ON THE MATHEMATICAL ANALYSIS OF THICK FLUIDS

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Abstract. In chemical engineering models, shear-thickening or dilatant fluids converge in the limit case to a class of incompressible fluids with a maximum admissible shear rate, the so-called thick fluids. These non-Newtonian fluids may be obtained, in particular, as the power limit of Ostwald-de Waele fluids, and may be formulated as a new class of evolution variational inequalities, in which the shear rate is bounded by a positive constant or, more generally, by a bounded positive function. We prove the existence, uniqueness and continuous dependence of solutions to this general class of thick fluids with variable threshold on the absolute value of the deformation rate tensor, which solutions belong to a time dependent convex set. For sufficiently large viscosity, we also show the asymptotic stabilization towards the unique steady state.

(*Dedicated to V. A. Solonnikov on the occasion of his* 80^{th} *birthday*)

1. Introduction

In spite of their importance in industrial processes, the current understanding of shear thickening in complex fluids, like colloidal dispersions, granular suspensions, cements or soft body armor, is far from being well achieved. Though the rheological survey paper [2] already put in evidence the experimental fact of abrupt rising of viscosity, sometimes discontinuously, once a critical shear stress is attained, recent research in this direction motivated a feature article in Physics Today [31], showing the increasing interest in shear-thickening fluids, in particular, in armor applications [15].

In the recent book [19], where the whole Chapter 8 is dedicated to the rheology of shear thickening, we can find examples of complex fluids where the viscosity may increase without increase in shear rate when the sample is subject to increasing shear stress, or in physical situations where discontinuous shear thickening may occur and, regardless the applied stress, a limiting shear rate is achieved and the fluid exhibits significant slip and "jamming".

On the other hand, non-Newtonian flow is often modeled with a power law for the relationship between shear stress and shear rate. In this work we shall consider the typical example

$$\mathbf{S}_p = \mathbf{S}_p(\mathbf{D}\boldsymbol{u}) = \left(|\mathbf{D}\boldsymbol{u}|^{p-2} + \mu|\mathbf{D}\boldsymbol{u}|^{q-2}\right)\mathbf{D}\boldsymbol{u}, \tag{1.1}$$

suggested by Ladyzhenskaya in [13] with $p = 2$, relating the strain velocity or deformation rate tensor, *i.e.*, the symmetric part of the velocity gradient $\mathbf{D}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, with the stress tensor $\mathbf{S}_p$. Here $\mu \geq 0$ is a viscosity constant.

For small powers, $1 < p, q < 2$, below the linear Newtonian behavior $p = q = 2$, the fluids exhibit shear thinning properties, while for higher degree of the nonlinearity, *i.e.*, for $p, q > 2$, the behavior is shear thickening (see [18], for instance). It is interesting to observe how particular theoretical treatments in applied books may raise enough stimulating discussions on the infinity degree of shear thickening behavior (see, for instance, pages 124 and 332 of [5]), although almost nothing is known for the case of the limit model $p \to \infty$.

Several critical state problems in physics have mathematical models with gradient constraints, like in elastoplastic deformations, magnetization of type-II superconductors, sandpile growth or formation of network of lakes and rivers, that lead to variational inequalities (see, for instance, [23]). It is well-known that, in the scalar case, the gradient constraint can be obtained as limits of nonlinear viscosity coefficients, namely in steady problems in elastoplastic torsion [10] or in fast/slow diffusion taken as the limit $p \to \infty$ in the p-Laplacian [3], as well as in time dependent problems in nonlinear diffusion [26] or in a sandpile model [1]. More complex cases with the gradient threshold depending on the solution have been also studied in problems related to a superconductivity model [24] and to diffusion-less models [25].

Recently this approach has been extended to a p-curl system in electromagnetism [21] and it is shown here to be also applicable to the generalized Navier-Stokes system. Thickening a dilatant fluid, *i.e.*, letting $p \to \infty$ in (1.1), we conclude that, in the limit, the symmetric part of the velocity gradient $\mathbf{D}\boldsymbol{u}$ must satisfy the constraint

$$|\mathbf{D}\boldsymbol{u}| \leq 1 \tag{1.2}$$

on the shear rate. In the limit we say the fluid is thick. More generally, we may consider more complex thick fluids with a variable shear rate threshold, by replacing the unit constraint in (1.2) by a positive function $\psi = \psi(x, t)$.

Let $\Omega \subset \mathbb{R}^d$, be a bounded domain, with Lipschitz boundary $\partial\Omega$, $d \geq 2$ and $Q_T = \Omega \times (0, T)$, $T > 0$. The classical system for an incompressible dilatant fluid for the velocity $\boldsymbol{u} = \boldsymbol{u}(x, t)$ and the pressure $\pi = \pi(x, t)$ reads, in Q_T:

$$\partial_t \boldsymbol{u} - \operatorname{div}(\mathbf{S}_p - \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla\pi = \boldsymbol{f}, \tag{1.3}$$

$$\operatorname{div} \boldsymbol{u} = 0. \tag{1.4}$$

Here $\partial_t \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t}$ and $\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ represents the usual convective term under the incompressibility condition (1.4).

The existence of unique weak solution to the Cauchy-Dirichlet problem, *i.e.*, to (1.3)–(1.4) with

$$\boldsymbol{u} = \mathbf{0} \ \text{ on } \partial\Omega \times (0, T), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \ \text{ in } \Omega, \tag{1.5}$$

for $p \geq \frac{d+2}{2}$, $1 < q < \infty$, $\mu \geq 0$ is well-known (see [14] and [16]). Several variants of generalized Newtonian fluids of the type (1.3)–(1.4) have been considered in the mathematical literature (see, for instance, the book [17] or the survey [18]) and the existence of weak solutions has been shown for $d = 3$ up to $p > \frac{6}{5}$ [7].

Thick flows may also be seen as limit of certain non-Newtonian flows, similarly to the case of Bingham fluids, which lead to variational inequalities in the limit of some approximations [8], [30] and correspond to flows with two phases: rigid if $|\mathbf{D}\boldsymbol{u}| = 0$ for $|\mathbf{S}| \leq \sigma_*$ and fluid for $|\mathbf{S}| > \sigma_* > 0$. Although quite different, this new type of thick flow with shear rate constraint has also two phases. Indeed, formally, the constraint

$$|\mathbf{D}\boldsymbol{u}(x,t)| \leq \psi(x,t) \tag{1.6}$$

for the velocity field $\boldsymbol{u}$ of the thick fluid divides the domain into two subdomains

$$Q_T = \big\{(x,t) : |\mathbf{D}\boldsymbol{u}(x,t)| < \psi(x,t)\big\} \cup \big\{(x,t) : |\mathbf{D}\boldsymbol{u}(x,t)| = \psi(x,t)\big\}. \tag{1.7}$$

In the first phase, below the critical threshold, we have the power law flow

$$\partial_t \boldsymbol{u} - \operatorname{div}\big(\mu|\mathbf{D}\boldsymbol{u}|^{q-2}\mathbf{D}\boldsymbol{u} - \boldsymbol{u}\otimes\boldsymbol{u}\big) + \nabla\pi = \boldsymbol{f} \quad \text{in} \quad \big\{|\mathbf{D}\boldsymbol{u}| < \psi\big\} \tag{1.8}$$

while in the second phase, the thick strain velocity $\mathbf{D}\boldsymbol{u}$ is conditioned by the two scalar equations

$$|\mathbf{D}\boldsymbol{u}(x,t)| = \psi(x,t) \qquad \text{and} \qquad \operatorname{div}\boldsymbol{u} = 0. \tag{1.9}$$

In this phase, if ψ is a positive constant, an applied stress increase in the thick fluid may increase also the viscosity without increase the shear rate. Although this limit rheology law is also of discontinuity power type it does not falls in the class treated in [11], although our problem can also be treated with the theory of maximal monotone operators.

