The obstacle - mass constraint problem for hyperbolic conservation laws. Solvability

Paulo Amorim[1,3], Wladimir Neves[1], and José Francisco Rodrigues[2]

May 20, 2014

Abstract

In this work we introduce the obstacle-mass constraint problem for a multidimensional scalar hyperbolic conservation law. We prove existence of an entropy solution to this problem by a penalization/viscosity method. The mass constraint introduces a nonlocal Lagrange multiplier in the penalized equation, giving rise to a nonlocal parabolic problem. We determine conditions on the initial data and on the obstacle function which ensure global in time existence of solution. These are not smoothness conditions, but relate to the propagation of the support of the initial data.

AMS Subject Classification. Primary: 35L65. Secondary: 35R35

Key words and phrases. Hyperbolic conservation law, Obstacle problem, Mass conservation, Nonlocal parabolic equation, Free boundary problem.

1 Introduction

Consider the Cauchy problem for a hyperbolic conservation law,

$$\begin{aligned} &H(u) \equiv \partial_t u + \operatorname{div} f(u) = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{1.1}$$

under the restrictions

$$0 \le u(t,x) \le \theta(t,x), \qquad \int_{\mathbb{R}^d} u(t,x)\,dx = 1, \quad t \ge 0. \tag{1.2}$$

Here, $\theta(t,x)$ is a given obstacle function, f is the flux function which is supposed smooth, the Cauchy data u_0 is such that $0 \le u_0(x) \le \theta(0,x)$, with $\int_{\mathbb{R}^n} u_0(x)\,dx = 1$. In all that follows, every solution u of the various problems we will consider will be nonnegative, this being a consequence of $f(0) = 0$ and the properties of the hyperbolic operator H.

Even without the mass constraint $\int_{\mathbb{R}^d} u\,dx = 1$, some sense must be given to the hyperbolic problem (1.1) under the obstacle constraint $u \le \theta$. This was done mainly by Lévi in a series of works [8, 9, 10], in the case of a Dirichlet problem, in which a viscous approximation was introduced with a penalization term enforcing a constraint of type $u \le \theta$. However, such a solution, while verifying $u \le \theta$, does not conserve mass. This reduces the applicability of that approach to problems where mass conservation is important, such as in porous media models with saturation arising in petroleum engineering and crowd or traffic dynamics (see, however, [3] for an application

[1]Instituto de Matemática, Universidade Federal do Rio de Janeiro, Av. Athos da Silveira Ramos 149, Centro de Tecnologia - Bloco C, Cidade Universitária - Ilha do Fundão, Caixa Postal 68530, 21941-909 Rio de Janeiro, RJ - Brasil.
[2]Universidade de Lisboa, CMAF/FCUL, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal.
[3]Corresponding author. E-mail: paulo@im.ufrj.br

of an obstacle problem enforcing mass loss). More examples of domains where hyperbolic obstacle problems may be applicable can be found in [5] and the references in [9]. Other references on hyperbolic obstacle problems include [2, 11, 13, 14], although we could cite many others. For an introduction to classical obstacle problems, we address the reader to the book of Kinderlehrer and Stampacchia [7], and also Rodrigues [12].

One way to understand Lévi's approach is to observe that (formally at least) a solution u to the obstacle problem $H(u) = 0$, $u \leq \theta$ actually verifies

$$\partial_t u + \operatorname{div} f(u) = -H(\theta)^- \chi_{\{u=\theta\}}, \tag{1.3}$$

where we define the positive and negative parts as $v^+ := \operatorname{ess\,sup}\{v, 0\}$, $v^- = (-v)^+$, and H is the operator defined in (1.1). The motivation for the above equation (in the linear case) can be found in Remark 4.12 in [14]. In fact, equation (1.3) means that u must solve the equation $H(u) = 0$ wherever u does not coincide with θ. On the other hand, on the coincidence set $\{u = \theta\}$ one must have $H(\theta) = -H^-(\theta)$, which is to say, $H(\theta) \leq 0$. This property is related to the uniqueness of the solution, and ensures that somehow information is not created in situations where u "emerges" from the obstacle θ, or, equivalently, that the information u carries is lost when u "disappears" into the obstacle θ.

It is clear that a solution to (1.1)–(1.2) does not exist (in general) if the obstacle θ is reached. Indeed, in that case, there are two mutually exclusive effects taking place: on the one hand, the evolution equation $H(u) = 0$ naturally conserves the total mass; on the other hand, the presence of the obstacle leads to mass loss. In this work we propose a mechanism designed to reconcile these two contradictory aspects. One classical way in which an integral constraint like the unit integral condition in (1.2) may be enforced, is to introduce a Lagrange multiplier into the equation (1.1), see for instance Caffarelli and Lin [4] for a related problem. We take this approach here. In that spirit, our problem may be posed as follows: we look for a pair (u, λ), with $\lambda(t)$ a function of t alone, such that u and λ verify (see below for the precise definition)

$$\begin{aligned} &\partial_t u + \operatorname{div} f(u) = \lambda(t) u, \\ &u(0, x) = u_0(x), \end{aligned} \tag{1.4}$$

with

$$0 \leq u(t, x) \leq \theta(t, x), \qquad \int_{\mathbb{R}^d} u(t, x)\, dx = 1, \quad t \geq 0. \tag{1.5}$$

To our knowledge, this procedure is completely new for scalar conservation laws. We shall see that, even while respecting the obstacle condition, the solution u conserves the total mass, which is physically relevant for real applications. Formally, and in parallel with (1.3), the solution u should verify

$$\partial_t u + \operatorname{div} f(u) = -H(\theta)^- \chi_{\{u=\theta\}} + u \int_{\mathbb{R}^d} H(\theta)^- \chi_{\{u=\theta\}}\, dx. \tag{1.6}$$

Therefore, setting $\lambda(t) = \int_{\mathbb{R}^d} H(\theta)^- \chi_{\{u=\theta\}}\, dx$ and integrating on $\mathbb{R}^d$ one finds

$$\frac{d}{dt} \int_{\mathbb{R}^d} u\, dx = \lambda(t) \Big(\int_{\mathbb{R}^d} u\, dx - 1 \Big),$$

giving $\int_{\mathbb{R}^d} u\, dx = 1$ for $t > 0$, as long as $\int_{\mathbb{R}^d} u_0\, dx = 1$.

The main goal in this work is to make precise the above formal reasoning. For that, we will introduce a nonlocal parabolic equation containing a penalization term to enforce the constraint $u \leq \theta$ (as in [9]), and a new, nonlocal Lagrange multiplier term designed to enforce the mass constraint. As we will see below, this is not trivial to achieve. The first problem which arises is the lack of global in time existence for a possible solution of the problem (1.4)–(1.5). The reason is explained in more detail in Section 2, but can be understood by pointing out that the Lagrange multiplier term λu, introduced to make the mass of u grow, cannot fulfill its mission if $u \leq \theta$ only takes the values 0 or θ, which may happen for conservation laws (since solutions are in general discontinuous). Therefore, there must always remain some mass of u strictly below θ and above zero. This problem is solved here by finding appropriate conditions on the initial data and on the obstacle only (which, importantly, involve no additional smoothness), as presented in Section 2.

In Section 3 we analyze the nonlocal parabolic problem which will serve as an approximation to the full problem (1.4)–(1.5). This section follows a standard strategy, namely a fixed point argument. We decided to present some details, since in the a priori estimates one must be careful due to the presence of the penalization and, especially, the nonlocal term. Next, in Section 4 we provide the key estimates which allow us in Section 5 to pass to the limit on the penalized nonlocal parabolic equation. This is one of the main parts of the paper, where we obtain estimates for the penalized problem which are independent of all parameters, and also where we prove (using the assumptions of Section 2) that these estimates hold for arbitrarily large time intervals.

The uniqueness of solution is not established here. Nevertheless, we conjecture that a well-posedness property is valid. Note that in [9], the uniqueness property is a delicate part of that paper, as is usual in the theory of hyperbolic conservation laws. The difficulty in reproducing usual uniqueness arguments (Kruzkov's doubling of variables) is mainly due to the fact that a solution to (1.4)–(1.5) actually consists of a pair (u, λ) (see Definition 1.2 below). Note, however, that in order to obtain our existence result, a careful and involved study of a nonlocal parabolic problem is necessary, needing in particular completely new assumptions on the data and delicate estimates. However, our method does not give an explicit or clear dependence of $\lambda(t)$ with respect to u. For these reasons we chose to leave for future work the interesting question of wellposedness.

Finally, it would be interesting to determine wether the methods in our paper can be extended to deal with more general (e.g., time dependent) mass constraints, hyperbolic systems of conservation laws, etc. Also, it would be of great interest for physical applications (even under smoothness assumptions, to keep the analysis less involved) to extend the results of this work to a general conservation law with space and time dependent flux function and source term.

1.1 Smoothness assumptions on the data

The initial data u_0 is taken in the space $(L^\infty \cap L^1)(\mathbb{R}^d)$, while the flux function f is taken in $(C^1(\mathbb{R}^d))^d$. In fact, to simplify the exposition, we also consider that u_0 has bounded variation, that is, $u_0 \in BV(\mathbb{R}^d)$.

The obstacle $\theta(t, x)$ is assumed to satisfy the following conditions: For all $t \geq 0$,

$$\begin{aligned} &\|\partial_t \theta(t)\|_{L^1(\mathbb{R}^d)} \leq C, \quad \|\partial_{x_i} \theta(t)\|_{L^1(\mathbb{R}^d)} \leq C, \\ &\|\partial_{tt} \theta(t)\|_{L^1(\mathbb{R}^d)} \leq C, \quad \|\partial_{x_i t} \theta(t)\|_{L^1(\mathbb{R}^d)} \leq C, \quad \|\partial_{x_i x_i} \theta(t)\|_{L^1(\mathbb{R}^d)} \leq C, \end{aligned} \tag{1.7}$$

$$\text{For each compact set } K, \text{ the function } t \mapsto \int_K \theta(t, x)\, dx \quad \text{is continuous,} \tag{1.8}$$

and

$$0 < \underline{\theta} \leq \theta, \tag{1.9}$$

for some constant $\underline{\theta}$. For instance, conditions (1.7)–(1.9) are satisfied if $\theta - c \in W^{2,1}(\mathbb{R}^{d+1})$ for some constant $c > 0$ such that $\theta - c \geq \underline{\theta} > 0$, and (1.8) follows by Sobolev's Embedding Theorem (i.e. $W^{2,1}(\mathbb{R}) \subset C^1(\mathbb{R})$). The proof of some estimates below require third derivatives of θ in space. However, only first and second derivatives of θ appear in the final statement of all estimates. for this reason, and to simplify the exposition, we assume without loss of generality throughout the paper that the obstacle θ is a smooth function (at least $W^{3,1}(\mathbb{R}^{d+1})$). The general case follows by standard regularization arguments. Note also that θ is not required to be bounded, at least when the dimension d is high enough.

