On a free boundary problem of magnetohydrodynamics for a viscous incompressible fluid not subjected to capillary forces

V.A.Solonnikov

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Dedicated to Prof. Hugo Beirao da Veiga to his 70-th birthday

Abstract. The paper is is concerned with a free boundary problem of magnetohydrodynamics for viscous incompressible fluid not subjected to capillary forces on the free boundary. It is shown that the problem is correctly posed in anisotropic Sobolev spaces.

1 Introduction

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The paper is concerned with a free boundary problem of magnetohydrodynamics for viscous incompressible electrically conducting fluid. It is assumed that the fluid occupies a variable domain Ω_{1t} with the boundary Γ_t that should be found together with the vector fields of velocity $\boldsymbol{v}(x,t)$, magnetic and electric fields $\boldsymbol{H}(x,t)$, $\boldsymbol{E}(x,t)$ and the pressure function $p(x,t)$. The fluid is surrounded by an infinite vacuum region $\Omega_{2t}=\mathbb{R}^3\setminus\overline{\Omega}_{1t}$, where the fields $\boldsymbol{H}$ and $\boldsymbol{E}$ vanishing at infinity should be found as well. At the initial moment of time $t=0$ the domains $\Omega_{i0}\equiv\Omega_i$, $i=1,2$, and the vector fields $\boldsymbol{v}(x,0)$, $\boldsymbol{H}(x,0)$ are given.

The governing equations are the Navier-Stokes equations with the magnetic field $\boldsymbol{H}$ and the Maxwell equations without displacement current (i.e., without the time derivative $\boldsymbol{E}_t$) - see [1,2]:

$$\begin{cases} \boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}-\nabla\cdot T(\boldsymbol{v},p)-\nabla\cdot T_M(\boldsymbol{H})=0, \\ \nabla\cdot\boldsymbol{v}(x,t)=0, \quad x\in\Omega_{1t}, \ \ t>0, \\ \mu\boldsymbol{H}_t=-rot\boldsymbol{E}, \quad \nabla\cdot\boldsymbol{H}=0, \qquad x\in\Omega_{1t}\cup\Omega_{2t}, \\ rot\boldsymbol{H}=\alpha(\boldsymbol{E}+\mu(\boldsymbol{v}\times\boldsymbol{H})), \quad x\in\Omega_{1t}, \ \ t>0, \\ rot\boldsymbol{H}=0, \quad \nabla\cdot\boldsymbol{H}=0, \quad \nabla\cdot\boldsymbol{E}=0, \quad x\in\Omega_{2t}. \end{cases} \tag{1.1}$$

These equations are supplemented by initial and jump conditions

$$\begin{cases} T(\boldsymbol{v},p)\boldsymbol{n}+[\boldsymbol{n}\cdot T_M(\boldsymbol{H})\boldsymbol{n}]=0, \quad V_n=\boldsymbol{v}\cdot\boldsymbol{n}, \quad x\in\Gamma_t, \\ \mathfrak{n}_t[\mu\boldsymbol{H}]+[\mathfrak{n}_x\times\boldsymbol{E}]=0, \quad x\in\Gamma_t, \\ [\mu\boldsymbol{H}\cdot\boldsymbol{n}]=0, \quad [\boldsymbol{H}_\tau]=0, \quad x\in\Gamma_t, \\ \boldsymbol{v}(x,0)=\boldsymbol{v}_0(x), \quad x\in\Omega_{10}, \quad \boldsymbol{H}(x,0)=\boldsymbol{H}_0(x), \quad x\in\Omega_{10}\cup\Omega_{20}. \end{cases} \tag{1.2}$$

Here, $T(\boldsymbol{v},p)$ is the viscous stress tensor: $T(\boldsymbol{v},p)=-pI+\nu S(\boldsymbol{v})$, $S(\boldsymbol{v})=\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T$ is the doubled rate-of-strain tensor, $T_M(\boldsymbol{H})=\mu(\boldsymbol{H}\otimes\boldsymbol{H}-\frac{1}{2}|\boldsymbol{H}|^2I)$ is the magnetic stress tensor, μ is a piece-wise constant function equal to μ_i in Ω_{it}, $\alpha=const>0$ in Ω_{1t}, $\alpha=0$ in

Ω_{2t}, $\boldsymbol{n}$ is the normal to exterior with respect to Ω_{1t}, V_n is the velocity of evolution of Γ_t in the direction $\boldsymbol{n}$, $\boldsymbol{H}_\tau = \boldsymbol{H} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{H})$ is the tangential component of $\boldsymbol{H}$. By $[u]$ we mean the jump of the function $u(x)$, given in $\overline{\Omega}_{it}$, $i = 1, 2$, on Γ_t: $[u]|_{\Gamma_t} = u^{(1)} - u^{(2)}$, $u^{(i)} = u(x,t)|_{x\in\overline{\Omega}_{it}}$. Finally, $\mathfrak{n}_x = (\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3)$ and $\mathfrak{n}_t$ are the components of the normal vector $\mathfrak{n}$ to the surface $\mathfrak{G} = \{x \in \Gamma_t, t > 0\}$ in $\mathbb{R}^4$.

The condition $\mathfrak{n}_t[\mu\boldsymbol{H}] + [\mathfrak{n}_x \times \boldsymbol{E}] = 0$ on Γ_t stems from the assumption that the equation $\boldsymbol{B}_t = -rot\boldsymbol{E}$, $\boldsymbol{B} = \mu\boldsymbol{H}$, is satisfied in the sense of the distributions theory in the space-time $\mathbb{R}^4$ (see [3,4]).

To separate out a unique $\boldsymbol{E}$, we need to impose on $\boldsymbol{E}$ a certain normalization restriction, because, together with $\boldsymbol{E}$, the vector field $\boldsymbol{E} + \nabla\omega(x,t) \equiv \boldsymbol{E}_1$, satisfies all the relations (1.1), (1.2), if ω is a solution of the Dirichlet problem

$$\nabla^2\boldsymbol{\omega}(x,t) = 0, \quad x \in \Omega_{2t}, \quad \omega|_{x\in\Gamma_t} = 0, \quad \omega \to 0, \quad |x| \to \infty,$$

and $\omega = 0$ in Ω_1. The normalization condition can be taken in the form

$$\int_{\Gamma_t} \boldsymbol{E}^{(2)} \cdot \boldsymbol{n} dS = 0 \tag{1.3}$$

(see more detail in Section 3).

Free boundary problems of magnetohydrodynamics are studied in [5] and then in [3,4,6,7], always with the positive coefficient of the surface tension σ. The surface tension is a strong regularizer of the problem that guarantees, in particular, more smooth free boundary in comparison with the case $\sigma = 0$. Therefore the scheme of analysis of the problem with $\sigma = 0$ is essentially modified in comparison with the capillary case, in particular, in the choice of basic functional spaces. It turns out that the vector field of velocity of the fluid should possess one spatial derivative more than the magnetic field. This implies modification of all the auxiliary technical estimates, in particular, of nonlinear terms. Another peculiarity of the present work is the fact that the solution can be extended on the infinite time interval $t > 0$. The method of extension is proposed in [8].

We consider the simplest problem studied in [5], but with the infinite domain Ω_{2t}. The solution is found in anisotropic Sobolev - Slobodetskii spaces with the exponent of summability $p = 2$. For simplicity, the domain Ω_{1t} is assumed to be simply connected, although the results extend to the case of multi-connected Ω_{1t}, as in [6].

Before stating the main result of the paper, we write the problem (1.1), (1.2) in fixed domains $\Omega_1 \cup \Omega_2$, $\Omega_i = \Omega_{i0}$, using the Lagrangian coordinates $\xi \in \Omega_1$ that are connected with the Eulerian coordinates $x \in \Omega_{1t}$ by

$$x = \xi + \int_0^t \boldsymbol{u}(\xi,t)d\xi \equiv X(\xi,t), \quad \xi \in \Omega_1, \tag{1.4}$$

where $\boldsymbol{u}(\xi,t) = \boldsymbol{v}(X(\xi,t),t)$. Since $\boldsymbol{v}(x,t)$ is defined only in Ω_{1t}, we extend $\boldsymbol{u}(\xi,t)$ from Ω_1 into $\mathbb{R}^3$ by means of a fixed extension operator and consider the mapping

$$x = \xi + \int_0^t \boldsymbol{u}^*(\xi,\tau)d\tau = X^*(\xi,t), \quad \xi \in \mathbb{R}^3, \tag{1.5}$$

where $\boldsymbol{u}^*$ is the extension of $\boldsymbol{u}$. We assume that the extension conserves the regularity properties of $\boldsymbol{u}$ (in the sense that inequalities (1.10) are satisfied) and $\boldsymbol{u}^*$ has a fixed compact support.

The mapping (1.4) transforms $\Omega_i \equiv \Omega_{i0}$, $i = 1, 2$, into Ω_{it}, at least for small t.

Let

$$\mathcal{L}(\boldsymbol{u}) \equiv \Big(\frac{\partial x}{\partial \xi}\Big), \quad L(\boldsymbol{u}) = det\mathcal{L}, \quad \widehat{\mathcal{L}} = L\mathcal{L}^{-1}$$

be the Jacobi matrix of the transformation (1.5), the Jacobian and the co-factors matrix, respectively. We set $A \equiv \widehat{\mathcal{L}}^T$; the index "$T$" means transposition. If $\xi \in \overline{\Omega}_1$, then $L(\xi, t) = 1$, $\mathcal{L} = \widehat{A}^T$.

We make the change of variables (1.5) in (1.1)-(1.2) and introduce new unknown functions

$$\boldsymbol{u}(\xi, t), \quad q(\xi, t) = p(X, t), \quad \boldsymbol{h}(\xi, t) = \widehat{\mathcal{L}}\widetilde{\boldsymbol{H}}(\xi, t), \quad \boldsymbol{e}(\xi, t) = \widehat{\mathcal{L}}\widetilde{\boldsymbol{E}}(\xi, t),$$

where

$$\widetilde{\boldsymbol{H}}(\xi, t) = \boldsymbol{H}(X^*(\xi, t), t), \quad \widetilde{\boldsymbol{E}}(\xi, t) = \boldsymbol{E}(X^*(\xi, t), t).$$

Since

$$\begin{aligned}
&\nabla_\xi \cdot \boldsymbol{h}(\xi, t) = L\nabla_x \cdot \boldsymbol{H}(x, t)|_{x=X^*(\xi,t)}, \quad \nabla_\xi \cdot \boldsymbol{e}(\xi, t) = L\nabla_x \cdot \boldsymbol{E}(x, t)|_{x=X^*(\xi,t)},\\
&rot_x\boldsymbol{H} = \frac{\mathcal{L}}{L} rot_y \mathcal{L}^T\widetilde{\boldsymbol{H}}, \quad \boldsymbol{H}_t|_{x=X^*} = \frac{\partial}{\partial t}\widetilde{\boldsymbol{H}} - (\mathcal{L}^{-1}\boldsymbol{u} \cdot \nabla)\widetilde{\boldsymbol{H}},\\
&\widetilde{\boldsymbol{H}} \cdot \boldsymbol{n}(X) = |A\boldsymbol{n}_0|^{-1}(\boldsymbol{h} \cdot \boldsymbol{n}_0), \quad \widetilde{\boldsymbol{H}} - \boldsymbol{n}(\widetilde{\boldsymbol{H}} \cdot \boldsymbol{n}) = A^T(\boldsymbol{h} - \frac{A^TA\boldsymbol{n}_0}{|A\boldsymbol{n}_0|^2}(\boldsymbol{h} \cdot \boldsymbol{n}_0)), \quad \xi \in \Gamma_0,
\end{aligned}$$

(cf. [5],), we obtain

$$\begin{cases}
\boldsymbol{u}_t - \nu\nabla_u^2\boldsymbol{u} + \nabla_u q - \nabla_u \cdot T_M(\mathcal{L}\boldsymbol{h}) = 0,\\
\nabla_u \cdot \boldsymbol{u} = 0, \quad \xi \in \Omega_1, \ \ t > 0,\\
T_u(\boldsymbol{u}, q)\boldsymbol{n}(X, t) + \big[T_M\big(\mathcal{L}\boldsymbol{h}\big)\boldsymbol{n}(X, t)\big] = 0, \quad \xi \in \Gamma_0,\\
\boldsymbol{u}(\xi, 0) = \boldsymbol{v}_0(\xi) \equiv \boldsymbol{u}_0(\xi), \quad \xi \in \Omega_1,
\end{cases} \tag{1.6}$$

$$\begin{cases}
\mu(\boldsymbol{h}_t - \widehat{\mathcal{L}}_t^T\frac{\mathcal{L}}{L}\boldsymbol{h} - \widehat{\mathcal{L}}^T(\boldsymbol{u} \cdot \nabla_u)\frac{\mathcal{L}}{L}\boldsymbol{h}) = -rot\mathcal{P}(\xi, t)\boldsymbol{e},\\
\xi \in \Omega_i, \quad i = 1, 2,\\
\mathcal{P}rot\mathcal{P}\boldsymbol{h} = \alpha(\mathcal{P}\boldsymbol{e} + \mu(\mathcal{L}^{-1}\boldsymbol{u} \times \boldsymbol{h})), \ \nabla \cdot \boldsymbol{h} = 0, \ \xi \in \Omega_1,\\
rot\mathcal{P}\boldsymbol{h} = 0, \quad \nabla \cdot \boldsymbol{h} = 0, \quad \nabla \cdot \boldsymbol{e} = 0, \quad \xi \in \Omega_2,
\end{cases} \tag{1.7}$$

$$\begin{cases}
[\mu\boldsymbol{h} \cdot \boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}_\tau] = (\frac{A^TA\boldsymbol{n}_0}{|A\boldsymbol{n}_0|^2} - \boldsymbol{n}_0))[\boldsymbol{h} \cdot \boldsymbol{n}_0],\\
[\boldsymbol{n}_0 \times \mathcal{P}\boldsymbol{e}] = (\boldsymbol{u} \cdot A\boldsymbol{n}_0)[\mu]\boldsymbol{h}_\tau, \quad \xi \in \Gamma_0,\\
\boldsymbol{h}, \boldsymbol{e} \to 0, \quad |\xi| \to \infty,\\
\boldsymbol{h}(\xi, 0) = \boldsymbol{h}_0(\xi), \quad \xi \in \Omega_i, \quad i = 1, 2, 3,
\end{cases} \tag{1.8}$$

where $\mathcal{P} = \mathcal{L}^T\mathcal{L}/L$,

$\nabla_u = \mathcal{L}^{-T}\nabla$ is the transformed gradient w.r. to x; ∇ is the gradient w.r.to ξ; $\nabla_u = A\nabla$ in Ω_1,

$\boldsymbol{n}(X) = A\boldsymbol{n}_0(\xi)/|A\boldsymbol{n}_0(\xi)|$,

$\boldsymbol{n}_0$: the exterior (w.r. to Ω_1) normal to Γ_0,

$T_u = -q + \nu S_u(\boldsymbol{u})$ is the transformed stress tensor;
$S_u(\boldsymbol{w}) = \nabla_u \boldsymbol{w} + (\nabla_u \boldsymbol{w})^T$ is the transformed rate-of-strain tensor.

We notice that the vector field

$$\boldsymbol{\Phi}(\xi,t) = \widehat{\mathcal{L}}_t^T \frac{\mathcal{L}}{L}\boldsymbol{h} + \widehat{\mathcal{L}}^T(\boldsymbol{u}\cdot\nabla_u)\frac{\mathcal{L}}{L}\boldsymbol{h} = \boldsymbol{h}_t - \widehat{\mathcal{L}}\boldsymbol{H}_t(x,t)|_{x=X^*}$$

is divergence free: $\nabla\cdot\boldsymbol{\Phi}(\xi,t) = 0$, $\xi\in\Omega_i$, $i=1,2$.

The next step is to separate the determination of $\boldsymbol{u}, q, \boldsymbol{h}$ from that of $\boldsymbol{e}$. We notice that (1.6), (1.7) imply

$$\begin{cases} \mu(\boldsymbol{h}_t - \boldsymbol{\Phi}) + \alpha^{-1} rot\mathcal{P}rot\mathcal{P}\boldsymbol{h} = rot\boldsymbol{J}, \quad \nabla\cdot\boldsymbol{h}(\xi,t) = 0, \quad \xi\in\Omega_1\cup\Omega_3, \\ rot\mathcal{P}\boldsymbol{h}(\xi,t) = 0, \quad \nabla\cdot\boldsymbol{h} = 0, \quad \xi\in\Omega_2, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \Big(\dfrac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{n}_0}{|\widehat{\mathcal{L}}^T\boldsymbol{n}_0|^2} - \boldsymbol{n}_0\Big)[\boldsymbol{h}\cdot\boldsymbol{n}_0], \quad \xi\in\Gamma_0, \\ \boldsymbol{h}(\xi,0) = \boldsymbol{h}_0(\xi), \quad \xi\in\Omega_1\cup\Omega_2, \end{cases} \tag{1.9}$$

where $\boldsymbol{J} = \mu_1\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}$ in Ω_1.

Equations (1.6), (1.9) constitute the main problem for $\boldsymbol{u}, q, \boldsymbol{h}$ that is solved in Sections 2 and 3. After this it is possible to reconstruct $\boldsymbol{e}(\xi,t)$ satisfying (1.6), (1.7).

