R})$ function, we begin by the study of a regularized system (with the same initial data):

$$\left\{\begin{array}{lcl} iu_t + u_{xx} & = & (w - \frac{c^2}{2})u + \frac{a}{2}(\phi+\epsilon)^p[(p+2)u + pe^{icx}\overline{u}] + e^{i\frac{c}{2}x}\phi v + \psi u \\ & & \\ v_t - 3\gamma(\psi^2 v)_x & = & 2Re(e^{-i\frac{c}{2}x}\phi u)_x, \end{array}\right. \tag{33}$$

where $\epsilon > 0$ if $p < 0$ ($\epsilon = 0$ otherwise).

We begin by proving the following result concerning this regularized system:

Proposition 5.1 *For each $p > -\frac{2}{3}$ there exists a unique solution*

$(u, v) \in (C([0, +\infty[; H^2) \cap C^1([0, +\infty[; L^2)) \times (C([0, +\infty[; H^1) \cap C^1([0, +\infty[; L^2))$

of system (33) with initial data $(u_0, v_0) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$.

Proof:

We follow the technique in [19],[33] and introduce an auxiliary system with non-local source which can be tackled by Kato's theory ([29], [30]). This is necessary in order to write the system (33) without derivative loss in the right-hand-side of the first equation (see [19] for details). Hence, we consider the system

$$\left\{\begin{array}{lcl} iF_t + F_{xx} & = & (w - \frac{c^2}{2} + \psi)F + a(\phi + \epsilon)^p[\frac{p}{2}(F + e^{icx}\overline{F}) + F] \\ & & +e^{i\frac{c}{2}x}\phi[3\gamma(\psi^2 v)_x + 2Re(e^{-i\frac{c}{2}x}\phi\tilde{u})_x] + \psi_t u \\ & & +e^{icx}\phi_t v + ap(\phi + \epsilon)^{p-1}\phi_t[\frac{p}{2}(u + e^{icx}\overline{u})] \\ v_t - 3\gamma(\psi^2 v)_x & = & 2Re(e^{-i\frac{c}{2}x}\phi\tilde{u})_x, \end{array}\right. \tag{34}$$

where

$$\left\{\begin{array}{lcl} u(x,t) & = & u_0(x) + \int_0^t F(x,s)ds, \\ \tilde{u}(x,t) & = & (\Delta - 1)^{-1}([(w - \frac{c^2}{2}) + \psi + \frac{a(p+2)}{2}(\phi + \epsilon)^p]u \\ & & +\frac{ap}{2}\phi^p e^{icx}\overline{u} + e^{i\frac{c}{2}x}\phi v - iF), \end{array}\right. \tag{35}$$

with initial data

$$F(.,0) = F_0 \in L^2(\mathbb{R}), \quad v(.,0) = v_0 \in H^1(\mathbb{R}). \tag{36}$$

Once we have, for a fixed $T > 0$, a solution

$$F \in C([0,T]; L^2) \cap C^1([0,T]; H^{-2}), \quad v \in C([0,T]; H^1) \cap C^1([0,T]; L^2) \tag{37}$$

for the problem (34)-(35)-(36), we can argue as in [19], Lemma 2.1, and show that (u, v) is the desired solution to system (33). We only sketch the argument, since it is similar to the one in [4] and [19].
First, we write (34) as a system of three equations, by decomposing F into its real and imaginary parts. This allows us to obtain a system with the abstract form

$$U_t + AU = g(t, U),\ U(.,0) = U_0, \tag{38}$$

with $U = (ReF, ImF, v)$ and $U_0 = (ReF_0, ImF_0, v_0)$, the corresponding initial data.

Following [4], [19], we decompose the operator

$$A = \begin{bmatrix} 0 & \Delta & 0 \\ -\Delta & 0 & 0 \\ 0 & 0 & -3\gamma[(\psi^2)_x + \psi^2 \frac{\partial}{\partial x}] \end{bmatrix}$$

in the form $SAS^{-1} = A + B$ for some operator B. In the present setting, we can choose

$$S = \begin{bmatrix} 1-\Delta & 0 & 0 \\ 0 & 1-\Delta & 0 \\ 0 & 0 & (1-\Delta)^{\frac{1}{2}} \end{bmatrix}$$