1.2 Entropy solutions to the obstacle-mass constraint problem

Here we recall some standard facts and terminology from hyperbolic conservation laws.

Definition 1.1. *A function $\eta \in C^1(\mathbb{R})$ is called an entropy for equation (1.4), with associated entropy flux $q \in C^1(\mathbb{R};\mathbb{R}^d)$, when for each $u \in \mathbb{R}$,*

$$q_j'(u) = \eta'(u) f_j'(u), \qquad (j = 1, \dots, d). \tag{1.10}$$

Also, we call $F(u) = (\eta(u), q(u))$ an entropy pair, and if η is convex we say that $F(u)$ is a convex entropy pair. Moreover, $F(u)$ is called a generalized entropy pair if it is the uniform limit of a family of entropy pairs over compact sets.

The Kruzkov entropies are the most important example of generalized convex entropy pairs, consisting of the following parametrized family

$$F(u, v) = (|u - v|, \operatorname{sgn}(u - v)(f(u) - f(v)), \qquad (v \in \mathbb{R}).$$

Next, we present in which sense a function $u(t, x)$ is a weak entropy solution of (1.4)–(1.5).

Definition 1.2. *Let θ be a function defined on $\mathbb{R}^{d+1}$, which is called an obstacle, verifying the conditions in (1.7)–(1.9). Let $u_0 \in (L^\infty \cap L^1 \cap BV)(\mathbb{R}^d)$ with $0 \le u_0(x) \le \theta(0, x)$ a.e., and $\int_{\mathbb{R}^d} u_0 = 1$. A pair (u, λ) is called an obstacle mass conserving weak entropy solution of the Cauchy problem (1.4)–(1.5) if for any $T > 0$:*

(i) The function u is in $L^\infty((0, T) \times \mathbb{R}^d)$ with $u(t) \in BV(\mathbb{R}^d)$ for a.a. $t \in [0, T]$, and the Lagrange multiplier λ is in $L^\infty(0, T; \mathbb{R}^+)$.

(ii) For each nonnegative test function $\varphi \in C_c^\infty((-\infty, T) \times \mathbb{R}^d)$, and any $k \in [0, 1]$

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} F(u(t, x), k\,\theta(t, x)) \cdot \nabla_{t,x} \varphi(t, x)\, dxdt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \Big(\lambda(t)\, u(t, x) - H(k\,\theta(t, x))\Big) \operatorname{sgn}(u(t, x) - k\,\theta(t, x))\, \varphi(t, x)\, dxdt \\ &\quad + \int_{\mathbb{R}^d} |u_0(x) - k\,\theta(0, x)|\, \varphi(0, x)\, dx \ge 0. \end{aligned} \tag{1.11}$$

(iii) For almost all $t \in (0, T)$, $\int_{\mathbb{R}^d} u(t)\, dx = 1$ and $u(t, x) \le \theta(t, x)$.

One observes that, as a consequence of Definition 1.2, the initial condition is assumed in $L^1(\mathbb{R}^d)$ strong sense:

$$\operatorname*{ess\,lim}_{t \to 0} \int_{\mathbb{R}^d} |u(t, x) - u_0(x)|\, dx = 0. \tag{1.12}$$

2 Preliminaries

In this section we make some remarks motivating our method. To begin, we describe our approach using a perturbed penalized problem.

2.1 An approach using a nonlocal penalization

For each $\varepsilon > 0$, and all $n \in \mathbb{N}$ we consider the following nonlocal perturbed parabolic problem

$$\begin{aligned} &\partial_t u_{n,\varepsilon} + \operatorname{div} f(u_{n,\varepsilon}) - \varepsilon \Delta u_{n,\varepsilon} = n\, u_{n,\varepsilon} \int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+ - n(u_{n,\varepsilon} - \theta)^+, \\ &u_{n,\varepsilon}(0, x) = u_0(x), \qquad \int_{\mathbb{R}^d} u_0\, dx = 1, \end{aligned} \tag{2.1}$$

as an approximation scheme to solve the problem (1.4),(1.5). Indeed, the last term in (2.1) is the usual term penalizing the mass of $u_{n,\varepsilon}$ above θ (see [9]), ensuring in the limit that the solutions $u_{n,\varepsilon}$ will stay below the obstacle θ. Moreover, we have introduced in (2.1) the nonlocal penalization term $n\, u_{n,\varepsilon} \int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+$, which implies the unit integral property. Indeed, integrating (2.1) on $\mathbb{R}^d$ and on $[0,T]$, applying the properties of the family $\{u_{n,\varepsilon}\}$, one finds

$$\int_{\mathbb{R}^d} u_{n,\varepsilon}\, dx - 1 \leq n \int_0^t \Big(\int_{\mathbb{R}^d} u_{n,\varepsilon}\, dx - 1 \Big) \Big(\int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+\, dx \Big)\, ds$$

which, using Gronwall's Lemma, yields $\int_{\mathbb{R}^d} u(t,x)\, dx = 1$ for $t \geq 0$. The computation above will be precisely described below. Still, this procedure alone does not ensure existence of solution outside of some small time interval, as the next section shows.

2.2 Counterexamples to global in time existence

Here we present some heuristic arguments which show that under very general conditions, one cannot hope that a limit u obtained by the penalized method (2.1) satisfies, at the same time, $u \leq \theta$ and $\int_{\mathbb{R}^d} u\, dx = 1$ except on a possibly small time interval. More precisely, we argue through heuristic arguments that, in general we do not expect that problem (1.4),(1.5) has a global in time solution (that is, on an arbitrary interval $(0,T)$), which preserves all the desired constraints.

Consider the case $d = 1$, $f \equiv 0$, with $u_0(x) = \chi_{(0,1)}(x)$, and suppose the obstacle is given by $\theta(t) = \frac{1}{2} + e^{-t}$.

If $t < \ln 2$, then $u(t,x) = u_0(x)$ is clearly the solution of (1.4),(1.5). But if $t > \ln 2$, then the only way that a possible solution of (1.4),(1.5) can satisfy the obstacle constraint *and* the unit mass constraint is for its support to expand, in order to compensate for the loss of mass due to the obstacle. And indeed, the goal of the Lagrange multiplier method is to allow for such a mass growth. But in the absence of flux terms, a Lagrange multiplier can never create mass in regions where $u \equiv 0$. Therefore, in this case, we see that a solution to the obstacle problem with mass conservation will not be global in time. In fact, no such solution exists for $t > \ln 2$, since condition (iii) in Definition 1.2 will be violated.

It is easy to produce similar counterexamples also in the case $f \neq 0$, using the linear advection equation and obstacles depending on x, in such a way that the compactly supported profile of u_0, moving by advection in the positive x direction will "collide" with a static obstacle having a decreasing profile.

The common thread of such counterexamples to global existence is the situation where the support of the solution u is carried into regions where the obstacle has mass less than one. In other words, we may say that if the support of u_0 is carried by the advection term into a region U_t, where the integral of θ (on that region) is less than 1, then, since the introduction of a Lagrange multiplier cannot make u grow outside U_t, there will be no Lagrange multiplier for which $\int_{\mathbb{R}^d} u\, dx = 1$, while verifying $u \leq \theta$.

Another way of summarizing this heuristic reasoning is to say that, some nonzero mass of u must remain below the obstacle at all times, so that the Lagrange multiplier has something to act on. Indeed, below we provide assumptions which ensure this property.

2.3 Assumptions on the data ensuring global well-posedness

In view of the previous discussion, some assumption on the data is needed in order to prevent the kind of situation described above. This is not trivial to ensure: as we pointed out, heuristic analysis suggests that some mass must remain below the obstacle at all times, so that the solution has "room to grow" in case the mass loss from the obstacle constraint becomes too great. But intuitively, this will always be the case for solutions of the penalized problem (2.1), due to the presence of the viscosity term, which tends to smooth out the solutions. Indeed, it is known that such a term will in general produce solutions with unbounded support, even for compactly supported data. Thus, for each value of the viscosity parameter ε, $u_{n,\varepsilon}$ is expected to have some nonzero mass below the obstacle. The trick is to develop an assumption which is somehow independent of ε, and which ensures that the eventual limit of $u_{n,\varepsilon}$ as $n \to \infty$ and $\varepsilon \to 0$ still has some mass below θ.

The solution we propose is to consider an auxiliary function v_γ, whose properties depend only on the data of the problem, having the property that, on the one hand, $v_\gamma \leq u_{n,\varepsilon}$, and on the other hand, the integral of θ remains greater than one on the set where v_γ is positve.

Assumption 2.1. Suppose the initial data u_0 and the obstacle θ satisfy the following hypothesis. There exist $0 < \gamma < \underline{\theta}$, and $\beta > 0$, such that, for each $t \in [0, T]$,

$$1 + \beta \leq \int_{\{v_\gamma > 0\}} \theta(t, x)\, dx \leq +\infty, \tag{2.2}$$

where $v_\gamma(t, x)$ is the unique entropy solution (see [6]) to the Cauchy problem for the homogenous conservation law

$$\begin{aligned} &\partial_t v_\gamma + \operatorname{div} f(v_\gamma) = 0, \\ &v_\gamma(0, x) = v_{0,\gamma}(x), \end{aligned} \tag{2.3}$$

with $v_{0,\gamma}(x) = \min(u_0(x), \gamma)$.