In the next section, we present the mathematical analysis of $p \to \infty$ in the constitutive law (1.1) with $q = 2$, obtaining an evolution variational inequality for the equation (1.8), *i.e.*, the Navier-Stokes equation with the constraint (1.2), which has a unique solution. In the following section, we extend the method of [24] and [21] to the variable shear rate threshold $\psi(x,t) > 0$, showing the well-posedness of the corresponding variational inequality with time dependent convex set. We prove the continuous dependence of the solution with respect to the data, including the threshold ψ, and, in the final section, we consider the asymptotic behavior towards the steady-state when $t \to \infty$ and the viscosity is sufficiently large to ensure the uniqueness of the stationary solution. In the following joint work [20] we extend these results to more general non-Newtonian fluids, including the power law fluids (1.1) with $q \neq 2$.

After presenting these results at the opening of the conference on “Mathematical Hydrodynamics and Parabolic Equations”, held in St. Petersburg, 11-14 September 2013, Harald Garcke called the attention of the author to the preprint [6] and Eduard Feireisl to the earlier work [28]. Both works consider also the constraint $\psi = 1$, the first one for the steady-state Stokes system adapting the methods of [9] and presenting numerical examples, and the second one proving the existence of weak solutions for non-Newtonian flows with rigid bodies in which the inhomogeneous viscosity μ is a solution of a transport equation.

2. Thickening a dilatant fluid ($p \to \infty$)

In order to work with the variational formulation for the dilatant model (1.1) with $q = 2$, we introduce the following notations:

$$\mathbb{H} = \boldsymbol{L}^2_\sigma(\Omega) = \text{closure of } \mathbb{J} \text{ in } \boldsymbol{L}^2(\Omega), \tag{2.1}$$

$$\mathbb{V}_r = \text{closure of } \mathbb{J} \text{ in } \boldsymbol{W}^{1,r}(\Omega), \tag{2.2}$$

where $\mathbb{J} = \{\boldsymbol{\varphi} \in \mathscr{C}^\infty_0(\Omega)^d : \operatorname{div}\boldsymbol{\varphi} = 0\}$ is the space of solenoidal, smooth vector functions with support in $\Omega \subset \mathbb{R}^d$, $W^{1,r}(\Omega)$ is a Sobolev space with $1 < r < \infty$, $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^d$ and $\boldsymbol{W}^{1,r}(\Omega) = W^{1,r}(\Omega)^d$.

We obtain the weak formulation by multiplying (1.3) by a test function $\boldsymbol{\varphi} \in \mathbb{J}$ and integration by parts in Ω. For *a.e.* $t \in (0,T)$, we obtain then

$$\int_\Omega \partial_t \boldsymbol{u} \cdot \boldsymbol{\varphi} + \int_\Omega \mathbf{S}_p(\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{\varphi} - \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}, \qquad \forall \boldsymbol{\varphi} \in \mathbb{V}_p, \tag{2.3}$$

where, by abuse of notation, the first integral is understood in the usual duality sense of $\mathbb{V}_p \subset \mathbb{H} \subset \mathbb{V}_p'$. Here we take $p \geq \frac{d+2}{2} \geq 2$ and $q > 1$ without loss of generality, since we are interested in $p \to \infty$.

Using variational methods (see [14], [16], [17] or [18] and their references), we may easily prove the following result.

Proposition 2.1. *For $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{u}_0 \in \mathbb{H}$, $p \geq \frac{d+2}{2} \geq 2$, there exists a unique solution $\boldsymbol{u} = \boldsymbol{u}_p(t)$ solving (2.3) for a.e. $t \in (0,T)$ in the class*

$$\boldsymbol{u} = \boldsymbol{u}_p \in \mathscr{C}\big([0,T]; \mathbb{H}\big) \cap L^p\big(0,T; \mathbb{V}_p\big), \tag{2.4}$$

with $\partial_t \boldsymbol{u} \in L^{p'}\big(0,T;\mathbb{V}_p'\big)$. In addition, if $\boldsymbol{u}_0 \in \mathbb{V}_p$ then

$$\partial_t \boldsymbol{u} = \partial_t \boldsymbol{u}_p \in L^2\big(0,T;\mathbb{H}\big) \subset \boldsymbol{L}^2(Q_T). \tag{2.5}$$

Proof. The existence of a weak solution follows by well-known methods using the *a priori* estimate

$$\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \int_0^t \|\mathbf{D}\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \leq C_d \left(\int_0^t \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} \right) \tag{2.6}$$

which is a consequence of formally testing (2.3) with $\boldsymbol{\varphi} = \boldsymbol{u}_p(t)$ and applying Poincaré's inequality. Note that the constant $C_d > 0$ in (2.6) is independent of $p \geq \frac{d+2}{2}$. This assumption also implies, by Sobolev's inequalities that $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^2(Q_T)$, also independently of p by (2.6). Hence, by testing (2.3) with $\boldsymbol{\varphi} = \partial_t \boldsymbol{u} = \partial_t \boldsymbol{u}_p$, using (2.6) and

$$-\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla \partial_t \boldsymbol{u} = \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \partial_t \boldsymbol{u} \leq C_p \|\boldsymbol{u}\|^2_{\boldsymbol{L}^{\frac{2p}{p-2}}(\Omega)} \|\mathbf{D}\boldsymbol{u}\|^2_{\boldsymbol{L}^p(\Omega)} + \frac{1}{3}\|\partial_t \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)},$$

we get the regularity (2.5) from the estimate, for *a.e.* $t > 0$,

$$\frac{1}{3}\int_0^t \|\partial_t \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \varphi_p(\boldsymbol{u}(t)) \leq \varphi_p(\boldsymbol{u}_0) + C_d' \left(\int_0^t \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} \right), \tag{2.7}$$

where, as in (2.6), the constant $C_d' > 0$ is also independent of p and we have set

$$\varphi_p(\boldsymbol{u}) = \frac{1}{p}\int_\Omega |\mathbf{D}\boldsymbol{u}|^p + \frac{\mu}{2}\int_\Omega |\mathbf{D}\boldsymbol{u}|^2. \tag{2.8}$$

The uniqueness of the solution is also well-known for the values of $p \geq \frac{d+2}{2}$. $\square$

Remarking that if $|\mathbf{D}\boldsymbol{u}_0| \leq 1$ in Ω we have $\varphi_p(\boldsymbol{u}_0) \leq C_0$, independent of p, we immediately conclude the following proposition.