Note that $S : Y = L^2 \times L^2 \times H^1 \to X = H^{-2} \times H^{-2} \times L^2$ is an isomorphism. The relevant properties of S (in particular the ones concerning the entry $(1-\Delta)^{\frac{1}{2}}$) can be found in [29], Section 8. Observe that the right-hand-side of (38) is linear in U, hence it is straightforward to derive the necessary estimates for the source term g and we may finally apply Theorem 2 in [30] (or Theorem 7.1 in [29]) and conclude with the existence of a unique pair (F, v) satisfying (34)-(35)-(36), which achieves the sketch of the proof. ■

We are now in position to prove the linearized stability result:

Proposition 5.2 *Let $p > -\frac{2}{3}$ and consider a special solution $(\tilde{u}, \tilde{v})$ to (5) satisfying (29)-(30).Then $(\tilde{u}, \tilde{v})$ is linearly stable in the sense that for any $T > 0$ and any initial data $(u_0, v_0) \in H^1 \times L^2$, the system (31) admits a unique weak solution $(u, v) \in L^\infty(0, T; H^1 \times L^2)$ such that*

$$\|(u, v)\|^2_{L^\infty(0,T;H^1\times L^2)} \leq G_T(\|(u_0, v_0)\|^2_{H^1\times L^2}), \tag{39}$$

where $G_T : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function vanishing at the origin. Moreover, if $(u_0, v_0) \in H^2 \times H^1$ and $p \geq 0$, (u, v) is a strong solution satisfying

$$(u, v) \in [C([0, T]; H^2) \cap C^1([0, T]; L^2)] \times [C([0, T]; H^1) \cap C^1([0, T]; L^2)]$$

and

$$\|(u, v)\|^2_{L^\infty(0,T;H^2\times H^1)} \leq G_T(\|(u_0, v_0)\|^2_{H^2\times H^1}). \tag{40}$$

Proof:

We consider, for fixed ϵ, the solution (u_ϵ, v_ϵ) of system (33) with initial data $(u_{0\epsilon}, v_{0\epsilon}) \in H^2 \times H^1$, with

$$(u_{0\epsilon}, v_{0\epsilon}) \to (u_0, v_0) \quad \text{in } H^1 \times L^2.$$

In what follows, for simplicity, we will drop the subscript ϵ. By multiplying the first equation in (33) by $\overline{u}$ (respectively by $\overline{u_t}$), taking the imaginary part (respectively the real part) and integrating, we get

$$\frac{1}{2}\frac{d}{dt}\int |u|^2 dx = \frac{ap}{2} Im \int (\phi+\epsilon)^p e^{i\frac{c}{2}x}\overline{u}^2 dx + Im \int e^{i\frac{c}{2}x}\phi v\overline{u} dx$$

and

$$\frac{d}{dt}\left\{\frac{1}{2}\int |u_x|^2 dx + \frac{1}{2}\left(w-\frac{c^2}{2}\right)\int |u|^2 dx + \frac{a(p+2)}{4}\int (\phi+\epsilon)^p |u|^2 dx + \frac{1}{2}\int \psi |u|^2 dx\right\} +$$

$$\frac{ap}{2}\int (\phi+\epsilon)^p Re\left(e^{icx}\overline{u}\frac{\partial \overline{u}}{\partial t}\right) dx + \int \phi Re\left(e^{i\frac{c}{2}x} v \frac{\partial \overline{u}}{\partial t}\right) dx.$$

We have

$$(\phi+\epsilon)^p Re\left(e^{icx}\overline{u}\frac{\partial \overline{u}}{\partial t}\right) = \frac{1}{2}(\phi+\epsilon)^p \frac{\partial}{\partial t} Re\left(e^{icx}\overline{u}^2\right)$$

$$= \frac{1}{2}\frac{d}{dt}\left\{(\phi+\epsilon)^p Re\left(e^{icx}\overline{u}^2\right)\right\} + pc(\phi+\epsilon)^{p-1}\phi' Re\left(e^{icx}\overline{u}^2\right)$$