For convenience, we also consider the following viscous perturbed problem

$$\begin{aligned} &\partial_t v_{\gamma,\varepsilon} + \operatorname{div} f(v_{\gamma,\varepsilon}) - \varepsilon \Delta v_{\gamma,\varepsilon} = 0, \\ &v_{\gamma,\varepsilon}(0, x) = v_{0,\gamma}(x). \end{aligned} \tag{2.4}$$

The existence, uniqueness and regularity assumptions on the family $\{v_{\gamma,\varepsilon}\}$, follows from the well-posedness theory for parabolic equations, see for instance [6].

Remark 2.2. 1. Note that Assumption 2.1 is a hypothesis on the initial data u_0, and the obstacle function θ. Indeed, it states that the support of the solution of the conservation law (2.3) cannot be carried into a region where the integral of θ is less than one. This is in agreement with the heuristic analysis presented before.

2. If (2.2) is verified for some γ_0, then it is verified for all $\gamma_0 \leq \gamma < \underline{\theta}$. This follows from the classical comparison property for hyperbolic conservation laws.

3. If the set of points where u_0 is positive has full measure, then Assumption 2.1 is automatically verified, since in that case the condition (2.2) is valid for every $0 < \gamma < \underline{\theta}$. This follows from the fact that each $v_{0,\gamma}$ will also be almost everywhere positive. From finite speed of propagation, the solutions v_γ will have the same property. To see this, consider a ball $B(r)$ of radius $r > 0$ centered around an arbitrary point of $\mathbb{R}^d$. Then we have that, for M large enough, $t > 0$, the solution $v_{\gamma,\varepsilon}(t, x)$ on $B(r)$ is influenced only by the values of $v_{0,\gamma}$ on $B(r + Mt)$. Let $c > 0$ be such that $v_{0,\gamma} \geq c$ on $B(r + (M + 1)t)$. Since c is a solution to the conservation law (2.3), the classical comparison property and domain of dependence arguments imply that, $v_{\gamma,\varepsilon}(t, x) \geq c > 0$ on $B(r)$.

4. It follows from Assumption 2.1 that the initial data u_0 has some mass below the obstacle θ, which will be useful later. Indeed, suppose not, hence $u_0(x) = 0$ or $u_0(x) = \theta$. Therefore, $v_{0,\gamma} = \gamma \chi_{\{u_0 > 0\}}$ and so, we would have

$$1 < 1 + \beta \leq \int_{\{v_{0,\gamma} > 0\}} \theta\, dx = \int_{\{u_0 > 0\}} \theta\, dx = \int_{\mathbb{R}^d} u_0\, dx = 1,$$

which is a contradiction.

3 Well-posedness for the nonlocal penalized problem

In this section, we establish well-posedness results for the nonlocal penalized parabolic problem introduced in (2.1). As we shall see, the analysis of this problem for each n and ε is not trivial, due to the presence of the nonlocal term. The main technical tool will be the Banach contraction principle.

For $T > 0$, define the space

$$W(0,T) := \{v : \mathbb{R}^d \times [0,T] \to \mathbb{R} : v \in L^2(0,T;H^1(\mathbb{R}^d)), \partial_t v \in L^2(0,T;H^{-1}(\mathbb{R}^d))\}.$$

One recalls that the space $W(0,T)$ is continuously imbedded into the space $C([0,T];L^2(\mathbb{R}^d))$. Moreover, for any $v \in W(0,T)$ the $\lim_{t\to 0} v(t) = v(0)$ is a well defined element of the space $L^2(\mathbb{R}^d)$.

Theorem 3.1. *Given $u_0 \in (L^\infty \cap L^1 \cap BV)(\mathbb{R}^d)$, for each $n \in \mathbb{N}, \varepsilon > 0$, there exists a unique solution*

$$u_{n,\varepsilon} \in L^\infty(0,T;H^1(\mathbb{R}^d)) \cap C([0,T];L^1(\mathbb{R}^d)),$$

of the nonlocal parabolic problem (2.1)*, in the sense that: For every $v \in H^1(\mathbb{R}^d)$, and for almost all $t \in (0,T)$,*

$$\begin{aligned} \langle \partial_t u_{n,\varepsilon}(t), v\rangle_{H^{-1}\times H^1} - \int_{\mathbb{R}^d} (f(u_{n,\varepsilon}(t)) - \varepsilon\nabla u_{n,\varepsilon}(t)) \cdot \nabla v\, dx \\ = n\int_{\mathbb{R}^d} u_{n,\varepsilon}(t)\, v\, dx \int_{\mathbb{R}^d} (u_{n,\varepsilon}(t) - \theta(t))^+\, dx - n\int_{\mathbb{R}^d} (u_{n,\varepsilon}(t) - \theta(t))^+\, v\, dx, \end{aligned} \tag{3.1}$$

and $\lim_{t\to 0} \int_{\mathbb{R}^d} \|u_{n,\varepsilon}(t) - u_0\|_{L^2(\mathbb{R}^d)} dx \to 0$. Moreover, this solution verifies for almost all $t \in (0,T)$, $\int_{\mathbb{R}^d} u_{n,\varepsilon}(t)dx = 1$.

Proof. 1. The theorem will be proved using the Banach contraction principle. To this end, we consider the approximate problem: given a function $\overline{v} \in C([0,T];L^1(\mathbb{R}^d))$, find $v \in W(0,T)$ such that, for all $w \in H^1(\mathbb{R}^d)$, and for almost all $t \in (0,T)$,

$$\begin{aligned} \langle \partial_t v(t), w\rangle_{H^{-1}\times H^1} - \int_{\mathbb{R}^d} (f(v(t)) - \varepsilon\nabla v(t)) \cdot \nabla w\, dx \\ = n\int_{\mathbb{R}^d} v(t)\, w\, dx \int_{\mathbb{R}^d} (\overline{v}(t) - \theta)^+\, dx - n\int_{\mathbb{R}^d} (v(t) - \theta)^+\, w\, dx. \end{aligned} \tag{3.2}$$

Moreover, $\lim_{t\to 0} \int_{\mathbb{R}^d} \|v(t) - u_0\|_{L^2(\mathbb{R}^d)} dx \to 0$. That is, v is a weak solution to the equation

$$\begin{aligned} &\partial_t v + \operatorname{div} f(v) - \varepsilon\Delta v = n\, v \int_{\mathbb{R}^d} (\overline{v} - \theta)^+\, dx - n(v-\theta)^+, \\ &v(0,x) = u_0(x). \end{aligned} \tag{3.3}$$

The proof that there exists a unique solution of (3.2) follows closely the one in [6, p.56], so we omit it. Note that (3.2) is a standard (local) parabolic problem.

2. Now, let us consider the mapping

$$\begin{aligned} \Phi : &C\big([0,T];L^1(\mathbb{R}^d)\big) \to W(0,T), \\ &\overline{v} \mapsto v \text{ solution of (3.2)}. \end{aligned} \tag{3.4}$$

Let $R > 1$. We want to show that, for T_0 sufficiently small, Φ is a contraction in the Banach space

$$\mathcal{E} := \{v \in C\big([0,T_0];L^1(\mathbb{R}^d)\big) : \sup_{t\in[0,T_0]} \|v(t)\|_{L^1(\mathbb{R}^d)} \le R\}. \tag{3.5}$$

Let v be the unique solution of problem (3.2). First of all, note that since $u_0 \ge 0$, we have $v \ge 0$. This follows from the fact that $v \equiv 0$ is a solution of the problem (3.2) and classical comparison arguments (see, in particular, Lemma 3.2 below). To follow, we establish the estimate

$$\int_{\mathbb{R}^d} v(t)\, dx \le e^{n\int_0^t \int_{\mathbb{R}^d} (\overline{v}-\theta)^+\, dx\, ds}. \tag{3.6}$$

Note that once (3.6) is proved, we find $\int_{\mathbb{R}^d} v(t)\,dx \le e^{ntR}$, and so, for $t \le T_0$ sufficiently small, $v \in \mathcal{E}$. To prove (3.6), we introduce the smooth positive functions $\psi_\rho : \mathbb{R}^d \to \mathbb{R}$ for large ρ, such that

$$\psi_\rho(x) = 1 \text{ if } |x| \le \rho/2, \quad \psi_\rho \text{ decays exponentially for } |x| \ge \rho,$$
$$|\nabla \psi_\rho| \le \frac{C\psi_\rho}{\rho}, \quad |\Delta \psi_\rho| \le \frac{C\psi_\rho}{\rho^2},$$

for some constant $C > 0$. Take $w = \psi_\rho$ in (3.2) to find, after discarding the last term on the right-hand side,

$$\begin{aligned} \langle \partial_t v(t), \psi_\rho \rangle_{H^{-1}\times H^1} &- \int_{\mathbb{R}^d} (f(v(t)) - \varepsilon \nabla v(t)) \cdot \nabla \psi_\rho \, dx \\ &\le n \int_{\mathbb{R}^d} v(t)\, \psi_\rho \, dx \int_{\mathbb{R}^d} (\overline{v}(t) - \theta)^+ \, dx. \end{aligned} \tag{3.7}$$

Now, for ρ sufficiently large,

$$\int_{\mathbb{R}^d} f(v) \cdot \nabla \psi_\rho + \varepsilon \nabla v \cdot \nabla \psi_\rho \, dx \le \int_{\mathbb{R}^d} M v |\nabla \psi_\rho| + \varepsilon\, v |\Delta \psi_\rho| \, dx \le \frac{C(M+\varepsilon)}{\rho} \int_{\mathbb{R}^d} v \psi_\rho \, dx.$$

Hence integrating (3.7) on $[0,t]$ (see [6, p.54]), we find

$$\int_{\mathbb{R}^d} v(t) \psi_\rho \, dx \le \int_{\mathbb{R}^d} u_0 \psi_\rho \, dx + \int_0^t \Big(C \frac{M+\varepsilon}{\rho} + n \int_{\mathbb{R}^d} (\overline{v}(s) - \theta)^+ \, dx \Big) \int_{\mathbb{R}^d} v(s) \psi_\rho \, dx \, ds.$$