Proposition 2.2. *The solution $\boldsymbol{u}_p$ satisfies the a priori estimates*

$$\|\boldsymbol{u}_p\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\mathbf{D}\boldsymbol{u}_p\|^p_{\boldsymbol{L}^p(Q_T)} + \mu\|\boldsymbol{u}_p\|^2_{\boldsymbol{L}^2(Q_T)} \leq C_0 \tag{2.9}$$

and, if in addition

$$\boldsymbol{u}_0 \in \mathbb{K}^1 \equiv \big\{\boldsymbol{w} \in \mathbb{V}_r : |\mathbf{D}\boldsymbol{w}| \leq 1 \text{ in } \Omega\big\}, \qquad \forall r < \infty, \tag{2.10}$$

it satisfies also

$$\|\partial_t \boldsymbol{u}_p\|^2_{\boldsymbol{L}^2(Q_T)} + \sup_{0<t<T} \frac{1}{p}\|\mathbf{D}\boldsymbol{u}_p(t)\|^p_{\boldsymbol{L}^p(\Omega)} \leq C_1, \tag{2.11}$$

where the constants $C_0, C_1 > 0$ *are independent of* $p \geq \frac{d+2}{2}$.

We may introduce the evolution variational inequality associated with the fixed convex set $\mathbb{K}^1 \subset \bigcap_{1<r<\infty} \mathbb{V}_r$, defined by (2.10) and corresponding to the limit problem for $\boldsymbol{v} = \boldsymbol{v}(t)$:

$$\int_\Omega \partial_t \boldsymbol{v} \cdot (\boldsymbol{w} - \boldsymbol{v}) + \mu \int_\Omega \mathbf{D}\boldsymbol{v} : \mathbf{D}(\boldsymbol{w} - \boldsymbol{v}) - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla(\boldsymbol{w} - \boldsymbol{v}) \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{v}), \qquad \forall \boldsymbol{w} \in \mathbb{K}^1, \tag{2.12}$$

which should hold for *a.e.* $t \in (0, T)$.

Theorem 2.1. *For any* $\mu \geq 0$, *if we denote by* $\boldsymbol{u}_p$ *the unique solution to* (2.3) *with* $\boldsymbol{f} \in \boldsymbol{L}^2(Q_T)$ *and initial condition* $\boldsymbol{u}_0 \in \mathbb{K}^1$, *we have, as* $p \to \infty$,

$$\boldsymbol{u}_p \to \boldsymbol{v} \quad \text{in } \mathscr{C}\big([0, T]; \boldsymbol{L}^2(\Omega)\big)\text{-strong} \cap L^r\big(0, T; \mathbb{V}_r\big)\text{-weak}, \tag{2.13}$$

$$\partial_t \boldsymbol{u}_p \rightharpoonup \partial_t \boldsymbol{v} \quad \text{in } \boldsymbol{L}^2(Q_T)\text{-weak}, \tag{2.14}$$

for any $r < \infty$, *where the limit* $\boldsymbol{v} = \boldsymbol{v}(t)$ *is in the class*

$$\boldsymbol{v} \in H^1\big(0, T; \boldsymbol{L}^2(\Omega)\big) \cap L^r\big(0, T; \mathbb{V}_r\big) \cap L^\infty\big(0, T; \mathbb{K}^1\big) \tag{2.15}$$

and satisfies (2.12) *for a.e.* $t \in (0, T)$ *and the initial condition* $\boldsymbol{v}(0) = \boldsymbol{u}_0$. *If* $\mu > 0$, *the solution of* (2.12) *in the class* (2.15) *is unique and the convergences* (2.13)–(2.14) *hold for the whole sequence.*

Proof. Since the estimates (2.9) and (2.11) are independent of p, (2.13) and (2.14) hold for at least a subsequence as $p \to \infty$. Hence for $r < s < p$, we have

$$\|\mathbf{D}\boldsymbol{v}\|_{\boldsymbol{L}^s(Q_T)} \leq \liminf_{p\to\infty} \|\mathbf{D}\boldsymbol{u}_p\|_{\boldsymbol{L}^s(Q_T)} \leq \limsup_{p\to\infty} |Q_T|^{\frac{1}{s}-\frac{1}{p}} \|\mathbf{D}\boldsymbol{u}_p\|_{\boldsymbol{L}^p(Q_T)} \leq |Q_T|^{\frac{1}{s}}, \qquad \forall s < \infty,$$

since, by the estimate (2.6), $\|\mathbf{D}\boldsymbol{u}_p\|_{\boldsymbol{L}^p(Q_T)} \leq C_0^{1/p}$. So, letting $s \to \infty$ we find $\|\mathbf{D}\boldsymbol{v}\|_{\boldsymbol{L}^\infty(Q_T)} \leq 1$ and consequently $\boldsymbol{v}$ is in the class (2.15).

By well-known compactness results, we may also assume that

$$\boldsymbol{u}_p \otimes \boldsymbol{u}_p \to \boldsymbol{v} \otimes \boldsymbol{v} \quad \text{in } \boldsymbol{L}^2(Q_T)\text{-strong.}$$

Integrating (2.3) in $t \in (0, T)$ with $\boldsymbol{\varphi} = \boldsymbol{w}(t) - \boldsymbol{u}_p(t)$, for an arbitrary $\boldsymbol{w} \in L^p\big(0, T; \mathbb{V}_p\big)$ such that $|\mathbf{D}\boldsymbol{w}| < 1$ in Q_T, and using the monotonicity of $\mathbf{S}_p(\mathbf{D}\boldsymbol{u}_p)$, we easily get

$$\int_{Q_T} \partial_t \boldsymbol{u}_p \cdot (\boldsymbol{w} - \boldsymbol{u}_p) + \int_{Q_T} |\mathbf{D}\boldsymbol{w}|^{p-2}\mathbf{D}\boldsymbol{w} : \mathbf{D}(\boldsymbol{w} - \boldsymbol{u}_p) + \mu \int_{Q_T} \mathbf{D}\boldsymbol{w} : \mathbf{D}(\boldsymbol{w} - \boldsymbol{u}_p) - \int_{Q_T} (\boldsymbol{u}_p \otimes \boldsymbol{u}_p) : \nabla(\boldsymbol{w} - \boldsymbol{u}_p) \geq \int_{Q_T} \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{u}_p).$$

In the limit $p \to \infty$, this yields

$$\int_{Q_T} \partial_t \boldsymbol{v} \cdot (\boldsymbol{w} - \boldsymbol{v}) + \mu \int_{Q_T} \mathbf{D}\boldsymbol{w} : \mathbf{D}(\boldsymbol{w} - \boldsymbol{v}) - \int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla(\boldsymbol{w} - \boldsymbol{v}) \geq \int_{Q_T} \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{v}).$$

which by density holds also for all $\boldsymbol{w} \in L^\infty\big(0, T; \mathbb{K}^1\big)$. By Minty's argument we conclude first that $\boldsymbol{v}$ satisfies the variational inequality (2.12) integrated in $t \in (0, T)$, for all such $\boldsymbol{w}$. By a well-known argument for parabolic variational inequalities (see [16]), this implies that $\boldsymbol{v}(t)$ also satisfies (2.12) for *a.e.* $t \in (0, T)$, proving the first part of Theorem 2.1.