Theorem 1. *Assume that $\Gamma_0\in W_2^{l+3/2}$, $l\in(3/2,2)$ and the vector field $\boldsymbol{u}^*(\xi,t)$ in (1.5) vanishes for large $|\xi|$: $|\xi|\geq R$ and satisfies the inequalities*

$$\|\boldsymbol{u}^*(\cdot,t)\|_{W_2^{2+l}(\mathbb{R}^3)} \leq c\|\boldsymbol{u}^*(\cdot,t)\|_{W_2^{2+l}(\Omega_1)}, \quad \|\boldsymbol{u}^*(\cdot,t)\|_{L_2(\mathbb{R}^3)} \leq c\|\boldsymbol{u}(\cdot,t)\|_{L_2(\Omega_1)}. \tag{1.10}$$

Then for arbitrary $\boldsymbol{u}_0\in W_2^{l+1}(\Omega_1)$, $\boldsymbol{h}_0\in W_2^{\lambda+1}(\Omega_i)$, $i=1,2$, $\lambda = l-1$, satisfying the compatibility and orthogonality conditions

$$\begin{aligned} &\nabla\cdot\boldsymbol{u}_0(\xi) = 0, \quad \xi\in\Omega_1, \quad (S(\boldsymbol{u}_0)\boldsymbol{n}_0)_\tau = 0, \quad \xi\in\Gamma_0, \\ &\nabla\cdot\boldsymbol{h}_0(\xi) = 0, \quad \xi\in\Omega_i, \quad i=1,2, \quad rot\boldsymbol{h}_0(\xi) = 0, \quad \xi\in\Omega_2, \\ &[\mu\boldsymbol{h}_0\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}_{0\tau}] = 0, \quad \xi\in\Gamma_0, \end{aligned} \tag{1.11}$$

$$\int_{\Omega_1}\boldsymbol{v}_0(\xi)d\xi = 0, \quad \int_{\Omega_1}\boldsymbol{v}_0\cdot\boldsymbol{\eta}_j(\xi)d\xi = 0, \quad j=1,2,3, \tag{1.12}$$

with $\boldsymbol{\eta}_j(\xi) = \boldsymbol{e}_j\times\boldsymbol{\xi}$, $\boldsymbol{e}_j = (\delta_{jk})_{j,k=1,2,3}$, and the smallness condition

$$\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_1)} + \sum_{i=1,2}\|\boldsymbol{h}_0\|_{W_2^{l+1}(\Omega_1)} \leq \epsilon \ll 1 \tag{1.13}$$

the problem (1.5), (1.9) has a unique solution $\boldsymbol{u}\in W_2^{2+l,1+l/2}(Q_T^1)$, $\nabla q\in W_2^{l,l/2}(Q_T^1)$, $\boldsymbol{h}_0\in W_2^{2+\lambda,1+\lambda/2}(Q_T^i)$, where $Q_T^i = \Omega_i\times(0,T)$, $T>0$. The solution is defined in an infinite time interval $t>0$ and satisfies the inequality

$$\begin{aligned} &\|e^{\beta t}\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(Q_\infty^1)} + \|e^{\beta t}\nabla q\|_{W_2^{l,l/2}(Q_\infty^1)} + \sum_{i=1}^2\|e^{\beta t}\boldsymbol{h}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_\infty^i)} \\ &\leq c(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_1)} + \sum_{i=1}^2\|\boldsymbol{h}_0\|_{W_2^{\lambda+1}(\Omega_i)}), \quad \beta = const > 0. \end{aligned} \tag{1.14}$$

It follows that $\Gamma_t = X\Gamma_0$ tends to a limit Γ_∞ as $t \to \infty$, and Γ_∞ is located in a neighborhood of Γ_0. Since $X(\cdot, t) \in W_2^{l+3/2}(\Gamma_0)$, there is no loss of smoothness of the free boundary Γ_t for $t > 0$.

2 Linear problems

The proof of Theorem 1 is based on the analysis of the following non-homogeneous linear problems:

$$\begin{cases} \boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(\xi,t), \ \ \nabla\cdot\boldsymbol{v} = f, \quad \xi\in\Omega_1, \ \ t>0, \\ T(\boldsymbol{v},p)\boldsymbol{n} = \boldsymbol{d}(\xi,t), \ \ \xi\in\Gamma_0, \\ \boldsymbol{v}(\xi,0) = \boldsymbol{v}_0(\xi), \quad \xi\in\Omega_1, \end{cases} \tag{2.1}$$

$$\begin{cases} \mu\boldsymbol{H}_t(\xi,t) + \alpha^{-1} rotrot\boldsymbol{H}(\xi,t) = \boldsymbol{G}(\xi,t), \quad \nabla\cdot\boldsymbol{H}(\xi,t) = 0, \quad \xi\in\Omega_1, \\ rot\boldsymbol{H}(\xi,t) = rot\boldsymbol{\ell}(\xi,t), \quad \nabla\cdot\boldsymbol{H}(\xi,t) = 0, \quad \xi\in\Omega_2, \\ [\mu\boldsymbol{H}\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{H}_\tau] = \boldsymbol{a}(\xi,t), \quad \xi\in\Gamma_0, \\ \boldsymbol{H}(\xi,0) = \boldsymbol{H}_0(\xi), \quad \xi\in\Omega_1\cup\Omega_2. \end{cases} \tag{2.2}$$

$$\begin{cases} rot\boldsymbol{h}(\xi) = \boldsymbol{k}(\xi), \quad \nabla\cdot\boldsymbol{h} = 0, \quad \xi\in\Omega_1\cup\Omega_2, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}_\tau] = \boldsymbol{a}, \quad \xi\in\Gamma_0, \\ \boldsymbol{h}\cdot\boldsymbol{n}(\xi) = 0, \quad \xi\in S. \end{cases} \tag{2.3}$$

Theorem 2. 1. *Assume that* $\Gamma_0 \in W_2^{3/2+l}$, $l\in(3/2,2)$, $\boldsymbol{f}\in W_2^{l,l/2}(Q_T^1)$ f *is given in the whole space* $\mathbb{R}^3$, *compactly supported and belongs to* $W_2^{l+1,0}(\mathbb{R}_T)$, $\mathbb{R}_T = \mathbb{R}^3\times(0,T)$, $f = \nabla\boldsymbol{F} + f'$, $\boldsymbol{F}, f' \in W_2^{0,l+1/2}(\mathbb{R}_T)$, $\boldsymbol{d}\in W_2^{1/2+l,1/4+l/2}(G_T)$, $G_T = \Gamma_0\times(0,T)$, $\boldsymbol{v}_0\in W_2^{l+1}(\Omega_1)$, *and let the compatibility conditions*

$$\nabla\cdot\boldsymbol{v}_0(\xi) = f(\xi,0), \quad \xi\in\mathcal{F}_1, \quad \nu(S(\boldsymbol{v}_0)\boldsymbol{n}_0)_\tau = \boldsymbol{d}_\tau(\xi,0), \quad \xi\in\mathcal{G}, \tag{2.4}$$

be satisfied. Then the problem (2.1) has a unique solution $\boldsymbol{v}, p$ *such that* $\boldsymbol{v}\in W_2^{2+l,1+l/2}(Q_T^1)$, $\nabla p\in W_2^{l,l/2}(Q_T^1)$, $p|_{\xi\in\Gamma_0}\in W_2^{l+1/2,l/2+1/4}(G_T)$, *and the solution satisfies the inequality*

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{l+2,1+l/2}(Q_T^1)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \\ &\leq c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)} + \|f\|_{W_2^{l+1,0}(\mathbb{R}_T)} + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(\mathbb{R}_T)} \\ &+ \|f'\|_{W_2^{0,1+l/2}(\mathbb{R}_T)} + \|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_1)}\Big). \end{aligned} \tag{2.5}$$

The constant $c(T)$ *in (2.5) is an increasing functions of* T.

2. If $\boldsymbol{f} = 0$, $f = 0$, $\boldsymbol{d} = 0$ *and the initial data satisfy the conditions (1.12), then the solution is defined for all* $t>0$ *and*

$$\begin{aligned} &\|e^{\beta t}\boldsymbol{v}\|_{W_2^{l+2,1+l/2}(Q_T^1)} + \|e^{\beta t}\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|e^{\beta t}p\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \\ &\leq c\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_1)} \quad \beta\in[0,b), \quad b>0. \end{aligned} \tag{2.6}$$

The condition $\nabla\cdot\boldsymbol{v}_0(\xi) = f(\xi,0)$, $\xi\in\Omega_1$ can be understood in a weak sense as $\int_{\mathcal{F}_1}(\boldsymbol{v}_0(y) - \boldsymbol{F}(y,0))\cdot\nabla\eta(y) + f'\eta)dy = 0$ for arbitrary smooth η such that $\eta|_{\Gamma_0} = 0$.

Theorem 3. *1. Assume that* $\Gamma_0\in W_2^{l+3/2}$, $\boldsymbol{k} = rot\boldsymbol{K}(\xi,t)$, $\boldsymbol{a} = [\boldsymbol{A}]$, $\boldsymbol{K},\boldsymbol{A}$ *are compactly supported vector fields from* $\cap_{i=1,2}W_2^{2+\lambda,1+\lambda/2}(Q_T^i)$, $\lambda = l-1$, *and*

$$[\boldsymbol{K}_\tau] = \boldsymbol{a}, \quad \boldsymbol{A}^{(1)}\cdot\boldsymbol{n}_0(\xi) = \boldsymbol{A}^{(2)}\cdot\boldsymbol{n}_0 = 0, \quad \xi\in\Gamma_0. \tag{2.7}$$

Then the problem (2.3) has a unique solution $\boldsymbol{h} \in W_2^{2+\lambda,1+\lambda/2}(Q_{Ti})$, $i=1,2$, *and it satisfies the inequality*

$$\sum_{i=1}^{2} \|\boldsymbol{h}\|_{W_2^{\lambda+2,\lambda/2+1}(Q_T^i)} \le c\sum_{i=1}(\|\boldsymbol{K}\|_{W_2^{\lambda+2,\lambda/2+1}(Q_T^i)} + \|\boldsymbol{A}\|_{W_2^{\lambda+2,\lambda/2+1}(Q_T^i)}). \tag{2.8}$$

2. *Let* $\boldsymbol{k}$ *be a square integrable function of* $\xi \in \mathbb{R}^3$,

$$\nabla\cdot\boldsymbol{k}^{(1)} = 0, \quad \xi\in\Omega_1, \quad \boldsymbol{k}\cdot\boldsymbol{n}_0 = 0, \quad \xi\in\Gamma_0,$$

$\boldsymbol{k}^{(2)} = 0$, *and* $\boldsymbol{a} = 0$. *Then the problem (2.3) has a unique solution from* $\cap_{i=1,2} W_2^1(\Omega_i)$ *and*

$$\sum_{i=1,2} \|\boldsymbol{h}\|_{W_2^1(\Omega_i)} \le c\|\boldsymbol{k}\|_{L_2(\Omega_1)} \tag{2.9}$$

Theorem 4. *1. Assume that the data of the problem (2.2) possess the following properties:* $\boldsymbol{G} \in W_2^{2+\lambda,1+\lambda/2}(Q_T^1)$, $\boldsymbol{H}_0 \in W_2^{1+\lambda}(\Omega_j)$, $j=1,2$, $\boldsymbol{\ell} \in W_2^{2+\lambda,1+\lambda/2}(Q_T^2)$, $\boldsymbol{a} \in W_2^{3/2+\lambda,3/4+\lambda/2}(G_T)$, *moreover,* $\boldsymbol{a} = [\boldsymbol{A}]$ *with* $\boldsymbol{A}^{(i)} \in W_2^{2+\lambda,1+\lambda/2}(Q_T^i)$ *satisfying (2.7), finally, let the compatibility conditions*

$$\begin{aligned} &\nabla\cdot\boldsymbol{G}(\xi,t) = 0, \quad \xi\in\Omega_1,\\ &\nabla\cdot\boldsymbol{H}_0(\xi) = 0, \quad \xi\in\Omega_1\cup\Omega_2, \quad rot\boldsymbol{H}_0(\xi) = rot\boldsymbol{\ell}(\xi,0), \quad \xi\in\Omega_2,\\ &[\mu\boldsymbol{H}_0\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{H}_{0\tau}] = \boldsymbol{a}(\xi,0) = [\boldsymbol{A}(\xi,0)], \quad \xi\in\Gamma_0 \end{aligned} \tag{2.10}$$

hold. Then the problem (2.2) has a unique solution $\boldsymbol{H} \in W_2^{2+\lambda,1+\lambda/2}(Q_T^i)$, $\quad i=1,2$, *and*

$$\begin{aligned} \sum_{i=1}^{2} \|\boldsymbol{H}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^i)} &\le c\Big(\|\boldsymbol{G}\|_{W_2^{\lambda,\lambda/2}(Q_T^1)} + \sum_{i=1}^{2}\|\boldsymbol{H}_0\|_{W_2^{1+\lambda}(\Omega_i)}\\ &+ \|\boldsymbol{\ell}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^2)} + \sum_{i=1}^{2}\|\boldsymbol{A}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^i)}\Big) \end{aligned} \tag{2.11}$$

2. *Moreover, if* $\ell = 0$, $\boldsymbol{a} = 0$, $\boldsymbol{G} = 0$, *then the solution is defined for all* $t > 0$ *and*

$$\sum_{i=1}^{3} \|e^{\alpha t}\boldsymbol{H}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^i)} \le c\sum_{i=1}^{3}\|\boldsymbol{H}_0\|_{W_2^{1+\lambda}(\Omega_0)}, \quad \alpha\in[0,a), \quad a>0. \tag{2.12}$$

We give the main ideas of the proof of Theorems 2 - 4, starting with the first statement of Theorem 2. The first step is the reduction of (2.1) to a similar problem with $f = 0$. Since $f, \boldsymbol{F}, f'$ are given in $\mathbb{R}^3$ and compactly supported, we can define an auxiliary vector field

$$\boldsymbol{w}_1(\xi,t) = \nabla\int_{\mathbb{R}^3} E(\xi-\eta)f(\eta,t)d\eta$$

satisfying the equation $\nabla\cdot\boldsymbol{w}_1 = f$. It is clear that

$$\|\boldsymbol{w}_1\|_{W_2^{2+\lambda,0}(\mathbb{R}_T)} \le c\|f\|_{W_2^{l+\lambda,0}(\mathbb{R}_T)}. \tag{2.13}$$

In addition, since

$$\boldsymbol{w}_1 = \nabla \int_{\mathbb{R}^3} (\nabla E(\xi - \eta) \cdot \boldsymbol{F}(\eta, t) + E(\xi - \eta) f'(\eta, t))) d\eta,$$

we have

$$\|\boldsymbol{w}_1\|_{L_2(\mathbb{R}^3)} \le c(\|\boldsymbol{F}\|_{L_2(\mathbb{R}^3)} + \|f'\|_{L_2(\mathbb{R}^3)}), \quad \forall t \in (0, T).$$

Applying this inequality to the time derivative $\boldsymbol{w}_{1,t}$ and to the finite difference $\Delta_t(-h)\boldsymbol{w}_{1,t} = \boldsymbol{w}_{1,t}(\xi, t-h) - \boldsymbol{w}_{1,t}(\xi, t)$, we obtain

$$\|\boldsymbol{w}_1\|_{W_2^{2+l,1+l/2}(\mathbb{R}_T)} \le c(\|\boldsymbol{F}\|_{W_2^{0,l+1/2}(\mathbb{R}_T)} + \|f'\|_{W_2^{0,l+1/2}(\mathbb{R}_T)}). \tag{2.14}$$

For the difference $\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{w}_1$ and p we get the problem

$$\begin{cases} \boldsymbol{v}_{1,t} - \nu \nabla^2 \boldsymbol{v}_1 + \nabla p = \boldsymbol{f}_1(\xi, t) = \boldsymbol{f} - \boldsymbol{w}_{1,t} + \nu \nabla^2 \boldsymbol{w}_1, \\ \nabla \cdot \boldsymbol{v}_1 = 0, \quad \xi \in \Omega_1, \ \ t > 0, \\ T(\boldsymbol{v}_1, p)\boldsymbol{n}_0 = \boldsymbol{d}_1(\xi, t) = \boldsymbol{d} - \nu S(\boldsymbol{w}_1)\boldsymbol{n}_0, \ \ \xi \in \Gamma_0, \\ \boldsymbol{v}_1(\xi, 0) = \boldsymbol{v}_1(\xi) = \boldsymbol{v}_0 - \boldsymbol{w}_1(\xi, 0), \quad \xi \in \Omega_1, \end{cases} \tag{2.15}$$

It is analyzed by repeating the arguments in [9, Theorem 2], which leads to the proof of the first statement of Theorem 2.

We notice that the reduction of the problem (2.1) to (2.14) is made in [9] in a different way that has required some additional smoothness of Γ_0.