Using Gronwall's inequality and $\int_{\mathbb{R}^d} u_0 \psi_\rho \, dx \le \int_{\mathbb{R}^d} u_0 \, dx = 1$, we obtain

$$\int_{\mathbb{R}^d} v(t) \psi_\rho \, dx \le e^{t \frac{M+\varepsilon}{\rho}} e^{n \int_0^t \int_{\mathbb{R}^d} (\overline{v} - \theta)^+ \, dx \, ds}.$$

Then, passing to the limit as $\rho \to \infty$ and applying the Monotone Convergence Theorem, we conclude that

$$\int_{\mathbb{R}^d} v(t) \, dx \le e^{n \int_0^t \int_{\mathbb{R}^d} (\overline{v} - \theta)^+ \, dx \, ds}. \tag{3.8}$$

3. Next, let u and v be solutions of (3.2) associated with $\overline{u}$ and $\overline{v}$, respectively. For convenience, we introduce the regularized sign function $\text{sgn}_\delta(u)$ as the continuous function which is linear for $0 \le |u| \le \delta$, and equal ± 1 otherwise. Also, we use the notations

$$I_\delta(u) = \int_0^u \text{sgn}_\delta(v) \, dv, \qquad u \ge 0,$$

and

$$(u)_\delta^+ = u \,\text{sgn}_\delta(u)^+,$$

both of which are Lipschitz approximations of the positive part u^+. Then, from (3.2) and taking $w = \text{sgn}_\delta(u-v)^+ \psi_\rho$, we find

$$\begin{aligned} \langle \partial_t I_\delta(u-v), \psi_\rho \rangle_{H^{-1}\times H^1} &= \int_{\mathbb{R}^d} (f(u) - f(v) - \varepsilon \nabla (u-v)) \cdot \nabla (\text{sgn}_\delta(u-v)^+ \psi_\rho) \, dx \\ &+ n \int_{\mathbb{R}^d} u \,\text{sgn}(u-v)_\delta^+ \psi_\rho \, dx \int_{\mathbb{R}^d} (\overline{u} - \theta)^+ \, dx \\ &- n \int_{\mathbb{R}^d} v \,\text{sgn}(u-v)_\delta^+ \psi_\rho \, dx \int_{\mathbb{R}^d} (\overline{v} - \theta)^+ \, dx \\ &- n \int_{\mathbb{R}^d} ((u-\theta)^+ - (v-\theta)^+) \,\text{sgn}_\delta(u-v)^+ \psi_\rho \, dx. \end{aligned} \tag{3.9}$$

Now, observe that for each $\delta \ge 0$, the algebraic inequality holds,

$$((u-\theta)^+ - (v-\theta)^+) \,\text{sgn}_\delta(u-v)^+ \ge 0,$$

which allows us to neglect the last term in (3.9). Moreover, regarding the remaining terms in (3.9), we find after integrating by parts,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} (f(u) - f(v) - \varepsilon \nabla(u-v)) \cdot \nabla(\operatorname{sgn}_\delta(u-v)^+ \psi_\rho)\, dx \\
=& \int_{\mathbb{R}^d} (f(u) - f(v)) \cdot \nabla(u-v) \operatorname{sgn}'_\delta(u-v)^+ \psi_\rho \, dx + \int_{\mathbb{R}^d} (f(u) - f(v)) \cdot \operatorname{sgn}_\delta(u-v)^+ \nabla\psi_\rho \, dx \\
&- \varepsilon \int_{\mathbb{R}^d} |\nabla(u-v)|^2 \operatorname{sgn}'_\delta(u-v)^+ \psi_\rho \, dx - \varepsilon \int_{\mathbb{R}^d} \nabla(u-v) \operatorname{sgn}_\delta(u-v)^+ \nabla\psi_\rho \, dx \\
\le& M \int_{\mathbb{R}^d} (u-v)^+ \operatorname{sgn}'_\delta(u-v)^+ |\nabla(u-v)| \psi_\rho \, dx + \frac{CM}{\rho} \int_{\mathbb{R}^d} (u-v)^+_\delta \psi_\rho \, dx \\
&+ \varepsilon \int_{\mathbb{R}^d} (u-v)^+ \operatorname{sgn}'_\delta(u-v)^+ |\nabla(u-v)| |\nabla\psi_\rho| \, dx + \frac{\varepsilon C}{\rho^2} \int_{\mathbb{R}^d} (u-v)^+_\delta \psi_\rho \, dx.
\end{aligned}
$$

Now, the first and third terms of the last expression are functions of time, which tends to zero with δ (for each fixed t), hence we denote them by $o(\delta, t)$. We have used the fact that the function $(u(t)-v(t))^+ \operatorname{sgn}'_\delta(u(t)-v(t))^+$ tends to 0 with δ for almost every $x \in \mathbb{R}^d$ and Lebesgue's Theorem. Thus we find

$$
\begin{aligned}
&\int_{\mathbb{R}^d} (f(u) - f(v) - \varepsilon \nabla(u-v)) \cdot \nabla(\operatorname{sgn}_\delta(u-v)^+ \psi_\rho)\, dx \\
&\qquad \le o(\delta, t) + \frac{C(M+\varepsilon)}{\rho} \int_{\mathbb{R}^d} (u-v)^+_\delta \psi_\rho \, dx.
\end{aligned}
$$

Also, from (3.9),

$$
\begin{aligned}
n &\int_{\mathbb{R}^d} u \operatorname{sgn}_\delta(u-v)^+ \psi_\rho \, dx \int_{\mathbb{R}^d} (\overline{u} - \theta)^+ \, dx - n \int_{\mathbb{R}^d} v \operatorname{sgn}_\delta(u-v)^+ \psi_\rho \, dx \int_{\mathbb{R}^d} (\overline{v} - \theta)^+ \, dx \\
&\le n \int_{\mathbb{R}^d} (u-v)^+_\delta \psi_\rho \, dx \int_{\mathbb{R}^d} (\overline{u} - \theta)^+ \, dx \\
&\quad + n \int_{\mathbb{R}^d} v \operatorname{sgn}_\delta(u-v)^+ \psi_\rho \, dx \int_{\mathbb{R}^d} (\overline{u} - \theta)^+ - (\overline{v} - \theta)^+ \, dx \\
&\le n \int_{\mathbb{R}^d} (u-v)^+_\delta \psi_\rho \, dx \int_{\mathbb{R}^d} (\overline{u} - \theta)^+ \, dx \\
&\quad + n \int_{\mathbb{R}^d} v \, dx \int_{\mathbb{R}^d} (\overline{u} - \overline{v})^+ \, dx.
\end{aligned}
$$

Therefore, integrating (3.9) on $[0, t]$ yields

$$
\begin{aligned}
\int_{\mathbb{R}^d} I_\delta(u-v) \psi_\rho(t) \, dx &\le \int_0^t o(\delta, s) \, ds + \int_0^t \frac{C(M+\varepsilon)}{\rho} \int_{\mathbb{R}^d} (u-v)^+_\delta \psi_\rho \, dx \, ds \\
&+ n \int_0^t \int_{\mathbb{R}^d} (u-v)^+_\delta \psi_\rho \, dx \int_{\mathbb{R}^d} (\overline{u} - \theta)^+ \, dx \, ds + n \int_0^t \int_{\mathbb{R}^d} v \, dx \int_{\mathbb{R}^d} (\overline{u} - \overline{v})^+ \, dx \, ds.
\end{aligned}
$$

Now, we apply the Monotone Convergence Theorem to take $\delta \to 0$, and use Gronwall's Lemma to get

$$
\begin{aligned}
\int_{\mathbb{R}^d} (u-v)^+ \psi_\rho(t) \, dx \le n \int_0^t &\Big(\int_{\mathbb{R}^d} v \, dx \Big) \Big(\int_{\mathbb{R}^d} (\overline{u} - \overline{v})^+ \, dx \Big) \, ds \\
&\times e^{n \int_0^t \int_{\mathbb{R}^d} (\overline{u} - \theta)^+ \, dx \, ds + t \frac{C(M+\varepsilon)}{\rho}}.
\end{aligned}
$$

Then, taking $\rho \to \infty$ (again by monotone convergence), we obtain the estimate

$$
\int_{\mathbb{R}^d} (u-v)^+(t) \, dx \le n \int_0^t \Big(\int_{\mathbb{R}^d} v \, dx \Big) \Big(\int_{\mathbb{R}^d} (\overline{u} - \overline{v})^+ \, dx \Big) \, ds \; e^{n \int_0^t \int_{\mathbb{R}^d} (\overline{u} - \theta)^+ \, dx \, ds}. \tag{3.10}
$$

Now we use the estimate (3.8) of $\int_{\mathbb{R}^d} v\,dx$ in (3.10) to find

$$\int_{\mathbb{R}^d} (u-v)^+(t)\,dx \le n \int_0^t \int_{\mathbb{R}^d} (\overline{u}-\overline{v})^+\,dx\,ds\, e^{n\int_0^t \int_{\mathbb{R}^d} (\overline{u}-\theta)^+ + (\overline{v}-\theta)^+\,dx\,ds}$$

or

$$\begin{aligned}\int_{\mathbb{R}^d} (u-v)^+(t)\,dx &\le nt \sup_{0\le t\le T_0} \|\overline{u}-\overline{v}\|_{L^1(\mathbb{R}^d)}\, e^{nt\sup_{0\le t\le T_0}(\|\overline{u}\|_{L^1(\mathbb{R}^d)}+\|\overline{v}\|_{L^1(\mathbb{R}^d)})} \\ &\le nt\|\overline{u}-\overline{v}\|_{\mathcal{E}}\, e^{2Rnt}.\end{aligned}$$

Recall the definition of the Banach space $\mathcal{E}$ in (3.5). By symmetry, we find an estimate equal to the previous one, but with $(v-u)^+$ instead of $(u-v)^+$. From $|a| = a^+ + (-a)^+$, we have

$$\int_{\mathbb{R}^d} |u-v|(t)\,dx \le 2nt\|\overline{u}-\overline{v}\|_{\mathcal{E}}\, e^{2Rnt}. \tag{3.11}$$