If $\mu > 0$, the uniqueness of $\boldsymbol{v}$ in the class (2.15) follows since $L^\infty\big(0, T; \mathbb{K}^1\big) \subset \boldsymbol{L}^\infty(Q_T)$ by Sobolev's inequalities, and we may control the convection terms of any two solutions $\boldsymbol{v}$ and $\widehat{\boldsymbol{v}}$ in terms of their difference $\boldsymbol{w} = \boldsymbol{v} - \widehat{\boldsymbol{v}}$

$$\begin{aligned} \left| \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v} - \widehat{\boldsymbol{v}} \otimes \widehat{\boldsymbol{v}}) : \nabla \boldsymbol{w} \right| &\leq \int_\Omega |(\boldsymbol{v} \otimes \boldsymbol{w} + \boldsymbol{w} \otimes \widehat{\boldsymbol{v}}) : \nabla \boldsymbol{w}| \\ &\leq 2M\sqrt{d}\, \|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq 2M\sqrt{d}\sqrt{2}\, \|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \|\mathbf{D}\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq \frac{4d}{\mu} M^2 \|\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\mu}{2} \|\mathbf{D}\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}, \end{aligned} \tag{2.16}$$

where $M > 0$ is an upper bound for $|\boldsymbol{v}|$ and $|\widehat{\boldsymbol{v}}|$. Hence Gronwall's inequality immediately implies the uniqueness of the solution of (2.12). □

Remark 2.1. Under the assumption $\boldsymbol{u}_0 \in \mathbb{K}^1$ it is clear that the constant C_1 in (2.11) is independent of $\mu \leq 1$. Therefore we can let $\mu \to 0$ in (2.12), *i.e.*, if $\boldsymbol{v}_\mu$ denote the solutions of (2.12) in the class (2.15), as in the proof of Theorem 2.1, we may conclude that for subsequences, $\boldsymbol{v}_\mu \to \boldsymbol{v}_0$ in $\mathscr{C}\big([0, T]; \boldsymbol{L}^2(\Omega)\big)$-strong $\cap$ $L^r\big(0, T; \mathbb{V}_r\big)$-weak $\cap\, H^1\big(0, T; \boldsymbol{L}^2(\Omega)\big)$-weak, where $\boldsymbol{v}_0$ is a solution of the Euler system (1.8) with $\mu = 0$ and the constraint (1.2). However in this inviscid limit the uniqueness argument fails and we do not know if $\boldsymbol{v}_0$ is unique.

3. Thick fluids with variable shear rate threshold

Let $\psi = \psi(x, t) : Q_T \to \mathbb{R}^+$ be a given function, such that

$$\psi \geq \alpha > 0 \qquad \text{and} \qquad \psi \in W^{1,\infty}(0, T; L^\infty(\Omega)), \tag{3.1}$$

so that, the following closed convex subset of $\mathbb{V}_2$ is nonempty for *a.e.* $t \in (0, T)$,

$$\mathbb{K}(t) \equiv \{\boldsymbol{w} \in \mathbb{V}_2 : |\mathbf{D}\boldsymbol{w}| \leq \psi(x, t) \quad a.e.\ x \in \Omega\}. \tag{3.2}$$

Motivated by the previous section, we introduce the following variational inequality for the thick flow with variable shear rate threshold and positive viscosity $\mu > 0$. Find a strong solution

$$\boldsymbol{u} \in L^\infty(0, T; \mathbb{V}_2) \cap H^1(0, T; \boldsymbol{L}^2(\Omega)), \tag{3.3}$$

such that

$$\boldsymbol{u}(t) \in \mathbb{K}(t) \qquad \text{for } a.e.\ \ t \in (0, T), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{3.4}$$

$$\int_\Omega \partial_t \boldsymbol{u}\cdot(\boldsymbol{w}-\boldsymbol{u}) + \mu\int_\Omega \mathbf{D}\boldsymbol{u} : \mathbf{D}(\boldsymbol{w}-\boldsymbol{u}) - \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla(\boldsymbol{w}-\boldsymbol{u}) \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u}), \qquad \forall\, \boldsymbol{w}\in\mathbb{K}(t), \quad a.e.\ t\in(0,T), \tag{3.5}$$

where we shall assume

$$\boldsymbol{f}\in \boldsymbol{L}^2(Q_T) \qquad \text{and} \qquad \boldsymbol{u}_0\in\mathbb{K}(0). \tag{3.6}$$

As in the scalar case of [27] and the curl system of [21] we may prove the following result.

Theorem 3.1. *Under the assumptions (3.1), $\mu>0$ and (3.6), there exists a unique strong solution $\boldsymbol{u}$ to the variational inequality (3.4)–(3.5) satisfying (3.3) and, as a consequence, also $\boldsymbol{u}\in\mathscr{C}^0([0,T];\mathscr{C}^{0,\gamma}(\bar\Omega)^d)$ for all $0\leq\gamma<1$.*

Proof. For a positive parameter $\varepsilon<1$, we consider the following family of continuous, bounded and increasing functions $\mu_\varepsilon:\mathbb{R}\to\mathbb{R}^+$, such that

$$\mu_\varepsilon(s) = \begin{cases} \mu & \text{if } s\leq 0,\\ \mu\, e^{s/\varepsilon} & \text{if } \varepsilon\leq s<\frac{1}{\varepsilon}-\varepsilon,\\ \mu\, e^{1/\varepsilon^2} & \text{if } s>\frac{1}{\varepsilon}.\end{cases}$$

Then, the approximate problem with penalized viscosity for $\boldsymbol{u}_\varepsilon=\boldsymbol{u}_\varepsilon(t)$, *a.e.* $t\in(0,T)$,

$$\int_\Omega \partial_t\boldsymbol{u}_\varepsilon\cdot\boldsymbol{\varphi} + \int_\Omega \mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2-\psi^2)\,\mathbf{D}\boldsymbol{u}_\varepsilon : \mathbf{D}\boldsymbol{\varphi} - \int_\Omega(\boldsymbol{u}_\varepsilon\otimes\boldsymbol{u}_\varepsilon):\nabla\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}, \qquad \forall\,\boldsymbol{\varphi}\in\mathbb{V}_2, \tag{3.7}$$

with initial condition $\boldsymbol{u}_\varepsilon(0)=\boldsymbol{u}_0$, by well-known nonlinear methods (see [16]) has solutions $\boldsymbol{u}_\varepsilon\in L^2(0,T;\mathbb{V}_2)\cap H^1(0,T;\boldsymbol{L}^2(\Omega))$ satisfying the following *a priori* estimates independently of $0<\varepsilon<1$:

$$\|\boldsymbol{u}_\varepsilon\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \mu\,\|\mathbf{D}\boldsymbol{u}_\varepsilon\|^2_{\boldsymbol{L}^2(Q_T)} \leq C_0, \tag{3.8}$$

$$\int_{Q_T}\mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2-\psi^2)\leq C_1, \tag{3.9}$$

$$\|\partial_t\boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(Q_T)}\leq C_2. \tag{3.10}$$