We pass to the second statement of Theorem 2. The exponential decay of the solution can be established using the energy inequality. By elementary calculation it is easy to verify that

$$\frac{d}{dt} \int_{\Omega_1} \boldsymbol{v}(\xi, t) d\xi = 0, \quad \frac{d}{dt} \int_{\Omega_1} \boldsymbol{v}(\xi, t) \cdot \boldsymbol{\eta}_i(\xi) d\xi = 0, \quad i = 1, 2, 3,$$

which implies

$$\int_{\Omega_1} \boldsymbol{v}(\xi, t) d\xi = 0, \quad \int_{\Omega_1} \boldsymbol{v}(\xi, t) \cdot \boldsymbol{\eta}_i(\xi) d\xi = 0. \tag{2.16}$$

By the energy relation and the Korn inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{v}(\cdot, t)\|^2_{L_2(\Omega_1)} + b\|\boldsymbol{v}(\cdot, t)\|^2_{L_2(\Omega_1)} \le 0, \quad b > 0$$

and, as a consequence,

$$\begin{aligned} &e^{bt} \|\boldsymbol{v}\|_{L_2(\Omega_1)} \le \|\boldsymbol{v}_0\|_{L_2(\Omega_1)}, \\ &\Big(\int_0^T e^{2\beta t} \|\boldsymbol{v}\|^2_{L_2(\Omega_1)} \Big)^{1/2} \le c\|\boldsymbol{v}_0\|_{L_2(\Omega_1)}, \quad \beta < b. \end{aligned} \tag{2.17}$$

The solution of the problem (2.1) satisfies the local (in time) estimate

$$\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_{t-1/2,t})} \le c\|\boldsymbol{v}\|_{L_2(Q_{t-1,t})}, \quad Q_{s,t} = \Omega_1 \times (s, t),$$

that is proved in the same way as (4.6) in [10] (cf. the proof of (2.26) below). From this estimate and (2.17) it is easy to deduce

$$\|e^{\beta t} \boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_{1/2,T})} \le c\|\boldsymbol{v}_0\|_{L_2(\Omega_1)}.$$

This inequality and (2.5) (for $T=1$) imply (2.6).

The first part of Theorem 3 is proved in [5] even in a more complicated case when Ω_2 is a bounded domain and $\Omega=\Omega_1\cup\Gamma_0\cup\Omega_2$ is multi-connected. The formula (3.18) in [3] (the representation of the solution of (2.3)) remains valid also for our case, since the functions $f, \boldsymbol{F}, f'$ are compactly supported (but the last term in this formula drops out, because $\mathbb{R}^3$ is simply connected). The same is true for the estimates (3.15), (3.17) that imply (2.8).

The second statement of the theorem is proved in [11]. The estimate (2.9) is obtained using the representation formula for the solution: $\boldsymbol{h}=\boldsymbol{h}'+\nabla\phi$,

$$\boldsymbol{h}'(\xi)=\frac{1}{4\pi}rot\int_{\Omega_1}\frac{\boldsymbol{k}(\eta)d\eta}{|\xi-\eta|},$$
$$\nabla^2\phi(\xi)=0,\quad \xi\in\Omega_1\cup\Omega_2,\quad [\phi]=0,\quad [\mu\frac{\partial\phi}{\partial n_0}]=-[\mu]\boldsymbol{h}'\cdot\boldsymbol{n}_0,\quad \xi\in\Gamma_0.$$

We pass to the first statement of Theorem 4. First of all, the problem (2.2) is reduced to a similar problem with $\ell=0$, $\boldsymbol{a}=0$. We extend $\boldsymbol{\ell}$ into the whole space $\mathbb{R}^3$ so that the extended field $\boldsymbol{\ell}^*$ satisfies

$$\|\boldsymbol{\ell}^*\|_{W_2^{2+\lambda,1+\lambda/2}(\mathbb{R}_T)}\leq\|\boldsymbol{\ell}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^2)},\tag{2.18}$$

and we solve the problem (2.3) with $\boldsymbol{k}=rot\boldsymbol{K}$, $\boldsymbol{K}=\boldsymbol{\ell}^*+\boldsymbol{A}$. By (2.8) and (2.18), the solution $\boldsymbol{h}_1$ satisfies

$$\sum_{i=1,2}\|\boldsymbol{h}_1\|_{W_2^{\lambda+2,\lambda+l/2}(Q_T^i)}\leq c(\|\boldsymbol{\ell}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^2)}+\sum_{i=1,2}\|\boldsymbol{A}^{(i)}\|_{W_2^{2+\lambda+1+\lambda/2}(Q_T^i)}).\tag{2.19}$$

For $\boldsymbol{h}=\boldsymbol{H}-\boldsymbol{h}_1$ we obtain the problem

$$\begin{cases}\mu\boldsymbol{h}_t(\xi,t)+\alpha^{-1}rotrot\boldsymbol{h}(\xi,t)=\boldsymbol{g}(\xi,t),\quad \nabla\cdot\boldsymbol{h}(\xi,t)=0,\quad \xi\in\Omega_1,\\ rot\boldsymbol{h}(\xi,t)=0,\quad \nabla\cdot\boldsymbol{h}(\xi,t)=0,\quad \xi\in\Omega_2,\\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}_0]=0,\quad [\boldsymbol{h}_\tau]=0,\quad \xi\in\Gamma_0,\\ \boldsymbol{h}(\xi,0)=\boldsymbol{H}_0(\xi)-\boldsymbol{h}_1(\xi,0)=\boldsymbol{h}_0(\xi),\quad \xi\in\Omega_1\cup\Omega_2,\end{cases}\tag{2.20}$$

where $\boldsymbol{g}=\boldsymbol{G}-\mu\boldsymbol{h}_{1,t}-\alpha^{-1}rotrot\boldsymbol{h}_1$. It has been studied in [12,11,5], in particular, in [5] the estimate

$$\sum_{i=1,2}\|\boldsymbol{h}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^i)}\leq c\Big(\|\boldsymbol{g}\|_{W_2^{\lambda,\lambda/2}(Q_T^1)}+\sum_{i=1}^2\|\boldsymbol{h}_0\|_{W_2^{1+\lambda}(\Omega_i)}\Big)\tag{2.21}$$

is obtained. Together with (2.19),this estimate implies (2.11).

Now we obtain the inequality (2.12) under the assumptions of the second part of Theorem 4. Let $\mathcal{H}^{(1)}$ be the space of the vector fields $\boldsymbol{\psi}\in W_2^1(\Omega_i)$, $i=1,2$, satisfying the conditions

$$\begin{aligned}&\nabla\cdot\boldsymbol{\psi}(\xi)=0,\quad \xi\in\Omega_i,\quad i=1,2,\quad rot\boldsymbol{\psi}=0,\quad \xi\in\Omega_2,\\ &[\mu\boldsymbol{\psi}\cdot\boldsymbol{n}_0]=0,\quad [\boldsymbol{\psi}_\tau]=0,\quad \xi\in\Gamma_0,\end{aligned}\tag{2.22}$$

The equations $rot\boldsymbol{\psi}=0,\quad \nabla\cdot\boldsymbol{\psi}=0$ in Ω_2 imply

$$\boldsymbol{\psi}(\xi)=\nabla\varphi(\xi),\quad \xi\in\Omega_2,$$

where φ is the solution of the Neumann problem

$$\nabla^2\varphi(\xi) = 0, \quad \xi \in \Omega_2, \quad \mu_2\frac{\partial\varphi}{\partial n} = \mu\boldsymbol{\psi}^{(1)}\cdot\boldsymbol{n}, \quad \xi \in \Gamma_0, \quad \varphi(\xi) \to 0, \quad |\xi| \to \infty.$$

Since $\nabla\varphi$ decays at infinity as $|x|^{-2}$, it belong to $L_2(\Omega_2)$.

In particular, the solutions of a homogeneous problem (2.20) satisfy (2.22).

For arbitrary $\boldsymbol{\psi} \in \mathcal{H}_0^{(1)}$ we define the function $\Phi(\xi)$, $\xi \in \Omega_1$, as the solution of the Dirichlet problem

$$\nabla^2\Phi(\xi) = 0, \quad \Phi(\xi) = \varphi(\xi), \quad \xi \in \Gamma_0. \tag{2.23}$$

We multiply the first equation in (2.20) (with $\boldsymbol{g} = 0$) by $\boldsymbol{\psi} - \nabla\Phi$, $\boldsymbol{\psi} \in \mathcal{H}_0^{(1)}$, integrate over Ω_1 and make use of the boundary condition $\boldsymbol{\psi}_\tau - \nabla_\tau\Phi|_{\Gamma_0} = 0$. After easy calculation (as in [5]) we obtain

$$\int_{\mathbb{R}^3}\mu\boldsymbol{H}_t\cdot\boldsymbol{\psi}(\xi)d\xi + \alpha^{-1}\int_{\Omega_1} rot\boldsymbol{H}\cdot rot\boldsymbol{\psi}d\xi = 0, \quad \forall\psi \in \mathcal{H}_0^{(1)}. \tag{2.24}$$

Setting $\boldsymbol{\psi} = \boldsymbol{H}(\xi, t)$ in (2.24) we obtain the energy relation

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\mu}\boldsymbol{H}(\cdot, t)\|^2_{L_2(\mathbb{R}^3)} + \alpha^{-1}\|rot\boldsymbol{H}\|^2_{L_2(\Omega_1)} = 0,$$

which implies, in view of (2.9),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\mu}\boldsymbol{H}(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + a\|\sqrt{\mu}\boldsymbol{H}(\cdot,t)\|^2_{L_2(\Omega_1)} \le 0, \quad a > 0,\\
&e^{at}\|\sqrt{\mu}\boldsymbol{H}(\cdot,t)\|_{L_2(\Omega_1)} \le \|\sqrt{\mu}\boldsymbol{H}_0\|_{L_2(\Omega_1)},\\
&\Big(\int_0^T e^{2\alpha t}\|\boldsymbol{H}\|^2_{L_2(\Omega_1)}dt\Big)^{1/2} \le c\|\boldsymbol{H}_0\|_{L_2(\Omega_1)}, \quad \alpha < a.
\end{aligned} \tag{2.25}$$

Moreover, $\boldsymbol{H}$ satisfies local in time estimate

$$\|\boldsymbol{H}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_{t-1/2})} \le c\rho^{-1}\|\boldsymbol{H}\|_{L_2(Q_{t-1})}, \tag{2.26}$$

that is proved by the same arguments as (4.6) in [10]. We reproduce a simple proof of (2.26). Let $\chi_\rho(\tau)$, $\rho \in [0, 1/2]$, be a smooth monotone function of τ equal to one for $\tau > t - 1 + \rho$ and to zero for $\tau \le t - 1 + \rho/2$ such that $|\chi'_\rho(\tau)| \le c\rho^{-1}$, $|\chi''_\rho(\tau)| \le c\rho^{-2}$ The vector field $\boldsymbol{H}_\rho(\xi,\tau) = \chi_\rho(\tau)\boldsymbol{H}(\xi,\tau)$ is a solution of the problem

$$\begin{cases}
\mu\boldsymbol{H}_{\rho,\tau}(\xi,\tau) + \alpha^{-1}rotrot\boldsymbol{H}_\rho(\xi,\tau) = \mu\boldsymbol{H}(\xi,\tau)\chi'_\rho(\tau),\\
\quad\nabla\cdot\boldsymbol{H}_\rho(\xi,\tau) = 0, \quad \xi \in \Omega_1, \quad \tau \in (t-1,t),\\
rot\boldsymbol{H}_\rho(\xi,\tau) = 0, \quad \nabla\cdot\boldsymbol{H}_\rho(\xi,\tau) = 0, \quad \xi \in \Omega_2\\
[\mu\boldsymbol{H}_\rho\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{H}_{\rho,\tau}] = 0, \quad \xi \in \Gamma_0,\\
\boldsymbol{H}_\rho(\xi, t-1) = 0, \quad \xi \in \Omega_1\cup\Omega_2
\end{cases} \tag{2.27}$$

By (2.11),

$$\sum_{i=1,2}\|\boldsymbol{H}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_{t-1+\rho,t})} \le c\rho^{-1-\lambda/2}\sum_{i=1,2}\|\boldsymbol{H}\|_{W_2^{\lambda,\lambda/2}(Q^i_{t-1+\rho/2,t})}.$$

We estimate the last norm of $\boldsymbol{H}$ by the interpolation inequality

$$\sum_{i=1,2}\|\boldsymbol{H}\|_{W_2^{\lambda,\lambda/2}(Q^i_{t-1+\rho/2,t})}\leq\sum_{i=1,2}(\vartheta\|\boldsymbol{H}\|_{W_2^{2+\lambda,1+\lambda/2}(Q^i_{t-1+\rho/2,t})}+c_1\vartheta^{-\lambda/2}\|\boldsymbol{H}\|_{L_2(Q^i_{t-1+\rho/2,t})})$$

and set $c\rho^{-1-\lambda/2}\vartheta=\epsilon_1$, which leads to

$$\sum_{i=1,2}\|\boldsymbol{H}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_{t-1+\rho,t})}\leq\epsilon_1\sum_{i=1,2}\|\boldsymbol{H}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_{t-1+\rho/2,t})}+c(\epsilon_1)\rho^{-(1+\lambda/2)^2}\|\boldsymbol{H}\|_{L_2(Q_{t-1+\rho/2})}.$$

This is equivalent to

$$N(\rho)\leq 2^{(1+\lambda/2)^2}\epsilon_1 N(\rho/2)+c(\epsilon_1)\|\boldsymbol{H}\|_{L_2(Q_{t-1+\rho/2})},$$

where $N(\lambda)=\rho^{(1+\lambda/2)^2}\sum_{i=1,2}\|\boldsymbol{H}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_{t-1+\rho,t})}$. If $2^{(1+\lambda/2)^2}\epsilon_1\leq 1/2$, then after some iterations we arrive at

$$N(\rho)\leq 2c(\epsilon_1)\|\boldsymbol{H}\|_{L_2(Q_{t-1,t})}.$$

Setting $\rho=1/2$ we obtain (2.26). From (2.25) and (2.26) we conclude, as in the preceding theorem, that

$$\sum_{i=1,2}\|e^{\alpha t}\boldsymbol{H}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_{1/2,T})}\leq c\|\boldsymbol{h}_0\|_{L_2(\mathbb{R}^3)}$$

and obtain (2.12).

3 Nonlinear problem

In this section we outline the main steps in the proof of Theorem 1. We write the problem (1.6), (1.9) in the form

$$\begin{cases} \boldsymbol{u}_t(\xi,t)-\nu\nabla^2\boldsymbol{u}+\nabla q=\nabla_u\cdot T_M(\mathcal{L}\boldsymbol{h})+\boldsymbol{l}_1(\boldsymbol{u},q,\boldsymbol{h}),\\ \nabla\cdot\boldsymbol{u}=l_2(\boldsymbol{u}),\quad \xi\in\Omega_1,\ t>0,\\ \Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0=\boldsymbol{l}_3(\boldsymbol{u}),\\ -q+\nu\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0(\xi)=l_4(\boldsymbol{u},\boldsymbol{h})-[\boldsymbol{n}\cdot T_M(\mathcal{L}\boldsymbol{h})\boldsymbol{n}],\quad \xi\in\Gamma_0,\\ \boldsymbol{u}(\xi,0)=\boldsymbol{u}_0(\xi),\quad \xi\in\Omega_1, \end{cases} \tag{3.1}$$

$$\begin{cases} \mu_1\boldsymbol{h}_t+\alpha_1^{-1}rotrot\boldsymbol{h}=\boldsymbol{l}_5(\boldsymbol{h},\boldsymbol{u}),\\ \nabla\cdot\boldsymbol{h}=0,\quad \xi\in\Omega_1,\\ rot\boldsymbol{h}=rot\boldsymbol{l}_6(\boldsymbol{h},\boldsymbol{u}),\quad \nabla\cdot\boldsymbol{h}=0,\quad \xi\in\Omega_2,\\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}_0]=0,\quad [\boldsymbol{h}_\tau]=\boldsymbol{l}_7(\boldsymbol{h},\boldsymbol{u}),\quad \xi\in\Gamma_0,\\ \boldsymbol{h}(\xi,0)=\boldsymbol{h}_0(\xi),\quad \xi\in\Omega_1\cup\Omega_2\cup\Omega_3, \end{cases} \tag{3.2}$$

where

$$\begin{cases} \boldsymbol{l}_1(\boldsymbol{u},q)=\nu(\nabla_u^2-\nabla^2)\boldsymbol{u}+(\nabla-\nabla_u)q,\\ l_2(\boldsymbol{u})=(I-A)\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{L}(\boldsymbol{u},\rho),\quad \boldsymbol{L}=(I-A^T)\boldsymbol{u},\quad \xi\in\Omega_1,\\ \boldsymbol{l}_3(\boldsymbol{u},\boldsymbol{h})=\Pi_0(\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0(y)-\Pi S_u(\boldsymbol{u})\boldsymbol{n}(X)),\\ l_4(\boldsymbol{u},\boldsymbol{h})=\nu(\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0-\boldsymbol{n}\cdot S_u(\boldsymbol{u})\boldsymbol{n}),\\ \boldsymbol{l}_5(\boldsymbol{h},\boldsymbol{u})=\alpha^{-1}rot(rot\boldsymbol{h}-\mathcal{P}rot\mathcal{P}\boldsymbol{h})+\mu_1\boldsymbol{\Phi}+\mu_1 rot(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}),\quad \xi\in\Omega_1,\\ \boldsymbol{l}_6(\boldsymbol{h},\boldsymbol{u})=(I-\mathcal{P})\boldsymbol{h},\quad \xi\in\Omega_2,\\ \boldsymbol{l}_7(\boldsymbol{h},\boldsymbol{u})=(\dfrac{A^TA\boldsymbol{n}_0}{|A\boldsymbol{n}_0|^2}-\boldsymbol{n}_0)[\boldsymbol{h}\cdot\boldsymbol{n}_0]=[\boldsymbol{A}(\boldsymbol{h},\boldsymbol{u})],\quad \xi\in\Gamma_0,\\ \boldsymbol{A}^{(i)}=(\dfrac{A^TA\boldsymbol{n}_0^*(y)}{|A^T\boldsymbol{n}_0^*|^2}-\dfrac{\boldsymbol{n}_0^*(y)}{|\boldsymbol{n}_0^*|^2})(\boldsymbol{n}_0^*\cdot\boldsymbol{h}^{(i)}),\quad i=1,2, \end{cases} \tag{3.3}$$