4. If $R > 1$, we deduce from (3.11) that there exists a $T_0 > 0$, such that for $t \le T_0$ the map Φ defined in (3.4) is a strict contraction. The first part of the Banach Contraction Principle tells us that, the sequence defined by $u^k = \Phi(u^{k-1})$ with $u^0 \in \mathcal{E}$ converges strongly in $\mathcal{E}$ towards some $u_{n,\varepsilon} \in \mathcal{E}$. Each u^k verifies equation (3.2) with u^{k-1} in place of $\overline{v}$. Taking $w = u^k$ in (3.2) gives

$$\begin{aligned}&\frac{1}{2}\int_0^t \partial_t (u^k)^2\,dx\,ds + \varepsilon \int_{\mathbb{R}^d} |\nabla u^k|^2\,dx\,ds + \int_{\mathbb{R}^d} \operatorname{div} f(u^k) u^k\,dx \\ &\quad\le n \int_{\mathbb{R}^d} (u^k)^2\,dx \Big(\int_{\mathbb{R}^d} (u^{k-1}-\theta)^+\,dx \Big) \\ &\quad\le nR \int_{\mathbb{R}^d} (u^k)^2\,dx,\end{aligned}$$

since $u^{k-1} \in \mathcal{E}$. Integrating on $(0,t), t \le T_0$ and applying Gronwall's lemma gives

$$u^k \in L^\infty(0,t;L^2(\mathbb{R}^d)), \quad u^k \in L^2(0,t;H^1(\mathbb{R}^d)), \quad t \le T_0,$$

uniformly in k. This allows us to conclude that the limit $u_{n,\varepsilon}$ is in $W(0,t)$ and so solves the problem (3.1), at least for some time T_0. Since functions in $W(0,T_0)$ are actually continuous on $[0,T_0]$ with values in $L^2(\mathbb{R}^d)$ (see [6, p.54]), the initial data u_0 is indeed assumed.

5. Finally, we show global in time existence. For this it will be sufficient to prove that $\int_{\mathbb{R}^d} u_{n,\varepsilon}(t)\,dx = 1$ for almost all $t \in [0,T_0)$. In (3.1) take ψ_ρ as test function to obtain easily

$$\begin{aligned}\frac{d}{dt}\langle u_{n,\varepsilon}(t), \psi_\rho\rangle &\le \frac{C(M+\varepsilon)}{\rho} \int_{\mathbb{R}^d} u_{n,\varepsilon}(t)\,\psi_\rho\,dx + n\int_{\mathbb{R}^d} u_{n,\varepsilon}(t)\,\psi_\rho\,dx \Big(\int_{\mathbb{R}^d} (u_{n,\varepsilon}(t)-\theta)^+\,dx \Big) \\ &\quad - n\int_{\mathbb{R}^d} (u_{n,\varepsilon}(t)-\theta)^+ \psi_\rho\,dx\end{aligned}$$

and so

$$\begin{aligned}\int_{\mathbb{R}^d} u_{n,\varepsilon}(t)\,\psi_\rho\,dx \le \int_{\mathbb{R}^d} u_0\,dx &+ \frac{C(M+\varepsilon)}{\rho}\int_0^t \int_{\mathbb{R}^d} u_{n,\varepsilon}(s)\,\psi_\rho\,dx\,ds \\ &+ n\int_0^t \int_{\mathbb{R}^d} u_{n,\varepsilon}(s)\,\psi_\rho\,dx \Big(\int_{\mathbb{R}^d} (u_{n,\varepsilon}(s)-\theta)^+\,dx \Big)\,ds \\ &- n\int_0^t \int_{\mathbb{R}^d} (u_{n,\varepsilon}(s)-\theta)^+ \psi_\rho\,dx\,ds.\end{aligned} \tag{3.12}$$

Since $u_{n,\varepsilon} \in \mathcal{E}$, we have $u_{n,\varepsilon}(t) \in L^1(\mathbb{R}^d)$. Now, we return to (3.12) and take $\rho \to \infty$ applying the Dominated Convergence Theorem to find

$$\begin{aligned}\int_{\mathbb{R}^d} u_{n,\varepsilon}\,dx \le \int_{\mathbb{R}^d} u_0\,dx &+ n\int_0^t \int_{\mathbb{R}^d} u_{n,\varepsilon}\,dx \Big(\int_{\mathbb{R}^d} (u_{n,\varepsilon}-\theta)^+\,dx \Big)\,ds \\ &- n\int_0^t \int_{\mathbb{R}^d} (u_{n,\varepsilon}-\theta)^+\,dx\,ds.\end{aligned}$$

Then, it follows that

$$\int_{\mathbb{R}^d} u_{n,\varepsilon}\,dx - 1 \le n\int_0^t \Big(\int_{\mathbb{R}^d} u_{n,\varepsilon}\,dx - 1\Big)\Big(\int_{\mathbb{R}^d} (u_{n,\varepsilon}-\theta)^+\,dx\Big)\,ds$$
$$\le nR\int_0^t \Big(\int_{\mathbb{R}^d} u_{n,\varepsilon}\,dx - 1\Big)\,ds,$$

and consequently, $\int_{\mathbb{R}^d} u_{n,\varepsilon}(t)\,dx = 1$, for almost all $t \le T_0$. Furthermore, since $u \in \mathcal{E}$, we may suppose, actually, that

$$\forall\, t \le T_0, \qquad \int_{\mathbb{R}^d} u_{n,\varepsilon}(t)\,dx = 1. \tag{3.13}$$

This completes the proof of Theorem 3.1. □

The next lemma compares the solution of the nonlocal problem (2.1) with the solution of the homogenous conservation law (2.4) with initial data $v_{0,\gamma}$. The key point is that, this comparison property is independent of n.

Lemma 3.2. *Let $u_{n,\varepsilon}$ be a solution of* (2.1)*, and let $v_{\gamma,\varepsilon}$ be a solution to the Cauchy problem for the viscous homogenous conservation law* (2.4)*. Then, $u_{n,\varepsilon} \ge v_{\gamma,\varepsilon}$. In particular, this comparison property holds for all n.*

Proof of the lemma. We drop the subscripts n, ε from $u_{n,\varepsilon}$ and γ, ε from $v_{\gamma,\varepsilon}$ during the proof. Subtract (2.1) from (2.4), multiply by $(v_{\gamma,\varepsilon} - u_{n,\varepsilon})^+$, and integrate on $\mathbb{R}^d$ to get (with $w = v_{\gamma,\varepsilon} - u_{n,\varepsilon}$)

$$\frac{d}{dt}\int_{\mathbb{R}^d} (w^+)^2\,dx \le -\int_{\mathbb{R}^d} \operatorname{div}(f(v)-f(u))(w)^+\,dx + \varepsilon\int_{\mathbb{R}^d} \Delta w\, w^+\,dx$$
$$- n\int_{\mathbb{R}^d} u\, w^+\,dx \int_{\mathbb{R}^d} (u-\theta)^+\,dx + n\int_{\mathbb{R}^d} (u-\theta)^+ w^+\,dx.$$

By the maximum principle, we have $v_{\gamma,\varepsilon} \le \underline{\theta}$, and so, if $u_{n,\varepsilon} \ge \theta$, then necessarily $u_{n,\varepsilon} > v_{\gamma,\varepsilon}$ and thus $w^+ = 0$. Therefore, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} (w^+)^2\,dx \le -\int_{\mathbb{R}^d} \operatorname{div}(f(v)-f(u))(w)^+\,dx - \varepsilon\int_{\mathbb{R}^d} \nabla w \nabla w^+\,dx$$
$$\le M\int_{\mathbb{R}^d} |w||\nabla w^+|\,dx - \varepsilon\int_{\mathbb{R}^d} |\nabla w^+|^2\,dx$$
$$= M\int_{\mathbb{R}^d} |w^+||\nabla w^+|\,dx - \varepsilon\int_{\mathbb{R}^d} |\nabla w^+|^2\,dx.$$

Integrating on $[0,t]$ for $t \le T$, using a weighted Young inequality and Gronwall's lemma, we conclude that $\int_{\mathbb{R}^d} (w^+)^2\,dx = 0$ and so $v \le u$ on $[0,T]$. This proves the lemma. □

4 Uniform estimates for the penalized nonlocal problem

In this section, we prove estimates for solutions of (2.1) independently of n. They will allow not only the necessary compactness properties on the sequence $(u_{n,\varepsilon})$ but also give a more precise characterization of the limit of $u_{n,\varepsilon}$ as $n \to \infty$, $\varepsilon \to 0$. So, in Theorem 4.1 we prove an estimate which ensures that, in the limit, the solution of the obstacle-mass constraint problem will indeed stay below the obstacle. For this, we need the result in Lemma 4.2 (whose proof is found at the end of this section), which states that the solutions $u_{n,\varepsilon}$ retain some mass below the obstacle, uniformly in n. Recall from the discussion in Section 2, that Assumption 2.1 was especially designed to ensure this type of property.

Then, in Theorem 4.3, we establish uniform (in n and ε) estimates for $u_{n,\varepsilon}$ in $W^{1,1}((0,T)\times\mathbb{R}^d)$. These estimates will allow us in the next section to obtain existence of a solution for problem (1.4)-(1.5), using the vanishing viscosity method.