Indeed, (3.8) is a simple consequence of setting $\boldsymbol{\varphi}=\boldsymbol{u}_\varepsilon$ in (3.7), that also yields

$$\int_{Q_T}\mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2-\psi^2)\,|\mathbf{D}\boldsymbol{u}_\varepsilon|^2\leq C_0.$$

Recalling $\psi\geq\alpha>0$ and $\mu_\varepsilon(s)=\mu$ for $s\leq0$ and $\mu_\varepsilon(s)\geq0$ for $s\geq0$, we conclude (3.9) from

$$\begin{aligned}\alpha^2\int_{Q_T}\mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2-\psi^2) &\leq \int_{Q_T}\mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2-\psi^2)\,\psi^2\\ &\leq \int_{\{|\mathbf{D}\boldsymbol{u}_\varepsilon|^2\leq\psi^2\}}\mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2-\psi^2)\,(\psi^2-|\mathbf{D}\boldsymbol{u}_\varepsilon|^2)+\int_{Q_T}\mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2-\psi^2)\,|\mathbf{D}\boldsymbol{u}_\varepsilon|^2\\ &\leq \mu\int_{Q_T}\psi^2+C_0=\alpha^2\,C_1.\end{aligned}$$

Setting $m_\varepsilon(s) = \int_0^s \mu_\varepsilon(\tau)\,d\tau$ and letting formally $\boldsymbol{\varphi} = \partial_t \boldsymbol{u}_\varepsilon$ in (3.7), we have

$$\begin{aligned}(3.11)\qquad \int_\Omega |\partial_t \boldsymbol{u}_\varepsilon|^2 + \frac{1}{2}\int_\Omega \partial_t m_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2 - \psi^2)& \\ = \int_\Omega \boldsymbol{f}\cdot\partial_t \boldsymbol{u}_\varepsilon - \int_\Omega \mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2 - \psi^2)\,\psi\,\partial_t\psi& \\ \leq \frac{1}{2}\int_\Omega |\partial_t \boldsymbol{u}_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\boldsymbol{f}|^2 + C_\psi\, C_1,&\end{aligned}$$

by (3.1) and (3.9). Integrating in time $t \in (0,T)$, we conclude (3.10) where $C_2 = C_2(\|\boldsymbol{f}\|_{\boldsymbol{L}^2}, \|\psi\|_{W^{1,\infty}(0,T;L^\infty(\Omega))}, \|\boldsymbol{u}_0\|_{\mathbb{V}_2})$ is independent of ε by the condition $|\mathbf{D}\boldsymbol{u}_0| \leq \psi(0)$ in Ω.

Hence, by compactness, we can take a subsequence $\varepsilon \to 0$, such that

$$\begin{aligned}\boldsymbol{u}_\varepsilon \to \boldsymbol{u} &\quad \text{in } \mathscr{C}([0,T];\boldsymbol{L}^2_\sigma(\Omega))\text{-strong},\\ \mathbf{D}\boldsymbol{u}_\varepsilon \rightharpoonup \mathbf{D}\boldsymbol{u} &\quad \text{in } \boldsymbol{L}^2(Q_T)\text{-weak},\\ \partial_t\boldsymbol{u}_\varepsilon \rightharpoonup \partial_t\boldsymbol{u} &\quad \text{in } \boldsymbol{L}^2(Q_T)\text{-weak},\end{aligned}$$

for some $\boldsymbol{u} \in H^1(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\mathbb{V}_2)$, with $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

By considering the partition of Q_T in the form

$$\begin{aligned}A_\varepsilon &= \left\{(x,t)\in Q_T : |\mathbf{D}\boldsymbol{u}_\varepsilon(x,t)|^2 - \psi^2(x,t) < \sqrt{\varepsilon}\right\},\\ B_\varepsilon &= \left\{(x,t)\in Q_T : \sqrt{\varepsilon} \leq |\mathbf{D}\boldsymbol{u}_\varepsilon(x,t)|^2 - \psi^2(x,t) \leq 1/\varepsilon\right\},\\ C_\varepsilon &= \left\{(x,t)\in Q_T : |\mathbf{D}\boldsymbol{u}_\varepsilon(x,t)|^2 - \psi^2(x,t) > 1/\varepsilon\right\},\end{aligned}$$

we conclude first $\boldsymbol{u}(t) \in \mathbb{K}(t)$, *a.e.* $t \in (0,T)$, *i.e.*, $|\mathbf{D}\boldsymbol{u}(t)| \leq \psi(t)$ *a.e.* in Q_T, from

$$\begin{aligned}\int_{Q_T} (|\mathbf{D}\boldsymbol{u}_\varepsilon|^2 - \psi^2)^+ &\leq \liminf_\varepsilon \int_{Q_T} (|\mathbf{D}\boldsymbol{u}_\varepsilon|^2 - \psi^2) \wedge \frac{1}{\varepsilon} \vee \sqrt{\varepsilon}\\ &\leq \liminf_\varepsilon \int_{A_\varepsilon} \sqrt{\varepsilon} + \liminf_\varepsilon \int_{B_\varepsilon\cup C_\varepsilon} \frac{1}{\varepsilon}\\ &\leq \lim_\varepsilon \left[\frac{C_1}{\varepsilon\,\eta}\,(e^{-1/\sqrt{\varepsilon}} + e^{-1/\varepsilon^2})\right] = 0\end{aligned}$$

by the estimate (3.9) and the definition of μ_ε implies $\mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2 - \psi^2) \geq \mu\, e^{1/\sqrt{\varepsilon}}$ in B_ε and $\mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2 - \psi^2) = \mu e^{1/\varepsilon^2}$ in C_ε.

Since μ_ε is monotone, for $\boldsymbol{w} = \boldsymbol{w}(t) \in \mathbb{K}(t)$ we have

$$\int_\Omega \mu_\varepsilon(|\mathbf{D}\boldsymbol{u}_\varepsilon|^2 - \psi^2)\,\mathbf{D}\boldsymbol{u}_\varepsilon : \mathbf{D}(\boldsymbol{w} - \boldsymbol{u}_\varepsilon) \leq \mu \int_\Omega \mathbf{D}\boldsymbol{w} : \mathbf{D}(\boldsymbol{w} - \boldsymbol{u}_\varepsilon),$$

and from (3.7), integrating in time over $0 < t < t+h < T$ we obtain in $Q_h = \Omega \times (t, t+h)$

$$\begin{aligned}\int_{Q_h} \partial_t\boldsymbol{u}_\varepsilon\cdot(\boldsymbol{w} - \boldsymbol{u}_\varepsilon) + \mu\int_{Q_h} \mathbf{D}\boldsymbol{w} : \mathbf{D}(\boldsymbol{w} - \boldsymbol{u}_\varepsilon) + \int_{Q_h} (\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \mathbf{D}\boldsymbol{w}&\\ \geq \int_{Q_h} \boldsymbol{f}\cdot(\boldsymbol{w} - \boldsymbol{u}_\varepsilon).&\end{aligned}$$