(recall that $c=0$ if $-\frac{2}{3}<p<0$ and $\epsilon=0$ if $p\geq 0$),

$$\phi Re\left(e^{i\frac{c}{2}x} v\frac{d\overline{u}}{dt}\right) = \frac{\partial}{\partial t}\left\{\phi Re\left(e^{i\frac{c}{2}x}v\overline{u}\right)\right\} + c\phi' Re\left(e^{i\frac{c}{2}x}v\overline{u}\right) - Re\left(e^{i\frac{c}{2}x}\phi\overline{u}\frac{\partial v}{\partial t}\right),$$

and, by the second equation in (31),

$$Re\left(e^{i\frac{c}{2}x}\phi\overline{u}\frac{\partial v}{\partial t}\right) = Re\left(3\gamma e^{i\frac{c}{2}x}\phi\overline{u}(\psi^2 v)_x\right) + 2Re\left(e^{i\frac{c}{2}x}\phi\overline{u}\right) Re\left(e^{-i\frac{c}{2}x}\phi u\right)_x,$$

and so

$$\int Re\left(e^{i\frac{c}{2}x}\phi\overline{u}\frac{\partial v}{\partial t}\right) dx = -3\gamma Re\int \left(e^{i\frac{c}{2}x}\phi\overline{u}\right)_x \psi^2 v dx.$$

Now, we also derive, from the second equation in (31),

$$\frac{1}{2}\frac{d}{dt}\int v^2 dx - 3\gamma\int (\psi^2 v)_x v dx = 2\int Re(e^{-i\frac{c}{2}x}\phi u)_x v dx.$$

Moreover,

$$\int (\psi^2 v)_x v dx = -\int \psi^2 v v_x dx = \frac{1}{2}\int (\psi^2)_x v^2 dx.$$

By applying Cauchy-Schwarz and Gronwall inequalities, it is now easy to obtain the following estimate for $t\in[0,T]$ and where $G_T:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function vanishing at the origin and indepedent of ϵ:

$$\|u_\epsilon(t)\|_{H^1}^2 + \|v_\epsilon(t)\|_{L^2}^2 \leq G_T(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2), \quad t\in[0,T]. \qquad (41)$$

The first part of the Theorem is now an easy consequence of (41) and (33), since, by (41), there exists a subsequence of $\{(u_\epsilon, v_\epsilon)\}$ (still denoted $\{(u_\epsilon, v_\epsilon)\}$) and $(u, v) \in L^\infty(0, T; H^1 \times L^2)$ such that

- $u_\epsilon \rightharpoonup u$ in $L^\infty(0, T; H^1)$ weak *;
- $v_\epsilon \rightharpoonup v$ in $L^\infty(0, T; L^2)$ weak *;
- (u, v) satisfies (39) and $(u_t, v_t) \in L^\infty(0, T; H^{-1} \times H^{-1})$.

Hence, $u \in C([0, T]; L^2)$, $v \in C([0, T]; H^{-1})$, $(u(0), v(0)) = (u_0, v_0)$ and (u, v) is a weak solution of (31). The uniqueness follows from (39).

In the case $p \geq 0$, we have $\phi^p \in C^2(\mathbb{R}) \cap W^{2,\infty}$ (cf. Remark 3.3), so we do not need to regularize ϕ: we can solve directly (31) for initial data $(u_0, v_0) \in H^2 \times H^1$. In this case we still obtain estimates of v_x, v_t, u_t and u_{xx} in L^2 to prove (40).
Differentiating the second equation of the system (31), multiplying by v_x, and after a few integrations by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int (v_x)^2 dx + \frac{15\gamma}{2}\int (\psi^2)_x (v_x)^2 dx = 3\int Re\left(e^{-i\frac{c}{2}x}\phi u\right)_{xx} v_x dx. \tag{42}$$

From (42) and the first equation in (31) we deduce, with $G_T : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous function vanishing at the origin:

$$\|v_x\|_2^2 \leq G_T(\|v_0\|_{H^1}^2)(\|u_{xx}\|_2^2 + \|u_x\|_2^2), \quad t \in [0, T]. \tag{43}$$

Now, the first equation and (41) gives

$$\|u_{xx}\|_2^2 \leq \|u_t\|_2^2 + G_T(\|(u_0, v_0)\|_{H^1 \times L^2}), \quad t \in [0, T]. \tag{44}$$

Finally, we differentiate with respect to time the first equation of (31), multiply by $\overline{u_t}$ and integrate the imaginary part to obtain

$$\frac{d}{dt}\|u_t\|_2^2 \leq C(\|u\|_2^2 + \|v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2), \quad t \in [0, T]. \tag{45}$$

From the second equation in (31) we also derive

$$\|v_t\|_2^2 \leq C(\|v\|_2^2 + \|v_x\|_2^2 + \|u\|_2^2 + \|u_x\|_2^2), \quad t \in [0, T]. \tag{46}$$

Applying Gronwall's inequality to (45) and, by (41), (43), (44) and (46), we obtain the estimate (40). ∎

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