$A=\widehat{\mathcal{L}}^T$, $\boldsymbol{n}_0^*\in W_2^{l+1}(\mathbb{R}^3)$ is the extension of $\boldsymbol{n}_0\in W_2^{l+1/2}(\Gamma_0)$ into $\mathbb{R}^3$ such that $|\boldsymbol{n}_0^*|\neq 0$, and

$$\Pi\boldsymbol{g}=\boldsymbol{g}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{g}),\quad \Pi_0\boldsymbol{g}=\boldsymbol{g}-\boldsymbol{n}_0(\boldsymbol{n}_0\cdot\boldsymbol{g}).$$

We solve the problem (3.1), (3.2) in a finite time interval $(0,T)$ with $T>1$ fixed later. As in [7,8], the solution is sought in the form

$$\boldsymbol{u}=\boldsymbol{u}'+\boldsymbol{u}'',\quad q=q'+q'',\quad \boldsymbol{h}=\boldsymbol{h}'+\boldsymbol{h}''. \tag{3.4}$$

As the first step, we define $\boldsymbol{u}''(\xi,0)=\boldsymbol{u}''_0(\xi)$ and $\boldsymbol{h}''(\xi,0)=\boldsymbol{h}''_0(\xi)$ satisfying the conditions

$$\begin{aligned} &\int_{\Omega_1}\boldsymbol{u}''_0 d\xi=\int_{\Omega_1}\boldsymbol{u}_0 d\xi=0,\\ &\int_{\Omega_1}\boldsymbol{u}''_0\cdot\boldsymbol{\eta}_i(\xi)d\xi=\int_{\Omega_1}\boldsymbol{u}_0\cdot\boldsymbol{\eta}_i(\xi)d\xi=0,\quad i=1,2,3, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\nabla\cdot \boldsymbol{h}_0^{''}(\xi)=\nabla\cdot\boldsymbol{h}_0=0,\quad \xi\in\Omega_1\cup\Omega_2,\quad rot\boldsymbol{h}_0^{''}(\xi)=rot\boldsymbol{l}_6(\boldsymbol{u}_0,\boldsymbol{h}_0)=0,\quad \xi\in\Omega_2,\\ &[\mu\boldsymbol{h}_0\cdot\boldsymbol{n}_0]=0,\quad [\boldsymbol{h}_{0,\tau}]=\boldsymbol{l}_7(\boldsymbol{u}_0,\boldsymbol{h}_0)=0,\quad \xi\in\Gamma_0 \end{aligned} \tag{3.6}$$

(we have used the compatibility conditions (1.11)). It is obvious that we can set $\boldsymbol{u}_0^{''}=0$, $\boldsymbol{h}_0^{''}=0$, $\boldsymbol{u}_0^{'}=\boldsymbol{u}_0$, $\boldsymbol{h}_0^{'}=\boldsymbol{h}_0$ and define $(\boldsymbol{u}',q',\boldsymbol{h}')$ as the solution of the problem

$$\begin{cases} \boldsymbol{u}'_t-\nu\nabla^2\boldsymbol{u}'+\nabla q'=0,\ \nabla\cdot\boldsymbol{u}'=0,\quad \xi\in\Omega_1,\ \ t>0,\\ T(\boldsymbol{u}',q')\boldsymbol{n}_0=0,\ \ \xi\in\Gamma_0,\\ \boldsymbol{u}'(\xi,0)=\boldsymbol{u}'_0(\xi),\quad \xi\in\Omega_1,\\ \mu\boldsymbol{h}'_t(\xi,t)+\alpha^{-1}rotrot\boldsymbol{h}'(\xi,t)=0,\quad \nabla\cdot\boldsymbol{h}'(\xi,t)=0,\quad \xi\in\Omega_1,\\ rot\boldsymbol{h}'(\xi,t)=0,\quad \nabla\cdot\boldsymbol{h}'(\xi,t)=0,\quad \xi\in\Omega_2,\\ [\mu\boldsymbol{h}'\cdot\boldsymbol{n}_0]=0,\quad [\boldsymbol{h}'_\tau]=0,\quad \xi\in\Gamma_0,\\ \boldsymbol{h}'(\xi,0)=\boldsymbol{h}'_0(\xi),\quad \xi\in\Omega_1\cup\Omega_2. \end{cases} \tag{3.7}$$

The functions $(\boldsymbol{u}'_0,\boldsymbol{h}'_0)$ satisfy the compatibility and orthogonality conditions (1.11), (1.12); in addition,

$$\|\boldsymbol{u}'_0\|_{W_2^{1+l}(\Omega_1)}+\sum_{i=1,2}\|\boldsymbol{h}'_0\|_{W_2^{1+\lambda}(\Omega_i)}=\|\boldsymbol{u}_0\|_{W_2^{1+l}(\Omega_1)}+\sum_{i=1,2}\|\boldsymbol{h}_0\|_{W_2^{1+\lambda}(\Omega_i)}. \tag{3.8}$$

By theorems 2 and 4, $(\boldsymbol{u}',q',\boldsymbol{h}')$ are defined for $t>0$ and

$$\begin{aligned} &\|e^{\beta t}\boldsymbol{u}'\|_{W_2^{2+l,1+l/2}(Q_T^1)}+\|e^{\beta t}\nabla q'\|_{W_2^{l,l/2}(Q_T^1)}+\sum_{i=1,2}\|e^{\beta t}\boldsymbol{h}'\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^i)}\\ &\le c(\|\boldsymbol{u}_0\|_{W_2^{1+l}(\Omega_1)}+\sum_{i=1,2}\|\boldsymbol{h}_0\|_{W_2^{1+\lambda}(\Omega_i)}) \end{aligned} \tag{3.9}$$

(in what follows we mean by β the minimal of positive exponents defined in Theorems 2 and 4). It follows that

$$e^{\beta T}(\|\boldsymbol{u}'(\cdot,T)\|_{W_2^{1+l}(\Omega_1)}+\sum_{i=1,2}\|\boldsymbol{h}'(\cdot,T)\|_{W_2^{1+\lambda}(\Omega_i)})\le c(\|\boldsymbol{u}_0\|_{W_2^{1+l}(\Omega_1)}+\sum_{i=1,2}\|\boldsymbol{h}_0\|_{W_2^{1+\lambda}(\Omega_i)}).$$

We fix T such that

$$ce^{-\beta T}\le\frac{1}{8}. \tag{3.10}$$

For $(\boldsymbol{u}^{''}, q^{''}, \boldsymbol{h}^{''})$ we have the problem

$$\left\{\begin{array}{l}\boldsymbol{u}^{''}_t(\xi,t) - \nu\nabla^2\boldsymbol{u}^{''} + \nabla q^{''}\\ \;= \nabla_{u'+u''} T_M(\widehat{A}^T(\boldsymbol{u}' + \boldsymbol{u}^{''})(\boldsymbol{h}' + \boldsymbol{h}^{''})) + \boldsymbol{l}_1(\boldsymbol{u}' + \boldsymbol{u}^{''}, q' + q^{''}, \boldsymbol{h}' + \boldsymbol{h}^{''}),\\ \nabla\cdot\boldsymbol{u}^{''} = l_2(\boldsymbol{u}' + \boldsymbol{u}^{''}), \quad \xi\in\Omega_1,\ t>0,\\ \Pi_0 S(\boldsymbol{u}^{''})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}' + \boldsymbol{u}^{''}),\\ \;-q^{''} + \nu\boldsymbol{n}_0\cdot S(\boldsymbol{u}^{''})\boldsymbol{n}_0(\xi) = l_4(\boldsymbol{u}' + \boldsymbol{u}^{''}, \boldsymbol{h}' + \boldsymbol{h}^{''})\\ \;+[\boldsymbol{n}\cdot T_M(\widehat{A}^T(\boldsymbol{u}' + \boldsymbol{u}^{''})(\boldsymbol{h}' + \boldsymbol{h}^{''})\boldsymbol{n}], \quad \xi\in\Gamma_0,\\ \boldsymbol{u}^{''}(\xi,0) = \boldsymbol{u}^{''}_0(\xi) = 0, \quad \xi\in\Omega_1,\\ \mu_1\boldsymbol{h}^{''}_t + \alpha_1^{-1} rotrot\boldsymbol{h}^{''} = \boldsymbol{l}_5(\boldsymbol{h}' + \boldsymbol{h}^{''}, \boldsymbol{u}' + \boldsymbol{u}^{''}),\\ \nabla\cdot\boldsymbol{h}^{''} = 0, \quad \xi\in\Omega_1,\\ rot\boldsymbol{h}^{''} = rot\boldsymbol{l}_6(\boldsymbol{h}' + \boldsymbol{h}^{''}, \boldsymbol{u}' + \boldsymbol{u}^{''}), \quad \nabla\cdot\boldsymbol{h}^{''} = 0, \quad \xi\in\Omega_2,\\ [\mu\boldsymbol{h}^{''}\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}^{''}_\tau] = \boldsymbol{l}_7(\boldsymbol{h}' + \boldsymbol{h}^{''}, \boldsymbol{u}' + \boldsymbol{u}^{''}), \quad \xi\in\Gamma_0,\\ \boldsymbol{h}^{''}(\xi,0) = \boldsymbol{h}^{''}_0(\xi) = 0, \quad \xi\in\Omega_1\cup\Omega_2,\end{array}\right. \tag{3.11}$$

solved by iterations, according to the scheme

$$\left\{\begin{array}{l}\boldsymbol{u}^{''}_{m+1,t}(\xi,t) - \nu\nabla^2\boldsymbol{u}^{''}_{m+1} + \nabla q^{''}_{m+1}\\ \;= \nabla_{u'+u^{''}_m} T_M(\widehat{A}^T(\boldsymbol{u}' + \boldsymbol{u}^{''}_m)(\boldsymbol{h}' + \boldsymbol{h}^{''}_m)) + \boldsymbol{l}_1(\boldsymbol{u}' + \boldsymbol{u}^{''}_m, q' + q^{''}_m, \boldsymbol{h}' + \boldsymbol{h}^{''}_m),\\ \nabla\cdot\boldsymbol{u}^{''}_{m+1} = l_2(\boldsymbol{u}' + \boldsymbol{u}^{''}_m), \quad \xi\in\Omega_1,\ t>0,\\ \Pi_0 S(\boldsymbol{u}^{''}_{m+1})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}' + \boldsymbol{u}^{''}_m),\\ \;-q^{''}_{m+1} + \nu\boldsymbol{n}_0\cdot S(\boldsymbol{u}^{''}_{m+1})\boldsymbol{n}_0(\xi) = l_4(\boldsymbol{u}' + \boldsymbol{u}^{''}_m, \boldsymbol{h}' + \boldsymbol{h}^{''}_m)\\ \;+[\boldsymbol{n}\cdot T_M(\widehat{A}^T(\boldsymbol{u}' + \boldsymbol{u}^{''}_m)(\boldsymbol{h}' + \boldsymbol{h}^{''}_m)\boldsymbol{n}], \quad \xi\in\Gamma_0,\\ \boldsymbol{u}^{''}_{m+1}(\xi,0) = \boldsymbol{u}^{''}_0(\xi) = 0, \quad \xi\in\Omega_1,\\ \mu_1\boldsymbol{h}^{''}_{m+1,t} + \alpha_1^{-1} rotrot\boldsymbol{h}^{''}_{m+1} = \boldsymbol{l}_5(\boldsymbol{h}' + \boldsymbol{h}^{''}_m, \boldsymbol{u}' + \boldsymbol{u}^{''}_m),\\ \nabla\cdot\boldsymbol{h}^{''}_{m+1} = 0, \quad \xi\in\Omega_1,\\ rot\boldsymbol{h}^{''}_{m+1} = rot\boldsymbol{l}_6(\boldsymbol{h}' + \boldsymbol{h}^{''}_m, \boldsymbol{u}' + \boldsymbol{u}^{''}_m), \quad \nabla\cdot\boldsymbol{h}^{''}_{m+1} = 0, \quad \xi\in\Omega_2,\\ [\mu\boldsymbol{h}^{''}_{m+1}\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}^{''}_{m+1,\tau}] = \boldsymbol{l}_7(\boldsymbol{h}' + \boldsymbol{h}^{''}_m, \boldsymbol{u}' + \boldsymbol{u}^{''}_m), \quad \xi\in\Gamma_0,\\ \boldsymbol{h}^{''}_{m+1}(\xi,0) = \boldsymbol{h}^{''}_0(\xi) = 0, \quad \xi\in\Omega_1\cup\Omega_2, \quad m=1,...\end{array}\right. \tag{3.12}$$

As the zero approximation, we take $\boldsymbol{u}^{''}_0 = 0$, $q^{''}_0 = 0$, $\boldsymbol{h}^{''}_0 = 0$.

The solvability of the problem (3.13) is established on the basis of Theorems 2,4 and the estimates of nonlinear terms in (3.3). Applying Theorem 2, we should think of $\boldsymbol{u}_m = \boldsymbol{u}' + \boldsymbol{u}^{''}_m$ as of vector fields extended from Ω_1 into $\mathbb{R}^3$ with preservation of class (which reduces to inequalities (1.10)). The estimates of $f = l_2(\boldsymbol{u}_m)$ and $\boldsymbol{F} = \boldsymbol{L}(\boldsymbol{u}_m)$ are obtained in Section 4.

Let $\boldsymbol{u}_m = \boldsymbol{u}' + \boldsymbol{u}''_m$, $q_m = q' + q''_m$, $\boldsymbol{h}_m = \boldsymbol{h}' + \boldsymbol{h}''_m$, $A_m = A(\boldsymbol{u}_m)$,

$$
\begin{aligned}
&Y(\boldsymbol{u}, q, \boldsymbol{h}) = \|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(Q_T^1)} + \|\nabla q\|_{W_2^{l,l/2}(Q_T^1)} + \|q\|_{W_2^{l,l/2}(G_T)} + \sum_{i=1,2} \|\boldsymbol{h}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^i)},\\
&Y_m = Y(\boldsymbol{u}_m, q_m, \rho_m), \quad Y'_m = Y(\boldsymbol{u}'_m, q'_m, \rho'_m), \quad Y''_m = Y(\boldsymbol{u}''_m, q''_m, \rho''_m),\\
&N_0 \equiv N(\boldsymbol{u}_0, \boldsymbol{h}_0) = \|\boldsymbol{u}_0\|_{W_2^{1+l}(\Omega_1)} + \sum_{i=1,2} \|\boldsymbol{h}_0\|_{W_2^{!+\lambda}(\Omega_i)},\\
&\boldsymbol{U}(\xi, t) = \int_0^t \boldsymbol{u}(\xi, \tau) d\tau, \quad \boldsymbol{U}_m(\xi, t) = \int_0^t \boldsymbol{u}_m(\xi, \tau) d\tau, \quad A_m = A(\boldsymbol{u}_m),\\
&Z(\boldsymbol{u}, q, \boldsymbol{h}) = \|\boldsymbol{l}_1\|_{W_2^{l,l/2}(Q_T^1)} + \|l_2\|_{W_2^{l+1,0}(Q_T^1)} + \|\boldsymbol{L}\|_{W_2^{0,l/2+1}(Q_T^1)}\\
&+ \|\boldsymbol{l}_3\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|l_4\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\
&+ \sum_{i=1,2} \|\boldsymbol{l}_5\|_{W_2^{\lambda,\lambda/2}(Q_T^i)} + \|\boldsymbol{l}_6\|_{W_2^{\lambda,\lambda/2}(Q_T^2)} + \|\boldsymbol{l}_7\|_{W_2^{\lambda+3/2,\lambda/2+3/4}(G_T)} + \sum_{i=1,2} \|\boldsymbol{A}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^i)}.
\end{aligned}
\tag{3.13}
$$

The following theorem is proved in Section 4.