We notice that $\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}):\nabla(\boldsymbol{w}-\boldsymbol{v}) = \int_\Omega(\boldsymbol{v}\otimes\boldsymbol{v}):\mathbf{D}\boldsymbol{w}$, for $\boldsymbol{v},\boldsymbol{w}\in\mathbb{V}_2$. Passing to the limit in (3.11) we obtain, using Minty's Lemma and $\boldsymbol{u}(t)\in\mathbb{K}(t)$, first

$$\int_{Q_h}\partial_t\boldsymbol{u}\cdot(\boldsymbol{w}-\boldsymbol{u})+\mu\int_{Q_h}\mathbf{D}\boldsymbol{u}:\mathbf{D}(\boldsymbol{w}-\boldsymbol{u})+\int_{Q_h}(\boldsymbol{u}\otimes\boldsymbol{u}):\mathbf{D}\boldsymbol{w} \geq \int_{Q_h}\boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u})$$

and, afterwards, multiplying by $1/h$ and letting $h\to 0$ and choosing test functions continuous in t, which is possible by assumption (3.1), we conclude that $\boldsymbol{u}=\boldsymbol{u}(t)$ satisfies also (3.5) for *a.e.* $t\in(0,T)$ (see also Remark 3.2 below). Finally, $\mathbb{K}(t)\subset\bigcap_{0<r<\infty}\mathbb{V}_r$, by Poincaré and Korn inequalities, implies $\boldsymbol{u}\in L^\infty(0,T;\mathbb{W}^{1,r}(\Omega))$, $\forall\, r<\infty$. The compactness results of [29] with Sobolev embeddings yields that in fact $\boldsymbol{u}\in\mathscr{C}^0([0,T];\mathscr{C}^{0,\gamma}(\overline{\Omega})^d)$ for all $0\leq\gamma<1$. □

Remark 3.1. As it is well-known, Korn's inequality establishes that there exists a constant $k_p>0$ such that

$$\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}\leq k_p\,\|\mathbf{D}\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \tag{3.12}$$

holds for all $\boldsymbol{w}\in\boldsymbol{W}_0^{1,p}(\Omega)^d$ and all $1<p<\infty$ (see [17, p. 196], for instance). However, it is also known that in the limits cases $p=1$ and $p=\infty$, it is false (see [4], for a recent proof and references). Therefore it is not possible to conclude from this theorem that all strong solutions to (3.4)–(3.5) have bounded spatial gradient, and therefore the limit regularity $\gamma=1$ is an open question. Observe that for $\boldsymbol{w}\in\mathbb{V}_2$ we may take the equality in (3.12) with $k_2=\sqrt{2}$.

Remark 3.2. As in [27] (see also Lemma 4.6 of [21]), if we give two thresholds ψ_1 and ψ_2 satisfying (3.1) and we denote by $\mathbb{K}_1(t)$ and $\mathbb{K}_2(t)$ their respective convex sets defined by (3.2), for any $\boldsymbol{w}_1\in\mathbb{K}_1(t)$, there exists $\boldsymbol{w}_2\in\mathbb{K}_2(t)$ such that

$$\|\mathbf{D}(\boldsymbol{w}_1-\boldsymbol{w}_2)(t)\|_{\boldsymbol{L}^2(\Omega)}\leq C_1\,\|\psi_1(t)-\psi_2(t)\|_{\boldsymbol{L}^\infty(\Omega)}\equiv C_1\,\beta(t). \tag{3.13}$$

Indeed, it is sufficient to take $\boldsymbol{w}_2(t)=\frac{\alpha\,\boldsymbol{w}_1(t)}{\alpha+\beta(t)}$ and $C_1\geq\frac{1}{\alpha}\|\mathbf{D}\boldsymbol{w}_1(t)\|_{\boldsymbol{L}^2(\Omega)}$. In particular, if we take $\psi_2(t)=\psi_1(t+h)$, for $|h|>0$, clearly the assumption (3.1) yields the continuity of $t\to\mathbb{K}_1(t)$ in a strong sense.

This remark yields a strong continuous dependence result for thick fluids in the case $\mu>0$.

Theorem 3.2. *Let $\boldsymbol{u}_i$ denote the solution to the variational inequality* (3.4)–(3.5) *with data* (3.6) *given by* $(\psi_i,\boldsymbol{f}_i,\boldsymbol{u}_{0i})$, $i=1,2$, *with both ψ_1 and ψ_2 satisfying* (3.1). *Then there exists a constant $C=C(T)>0$ such that*

$$\|\boldsymbol{u}_1-\boldsymbol{u}_2\|^2_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))}+\|\mathbf{D}(\boldsymbol{u}_1-\boldsymbol{u}_2)\|^2_{\boldsymbol{L}^2(Q_T)} \leq C\left(\|\boldsymbol{f}_1-\boldsymbol{f}_2\|^2_{\boldsymbol{L}^2(Q_T)}+\|\boldsymbol{u}_{01}-\boldsymbol{u}_{02}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\psi_1-\psi_2\|_{L^\infty(Q_T)}\right). \tag{3.14}$$

Proof. In (3.5) for $\boldsymbol{u}_i$ we set $\boldsymbol{w}_i=\rho(t)\boldsymbol{u}_j(t)\in\mathbb{K}_i(t)$ for $i,j=1,2$, $j\neq i$ and with $\rho(t)=\frac{\alpha}{\alpha+\beta(t)}$, $\beta(t)=\|\psi_1(t)-\psi_2(t)\|_{L^\infty(\Omega)}$. By addition, we obtain for $\boldsymbol{v}=\boldsymbol{u}_1-\boldsymbol{u}_2$ and *a.e.* $t>0$:

$$\int_\Omega\partial_t\boldsymbol{v}\cdot\boldsymbol{v}+\mu\int_\Omega|\mathbf{D}\boldsymbol{v}|^2\leq\int_\Omega(\boldsymbol{f}_1-\boldsymbol{f}_2)\cdot\boldsymbol{v}+\Theta(t)+\Upsilon(t), \tag{3.15}$$

where

$$\Theta(t) = \int_\Omega \Big\{ \partial_t \boldsymbol{u}_1 \cdot (\boldsymbol{w}_1 - \boldsymbol{u}_2) + \mu\, \mathbf{D}\boldsymbol{u}_1 : \mathbf{D}(\boldsymbol{w}_1 - \boldsymbol{u}_2) + \boldsymbol{f}_1 \cdot (\boldsymbol{u}_2 - \boldsymbol{w}_1) \\ + \partial_t \boldsymbol{u}_2 \cdot (\boldsymbol{w}_2 - \boldsymbol{u}_1) + \mu\, \mathbf{D}\boldsymbol{u}_2 : \mathbf{D}(\boldsymbol{w}_2 - \boldsymbol{u}_1) + \boldsymbol{f}_2 \cdot (\boldsymbol{u}_1 - \boldsymbol{w}_2) \Big\}$$

and

$$\begin{aligned}\Upsilon(t) &= \int_\Omega \Big\{ (\boldsymbol{u}_1 \otimes \boldsymbol{u}_1) : \nabla(\boldsymbol{w}_1 - \boldsymbol{u}_1) + (\boldsymbol{u}_2 \otimes \boldsymbol{u}_2) : \nabla(\boldsymbol{w}_2 - \boldsymbol{u}_2) \Big\} \\ &= \int_\Omega \Big\{ (\boldsymbol{u}_1 \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{u}_2) : \nabla \boldsymbol{v} \\ &\quad + (\boldsymbol{u}_1 \otimes \boldsymbol{u}_1) : (\rho - 1)\, \nabla \boldsymbol{u}_2 + (\boldsymbol{u}_2 \otimes \boldsymbol{u}_2) : (\rho - 1)\, \nabla \boldsymbol{u}_1 \Big\}.\end{aligned}$$