4.1 Main estimates independent of n

Theorem 4.1. *Let $T > 0$ be arbitrary. Suppose u_0 satisfies the hypothesis (2.2), $\int_{\mathbb{R}^d} u_0 dx = 1$, and let $\{u_{n,\varepsilon}\}$ be the family of solutions of the nonlocal parabolic problem (3.1). Then, there exists a constant $\alpha > 0$ depending on T, u_0 and θ, but not on n (given by Lemma 4.2 below), such that for all $\varepsilon > 0$ sufficiently small, and a.a. $t \in (0, T)$*

$$\int_{\mathbb{R}^d} (u_{n,\varepsilon}(t) - \theta(t))^+ \, dx \leq \frac{C_\theta}{\alpha n}, \tag{4.1}$$

where C_θ only depends on θ but not on n or ε, and

$$\|u_{n,\varepsilon}(t)\|_{L^\infty(\mathbb{R}^d)} \leq e^{t\frac{C_\theta}{\alpha}}. \tag{4.2}$$

To prove Theorem 4.1, we consider the following key result:

Lemma 4.2. *Under the same conditions of Theorem 4.1, there exists a constant $\alpha > 0$ depending on T, u_0 and θ, but not on n, such that the estimate is valid:*

$$\inf_{0 \leq t \leq T} \int_{\{u_{n,\varepsilon} < \theta\}} u_{n,\varepsilon}(t) \, dx \geq \alpha. \tag{4.3}$$

We also have

Theorem 4.3. *Suppose the initial data u_0 is in $(L^\infty \cap BV)(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} u_0 \, dx = 1$. The solution $u_{n,\varepsilon}$ of the nonlocal penalized parabolic problem (4.5) with regularized initial data satisfies*

$$u_{n,\varepsilon} \in W^{1,1}((0,T) \times \mathbb{R}^d), \qquad \textit{uniformly in } \varepsilon, n.$$

More precisely, for each $n \in \mathbb{N}$, and $\varepsilon > 0$, and almost all $t \in (0, T)$

$$\|\partial_t u_{n,\varepsilon}(t)\|_{L^1(\mathbb{R}^d)} \leq \big(C\,\mathrm{TV}(u_0) + C_\theta(t)\big)e^{t\frac{C_\theta}{\alpha}},$$
$$\|\nabla u_{n,\varepsilon}(t)\|_{L^1(\mathbb{R}^d)} \leq \big(\mathrm{TV}(u_0) + C_\theta(t)\big)e^{t\frac{C_\theta}{\alpha}},$$

with α as in Theorem 4.1, and C_θ depending on θ but not on n or ε.

Remark 4.4. Let us comment briefly on the results of Lemma 4.2 and Theorem 4.1. The estimate (4.3) states that, for $t \in [0, T]$, the function $u_{n,\varepsilon}$ retains some mass below the obstacle θ, *uniformly in n*, and it is the most delicate estimate in this work. The key property (2.2) in Assumption 2.1 is used to prove the estimate (4.3) only, which in turn ensures the property (4.1). This last estimate ensures that as $n \to \infty$ the mass above the obstacle θ of the solutions $u_{n,\varepsilon}$ vanishes.

Also, although a smoother initial data is required for Theorem 4.3, when passing to the limit $n \to \infty, \varepsilon \to 0$ this requirement can be eliminated in a completely standard way. We omit this straightforward procedure (found, e.g., in [6]) for the sake of clarity.

Now we prove Theorems 4.1 and 4.3, leaving the proof of Lemma 4.2 to the end of this section.

Proof of Theorem 4.1. We prove the estimate (4.1). Multiply equation (2.1) by $\mathrm{sgn}_\delta(u_{n,\varepsilon} - \theta)^+$ and integrate over $\mathbb{R}^d$ to find after some manipulations

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} I_\delta(u_{n,\varepsilon} - \theta)^+ \, dx &+ \int_{\mathbb{R}^d} \mathrm{div}(f(u_{n,\varepsilon}) - f(\theta))\,\mathrm{sgn}_\delta(u_{n,\varepsilon} - \theta)^+ \, dx \\
&+ \varepsilon \int_{\mathbb{R}^d} \mathrm{sgn}'_\delta(u_{n,\varepsilon} - \theta)^+ |\nabla(u_{n,\varepsilon} - \theta)|^2 \, dx \\
&= n \int_{\mathbb{R}^d} u_{n,\varepsilon}\,\mathrm{sgn}_\delta(u_{n,\varepsilon} - \theta)^+ \, dx \int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+ \, dx \\
&- n \int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+ \, dx - \int_{\mathbb{R}^d} (H(\theta) - \varepsilon\Delta\theta)\,\mathrm{sgn}_\delta(u_{n,\varepsilon} - \theta)^+ \, dx.
\end{aligned}$$

The second term on the left-hand side tends to zero as $\delta \to 0$, by a (classical) calculation similar to the one after (3.9) above, while the third is non-negative. Set

$$\varphi(t) = \int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+(t)\,dx.$$

We have that

$$\begin{aligned} 1 - \int_{\mathbb{R}^d} u_{n,\varepsilon}\,\mathrm{sgn}(u_{n,\varepsilon} - \theta)^+\,dx &= \int_{\mathbb{R}^d} u_{n,\varepsilon}\,dx - \int_{\mathbb{R}^d} u_{n,\varepsilon}\chi_{\{u_{n,\varepsilon} > \theta\}}\,dx \\ &= \int_{\mathbb{R}^d} u_{n,\varepsilon}\chi_{\{u_{n,\varepsilon} \le \theta\}}\,dx \ge \int_{\mathbb{R}^d} u_{n,\varepsilon}\chi_{\{u_{n,\varepsilon} < \theta\}}\,dx \end{aligned} \tag{4.4}$$

and from (4.3), $\int_{\mathbb{R}^d} u_{n,\varepsilon}\chi_{\{u_{n,\varepsilon} < \theta\}}\,dx \ge \alpha$. Therefore, we have

$$\begin{aligned} \varphi'(t) &\le n\varphi(t)\Big(\int_{\mathbb{R}^d} u_{n,\varepsilon}\,\mathrm{sgn}(u_{n,\varepsilon} - \theta)^+\,dx - 1\Big) - \int_{\mathbb{R}^d} (H(\theta) - \varepsilon\Delta\theta)\,\mathrm{sgn}(u_{n,\varepsilon} - \theta)^+\,dx \\ &\le -n\varphi(t)\int_{\mathbb{R}^d} u_{n,\varepsilon}\chi_{\{u_{n,\varepsilon} < \theta\}}\,dx - \int_{\mathbb{R}^d} (H(\theta) - \varepsilon\Delta\theta)\,\mathrm{sgn}(u_{n,\varepsilon} - \theta)^+\,dx \\ &\le -\alpha n\varphi(t) - \int_{\mathbb{R}^d} (H^-(\theta) - \varepsilon\Delta\theta)\,\mathrm{sgn}(u_{n,\varepsilon} - \theta)^+\,dx. \end{aligned}$$

Thus, for appropriate C_θ not depending on n or ε,

$$\begin{aligned} &\big(e^{\alpha n t}\varphi(t)\big)' \le C_\theta e^{\alpha n t} \\ \Rightarrow\ &\varphi(t) \le C_\theta \int_0^t e^{\alpha n(s-t)}\,ds \le C_\theta \frac{1 - e^{-\alpha n t}}{\alpha n} \le \frac{C_\theta}{\alpha n}, \end{aligned}$$

which proves the estimate (4.1), or rather, a slightly more precise version of (4.1) ensuring that $\varphi(t) \to 0$ as $t \to 0$.

We will now use (4.1) to prove the pointwise estimate (4.2). We follow the same procedure as in Lemma 3.2, by proving that $(u_{n,\varepsilon} - e^{t\frac{C_\theta}{\alpha}})^+$ vanishes. We omit the details of this standard computation. This concludes the proof of Theorem 4.1. ☐

Proof of Theorem 4.3. First of all, we note that it is easy to see, using classical arguments similar to [6, p.61], that for each ε, n the function $u_{n,\varepsilon}$ satisfies

$$u_{n,\varepsilon} \in L^2(0,T;H^2(\mathbb{R}^d)), \quad \partial_t u_{n,\varepsilon} \in L^2(0,T;H^1(\mathbb{R}^d)),$$

as long as f is a C^1 function and the initial data u_0 is in $H^1(\mathbb{R}^d)$. This allows us to write the equation (3.1) in strong form,

$$\begin{aligned} &\partial_t u_{n,\varepsilon} + \mathrm{div}\, f(u_{n,\varepsilon}) - \varepsilon\Delta u_{n,\varepsilon} = n\, u_{n,\varepsilon}\int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+\,dx - n(u_{n,\varepsilon} - \theta)^+, \\ &u_{n,\varepsilon}(0,x) = u_0(x). \end{aligned} \tag{4.5}$$

Now, we obtain a uniform estimate of $\|\nabla u_{n,\varepsilon}\|_{L^1(\mathbb{R}^d)}$. to this end, differentiate (4.5) in the direction x_i, $i = 1, \dots, d$ [1] to get after summing and subtracting terms,

$$\begin{aligned} &\partial_t\partial_i(u_{n,\varepsilon} - \theta) + \partial_i\,\mathrm{div}(f(u_{n,\varepsilon}) - f(\theta)) - \varepsilon\partial_i\Delta(u_{n,\varepsilon} - \theta) \\ &\quad = n\partial_i u_{n,\varepsilon}\int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+\,dx - n\partial_i(u_{n,\varepsilon} - \theta)^+ - \partial_i H_\varepsilon(\theta). \end{aligned}$$

[1] The following calculation should actually be performed using discrete derivatives and then passing to the limit on the discretization, but we omit these details, which may be found in any PDE textbook.