Theorem 5. *Let $\boldsymbol{u} \in W_2^{1+l,1+l/2}(Q_T^1)$, $\boldsymbol{h} \in W_2^{1+\lambda,1+\lambda/2}(Q_T^i)$, $i = 1, 2$ satisfy the condition*

$$
\sup_{t<T} \|\boldsymbol{U}(\cdot, t)\|_{W_2^{2+l}(\Omega_1)} \le \delta_1 \ll 1. \tag{3.14}
$$

Then

$$
\begin{aligned}
&Z(\boldsymbol{u}, q, \boldsymbol{h}) + \|A\nabla \cdot T_M(\widehat{A}^T \boldsymbol{h})\|_{W_2^{l,l/2}(Q_T^1)} + \|\boldsymbol{n} \cdot T_M(\widetilde{A}^T \boldsymbol{h})\boldsymbol{n}\|_{W_2^{1+1/2,1/2+l/4}(G_T)}\\
&\le c(\delta_1 Y(\boldsymbol{u}, q, \boldsymbol{h}) + Y^2(\boldsymbol{u}, q, \boldsymbol{h}) + Y^3(\boldsymbol{u}, q, \boldsymbol{h}) + Y^4(\boldsymbol{u}, q, \boldsymbol{h})).
\end{aligned}
\tag{3.15}
$$

If Y_m is finite and $\boldsymbol{U}_m$ satisfies (3.14), then the problem (3.13) has a unique solution, and

$$
\begin{aligned}
Y''_{m+1} &\le c(Z(\boldsymbol{u}_m, q_m, \boldsymbol{h}_m) + \|A_m \nabla \cdot T_M(\widehat{A}_m^T \boldsymbol{h}_m)\|_{W_2^{l,l/2}(Q_T^1)}\\
&+ \|[\boldsymbol{n}_m \cdot T_M(\widetilde{A}_m^T \boldsymbol{h}_m)\boldsymbol{n}_m]\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + N(\boldsymbol{u}''_0, \boldsymbol{h}''_0))\\
&\le c(\delta_1 Y_m + Y_m^2 + Y_m^3 + Y_m^4 + \epsilon N(\boldsymbol{u}_0, \boldsymbol{h}_0)).
\end{aligned}
\tag{3.16}
$$

It follows that

$$
Y''_{m+1} \le c_1(\delta_1 Y''_m + Y''^2_m + Y''^3_m + Y''^4_m + (\epsilon + \delta_1)N_0)
$$

and

$$
Y''_{m+1} \le 2c_1(\epsilon + \delta_1)N_0, \tag{3.17}
$$

provided that

$$
Y''_m \le 2c_1 \epsilon_1 N_0, \tag{3.18}
$$

$$
2\delta_1 c_1 + 4c_1 \epsilon_1 N_0 + 8c_1^3 \epsilon_1^2 N_0^2 + 16 c_1^4 \epsilon_1^3 N_0^3 \le 1, \tag{3.19}
$$

$\epsilon_1 = \epsilon + \delta_1$. The functions $(\boldsymbol{u}''_1, q''_1, \boldsymbol{h}''_1)$ that are the solution of (3.13) with $\boldsymbol{u}''_0 = 0$, $q''_0 = 0$, $\boldsymbol{h}''_0 = 0$, satisfy the inequality

$$
Y(\boldsymbol{u}''_1, q''_1, \boldsymbol{h}''_1) \le c(\delta_1 Y' + Y'^2 + Y'^3 + Y'^4) \le c_0(\delta_1 + \epsilon)N_0.
$$

If

$$c_0 \le 2c_1, \tag{3.20}$$

then (3.18) holds for $m = 1$ and, as a consequence, for all $m = 1, ...,$

By estimating the differences $(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, q_{m+1} - q_m, \boldsymbol{h}_{m+1} - \boldsymbol{u}_m)$, as in [5], we show that the sequence $(\boldsymbol{u}''_m, q''_m, \boldsymbol{h}''_m)$ is convergent to the unique solution of (3.12) and

$$Y(\boldsymbol{u}'', q'', \boldsymbol{h}'') \le 2c_1\epsilon_1 N_0. \tag{3.21}$$

Hence the problem (3.1), (3.2) is solvable in the time interval $(0, T)$, and the solution satisfies the estimate

$$Y(\boldsymbol{u}, q, \boldsymbol{h}) \le cN(\boldsymbol{u}_0, \boldsymbol{h}_0). \tag{3.22}$$

Now we extend the solution into the time interval $t > 0$. In the Eulerian coordinates, $(\boldsymbol{v}, p, \boldsymbol{H})$ satisfy the relations

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla \cdot T(\boldsymbol{v}, p) - \nabla \cdot T_M(\boldsymbol{H}) = 0, \\ \nabla \cdot \boldsymbol{v}(x, t) = 0, \quad x \in \Omega_{1t}, \ \ t > 0, \\ \boldsymbol{v}(x, 0) = \boldsymbol{v}_0(x), \quad x \in \Omega_1 \end{cases} \tag{3.23}$$

We verify that these relations imply

$$\int_{\Omega_{1t}} \boldsymbol{v}(x, t)dx = 0, \quad \int_{\Omega_{1t}} \boldsymbol{v}(x, t) \cdot \boldsymbol{\eta}_i(x)dx = 0, \quad i = 1, 2, 3. \tag{3.24}$$

If $\boldsymbol{\eta} = \boldsymbol{e}_i$ or $\boldsymbol{\eta} = \boldsymbol{\eta}_i(x)$, then

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega_{1t}} \boldsymbol{v}(x, t) \cdot \boldsymbol{\eta}dx = \int_{\Omega_{1t}} (\boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{\eta}dx = \int_{\Omega_{1t}} (\nabla \cdot (T(\boldsymbol{v}, p) + T_M(\boldsymbol{H}))\boldsymbol{\eta}dx \\ &= \int_{\Gamma_t} (T(\boldsymbol{v}, p) + T_M(\boldsymbol{H}^{(1)}))\boldsymbol{n} \cdot \boldsymbol{\eta}dS = \int_{\Gamma_t} T_M(\boldsymbol{H}^{(2)})\boldsymbol{n} \cdot \boldsymbol{\eta}dS \\ &= -\int_{\Omega_{2t}} \nabla \cdot T_M(\boldsymbol{H}) \cdot \boldsymbol{\eta}dx + \lim_{R \to \infty} \int_{|x|=R} T_M(\boldsymbol{H})\boldsymbol{n} \cdot \boldsymbol{\eta}dS = 0. \end{aligned}$$

Hence the conditions (3.24) hold, i.e.,

$$\int_{\Omega_1} \boldsymbol{u}(\xi, t)d\xi = 0, \quad \int_{\Omega_1} \boldsymbol{u}(\xi, t) \cdot \boldsymbol{\eta}_i(\xi)d\xi = -\int_{\Omega_i} \boldsymbol{u}(\xi, t) \cdot \boldsymbol{\eta}_i(\boldsymbol{U}(\xi, t))d\xi, \quad i = 1, 2, 3. \tag{3.25}$$

We proceed by constructing the solution of (3.12) step by step, for $t \in (jT, (j+1)T)$, $j = 1, ...$ Assume that the solution is found for $t \le kT$ and that

$$N_j \equiv N(\boldsymbol{u}(\xi, jT), \boldsymbol{h}(\xi, jT)) \le \epsilon, \quad j = 1, 2, ..., k, \tag{3.26}$$

$$U_k = \sup_{t < kT} \|\boldsymbol{U}\|_{W_2^{2+l}} \le \delta_1 \ll 1. \tag{3.27}$$

We consider the problem (...) in the time interval $(kT, (k+1)T)$ taking $(\boldsymbol{u}(\xi, kT), \boldsymbol{h}(\xi, kT)) \equiv (\boldsymbol{v}_k(\xi), \boldsymbol{h}_k(\xi))$ as the initial data. The solution is sought in the form (...) where $(\boldsymbol{u}', q', \boldsymbol{h}')$ and

$(\boldsymbol{u}'', q'', \boldsymbol{h}'')$ are found as the solutions of the problems

$$\begin{cases} \boldsymbol{u}'_t(\xi,t) - \nu\nabla^2\boldsymbol{u}' + \nabla q' = 0, \quad \nabla\cdot\boldsymbol{u}' = 0, \quad \xi\in\Omega_1, \quad t > kT, \\ S(\boldsymbol{u}')\boldsymbol{n}_0 = 0, \quad \xi\in\Gamma_0, \\ \mu_1\boldsymbol{h}'_t + \alpha_1^{-1}rotrot\boldsymbol{h}' = 0, \quad \nabla\cdot\boldsymbol{h}' = 0, \quad \xi\in\Omega_1, \\ rot\boldsymbol{h}' = 0, \quad \nabla\cdot\boldsymbol{h}' = 0, \quad \xi\in\Omega_2, \\ [\mu\boldsymbol{h}'\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}'_\tau] = 0, \quad \xi\in\Gamma_0, \\ \boldsymbol{u}'(\xi,kT) = \boldsymbol{u}'_k(\xi), \quad \xi\in\Omega_1, \quad \boldsymbol{h}'(\xi,kT) = \boldsymbol{h}'_k(\xi), \quad \xi\in\Omega_1\cup\Omega_2, \end{cases} \tag{3.28}$$

$$\begin{cases} \boldsymbol{u}''_t(\xi,t) - \nu\nabla^2\boldsymbol{u}'' + \nabla q'' = \nabla_u\cdot T_M(\widehat{A}^T(\boldsymbol{u})(\boldsymbol{h}'+\boldsymbol{h}'') + \boldsymbol{l}_1(\boldsymbol{u}'+\boldsymbol{u}'', q'+q'', \boldsymbol{h}'+\boldsymbol{h}''), \\ \nabla\cdot\boldsymbol{u}'' = l_2(\boldsymbol{u}'+\boldsymbol{u}''), \quad \xi\in\Omega_1, \ t > kT, \\ \Pi_0 S(\boldsymbol{u}'')\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}'+\boldsymbol{u}''), \\ -q'' + \nu\boldsymbol{n}_0\cdot S(\boldsymbol{u}'')\boldsymbol{n}_0(\xi) \\ = l_4(\boldsymbol{u}'+\boldsymbol{u}'', \boldsymbol{h}'+\boldsymbol{u}'') + [\boldsymbol{n}\cdot T_M(\widehat{A}^T(\boldsymbol{u})(\boldsymbol{h}'+\boldsymbol{h}'')\boldsymbol{n}], \quad \xi\in\Gamma_0, \\ \mu_1\boldsymbol{h}''_t + \alpha_1^{-1}rotrot\boldsymbol{h}'' = \boldsymbol{l}_5(\boldsymbol{h}'+\boldsymbol{h}'', \boldsymbol{u}'+\boldsymbol{u}''), \\ rot\boldsymbol{h}'' = rot\boldsymbol{l}_6(\boldsymbol{h}'+\boldsymbol{h}'', \boldsymbol{u}'+\boldsymbol{u}''), \quad \nabla\cdot\boldsymbol{h}'' = 0, \quad \xi\in\Omega_2, \\ [\mu\boldsymbol{h}''\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}''_\tau] = \boldsymbol{l}_7(\boldsymbol{h}'+\boldsymbol{h}'', \boldsymbol{u}'+\boldsymbol{u}''), \quad \xi\in\Gamma_0, \\ \boldsymbol{u}''(\xi,kT) = \boldsymbol{u}''_k(\xi), \quad \xi\in\Omega_1, \quad \boldsymbol{h}''(\xi,kT) = \boldsymbol{h}_k(\xi), \quad \xi\in\Omega_1\cup\Omega_2 \end{cases} \tag{3.29}$$

with $\boldsymbol{u}'_k = \boldsymbol{u}_k - \boldsymbol{u}''_k$, $\boldsymbol{h}'_k = \boldsymbol{h}_k - \boldsymbol{h}''_k$ and with $(\boldsymbol{u}''_k, \boldsymbol{h}''_k)$ possessing the properties

$$\begin{aligned} &\int_{\Omega_1}\boldsymbol{u}''_k(\xi)d\xi = 0, \quad \int_{\Omega_1}\boldsymbol{u}''_k\cdot\boldsymbol{\eta}_i(\xi)d\xi = -\int_{\Omega_1}\boldsymbol{u}_k\cdot\boldsymbol{\eta}_i(\boldsymbol{U}(\xi,kT)d\xi, \quad i=1,2,3, \\ &\nabla\cdot\boldsymbol{u}''_k = (I - A(\boldsymbol{u}_k))\nabla\cdot\boldsymbol{u}_k, \quad (S(\boldsymbol{u}''_k)\boldsymbol{n}_0)_\tau = \boldsymbol{l}_3(\boldsymbol{u}_k), \\ &\nabla\cdot\boldsymbol{h}''_k(\xi) = 0, \quad \xi\in\Omega_1\cup\Omega_2, \quad rot\boldsymbol{h}''_k(\xi) = rot\boldsymbol{l}_6(\boldsymbol{u}_k, \boldsymbol{h}_k), \\ &[\mu\boldsymbol{h}''_k\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{h}''_{k,\tau}] = \boldsymbol{l}_7(\boldsymbol{u}_k, \boldsymbol{h}_k), \end{aligned} \tag{3.30}$$

$$\|\boldsymbol{u}''_k\|_{W_2^{1+l}(\Omega_1)} + \sum_{i=1,2}\|\boldsymbol{h}''_k\|_{W_2^{1+\lambda}(\Omega_i)} \le c(\epsilon+\delta_1)(\|\boldsymbol{u}_k\|_{W_2^{1+l}(\Omega_1)} + \sum_{i=1,2}\|\boldsymbol{h}_k\|_{W_2^{1+\lambda}(\Omega_i)}). \tag{3.31}$$

The construction of $(\boldsymbol{u}''_k, \boldsymbol{h}''_k)$ is carried out with the help of the following theorem.

Theorem 6. *For arbitrary* $\boldsymbol{\lambda}, \boldsymbol{l}\in\mathbb{R}^3$, $r\in W_2^l(\Omega_1)$, $\boldsymbol{d}\in W_2^{l-1/2}(\Gamma_0)$, $\boldsymbol{a}\in W_2^{1/2+\lambda}(\Gamma_0)$ *such that* $\boldsymbol{d}_\tau = 0$, $\boldsymbol{a}_\tau = 0$, *and* $\boldsymbol{\ell}\in W_2^{1+\lambda}(\Omega_2)$, *there exist* $\boldsymbol{w}\in W_2^{1+l}(\Omega_1)$ *and* $\boldsymbol{g}\in W_2^{1+\lambda}(\Omega_i)$, $i=1,2$, *satisfying the relations*

$$\begin{aligned} &\int_{\Omega_1}\boldsymbol{w}(\xi)d\xi = \boldsymbol{\lambda}, \quad \int_{\Omega_1}\boldsymbol{w}\cdot\boldsymbol{\eta}_i(\xi)d\xi = \boldsymbol{l}, \quad i=1,2,3, \\ &\nabla\cdot\boldsymbol{w} = r(\xi), \quad \xi\in\Omega_1, \quad \nu(S(\boldsymbol{u}_0)\boldsymbol{n}_0)_\tau = \boldsymbol{d}(\xi), \quad \xi\in\Gamma_0, \end{aligned} \tag{3.32}$$

$$\begin{aligned} &\nabla\cdot\boldsymbol{g}(\xi) = 0, \quad \xi\in\Omega_1\cup\Omega_2, \quad rot\boldsymbol{g}(\xi) = rot\boldsymbol{\ell}(\xi), \quad \xi\in\Omega_2, \\ &[\mu\boldsymbol{g}\cdot\boldsymbol{n}_0] = 0, \quad [\boldsymbol{g}_\tau] = \boldsymbol{a}(\xi), \quad \xi\in\Gamma_0, \end{aligned} \tag{3.33}$$

and the inequalities

$$\|\boldsymbol{w}\|_{W_2^{1+l}(\Omega_1)} \leq c(|\boldsymbol{\lambda} + |\boldsymbol{l}| + \|r\|_{W_2^l(\Omega)} + \|\boldsymbol{d}\|_{W_2^{l-1/2}(\Gamma_0)}), \tag{3.34}$$

$$\sum_{i=1,2} \|\boldsymbol{g}\|_{W_2^{1+\lambda}(\Omega_i)} \leq c(\|\boldsymbol{\ell}\|_{W_2^{1+\lambda}(\Omega_2)} + \|\boldsymbol{a}\|_{W_2^{1/2+\lambda}(\Gamma_0)}). \tag{3.35}$$

Proof. We set

$$\boldsymbol{w} = \boldsymbol{W}(\xi) + \sum_{i=1}^{3} c_i \boldsymbol{e}_i + \sum_{i=1}^{3} d_i \boldsymbol{\eta}_i(\xi),$$

with $\boldsymbol{W}$ satisfying

$$\begin{aligned} &-\nabla^2 \boldsymbol{W}(\xi) + \nabla P(\xi) = \boldsymbol{F}, \quad \nabla \cdot \boldsymbol{W}(\xi) = r(\xi), \quad \xi \in \Omega_1, \\ &T(\boldsymbol{W}, P)\boldsymbol{n}_0(\xi) = \boldsymbol{d}(\xi), \quad \xi \in \Gamma_0, \end{aligned} \tag{3.36}$$

where $\boldsymbol{F}$ is the constant vector such that

$$|\Omega_1|\boldsymbol{F} = \int_{\Omega_1} \nabla r(\xi) d\xi - \int_{\Gamma_0} \boldsymbol{d}(\xi) dS.$$

Then the problem (3.36) has a solution satisfying the estimate

$$\|\boldsymbol{W}\|_{W_2^{1+l}(\Omega_1)} \leq c(\|r\|_{W_2^l(\Omega)} + \|\boldsymbol{d}\|_{W_2^{l-1/2}(\Gamma_0)}).$$

(the regularity of the boundary Γ_0: $\Gamma_0 \in W_2^{3/2+l}$ is sufficient for this). The constants c_i and d_i are chosen so that (3.32) holds and

$$\|\boldsymbol{w}\|_{W_2^{1+l}(\Omega_1)} \leq c(\|\boldsymbol{W}\|_{W_2^{1+l}(\Omega_1)} + |\boldsymbol{\lambda}| + |\boldsymbol{l}|) \leq c(|\boldsymbol{\lambda} + |\boldsymbol{l}| + \|r\|_{W_2^l(\Omega)} + \|\boldsymbol{d}\|_{W_2^{l-1/2}(\Gamma_0)}).$$