Since $\boldsymbol{f}_i, \partial_t \boldsymbol{u}_i$ are bounded in $\boldsymbol{L}^2(Q_T)$ and $\boldsymbol{u}_i, \mathbf{D}\boldsymbol{u}_i$ in $\boldsymbol{L}^\infty(Q_T)$, arguing as in (2.16) and using Remark 3.2, we find that

$$\int_0^t \Theta(\tau)\, d\tau \le C\, \|\psi_1 - \psi_2\|_{L^\infty(Q_T)}$$

and

$$\int_0^t \Upsilon(\tau)\, d\tau \le \frac{\mu}{2} \int_0^t \int_\Omega |\mathbf{D}\boldsymbol{v}|^2 + C_\mu \int_0^t \int_\Omega |\boldsymbol{v}|^2 + C'\, \|\psi_1 - \psi_2\|_{L^\infty(Q_T)}.$$

Hence, (3.15) follows easily by the integral form of Gronwall's inequality. □

4. The Asymptotic Behavior in Time

We consider first the steady-state problem for given

$$\boldsymbol{f}_\infty \in \boldsymbol{L}^2(\Omega),\ \psi_\infty \in L^\infty(\Omega) \quad \text{with} \quad \psi_\infty = \psi_\infty(x) \ge \alpha > 0 \ \ a.e. \text{ in } \Omega, \tag{4.1}$$

and the convex set

$$\mathbb{K}_\infty = \Big\{ \boldsymbol{v} \in \mathbb{V}_2 :\ |\mathbf{D}\boldsymbol{v}(x)| \le \psi_\infty(x) \ \ a.e.\ x \in \Omega \Big\}. \tag{4.2}$$

Theorem 4.1. *For any $\mu \ge 0$, there exists at least a solution $\boldsymbol{u}_\infty$ such that*

$$\boldsymbol{u}_\infty \in \mathbb{K}_\infty \cap \mathscr{C}^{0,\gamma}(\overline{\Omega})^d \qquad (0 \le \gamma < 1), \tag{4.3}$$

$$\mu \int_\Omega \mathbf{D}\boldsymbol{u}_\infty : \mathbf{D}(\boldsymbol{w} - \boldsymbol{u}_\infty) - \int_\Omega (\boldsymbol{u}_\infty \otimes \boldsymbol{u}_\infty) : \nabla(\boldsymbol{w} - \boldsymbol{u}_\infty) \\ \ge \int_\Omega \boldsymbol{f}_\infty \cdot (\boldsymbol{w} - \boldsymbol{u}_\infty), \qquad \forall\, \boldsymbol{w} \in \mathbb{K}_\infty. \tag{4.4}$$

Moreover, there exists $\mu_d = \mu_d(\boldsymbol{f}_\infty, \psi_\infty, \Omega) > 0$ for which the solution is unique for $\mu > \mu_d$.

Proof. Since $\mathbb{K}_\infty$ is a bounded, closed and non empty convex subset of $\mathbb{V}_2$ and $A : \mathbb{K}_\infty \to \mathbb{V}_2'$ given by

$$\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \mu \int_\Omega \mathbf{D}\boldsymbol{v} : \mathbf{D}\boldsymbol{w} - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{w}, \qquad \boldsymbol{v} \in \mathbb{K}_\infty, \quad \boldsymbol{w} \in \mathbb{V}_2,$$

clearly defines a pseudo-monotone operator (for all $\mu \ge 0$), the existence result is an immediate consequence of the general theory of variational inequalities (see, for instance, Theorem 8.1, p. 245 of [16]). Since also $\mathbb{K}_\infty \subset \bigcap_{1<r<\infty} \mathbb{V}_r$, the $\mathscr{C}^{0,\gamma}$ regularity follows by Sobolev's inequalities.

The uniqueness for large μ follows essentially as in the case of stationary Navier–Stokes with low Reynold's numbers (see [14]).

Indeed, the difference $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ of two solutions with the same data $\boldsymbol{f}_\infty$ and ψ_∞, satisfies

$$\begin{aligned} \mu \int_\Omega |\mathbf{D}\boldsymbol{w}|^2 &\leq \int_\Omega (\boldsymbol{u}_1 \otimes \boldsymbol{u}_1 - \boldsymbol{u}_2 \otimes \boldsymbol{u}_2) : \nabla \boldsymbol{w} \\ &\leq \int_\Omega |(\boldsymbol{u}_1 \otimes \boldsymbol{w}) : \nabla \boldsymbol{w}| \\ &\leq M\sqrt{d}\, \|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \leq 2M\sqrt{d}\,\lambda\, \|\mathbf{D}\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}, \end{aligned} \tag{4.5}$$

where, as in (2.16), M is an upper bound for $|\boldsymbol{u}_1|$ and $\lambda > 0$ is a Poincaré's constant.

Hence, the uniqueness follows if

$$\mu > \mu_d \equiv 2M\sqrt{d}\,\lambda > 0. \tag{4.6}$$

□

Remark 4.1. If we choose the Poincaré's constant $\lambda = 1/\sqrt{\lambda_1}$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$, by letting $\boldsymbol{w} = \boldsymbol{0}$ in (4.4) we get the *a priori* estimate for $\boldsymbol{u}_\infty$

$$\|\nabla \boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)} = \sqrt{2}\, \|\mathbf{D}\boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{\sqrt{2}\,\lambda}{\mu}\, \|\boldsymbol{f}_\infty\|_{\boldsymbol{L}^2(\Omega)},$$

and using Ladyzhenskaya's inequalities for $d = 2, 3$ we can in fact use the same uniqueness criteria for uniqueness to the solution of (4.4) as in Navier–Stokes stationary problem. For instance, for $d = 3$ we have $2\sqrt{6}\,\lambda^3\,\mu^{-2}\,\|\boldsymbol{f}_\infty\|_{\boldsymbol{L}^2(\Omega)} < 1$ (see [14, p.118]). However, for large $\boldsymbol{f}_\infty$ we may have smaller μ_d if ψ_∞ is also smaller.

For the asymptotic result we use the following well-known Lemma (see, for instance, [12, p.286]).