After multiplying by $\operatorname{sgn}(\partial_i(u_{n,\varepsilon}-\theta))$ and integrating on $\mathbb{R}^d$ we find

$$\begin{aligned}\int_{\mathbb{R}^d} \partial_t|\partial_i(u_{n,\varepsilon}-\theta)|\,dx &+ \int_{\mathbb{R}^d} \big(\partial_i \operatorname{div}(f(u_{n,\varepsilon})-f(\theta)) - \varepsilon\partial_i\Delta(u_{n,\varepsilon}-\theta)\big)\operatorname{sgn}\partial_i(u_{n,\varepsilon}-\theta)\,dx \\ &= n\int_{\mathbb{R}^d} \partial_i u_{n,\varepsilon}\operatorname{sgn}\partial_i(u_{n,\varepsilon}-\theta)\,dx \int_{\mathbb{R}^d}(u_{n,\varepsilon}-\theta)^+\,dx \\ &\quad - n\int_{\mathbb{R}^d}|\partial_i(u_{n,\varepsilon}-\theta)^+|\,dx - \int_{\mathbb{R}^d}\partial_i H_\varepsilon(\theta)\operatorname{sgn}\partial_i(u_{n,\varepsilon}-\theta)\,dx.\end{aligned}$$

The second integral on the left-hand side actually gives a nonnegative contribution [6, p.64], as well as the last term on the right-hand side, and so may be neglected. Therefore, using the estimate (4.1) and integrating on $[0,T]$ we get

$$\begin{aligned}\int_{\mathbb{R}^d}|\partial_i(u_{n,\varepsilon}-\theta)|\,dx \le & \int_{\mathbb{R}^d}|\partial_i(u_0-\theta(0))|\,dx \\ & + \frac{C_\theta}{\alpha}\int_0^T\int_{\mathbb{R}^d}|\partial_i u_{n,\varepsilon}|\,dx\,dt + \int_0^T\int_{\mathbb{R}^d}|\partial_i H_\varepsilon(\theta)|\,dx\,dt.\end{aligned}$$

Finally, writing $|\partial_i u_{n,\varepsilon}| \le |\partial_i(u_{n,\varepsilon}-\theta)| + |\partial_i\theta|$ we find

$$\begin{aligned}\int_{\mathbb{R}^d}|\partial_i u_{n,\varepsilon}|\,dx \le & \int_{\mathbb{R}^d}|\partial_i u_0|\,dx + \frac{C_\theta}{\alpha}\int_0^T\int_{\mathbb{R}^d}|\partial_i u_{n,\varepsilon}|\,dx\,dt \\ & + \int_0^T\int_{\mathbb{R}^d}|\partial_i H_\varepsilon(\theta)|\,dx\,dt + \int_{\mathbb{R}^d}|\partial_i\theta(0)| + |\partial_i\theta|\,dx.\end{aligned}$$

Since the last term can be bounded by a constant $C_\theta(t)$ independent of ε or n, (recall the smoothness assumptions on θ, (1.7)) we find after applying Gronwall's inequality that

$$\|\nabla u_{n,\varepsilon}(t)\|_{L^1(\mathbb{R}^d)} \le \big(\|\nabla u_0\|_{L^1(\mathbb{R}^d)} + C_\theta(t)\big)e^{t\frac{C_\theta}{\alpha}},$$

for some (possibly larger) constant C_θ, independently of ε and n. Thus,

$$\nabla u_{n,\varepsilon} \in L^\infty(0,T;L^1(\mathbb{R}^d)) \qquad \text{uniformly in } n,\varepsilon. \tag{4.6}$$

Next, we obtain a uniform estimate of $\|\partial_t u_{n,\varepsilon}(t)\|_{L^1(\mathbb{R}^d)}$. Differentiate the equation (4.5) in t (after adding and subtracting terms with θ, as done previously), multiply by $\operatorname{sgn}\partial_t(u_{n,\varepsilon}-\theta)$ and integrate on $\mathbb{R}^d$. The flux and viscosity terms both vanish or have the convenient sign, as in the previous estimate. We find

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|\partial_t(u_{n,\varepsilon}-\theta)|\,dx \le & \; n\int_{\mathbb{R}^d}\partial_t u_{n,\varepsilon}\operatorname{sgn}\partial_t(u_{n,\varepsilon}-\theta)\,dx\int_{\mathbb{R}^d}(u_{n,\varepsilon}-\theta)^+\,dx \\ & + n\int_{\mathbb{R}^d}u_{n,\varepsilon}\operatorname{sgn}\partial_t(u_{n,\varepsilon}-\theta)\,dx\int_{\mathbb{R}^d}\partial_t(u_{n,\varepsilon}-\theta)^+\,dx - n\int_{\mathbb{R}^d}|\partial_t(u_{n,\varepsilon}-\theta)^+|\,dx \\ & + \int_{\mathbb{R}^d}|\partial_t H_\varepsilon(\theta)|\,dx.\end{aligned}$$

So, using (4.1) and $\int_{\mathbb{R}^d} u\operatorname{sgn}\partial_t(u-\theta)\,dx \le 1$,

$$\frac{d}{dt}\int_{\mathbb{R}^d}|\partial_t(u_{n,\varepsilon}-\theta)|\,dx \le \frac{C_\theta}{\alpha}\int_{\mathbb{R}^d}|\partial_t u_{n,\varepsilon}|\,dx + \int_{\mathbb{R}^d}|\partial_t H_\varepsilon(\theta)|\,dx.$$

Therefore, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}|\partial_t(u_{n,\varepsilon}-\theta)|\,dx \le & \int_{\mathbb{R}^d}|\partial_t(u_0-\theta(0))|\,dx \\ & + \frac{C_\theta}{\alpha}\int_0^T\int_{\mathbb{R}^d}|\partial_t u_{n,\varepsilon}|\,dx\,dt + \int_0^T\int_{\mathbb{R}^d}|\partial_t H_\varepsilon(\theta)|\,dx\,dt\end{aligned}$$

and

$$\int_{\mathbb{R}^d} |\partial_t u_{n,\varepsilon}|\,dx \le \int_{\mathbb{R}^d} |\partial_t u_0|\,dx + \frac{C_\theta}{\alpha}\int_0^T\int_{\mathbb{R}^d} |\partial_t u_{n,\varepsilon}|\,dx\,dt \\ + \int_0^T\int_{\mathbb{R}^d} |\partial_t H_\varepsilon(\theta)|\,dx\,dt + \int_{\mathbb{R}^d} |\partial_t\theta| + |\partial_t\theta(0)|dx.$$

Now, using the equation one obtains

$$\int_{\mathbb{R}^d} |\partial_t u_0|\,dx \le M\|\nabla u_0\|_{L^1(\mathbb{R}^d)} + \varepsilon\|\Delta u_0\|_{L^1(\mathbb{R}^d)}. \tag{4.7}$$

As in [6, p.68], we consider a smoothing of u_0 such that $\varepsilon\|\Delta u_0\|_{L^1(\mathbb{R}^d)} \le C\|\nabla u_0\|_{L^1(\mathbb{R}^d)}$ for some universal constant depending only on the dimension d. Thus

$$\|\partial_t u_{n,\varepsilon}\|_{L^1(\mathbb{R}^d)} \le C\|\nabla u_0\|_{L^1(\mathbb{R}^d)} + \frac{C_\theta}{\alpha}\int_0^T \|\partial_t u_{n,\varepsilon}\|_{L^1(\mathbb{R}^d)}\,dt \\ + \int_0^T \|\partial_t H_\varepsilon(\theta)\|_{L^1(\mathbb{R}^d)}\,dt + \|\partial_t\theta\|_{L^1(\mathbb{R}^d)} + \|\partial_t\theta(0)\|_{L^1(\mathbb{R}^d)}$$

and finally

$$\|\partial_t u_{n,\varepsilon}(t)\|_{L^1(\mathbb{R}^d)} \le \big(C\|\nabla u_0\|_{L^1(\mathbb{R}^d)} + C_\theta(t)\big)e^{t\frac{C_\theta}{\alpha}}.$$

This concludes the proof of Theorem 4.3. □

4.2 Proof of Lemma 4.2

The idea of the proof is the following: as discussed in Section 2, Assumption 2.1 is designed to ensure that the support of $u_{n,\varepsilon}$ always travels into regions where the integral of θ is greater than one. In view of this, and the fact (established in (3.13)) that the total mass of $u_{n,\varepsilon}$ is one, necessarily $u_{n,\varepsilon}$ cannot have all its nonzero values above θ, otherwise Assumption 2.1 would be violated. Therefore, $u_{n,\varepsilon}$ must retain some mass below θ, which is the claim in (4.3). We now make precise this statement, using a contradiction argument.

1. Suppose (4.3) is false. Then, there are sequences $\varepsilon_j \to 0$, $t_j \in (0,T]$, $n_j \to \infty$, such that

$$\int_{\mathbb{R}^d} u_j \chi_{\{u_j<\theta\}}(t_j)\,dx < \frac{1}{j}, \tag{4.8}$$

where $u_j \in L^1(\mathbb{R}^d)$ is the solution u_{n_j,ε_j} of equation (2.1) (in the sense of Theorem 3.1), at time t_j (so, $u_j(x) = u_{n_j,\varepsilon_j}(t_j,x)$). Upon extraction of a subsequence (which here, and in what follows, we do not relabel), we may suppose $t_j \to t^*$ for some $t^* \in (0,T]$ as $j \to \infty$. Observe that due to Remark 2.2 we ensure that $t^* > 0$. Thus, if we set $w_j(x) := u_j(x)\chi_{\{u_j(x)<\theta(t_j,x)\}} \in L^1(\mathbb{R}^d)$, then (4.8) gives $w_j \to 0$ as $j \to \infty$ in $L^1(\mathbb{R}^d)$, since u_j is nonnegative.

Let $v_j \in L^1(\mathbb{R}^d)$ denote the (smooth) solution of the viscous problem (2.4) with viscosity parameter $\varepsilon = \varepsilon_j$, at time t_j. That is, $v_j(x) = v_{\gamma,\varepsilon_j}(t_j,x)$ in (2.4). According to the comparison Lemma 3.2, we have $v_j \le u_j$, and so $v_j\chi_{\{u_j<\theta(t_j)\}} \le w_j$ for a.e. $x \in \mathbb{R}^d$. From $w_j \to 0$ in $L^1(\mathbb{R}^d)$ we obtain

$$v_j\chi_{\{u_j<\theta(t_j)\}} \to 0 \text{ in } L^1(\mathbb{R}^d) \tag{4.9}$$

as $j \to \infty$.