The vector field $\boldsymbol{g}(\xi)$ can be defined as $\boldsymbol{g}(\xi) = \boldsymbol{g}^*(\xi, 0)$ where $\boldsymbol{g}^*$ is the solution of

$$\begin{aligned} &rot\boldsymbol{g}^*(\xi,t) = rot(\boldsymbol{\ell}^*(\xi,t) + \boldsymbol{a}^*(\xi,t)), \quad \nabla \cdot \boldsymbol{g}^*(\xi,t) = 0 \quad \xi \in \Omega_1 \cup \Omega_2, \\ &[\boldsymbol{g}^*_\tau] = \boldsymbol{a}^*(\xi), \quad \xi \in \Gamma_0, \end{aligned} \tag{3.37}$$

$\boldsymbol{\ell}^* \in W_2^{2+\lambda,1+\lambda/2}(\mathbb{R}_T)$ is the extension of $\boldsymbol{\ell} \in W_2^{1+\lambda}(\Omega_2)$ into $\mathbb{R}_T$ vanishing for large $|\xi|$ and satisfying the inequality

$$\|\boldsymbol{\ell}^*\|_{W_2^{2+\lambda,1+\lambda/2}(\mathbb{R}_T)} \leq c\|\boldsymbol{\ell}\|_{W_2^{1+\lambda}(\Omega_2)}, \tag{3.38}$$

whereas $\boldsymbol{a}^*$ is the extension of $\boldsymbol{a}$ such that $\boldsymbol{a}^*(\xi,t) = 0$ for $\xi \in \Omega_2$, $\boldsymbol{a}^* \cdot \boldsymbol{n}_0 = 0$ for $\xi \in \Gamma_0$ and

$$\|\boldsymbol{a}^*\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^1)} \leq c\|\boldsymbol{a}\|_{W_2^{1+\lambda}(\Gamma_0)}. \tag{3.39}$$

By (2.8),

$$\begin{aligned} &\sum_{i=1,2} \|\boldsymbol{g}\|_{W_2^{1+\lambda}(\Omega_i)} \leq c \sum_{i=1,2} \|\boldsymbol{g}^*\|_{W_2^{2+\lambda,1+\lambda}(Q_T^i)} \leq c(\|\boldsymbol{\ell}^*\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T)} \\ &+ \|\boldsymbol{a}^*\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^2)}) \leq c(\|\boldsymbol{\ell}\|_{W_2^{1+\lambda}(\Omega_2)} + \|\boldsymbol{a}\|_{W_p^{1/2+\lambda}(\Gamma_0)}), \end{aligned} \tag{3.40}$$

q.e.d.

Thus, we have constructed $(\boldsymbol{u}_k'', \boldsymbol{h}_k'')$ satisfying (3.30) and (3.31).

The functions $(\boldsymbol{u}', q', \boldsymbol{h}')$ satisfy the inequality similar to (3.9) and, as a consequence,

$$N'_{k+1} \equiv N(\boldsymbol{u}'_{k+1}, \boldsymbol{h}'_{k+1}) \le c_0 e^{-\beta T} N(\boldsymbol{u}_k, \boldsymbol{h}_k) \equiv e^{-\beta T} N_k. \tag{3.41}$$

To solve (3.29), we make use of the estimate of nonlinear terms

$$Z^{(k)} \le c(\delta_1 Y^{(k)} + (Y^{(k)})^2 + (Y^{(k)})^3 + (Y^{(k)})^4), \tag{3.42}$$

where $Z^{(k)}$ and $Y^{(k)}$ are the expressions defined in (3.13) and calculated in the cylindrical domains $Q^i_{kT,(k+1)T}$. The inequality (3.42) follows from the estimates obtained in Section 4 and from the remark at the end of the section.

Repeating the above arguments, we prove the existence of a unique solution of the problem (3.29) and the estimates

$$Y^{(k)}(\boldsymbol{u}, q, \boldsymbol{h}) \le cN(\boldsymbol{u}_k, \boldsymbol{h}_k),$$
$$N_{k+1} \le \frac{1}{4} N_k.$$

The same estimates are true for $t \in (jT, (j+1)T)$, $j < k$.

If β is so small that $e^{\beta T} < 4$, then

$$(\sum_{j=1}^k Y^{(j)2} e^{2j\beta})^{1/2} \le c(\sum_{j=1}^k N_j^2 e^{2j\beta})^{1/2} \le cN_0, \tag{3.43}$$

which yields a uniform estimate for U_k:

$$U_k \le c(\sum_{j=1}^k Y^{(j)2} e^{2j\beta})^{1/2} \le cN_0 \le c\epsilon.$$

This shows that δ_1 is small for small ϵ. Letting $k \to \infty$ in (...), we obtain the inequality equivalent to (1.14). This completes the proof of Theorem 1.

In conclusion, we outline the scheme of reconstruction of $\boldsymbol{e}(\xi, t)$ satisfying (1.7), (1.8) with $\boldsymbol{u}, \boldsymbol{h}, q$ found above. We follow the approach in [4]. Let $\boldsymbol{e}_1(\xi, t)$ be a vector field such that

$$\begin{aligned} &\nabla \cdot \boldsymbol{e}_1(\xi, t) = 0, \quad \xi \in \Omega_1 \cup \Omega_2, \\ &[\boldsymbol{n}_0 \times \mathcal{P}\boldsymbol{e}_1] = \Psi, \quad [\boldsymbol{n}_0 \cdot \boldsymbol{e}_1] = 0, \quad x \in \Gamma_0. \end{aligned} \tag{3.44}$$

(in fact, we can set $\boldsymbol{e}_1 = 0$ in Ω_2 and define $\boldsymbol{e}_1^{(1)}$ as the solution of

$$\nabla \cdot \boldsymbol{e}_1^{(1)} = 0, \quad \xi \in \Omega_1, \quad \boldsymbol{n}_0 \times \mathcal{P}\boldsymbol{e}_1^{(1)} = \boldsymbol{\Psi}, \quad \boldsymbol{n}_0 \cdot \boldsymbol{e}_1^{(1)} = 0, \quad \xi \in \Gamma_0).$$

Since Γ_t is close to Γ_0, such a vector field can be constructed. Next, we solve the problem

$$rot\boldsymbol{\mathcal{E}} = -\mu(\boldsymbol{h}_t - \boldsymbol{\Phi}) - rot\mathcal{P}\boldsymbol{e}_1, \quad \nabla \cdot \mathcal{P}^{-1}\boldsymbol{\mathcal{E}} = 0, \quad y \in \Omega, \quad [\boldsymbol{\mathcal{E}}] = 0, \quad y \in \Gamma_0. \tag{3.45}$$

By Proposition 1 in [3] (proved by Dr. N.Filonov) the equation $\nabla\cdot\boldsymbol{h}(\xi,t)=0,\quad \xi\in\Omega_1\cup\Omega_2$ and $[\boldsymbol{n}_0\times\mathcal{P}\boldsymbol{e}_1]=\boldsymbol{\Psi}$ on Γ_0 imply $(\mu(h_t(z,t)-\boldsymbol{\Phi}(z,t))+rot\mathcal{P}\boldsymbol{e}_1)\cdot\boldsymbol{n}_0=0$, $\xi\in\Gamma_0$, so the problem (3.45) is solvable. The solution is expressed by the formula

$$\begin{aligned}
&\boldsymbol{\mathcal{E}}(\xi,t)=\boldsymbol{\mathcal{E}}_1(\xi,t)+\nabla Z(\xi,t),\\
&\boldsymbol{\mathcal{E}}_1(\xi,t)=-\frac{1}{4\pi}rot\int_\Omega\frac{\mu(\boldsymbol{h}_t(z,t)-\boldsymbol{\Phi}(z,t))+rot\mathcal{P}\boldsymbol{e}_1(z,t)}{|\xi-z|}dz,\\
&\nabla\cdot\mathcal{P}^{-1}\nabla Z(\xi,t)=-\nabla\cdot\mathcal{P}^{-1}\boldsymbol{\mathcal{E}}_1,\quad \xi\in\mathbb{R}^3,\quad [Z]=0,\quad [\boldsymbol{n}_0\cdot\mathcal{P}\nabla Z]=0,\quad \xi\in\Gamma_0.
\end{aligned}$$

Now we set

$$\mathcal{P}\boldsymbol{e}=\alpha_1^{-1}\mathcal{P}rot\mathcal{P}\boldsymbol{h}-\mu_1(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}),\quad y\in\Omega_1, \tag{3.46}$$

which implies

$$\mathcal{P}\boldsymbol{e}=\boldsymbol{\mathcal{E}}+\mathcal{P}\boldsymbol{e}_1+\nabla\chi_1,\quad \xi\in\Omega_1 \tag{3.47}$$

with a certain single-valued χ_1. Finally, we set

$$\mathcal{P}\boldsymbol{e}=\boldsymbol{\mathcal{E}}+\mathcal{P}\boldsymbol{e}_1+\nabla\chi_2+C(t)\nabla w(\xi)=\boldsymbol{\mathcal{E}}+\nabla\chi_2+C(t)\nabla w(\xi),\quad \xi\in\Omega_2, \tag{3.48}$$

where w and χ_2 are solutions to the problems

$$\nabla\cdot\mathcal{P}^{-1}\nabla\chi_2(y,t)=0,\quad y\in\Omega_2,\quad \chi_2(\xi,t)=\chi_1(\xi,t),\quad \xi\in\Gamma_0, \tag{3.49}$$

$$\nabla\cdot\mathcal{P}^{-1}\nabla w(\xi)=0,\quad \xi\in\Omega_2,\quad w(\xi,t)=1,\quad \xi\in\Gamma_0,\quad w\to 0,\quad |\xi|\to\infty. \tag{3.50}$$

For determination of $C(t)$ we should impose on $\boldsymbol{e}^{(2)}(\xi,t)$ a normalization condition, for instance, $\int_{\Gamma_0}\boldsymbol{e}^{(2)}\cdot\boldsymbol{n}_0dS=0$, which coincides with $\int_{\Gamma_t}\boldsymbol{E}^{(2)}\cdot\boldsymbol{n}dS=0$. Since

$$I=\int_{\Gamma_0}\boldsymbol{n}_0\cdot\nabla w dS\neq 0,$$

we have

$$C(t)=-I^{-1}\int_{\Gamma_0}(\boldsymbol{\mathcal{E}}^{(2)}+\nabla\chi_2(\xi,t))\cdot\boldsymbol{n}_0\xi)dS. \tag{3.51}$$

We pass to the estimates. The vector field $\boldsymbol{e}_1^{(1)}$ can be constructed so that

$$\|\boldsymbol{e}_1^{(1)}\|_{W_2^1(\Omega_1)}\leq c\|\boldsymbol{\Psi}\|_{W_2^{1/2}(\Gamma_0)},\quad \|\boldsymbol{e}_1^{(1)}\|_{W_2^{1+\lambda}(\Omega_1)}\leq c\|\boldsymbol{\Psi}\|_{W_2^{1/2+\lambda}(\Gamma_0)}, \tag{3.52}$$

$\forall t>0$. Applying the first inequality to the finite difference $\boldsymbol{e}_1^{(1)}(\xi,t-h)-\boldsymbol{e}_1^{(1)}(\xi,t)$, we easily deduce from (3.52) the estimate

$$\begin{aligned}
&\|e^{\beta t}\boldsymbol{e}_1^{(1)}\|_{W_2^{1+\lambda,0}(Q^1_\infty)}+\|e^{\beta t}\boldsymbol{e}_1^{(1)}\|_{W_2^{\lambda/2}(0,\infty;W_2^1(\Omega_1))}\\
&\leq c(\|e^{\beta t}\boldsymbol{\Psi}\|_{W_2^{1/2+\lambda,0}(G_\infty)}+\|e^{\beta t}\boldsymbol{\Psi}\|_{W_2^{\lambda/2}(0,\infty;W_2^{1/2}(\Gamma_0))}.
\end{aligned} \tag{3.53}$$

Let $\boldsymbol{\Phi}_1(\xi,t)=\mu(\boldsymbol{h}_t(z,t)-\boldsymbol{\Phi}(z,t))+rot\mathcal{P}\boldsymbol{e}_1(z,t)$. By the Calderon-Zygmund theorem,

$$\|\nabla\boldsymbol{\mathcal{E}}_1\|_{L_2(\mathbb{R}^3)}\leq c\|\boldsymbol{\Phi}_1\|_{L_2(\mathbb{R}^3)}, \tag{3.54}$$

in addition,

$$\sum_{i=1,2}\|\nabla\boldsymbol{\mathcal{E}}_1\|_{W_2^\lambda(\mathbb{R}^3)}\leq c\sum_{i=1,2}\|\boldsymbol{\Phi}_1\|_{W_2^\lambda(\mathbb{R}^3)}. \tag{3.55}$$

As for the L_2-norm of $\boldsymbol{\mathcal{E}}_1$, it can be estimated only in a finite domain, for instance, in $\Omega^*=$ $supp\boldsymbol{u}^*$. We have

$$\|\boldsymbol{\mathcal{E}}_1\|_{L_2(\Omega^*)}\leq c\|\boldsymbol{\Phi}_1\|_{L_2(\mathbb{R}^3)}. \tag{3.56}$$

Applying (3.55) and (3.54) to the finite differences $\Delta_t(-h)\nabla\boldsymbol{\mathcal{E}}_1$ and $\Delta_t(-h)\boldsymbol{\mathcal{E}}_1$, we conclude that

$$\sum_{i=1,2}\|e^{\beta t}\nabla\boldsymbol{\mathcal{E}}_1\|_{W_2^{\lambda,\lambda/2}(Q_\infty)}+\|e^{\beta t}\boldsymbol{\mathcal{E}}_1\|_{W_2^{0,\lambda/2}(Q^*_\infty)}\leq c\sum_{i=1,2}\|e^{\beta t}\boldsymbol{\Phi}_1\|_{W_2^{\lambda,\lambda/2}(Q_\infty)}, \tag{3.57}$$

where $Q^*_\infty=\Omega^*_1\times(0,\infty)$.

We proceed with the estimate of Z keeping in mind that $\mathcal{P}-I$ has a compact support Ω^*. Let

$$\overline{Z}(\xi,t)=Z(\xi,t)-|\Omega^*|^{-1}\int_{\Omega^*}Z(\eta,t)d\eta.$$

We have

$$\int_{\Omega^*}\mathcal{P}^{-1}\nabla Z\cdot\nabla Zd\eta=-\int_{\Omega^*}\mathcal{P}^{-1}\boldsymbol{\mathcal{E}}_1\nabla Zd\eta+\int_{\partial\Omega^*}\mathcal{P}^{-1}\boldsymbol{\mathcal{E}}_1\cdot\boldsymbol{n}\overline{Z}dS \tag{3.58}$$

Due to the Poincare inequality

$$\|\overline{Z}\|_{L_2(\partial\Omega^*)}\leq c\|\nabla Z\|_{L_2(\Omega^*)},$$

we deduce from (3.58)

$$\|\nabla Z\|_{L_2(\Omega^*)}\leq c\|\boldsymbol{\mathcal{E}}_1\|_{W_2^1(\Omega^*)}.$$

Moreover, in view of the equation

$$\nabla^2 Z=\nabla\cdot(I-\mathcal{P}^{-1})\nabla Z-\nabla\cdot\mathcal{P}^{-1}\mathcal{E}_1$$

and the estimate

$$\|I-\mathcal{P}^{-1}\|_{W_2^{1+l}(\Omega_i)}\leq c\delta_1,\quad i=1,2,$$

we have

$$\|D^2Z\|_{L_2(\mathbb{R}^3)}\leq c(\|\nabla\boldsymbol{\mathcal{E}}_1\|_{L_2(\mathbb{R}^3)}+\|\boldsymbol{\mathcal{E}}_1\|_{L_2(\Omega^*)})$$

and

$$\begin{aligned}\sum_{i=1,2}\|D^2Z\|_{W_2^\lambda(\Omega_i)}&\leq c(\sum_{i=1,2}\|\nabla\boldsymbol{\mathcal{E}}_1\|_{W_2^\lambda(\Omega_i)}+\|\boldsymbol{\mathcal{E}}_1\|_{W_2^\lambda(\Omega^*)})\\&\leq c(\sum_{i=1,2}\|\nabla\boldsymbol{\mathcal{E}}_1\|_{W_2^\lambda(\Omega_i)}+\|\boldsymbol{\mathcal{E}}_1\|_{L_2(\Omega^*)}).\end{aligned}$$