Lemma 4.1. *Let $\zeta = \zeta(t) \geq 0$ be locally absolutely continuous and $\xi = \xi(t) \geq 0$ be a locally integrable function. If $\omega > 0$ and*

$$\zeta'(t) + \omega\,\zeta(t) \leq \xi(t), \qquad t > 0,$$

then

$$\zeta(t+s) \leq e^{-\omega t}\zeta(s) + \frac{1}{1-e^{-\omega}} \left[\sup_{\tau \geq s} \int_\tau^{\tau+1} \xi(\sigma)\, d\sigma\right], \qquad \forall\, t, s > 0. \tag{4.7}$$

Using Remark 3.2, we may now prove the following asymptotic stabilization as $t \to \infty$. Set

$$\beta(t) = \|\psi(t) - \psi_\infty\|_{L^\infty(\Omega)} \qquad \text{and} \qquad \varphi(t) = \int_t^{t+1} \int_\Omega |f(\tau) - f_\infty|\, d\tau.$$

Theorem 4.2. *Let $\boldsymbol{u} = \boldsymbol{u}(t)$ be the solution of* (3.4)–(3.5) *with the assumptions* (3.1)*,*

$$\boldsymbol{f} \in L^\infty(0, \infty; \boldsymbol{L}^2(\Omega)) \qquad \textit{and} \qquad \psi \in W^{1,\infty}(0, \infty; L^\infty(\Omega)), \tag{4.8}$$

and $\boldsymbol{u}_\infty$ the unique solution of (4.4) *under the assumption* (4.6)*, i.e.,*

$$\mu > \mu_d = 2\,\lambda\sqrt{d}\, \|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^\infty(\Omega)}, \tag{4.9}$$

where $\lambda > 0$ *is as in Remark 4.1. Let* $\gamma > \frac{1}{2}$ *and*

$$\beta(t) = O(t^{-\gamma}), \quad \varphi(t) \to 0 \qquad as \quad t \to \infty. \tag{4.10}$$

Then

$$\zeta(t) = \|\boldsymbol{u}(t) - \boldsymbol{u}_\infty\|^2_{\boldsymbol{L}^2(\Omega)} \underset{t\to\infty}{\longrightarrow} 0 \quad and \quad \eta(t) = \|\boldsymbol{u}(t) - \boldsymbol{u}_\infty\|_{\mathscr{C}^0(\overline{\Omega})^d} \underset{t\to\infty}{\longrightarrow} 0. \tag{4.11}$$

If also $\varphi(t) = O(t^{-\eta})$ *then*

$$\zeta(t) = O(t^{-\sigma}) \quad and \quad \eta(t) = O(t^{-\tau}),$$

with $\sigma = \min(\eta, \gamma - \frac{1}{2}) > (d+2)\tau > 0$*, and if, for some* $\nu > 0$*,* $\beta(t) = O(e^{-\nu t})$ *and* $\varphi(t) = O(e^{-\nu t})$ *then*

$$\zeta(t) = O(e^{-\delta t}) \quad and \quad \eta(t) = O(e^{-\epsilon t})$$

for any δ *and* ϵ*, such that,* $0 < (d+2)\epsilon < \delta < \min(\nu, (\mu - \mu_d)/\lambda^2)$*.*

Proof. As in [27], we observe that the assumption (4.7) implies that there exists a constant $C_\infty = C_\infty(\boldsymbol{u}_0, \boldsymbol{f}, \psi)$ independent of $T \geq 1$, such that

$$\|\partial_t \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_T)} \leq C_\infty T. \tag{4.12}$$

In fact, since $\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(Q_T)} \leq T\,\|\boldsymbol{f}\|^2_{L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))}$, from the proof of Theorem 3.1, in particular, from (3.11), we easily conclude that both constants C_0 and C_1 grow linearly with T.

Now, setting $\rho(t) = \frac{\alpha}{\alpha+\beta(t)}$, we take $\boldsymbol{w} = \rho(t)\,\boldsymbol{u}_\infty \in \mathbb{K}(t)$ in (3.5) and $\boldsymbol{w} = \rho(t)\,\boldsymbol{u}(t) \in \mathbb{K}_\infty$ in (4.4), in order to get for $\boldsymbol{v} = \boldsymbol{u}(t) - \boldsymbol{u}_\infty$ and *a.e.* $t > 0$

$$\int_\Omega \partial_t \boldsymbol{v} \cdot \boldsymbol{v} + \mu \int_\Omega |\mathbf{D}\boldsymbol{v}|^2 \leq \int_\Omega (\boldsymbol{f}(t) - \boldsymbol{f}_\infty) \cdot \boldsymbol{v} + \Upsilon(t) + \Theta(t) + \Sigma(t), \tag{4.13}$$

where we have the following estimates:

$$\Upsilon(t) = \rho(t) \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{u}_\infty \otimes \boldsymbol{u}_\infty) : \nabla \boldsymbol{v} \leq 2\,\lambda\,\sqrt{d}\,\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^\infty(\Omega)} \int_\Omega |\mathbf{D}\boldsymbol{v}|^2$$

analogously to (4.4), since $0 < \rho(t) \leq 1$, with $l = \|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)}$,

$$\Theta(t) = (\rho(t) - 1) \int_\Omega \partial_t \boldsymbol{u} \cdot \boldsymbol{u}_\infty \leq l\beta(t)\,\|\partial_t \boldsymbol{u}(t)\|_{\boldsymbol{L}^2(\Omega)}$$

since $0 \leq 1 - \rho(t) \leq \beta(t)$, and

$$\Sigma(t) = (1 - \rho(t)) \int_\Omega (\boldsymbol{f} \cdot \boldsymbol{u}_\infty + \boldsymbol{f}_\infty \cdot \boldsymbol{u} + 2\,\mu\,\mathbf{D}\boldsymbol{u} : \mathbf{D}\boldsymbol{u}_\infty) \leq k\,\beta(t)$$

where the constant $k > 0$ depends on the data only through the assumptions (4.1) and (4.8).

From (4.9) and recalling $\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty(Q_\infty)} = M < \infty$ we conclude that we may apply the Lemma above to $\zeta(t) = \int_\Omega |\boldsymbol{v}(t)|^2$ with $\omega = (\mu - \mu_d)/\lambda^2 > 0$ and

$$\xi(t) = 2M \int_\Omega |\boldsymbol{f}(t) - \boldsymbol{f}_\infty| + 2\,\beta(t)\,\left(k + l\,\|\partial_t \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\right),$$

since (4.12) yields $\left(\int_t^{t+1}\!\int_\Omega |\partial_t \boldsymbol{u}(\tau)|^2\,d\tau\right)^{1/2} = O(t^{1/2})$ as $t \to \infty$. Hence the Theorem follows easily: first for $\zeta(t)$, by applying the assumptions to (4.7); then for $\eta(t)$ by applying the Gagliardo-Nirenberg interpolation equality [22]

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} \leq C_1 \|\nabla \boldsymbol{v}\|^a_{\boldsymbol{L}^r(\Omega)} \|\boldsymbol{v}\|^{1-a}_{\boldsymbol{L}^2(\Omega)} + C_2 \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$$

to the function $\boldsymbol{v} = \boldsymbol{u}(t) - \boldsymbol{u}_\infty \in \bigcap_{d<r<\infty} \mathbb{V}_r \subset \mathscr{C}^0(\overline{\Omega})^d$, with $a = a(r) = \frac{rd}{rd+2(r-d)} > 0$. Since $0 < 1 - a(r) < \frac{2}{d+2}$ for $d < r < \infty$, by recalling (3.12) and the assumptions, the conclusion follows by simple calculations. □

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