By virtue of this and the preceding inequality applied to $\Delta_t(-h)Z$, there holds

$$\begin{aligned}&\sum_{i=1,2}\|e^{\beta t}D^2Z\|_{W_2^{\lambda,\lambda/2}(Q^i_\infty)}+\|e^{\beta t}\nabla Z\|_{W_2^{0,\lambda/2}(Q^*_\infty)}\\&\leq c(\sum_{i=1,2}\|e^{\beta t}\nabla\boldsymbol{\mathcal{E}}_1\|_{W_2^{\lambda,\lambda/2}(Q^i_\infty)}+\|e^{\beta t}\boldsymbol{\mathcal{E}}_1\|_{W_2^{0,\lambda/2}(Q^*_\infty)}),\end{aligned} \tag{3.59}$$

hence

$$\sum_{i=1,2}\|e^{\beta t}\nabla\boldsymbol{\mathcal{E}}\|_{W_2^{\lambda,\lambda/2}(Q^i_\infty)}+\|e^{\beta t}\boldsymbol{\mathcal{E}}\|_{W_2^{0,\lambda/2}(Q^*_\infty)}\le c\sum_{i=1,2}\|e^{\beta t}\boldsymbol{\Phi}_1\|_{W^{\lambda,\lambda/2}(Q^i_\infty)}. \tag{3.60}$$

Now, using formulas (3.46) - (3.48) for $\boldsymbol{e}$ and (3.51) for $C(t)$, we obtain

$$\begin{aligned}
&\|e^{\beta t}\nabla\boldsymbol{e}\|_{W_2^{\lambda,\lambda/2}(Q^1_\infty)}+\|e^{\beta t}\boldsymbol{e}\|_{W_2^{0,\lambda/2}(Q^1_\infty)}\\
&\le c(\|e^{\beta t}\nabla\boldsymbol{\Phi}_2\|_{W_2^{\lambda,\lambda/2}(Q^1_\infty)}+\|e^{\beta t}\boldsymbol{\Phi}_2\|_{W_2^{0,\lambda/2}(Q_{\infty,1})}),\\
&\|e^{\beta t}D^2\chi_1\|_{W_2^{\lambda,\lambda/2}(Q^1_\infty)}+\|e^{\beta t}\nabla\chi_1\|_{W_2^{0,\lambda/2}(Q^1_\infty)}\le c(\sum_{i=1,2}\|e^{\beta t}\boldsymbol{\Phi}_1\|_{W_2^{\lambda,\lambda/2}(Q^i_\infty)}\\
&+\|e^{\beta t}\nabla\boldsymbol{\Phi}_2\|_{W_2^{\lambda,\lambda/2}(Q^1_\infty)}+\|e^{\beta t}\boldsymbol{\Phi}_2\|_{W_2^{0,\lambda/2}(Q_{\infty,1})}),\\
&\|e^{\beta t}D^2\chi_2\|_{W_2^{\lambda,\lambda/2}(Q^2_\infty)}+\|e^{\beta t}\nabla\chi_2\|_{W_2^{0,\lambda/2}(Q^2_\infty)}\\
&\le c(\|e^{\beta t}\chi_1\|_{W_2^{1/2+\lambda,0}(G_\infty)}+\|e^{\beta t}\chi_1\|_{W_2^{\lambda/2}(0,T;W_2^{1/2}(\Gamma_0)}\\
&+\|e^{\beta t}D^2\chi_1\|_{W_2^{\lambda,\lambda/2}(Q^1_\infty)}+\|e^{\beta t}\nabla\chi_1\|_{W_2^{0,\lambda/2}(Q^1_\infty)}),\\
&\|e^{\beta t}C\|_{W_2^{\lambda/2}(\mathbb{R}_+)}\le c(\|e^{\beta t}\boldsymbol{\mathcal{E}}_2\|_{W_2^{\lambda/2}(0,\infty;W_2^1(\Omega_2)}+\|e^{\beta t}\nabla\chi_2\|_{W_2^{\lambda/2}(0,T;W_2^1(\Omega_2)})\\
&\|e^{\beta t}\nabla\boldsymbol{e}\|_{W_2^{\lambda,\lambda/2}(Q^2_\infty)}+\|e^{\beta t}\boldsymbol{e}\|_{W_2^{0,\lambda/2}(Q^*_\infty)}\\
&\le c(\|e^{\beta t}\nabla\boldsymbol{\mathcal{E}}\|_{W_2^{\lambda,\lambda/2}(Q^2_\infty)}+\|e^{\beta t}\boldsymbol{\mathcal{E}}\|_{W_2^{0,\lambda/2}(Q^*_\infty)}\\
&+\|e^{\beta t}D^2\chi_2\|_{W_2^{\lambda,\lambda/2}(Q^2_\infty)}+\|e^{\beta t}\nabla\chi_2\|_{W_2^{0,\lambda/2}(Q^2_\infty)}),
\end{aligned} \tag{3.61}$$

where $\boldsymbol{\Phi}_2=\alpha_1^{-1}\mathcal{P}rot\mathcal{P}\boldsymbol{h}-\mu_1(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})$. Using the results of the next section for the estimates of $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$, one can show that $\boldsymbol{e}$ possesses a finite norm

$$\sum_{i=1,2}\|e^{\beta t}\nabla\boldsymbol{e}\|_{W_2^{\lambda,\lambda/2}(Q^i_\infty)}+\|e^{\beta t}\boldsymbol{e}\|_{W_2^{0,\lambda/2}(Q^*_\infty)},$$

that is estimated by the data of the problem.

4 Estimates of nonlinear terms

In this section the proof of Theorem 5 is presented. We estimate all the non-linear terms in (3.3).

We start with some auxiliary inequalities for the product of two functions $u(x), v(x)$ given in a domain $\mathcal{D} \subset \mathbb{R}^n$. We shall use the relations

$$\begin{aligned} &\|uv\|_{W_2^l(\mathcal{D})} \leq c(\|u\|_{W_2^l(\mathcal{D})}\|v\|_{W_2^{3/2+\eta}(\mathcal{D})} + \|u\|_{W_2^{3/2+\eta}(\mathcal{D})}\|v\|_{W_2^l(\mathcal{D})}) \\ &\leq c\|u\|_{W_2^l(\mathcal{D})}\|v\|_{W_2^l(\mathcal{D})}, \quad if \quad 2l > n. \\ &\|uv\|_{W_2^l(\mathcal{D})} \leq c\|u\|_{W_2^l(\mathcal{D})}\|v\|_{W_2^{n/2+\eta}(\mathcal{D})}, \quad \eta > 0, \quad or \\ &\|uv\|_{W_2^l(\mathcal{D})} \leq c\|v\|_{W_2^l(\mathcal{D})}\|u\|_{W_2^{n/2+\eta}(\mathcal{D})}, \quad if \quad 2l \leq n. \end{aligned} \tag{4.1}$$

In addition, there holds

$$\|uv\|_{W_2^1(\mathcal{D})} \leq c(\sup_{\mathcal{D}} |u(x)|\|\boldsymbol{v}\|_{W_2^1(\mathcal{D})} + \sup_{\mathcal{D}} |v(x)|\|u\|_{W_2^1(\mathcal{D})}). \tag{4.2}$$

We proceed by estimating the nonlinear terms in (3.3) under the assumption that the inequality (3.12) holds. The elements of the matrix A are homogeneous quadratic functions of the elements of the matrix $\mathcal{L} = I + \nabla \boldsymbol{U}$, hence

$$\begin{aligned} &\sup_{t<T} \|I - A\|_{W_2^{1+l}(\Omega)} \leq c \sup_{t<T} \|\boldsymbol{U}\|_{W_2^{2+l}(\Omega_1)} \leq c\delta_1, \\ &\sup_{t<T} \|A\|_{W_2^{1+l}(\Omega)} \leq c. \end{aligned} \tag{4.3}$$

We estimate the term $(I - A)\nabla q$. By (4.1) and (3.12),

$$\|(I - A)\nabla q\|_{W_2^{l,0}(Q_T^1)} \leq c \sup_{t<T} \|I - A\|_{W_2^l(\Omega)}\|\nabla q\|_{W_2^{l,0}(Q_T^1)} \leq c\delta_1\|\nabla q\|_{W_2^{l,0}(Q_T^1)}. \tag{4.4}$$

We also need to estimate

$$\|(I - A)\nabla q\|_{W_2^{0,l/2}(Q_T^1)} = \|(I - A)\nabla q\|_{L_2(Q_T^1)} + \|(I - A)\nabla q\|_{\dot{W}_2^{0,l/2}(Q_T^1)},$$

where

$$\|u\|_{\dot{W}_2^{0,l/2}(Q_T^1)} = \Big(\int_0^1 \frac{dh}{h^{1+l}} \int_h^T \|\Delta_t(-h)u(\xi, t)\|^2_{L_2(\Omega_1)} dt\Big)^{1/2}, \quad \Delta_t(-h)u = u(\xi, t - h) - u(\xi, t)$$

(we assume that $T \geq 1$). Since

$$|\Delta_t(-h)(I - A)| \leq c \int_0^h |\nabla \boldsymbol{u}(\xi, t - \tau)| d\tau \leq ch \sup_{Q_{t-h,t}^1} |\nabla \boldsymbol{u}(\xi, \tau)|, \tag{4.5}$$

we obtain

$$\begin{aligned} &\|(I - A)\nabla q\|_{\dot{W}_2^{0,l/2}(Q_T^1)} \leq c(\sup_{\Omega_1} |I - A|\|\nabla q\|_{\dot{W}_2^{0,1/2}(Q_T^1)} + \sup_{Q_T^1} |\nabla \boldsymbol{u}(\xi, t)|\|\nabla q\|_{L_2(Q_T^1)}) \\ &\leq c(\delta_1\|\nabla q\|_{W_2^{0,1/2}(Q_T^1)} + \|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(Q_T^1)}\|\nabla q\|_{W_2^{1,0}(Q_T^1)}). \end{aligned} \tag{4.6}$$

We proceed by estimating the norm of

$$(\nabla_u^2-\nabla^2)\boldsymbol{u}=\big((A-I)\nabla\cdot A\nabla+\nabla\cdot(A-I)\nabla\big)\boldsymbol{u}$$

We have

$$\|(\nabla_u^2-\nabla^2)\boldsymbol{u}\|_{W_2^{l,0}(Q_T^1)}\le c\delta_1\|\nabla\cdot A\nabla\boldsymbol{u}\|_{W_2^{l,0}(Q_T^1)}+\|(A-I)\nabla\boldsymbol{u}\|_{W_2^{1+l,0}Q_T^1)}$$
$$\le c\delta_1\|\boldsymbol{u}\|_{W_2^{2+l,0}(Q_T^1)},$$

in addition, we need to estimate $\|(\nabla_u^2-\nabla^2)\boldsymbol{u}\|_{\dot{W}_2^{0,l/2}(Q_T^1)}$. Since

$$\|\Delta_t(-h)(A-I)\nabla\cdot A\nabla\boldsymbol{u}\|_{L_2(\Omega_1)}\le c(\delta_1\|\Delta_t(-h)A\nabla\boldsymbol{u}\|_{W_2^1(\Omega_1)}$$
$$+\|\Delta_t(-h)(A-I)\nabla\boldsymbol{u}\|_{W_2^1(\Omega_1)}+\|\int_0^h\frac{\partial A}{\partial t}d\tau\nabla\cdot A\nabla\boldsymbol{u}\|_{L_2(\Omega_1)},$$
$$\|\Delta_t(-h)\nabla\cdot(A-I)\nabla\boldsymbol{u}\|_{L_2(\Omega_1)}\le c\|\Delta_t(-h)A\nabla\boldsymbol{u}\|_{W_2^1(\Omega_1)},$$

it holds

$$\|\Delta_t(-h)(\nabla_u^2-\nabla^2)\boldsymbol{u}\|_{L_2(\Omega_1)}\le c\delta_1\|\Delta_t(-h)D^2\boldsymbol{u}\|_{L_2(\Omega_1)}+ch\sup_{t<T}\|\boldsymbol{U}\|_{W_2^{2+l}(\Omega_1)}\|\boldsymbol{u}\|_{W_2^{2,0}(Q_T^1)}$$

and

$$\|(\nabla_u^2-\nabla^2)\boldsymbol{u}\|_{W_2^{0,l/2}(Q_T^1)}\le c(\delta_1\|\boldsymbol{u}\|_{W_2^{l/2}(0,T;W_2^2(\Omega_1)}+\sup_{t<T}\|\boldsymbol{U}\|_{W_2^{l+2}(\Omega_1)}\|\boldsymbol{u}\|_{W_2^{2,0}(Q_T^1)}).$$

The inequality

$$\|l_2(\boldsymbol{u})\|_{W_2^{2+l,0}(Q_T^1)}\le c\delta_1\|\boldsymbol{u}\|_{W_2^{2+l,0}(Q_T^1)} \tag{4.7}$$

is proved in the same way as (4.4), whereas

$$\|\boldsymbol{L}(\boldsymbol{u})\|_{W_2^{0,1+l/2}(Q_T^1)}\le\|(I-A^T)\boldsymbol{u}\|_{L_2(\Omega_T^1)}+\|(I-A)\boldsymbol{u}_t\|_{\dot{W}_2^{0,1/2}(Q_T^1)}+\|\nabla\boldsymbol{u}\boldsymbol{u}\|_{\dot{W}_2^{0,1/2}(Q_T^1)}. \tag{4.8}$$

The first norm in the right hand side is estimated as in (4.7), the second is treated as in (4.5), and the last one is estimated with the help of

$$\|\Delta_t(-h)(\nabla\boldsymbol{u}\boldsymbol{u})\|_{L_2(\Omega_1)}$$
$$\le\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{L_2(\Omega_1)}\sup_{\Omega_1}|\boldsymbol{u}(\xi,t)|+\sup_{\Omega_1}|\nabla\boldsymbol{u}|\int_0^h\|\boldsymbol{u}_t(\cdot,t-\tau)\|_{L_2(\Omega_1)}d\tau:$$

$$\|\nabla\boldsymbol{u}\boldsymbol{u}\|_{\dot{W}_2^{0,l/2}(Q_T^1)}\le c(\sup_{Q_T^1}|\boldsymbol{u}(\xi,t)|\|\nabla\boldsymbol{u}\|_{W_2^{l/2}(0,T;W_2^1(\Omega_1))}+\|\boldsymbol{u}_t\|_{L_2(Q_T^1)}\sup_{Q_T^1}|\nabla\boldsymbol{u}(\xi,t)|). \tag{4.9}$$

We pass to the estimates of $\boldsymbol{l}_3(\boldsymbol{u})$ and $l_4(\boldsymbol{u})$. At first we consider the expression

$$S(\boldsymbol{u})\boldsymbol{n}_0-S_u(\boldsymbol{u})\boldsymbol{n}=(S(\boldsymbol{u})-S_u(\boldsymbol{u}))\boldsymbol{n}+S(\boldsymbol{u})(\boldsymbol{n}_0-\boldsymbol{n}).$$

Let $\boldsymbol{n}_0^*\in W_2^{1+l}(\mathbb{R}^3)$ be the extension of $\boldsymbol{n}_0\in W_2^{1/2+l}(\Gamma_0)$ in $\mathbb{R}^3$ and $\boldsymbol{n}^*=A\boldsymbol{n}_0^*/|A\boldsymbol{n}_0^*|$. We have

$$\|S(\boldsymbol{u})\boldsymbol{n}_0-S_u(\boldsymbol{u})\boldsymbol{n}\|_{W_2^{1/2+l,0}(G_T)}\le c\|S(\boldsymbol{u})\boldsymbol{n}_0^*-S_u(\boldsymbol{u})\boldsymbol{n}^*\|_{W_2^{1+l,0}(Q_T^1)}$$
$$\le c(\|(S(\boldsymbol{u})-S_u(\boldsymbol{u}))\boldsymbol{n}^*\|_{W_2^{1+l,0}(Q_T^1)}+\|S(\boldsymbol{u})(\boldsymbol{n}_0^*-\boldsymbol{n}^*)\|_{W_2^{1+l,0}(Q_T^1)})\le c\delta\|\boldsymbol{u}\|_{W_2^{2+l,0}(Q_T^1)}. \tag{4.10}$$

Moreover, from

$$\|\frac{\partial}{\partial t}(S(\boldsymbol{u})\boldsymbol{n}_0 - S_u(\boldsymbol{u})\boldsymbol{n})\|_{L_2(G_T)} \leq \|S(\boldsymbol{u}_t)\boldsymbol{n}_0 - S_u(\boldsymbol{u}_t)\boldsymbol{n}\|_{L_2^{(}G_T)} + \|S'_u(\boldsymbol{u})\boldsymbol{n}\|_{L_2(G_T)}$$
$$+ \|S(\boldsymbol{u}_t)(\boldsymbol{n}_0 - \boldsymbol{n})\|_{L_2(G_T)} + \|S_u(\boldsymbol{u})\boldsymbol{n}_t\|_{L_2(G_T)},$$

where $S'_u(\boldsymbol{u}) = (A_t^{-T}\nabla\boldsymbol{u}) + (A_t^{-T}\nabla\boldsymbol{u})^T$, it follows that

$$\|\frac{\partial}{\partial t}(S(\boldsymbol{u})\boldsymbol{n}_0 - S_u(\boldsymbol{u})\boldsymbol{n})\|_{L_2(G_T)} \leq c(\delta_1\|\nabla\boldsymbol{u}_t\|_{L_2(G_T)} + \sup_{Q_T^1}|\nabla\boldsymbol{u}|\|\nabla\boldsymbol{u}\|_{L_2(G_T)}),$$

$$\|\frac{\partial}{\partial t}(S(\boldsymbol{u})\boldsymbol{n}_0 - S_u(\boldsymbol{u})\boldsymbol{n})\|_{\dot{W}_2^{0,\lambda/2-1/4}(G_T)}$$
$$\leq c(\delta_1\|\nabla\boldsymbol{u}\|_{\dot{W}_2^{0,\lambda/2-1/4}(G_T)} + (1+\sup_{Q_T^1}|\nabla\boldsymbol{u}|)\sup_{Q_T^1)}|\nabla\boldsymbol{u}|\|\nabla\boldsymbol{u}\|_{L_2(G_T)})$$

and

$$\|\frac{\partial}{\partial t}(S(\boldsymbol{u})\boldsymbol{n}_0 - S_u(\boldsymbol{u})\boldsymbol{n})\|_{W_2^{0,\lambda/2-1/4}(G_T)}$$
$$\leq c(\delta_1\|\nabla\boldsymbol{u}\|_{W_2^{0,\lambda/2-1/4}(G_T)} + (1+\sup_{Q_T^1}|\nabla\boldsymbol{u}|)\sup_{Q_T^1)}|\nabla\boldsymbol{u}|\|\nabla\boldsymbol{u}\|_{L_2(G_T)}) \tag{4.11}$$

Similar estimates hold for $\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0 - \boldsymbol{n}\cdot S_u(\boldsymbol{u})\boldsymbol{n}$ and

$$\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0 - \Pi S_u(\boldsymbol{u})\boldsymbol{n} = (I - \boldsymbol{n}_0\otimes\boldsymbol{n}_0)S(\nabla\boldsymbol{u})\boldsymbol{n}_0 - (I - \boldsymbol{n}\otimes\boldsymbol{n})S_u(\nabla\boldsymbol{u})\boldsymbol{n}.$$

From (4.10), (4.11) is is easy to deduce

$$\|\boldsymbol{l}_3(\boldsymbol{u})\|_{W_2^{1/2+l,1/4+l/2}(G_T)} + \|l_4(\boldsymbol{u})\|_{W_2^{1/2+l,1/4+l/2}(G_T)}$$
$$\leq c(\delta_1\|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(Q_T^1)} + \|\boldsymbol{u}\|^2_{W_2^{1+l,1/2}(Q_T^1)} + \|\boldsymbol{u}\|^3_{W_2^{1+l,1/2}(Q_T^1)}). \tag{4.12}$$

The next step is the estimate of $\boldsymbol{l}_5(\boldsymbol{u},\boldsymbol{h})$, $\boldsymbol{l}_6(\boldsymbol{u},\boldsymbol{h})$ and $\boldsymbol{A}^{(i)}$. By (4.1),

$$\|(I-\mathcal{P})\boldsymbol{h}\|_{W_2^{2+\lambda,0}(Q_T^i)} \leq c\sup_{t<T}\|I-\mathcal{P}\|_{W_2^{1+\lambda}(\Omega_i)}\|\boldsymbol{h}\|_{W_2^{2+\lambda,0}(Q_T^i)} \leq c\delta_1\|\boldsymbol{h}\|_{W_2^{2+\lambda,0}(Q_T^i)}; \tag{4.13}$$

in addition,

$$\|\mathcal{P}_t\boldsymbol{h}\|_{L_2(Q_T^i)} + \|(I-\mathcal{P})\boldsymbol{h}_t\|_{L_2(\Omega)} \leq c(\delta_1\|\boldsymbol{h}_t\|_{L_p(Q_T^i)} + \sup_{Q_T^i}|\boldsymbol{h}|\|\nabla\boldsymbol{u}\|_{L_2(Q_T^i)}). \tag{4.14}$$

We proceed by estimating the $\dot{W}_2^{0,\lambda/2}$-norm of $P_t\boldsymbol{h}$ in the cylinder Q_T^1 where $L=1$ and $\mathcal{P}_t = \mathcal{L}_t^T\mathcal{L} + \mathcal{L}^T\mathcal{L}_t$ (the formula

$$\mathcal{P}_t = \frac{\mathcal{L}_t^T\mathcal{L} + \mathcal{L}^T\mathcal{L}_t}{L} - \frac{\mathcal{L}^T\mathcal{L}}{L^2}L_t, \quad \xi\in\Omega_2,$$

leads to the same result). We have

$$\|\Delta_t(-h)\mathcal{P}_t\|_{L_2(\Omega_1)} \leq c(\|\Delta_t(-h)\mathcal{L}_t\|_{L_2(\Omega_1)} + \|\mathcal{L}^T\Delta_t(-h)\mathcal{L}\|_{L_2(\Omega_1)}$$
$$\leq c(\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{L_2(\Omega_1)} + h\sup_{Q_{t-h,t}^1}|\nabla\boldsymbol{u}|\|\nabla\boldsymbol{u}\|_{L_2(\Omega_1)},$$

hence

$$\|(\Delta_t(-h)\mathcal{P})_t\boldsymbol{h}\|_{L_2(\Omega_1)}+\|\mathcal{P}_t\Delta_t(-h)\boldsymbol{h}\|_{L_2(\Omega_1)}$$
$$\leq c\big(\sup_{\Omega_1}|\boldsymbol{h}|(\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{L_2(\Omega_1)}+h\sup_{Q^1_{t-h,t}}|\nabla\boldsymbol{u}|\|\nabla\boldsymbol{u}\|_{L_2(\Omega_1)})+h\sup_{Q^1_{t-h,t}}|\nabla\boldsymbol{u}|\|\boldsymbol{h}_t\|_{L_2(\Omega_1)}\big),$$

which yields

$$\begin{aligned}\|\mathcal{P}_t\boldsymbol{h}\|_{\cdot W_2^{0,\lambda/2}(Q_T^1)}&\leq c\sup_{Q_T^1}|\boldsymbol{h}(\xi,t)|(\|\nabla\boldsymbol{u}\|_{W_2^{0,\lambda/2}(Q_T^1)}\\&+\sup_{Q_T^1}|\nabla\boldsymbol{u}|\|\nabla\boldsymbol{u}\|_{L_2(Q_T^1)})+c\sup_{Q_T^1}|\nabla\boldsymbol{u}|\|\boldsymbol{h}_t\|_{L_2(Q_T^1)}.\end{aligned}\tag{4.15}$$

In addition, it holds

$$\|(\Delta_t(-h)\mathcal{P})\boldsymbol{h}_t\|_{L_2(\Omega_i)}+\|(\mathcal{P}-I)\Delta_t(-h)\boldsymbol{h}_t\|_{L_2(\Omega_i)}$$
$$\leq c(\delta_1\|\Delta_t(-h)\boldsymbol{h}\|_{L_2(\Omega_i)}+\|\boldsymbol{h}_t\int_0^h\nabla\boldsymbol{u}(\cdot,t-\tau)d\tau\|_{L_2(\Omega_i)},$$

and, as a consequence,

$$\|(I-\mathcal{P})\boldsymbol{h}_t\|_{W_2^{0,\lambda/2}(Q_T^i)}\leq c(\delta_1\|\boldsymbol{h}\|_{W_2^{0,1+\lambda/2}(Q_T^i)}+\sup_{Q_T^i}|\nabla\boldsymbol{u}^*|\|\boldsymbol{h}_t\|_{L_2(Q_T^i)}).\tag{4.16}$$

Together with (4.13)-(4.15), this inequality yields the desired estimate of the norm of $(I-\mathcal{P})\boldsymbol{h}$, that is controlled by the right hand side of (3.13). The same is true for the $W_2^{\lambda,\lambda/2}$- norms of $\boldsymbol{l}_6$, $rot(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})$, $\boldsymbol{\Phi}$ (the last two terms are of a lower order).

Since the expression

$$\frac{A^TA\boldsymbol{n}^*}{|A\boldsymbol{n}^*|^2}-\frac{\boldsymbol{n}^*}{|\boldsymbol{n}^*|^2}$$

has the same properties as $I-\mathcal{L}$, $I-\mathcal{P}$, the norm

$$\|\frac{A^TA\boldsymbol{n}_0^*}{|A\boldsymbol{n}_0^*|^2}-\frac{\boldsymbol{n}_0^*}{|\boldsymbol{n}_0^*|^2})\mu\boldsymbol{h}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^i)}$$

is also estimated as $\|(I-\mathcal{P})\boldsymbol{h}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^i)}$ above.

We proceed by estimating the norms of $\nabla_u\cdot T_M(\widetilde{\boldsymbol{h}})$ and $\boldsymbol{n}\cdot T_M(\widetilde{\boldsymbol{h}})\boldsymbol{n}$, where $\boldsymbol{n}=A\boldsymbol{n}_0/|A\boldsymbol{n}_0|$, $\widetilde{\boldsymbol{h}}=\widehat{A}^T\boldsymbol{h}=\mathcal{L}\boldsymbol{h}$. We start with finding a bound for $\mathcal{L}\boldsymbol{h}$ in terms of $\boldsymbol{h}$.

We have

$$\|\widetilde{\boldsymbol{h}}\|_{W_2^{2+\lambda,0}(Q_T^1)}\leq c\sup_{t<T}\|\mathcal{L}\|_{W_2^{2+\lambda}(\Omega_1)}\|\boldsymbol{h}\|_{W_2^{2+\lambda,0}(Q_T^1)}\leq c\|\boldsymbol{h}\|_{W_2^{2+\lambda,0}(Q_T^1)},$$
$$\|(\mathcal{L}\boldsymbol{h})_t\|_{L_2(Q_T^1)}\leq c(\|\boldsymbol{h}_t\|_{L_2(Q_T^1)}+\sup_{Q_T^1}|\boldsymbol{h}|\|\nabla\boldsymbol{u}\|_{L_2(Q_T^1)})$$
$$\|\Delta_t(-h)\mathcal{L}_t\boldsymbol{h}\|_{L_2(\Omega_1)}\leq c\|\int_0^h\nabla\boldsymbol{u}(\cdot,t-\tau)d\tau\boldsymbol{h}\|_{L_2(\Omega_1)},$$

which implies

$$\|(\widetilde{\boldsymbol{h}})_t\|_{\dot W_2^{0,\lambda/2}(Q_T^1)}\leq c(\|\boldsymbol{h}_t\|_{W_2^{0,\lambda/2}(Q_T^1)}+\|\nabla\boldsymbol{u}\|_{L_2(Q_T^1)}\sup_{Q_T^1}|\boldsymbol{h}(\xi,t)|).\tag{4.17}$$

Hence

$$\|\widetilde{\boldsymbol{h}}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^1)} \le c(\|\boldsymbol{h}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^1)} + \|\nabla \boldsymbol{u}\|_{L_2(Q_T^1)} \sup_{Q_T^1} |\boldsymbol{h}(\xi,t)|). \tag{4.18}$$

We continue the estimate of $\nabla_u \cdot T_M(\widetilde{\boldsymbol{h}})$:

$$\|A\nabla \cdot T_M(\widetilde{\boldsymbol{h}}\|_{W_2^{l,0}(Q_T^1)} \le c\|\nabla \widetilde{\boldsymbol{h}}\widetilde{\boldsymbol{h}}\|_{W_2^{l,0}(Q_T^1)} \le c \sup_{t<T} \|\boldsymbol{h}\|_{W_2^{1+\lambda}(\Omega_1)} \|\boldsymbol{h}\|_{W_2^{2+\lambda,0}(Q_T^1)}, \tag{4.19}$$

$$\begin{aligned}
&\|\Delta_t(-h)T_M(\widetilde{\boldsymbol{h}})\|_{L_2(\Omega_1)} \le c(\|\Delta_t(-h)\nabla\widetilde{\boldsymbol{h}}\|_{L_2(\Omega)} \sup_{\Omega_1} |\widetilde{\boldsymbol{h}}| + \|\Delta_t(-h)\widetilde{\boldsymbol{h}}\|_{L_6(\Omega_1)} \|\nabla\widetilde{\boldsymbol{h}}\|_{L_3(\Omega_1)}) \\
&\le c(\|\Delta_t(-h)\widetilde{\boldsymbol{h}}\|_{W_2^1(\Omega_1)} \|\widetilde{\boldsymbol{h}}\|_{W_2^{1+\lambda}(\Omega_1)}, \\
&\|(\Delta_t(-h)A)T_M(\widetilde{\boldsymbol{h}})\|_{L_2(\Omega_1)} \le ch \sup_{Q_{t-h,t}} |\nabla \boldsymbol{u}| \sup_{\Omega_1} |\boldsymbol{h}| \|\nabla \boldsymbol{h}\|_{L_2(\Omega_1)},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\|A\nabla \cdot T_M(\widetilde{\boldsymbol{h}})\|_{W_2^{0,l/2}(Q_T^1)} \le c(\|\widetilde{\boldsymbol{h}}\|_{W_2^{2+\lambda,1+\lambda/2}(Q_T^1)} \sup_{t<T} \|\boldsymbol{h}\|_{W_2^{1+\lambda}(\Omega_1)} \\
&+ \sup_{t<T} \|\boldsymbol{u}\|_{W_2^{1+l}(\Omega_1)} \sup_{t<T} \|\boldsymbol{h}\|_{W_2^{1+\lambda}(\Omega_1)} \|\nabla \boldsymbol{h}\|_{L_2(q)T^1)}),
\end{aligned} \tag{4.20}$$

and the norm of $\widetilde{\boldsymbol{h}}$ is estimated by (4.18).

Finally, we estimate $\|\boldsymbol{n} \cdot T_M(\widetilde{\boldsymbol{h}})\boldsymbol{n}\|_{W_2^{l+1/2,l/2+1/4}(G_T)}$. By (4.1),

$$\begin{aligned}
&\|\boldsymbol{n} \cdot T_M(\widetilde{\boldsymbol{h}})\boldsymbol{n}\|_{W_2^{l+1/2,0}(G_T)} \le c\|\boldsymbol{n}^* \cdot T_M(\widetilde{\boldsymbol{h}})\boldsymbol{n}^*\|_{W_2^{l+1,0}(Q_T^1)} \\
&\le c\|T_M(\widetilde{\boldsymbol{h}})\|_{W_2^{2+\lambda,0}(Q_T^1)} \le c \sup_{t<T} \|\boldsymbol{h}\|_{W_2^{1+\lambda}(\Omega_1)} \|\boldsymbol{h}\|_{W_2^{2+\lambda,0}(Q_T^1)}, \\
&\|\frac{\partial}{\partial t}(\boldsymbol{n} \cdot T_M(\widetilde{\boldsymbol{h}})\boldsymbol{n})\|_{L_2(G_T)} \le c(\sup_{G_T} |\boldsymbol{h}| \|\widetilde{\boldsymbol{h}}_t\|_{L_2(G_T)} + \sup_{Q_T^1} |\boldsymbol{h}|^2 \|\nabla \boldsymbol{u}\|_{L_2(G_T)}),
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
&\|\Delta_t(-h)T_M(\widetilde{\boldsymbol{h}})_t\|_{L_2(\Gamma_0)} \le c(\|\Delta_t(-h)\widetilde{\boldsymbol{h}}_t\|_{L_2(\Gamma_0)} \sup_{\Gamma_0} |\boldsymbol{h}| + \|\widetilde{\boldsymbol{h}}_t\|_{L_2(\Gamma_0)} \sup_{\Gamma_0} |\Delta_t(-h)\widetilde{\boldsymbol{h}}|, \\
&\|(\Delta_t(-h)\boldsymbol{n}_t) \cdot T_M(\widetilde{\boldsymbol{h}})\boldsymbol{n}\|_{L_2(\Gamma_0)} \le c \sup_{\Gamma_0} |\boldsymbol{h}|^2 \|\Delta_t(-h)\nabla \boldsymbol{u}\|_{L_2(\Gamma_0)}, \\
&\|(\Delta_t(-h)\boldsymbol{n})T_M(\widetilde{\boldsymbol{h}})\boldsymbol{n}_t\|_{L_2(\Gamma_0)} \le ch \sup_{Q_{t-h,t}^1} |\nabla \boldsymbol{u}| \sup_{\Gamma_0} |\boldsymbol{h}|^2 \|\nabla \boldsymbol{u}\|_{L_2(\Gamma_0)},
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\boldsymbol{n} \cdot T_M(\widetilde{\boldsymbol{h}})\boldsymbol{n}\|_{W_2^{0,l/2+1/4}(G_T)} \le c(\sup_{G_T)} |\boldsymbol{h}| \|\boldsymbol{h}_t\|_{W_2^{0,\lambda/2-1/4}(G_T)} + \sup_{G_T} |\boldsymbol{h}|^2 \|\nabla \boldsymbol{u}\|_{W_2^{0,\lambda/2-1/4}(G_T)} \\
&+ \|\boldsymbol{h}_t\|_{L_2(G_T)} \|\widetilde{\boldsymbol{h}}_{W_2^{\lambda/2-1/4}(0,T;W_2^{1+\eta}(\Gamma_0))} + \sup_{G_T} \nabla \boldsymbol{u}| \sup_{G_T} |\boldsymbol{h}|^2 \|\nabla \boldsymbol{u}\|_{L_2(G_T)})
\end{aligned} \tag{4.22}$$

It is clear that $\|\boldsymbol{n} \cdot T_M(\widetilde{\boldsymbol{h}})\boldsymbol{n}\|_{W_2^{l+1/2,l/2+1/4}(G_T)}$ is controlled by the right hand side of (3.15).

Remark. We have estimated all the nonlinear terms in (3.3) in the time interval $(0,T)$, but the same estimates hold for $t \in (kT,(k+1)T)$, provided that the condition (3.14) is satisfied for $t \le (k+1)T$.

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