2. We have $v_j(t_j) \to v(t^*)$ in $L^1(\mathbb{R}^d)$ as $j \to \infty$, with v solving (2.3). Indeed, according to standard results concerning the vanishing viscosity approximation of hyperbolic conservation laws and the continuity in time of the viscous approximations (see, for instance, [6]), we have

$$\|v_j(t_j) - v(t^*)\|_{L^1(\mathbb{R}^d)} \le \|v_j(t_j) - v_j(t^*)\|_{L^1(\mathbb{R}^d)} + \|v_j(t^*) - v(t^*)\|_{L^1(\mathbb{R}^d)} \to 0$$

as $j \to \infty$. Also, we have $\chi_{\{u_j(x)<\theta(t_j,x)\}} \overset{*}{\rightharpoonup} \xi$ in $L^\infty(\mathbb{R}^d)$, for some $\xi \in L^\infty(\mathbb{R}^d)$. Thus, (4.9) gives $v(t^*)\xi = 0$ a.e. on $\mathbb{R}^d$. Therefore,

$$\xi = 0 \text{ a.e. on } \{x \in \mathbb{R}^d | v(x,t^*) > 0\}, \tag{4.10}$$

which we abbreviate to $\{v(t^*) > 0\}$. Now, observe that a sequence of nonnegative functions weakly converging to zero also converges strongly in L^1_{loc}. Since $\chi_{\{u_j<\theta(t_j)\}} \geq 0$, we conclude from $\chi_{\{u_j<\theta(t_j)\}} \overset{*}{\rightharpoonup} \xi$ and (4.10) that actually

$$\chi_{\{u_j<\theta(t_j)\}} \to 0 \text{ strongly in } L^1_{\text{loc}}(\{v(t^*) > 0\})$$

and a.e. on $\{v(t^*) > 0\}$, as $j \to \infty$.

3. Let B_R denote the ball of radius $R > 0$ centered on the origin. Let $\delta > 0$ to be chosen later. According to Egorov's Theorem, There exists a set $J_\delta \subset (\{v(t^*) > 0\} \cap B_R)$ such that $|J_\delta| \leq \delta$ and $\chi_{\{u_j<\theta(t_j)\}} \to 0$ uniformly on $V_\delta := (\{v(t^*) > 0\} \cap B_R) \setminus J_\delta$ as $j \to \infty$. Since $\chi_{\{u_j<\theta(t_j)\}}$ only takes the values 0 and 1, this means that for sufficiently large j, we must have $u_j(x) > \theta(t_j,x)$ a.e. on V_δ. Therefore,

$$\int_{V_\delta} u_j(x)\,dx > \int_{V_\delta} \theta(t_j,x)\,dx = \int_{\{v(t^*)>0\}\cap B_R} \theta(t_j,x)\,dx - \int_{J_\delta} \theta(t_j,x)\,dx. \tag{4.11}$$

Now, from (2.2) in Assumption 2.1, we deduce that for large enough R,

$$\int_{\{v(t^*)>0\}\cap B_R} \theta(t^*,x)\,dx > 1 + \beta/2,$$

and, by the L^1 continuity property (1.8),

$$\int_{\{v(t^*)>0\}\cap B_R} \theta(t_j,x)\,dx > 1 + \beta/2$$

for sufficiently large j. On the other hand, from Lebesgue's theorem and (1.8), we see that since θ is locally integrable, we have $\int_{J_\delta} \theta(t^*,x)\,dx \to 0$ when $\delta \to 0$. Therefore, we choose δ small enough such that

$$\int_{J_\delta} \theta(t^*,x)\,dx \leq \frac{\beta}{8}.$$

Again using (1.8), we find for sufficiently large j

$$\begin{aligned}\int_{J_\delta} \theta(t_j,x)\,dx &\leq \int_{B_R} |\theta(t_j,x) - \theta(t^*,x)|\,dx + \frac{\beta}{8}\\ &\leq \frac{\beta}{8} + \frac{\beta}{8} = \frac{\beta}{4}.\end{aligned}$$

We conclude from (4.11) and from the unit integral property (3.13) that

$$1 \geq \int_{V_\delta} u_j(x)\,dx > 1 + \frac{\beta}{2} - \frac{\beta}{4} = 1 + \frac{\beta}{4},$$

which is a contradiction. Thus (4.8) cannot hold and so (4.3) is proven. This concludes the proof of Lemma 4.2. □

5 Solvability of the obstacle-mass constraint problem

In this section, we establish existence of an entropy solution for problem (1.4)–(1.5), in the sense of Definition 1.2, by the vanishing viscosity method. The main result of this paper is the following:

Theorem 5.1. *Let $u_0 \in (L^\infty \cap BV)(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} u_0\, dx = 1$, and let $\theta(t,x)$ be an obstacle function. Suppose that u_0, θ verify* (1.7)–(1.9) *and Assumption 2.1. Then, there exists an entropy solution to the hyperbolic obstacle-mass constraint problem* (1.4)–(1.5) *in the sense of Definition 1.2.*

Proof of Theorem 5.1. 1. First, for $\varepsilon > 0$ and $n \in \mathbb{N}$ we consider the nonlocal penalized viscous problem (2.1), which we repeat here for convenience:

$$\partial_t u_{n,\varepsilon} + \operatorname{div} f(u_{n,\varepsilon}) - \varepsilon \Delta u_{n,\varepsilon} = n\, u_{n,\varepsilon} \int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+ - n(u_{n,\varepsilon} - \theta)^+,$$
$$u_{n,\varepsilon}(0,x) = u_0(x), \qquad \int_{\mathbb{R}^d} u_0\, dx = 1.$$

For $\varphi \in C_c^\infty((-\infty,T) \times \mathbb{R}^d)$ and η an entropy (assumed C^2 without loss of generality), multiply (2.1) by $\varphi\, \eta'(u_{n,\varepsilon} - k\theta)$ and integrate in $(0,T) \times \mathbb{R}^d =: \Pi_T$. We obtain

$$\begin{aligned}
&- \iint_{\Pi_T} \eta(u_{n,\varepsilon} - k\theta)\, \varphi_t \, dxdt + \iint_{\Pi_T} \eta'(u_{n,\varepsilon} - k\theta)\, \varphi\, \partial_t(k\theta)\, dxdt \\
&- \iint_{\Pi_T} \eta'(u_{n,\varepsilon} - k\theta)\big(f(u_{n,\varepsilon}) - f(k\theta)\big) \cdot \nabla\varphi \, dxdt + \iint_{\Pi_T} \eta'(u_{n,\varepsilon} - k\theta)\, \varphi \operatorname{div} f(k\theta)\, dxdt \\
&- \iint_{\Pi_T} \varepsilon\, \Delta\eta(u_{n,\varepsilon} - k\theta)\, \varphi\, dxdt - \iint_{\Pi_T} \varepsilon\, \Delta(k\theta)\, \eta'(u_{n,\varepsilon} - k\theta)\, \varphi\, dxdt \\
&- \int_{\mathbb{R}^d} |u_0(x) - k\,\theta(0,x)|\, \varphi(0,x)\, dx \\
&= - \iint_{\Pi_T} \varepsilon\, \eta''(u_{n,\varepsilon} - k\theta)\, |\nabla(u_{n,\varepsilon} - k\theta)|^2\, \varphi\, dxdt \\
&\quad + \iint_{\Pi_T} \Big(n\, u_{n,\varepsilon} \int (u_{n,\varepsilon} - \theta)^+ \, dx - n(u_{n,\varepsilon} - \theta)^+\Big)\, \eta'(u_{n,\varepsilon} - k\theta)\, \varphi\, dxdt.
\end{aligned}$$

Neglecting the negative terms on the right-hand side, it follows that in the sense of distributions

$$\begin{aligned}
&\partial_t \eta(u_{n,\varepsilon} - k\theta) + \operatorname{div}\Big(\eta'(u_{n,\varepsilon} - k\theta)\big(f(u_{n,\varepsilon}) - f(k\theta)\big)\Big) - \varepsilon\Delta\eta(u_{n,\varepsilon} - k\theta) \\
&\qquad \le n\, u_{n,\varepsilon}\, \eta'(u_{n,\varepsilon} - k\theta) \int_{\mathbb{R}^d} (u_{n,\varepsilon} - \theta)^+ \, dx - \eta'(u_{n,\varepsilon} - k\theta)\big(H(k\theta) - \varepsilon\Delta(k\theta)\big),
\end{aligned} \tag{5.1}$$

which incidentally motivates the precise formulation in Definition 1.2.

2. Now, we define for almost all $t \in (0,T)$, $\lambda_{n,\varepsilon}(t) := n \int_{\mathbb{R}^d} (u_{n,\varepsilon}(t) - \theta(t))^+ \, dx$. According to the estimate (4.1), we have that $\lambda_{n,\varepsilon}(t)$ is uniformly bounded for a.a. $t \in (0,T)$. Thus (if necessary taking a subsequence), $\lambda_{n,\varepsilon}(t)$ converges weak-star in $L^\infty(0,T)$ to some $\lambda(t)$.

3. Now, with the inequality (5.1) in hand, and the estimates collected in previous sections, it is a standard matter to pass to limit and obtain an entropy solution. Indeed, using standard compactness results (see, e.g., the totally similar procedure in [6, p.70]), the family $(u_{n,\varepsilon})$ has a subsequence (which we do not relabel) converging a.e. on Π_T and in $L^1_{\text{loc}}((0,T) \times \mathbb{R}^d)$ to some $u \in L^\infty((0,T) \times \mathbb{R}^d)$. The gradient estimate in Theorem 4.3 ensures that $u(t) \in BV(\mathbb{R}^d)$ for a.a. $t \in (0,T)$. Note that Theorem 4.3 requires that the initial data is smooth enough, so we use a mollification of u_0 depending on ε. The procedure to obtain u_0 in the limit is exactly the same as in [6], so we omit it for the sake of simplicity. Moreover, from item 2 we see that the first term on the right-hand side of (5.1) converges to $u\lambda(t)\eta'(u - k\theta)$ weak-star in $L^\infty\big((0,T) \times \mathbb{R}^d\big)$, which is enough to pass to the limit on (5.1). Thus (u, λ) is a solution of problem (1.4)–(1.5) according to Definition 1.2. This completes the proof of Theorem 5.1. ☐

6 Acknowledgements

The work of Wladimir Neves is partially supported by Pronex-FAPERJ through the grant E-26/ 110.560/2010 entitled "*Nonlinear Partial Differential Equations*", and also by CNPq through the grants 484529/2013-7, 308652/2013-4.

References

[1] L. Barthélemy, Problème d'obstacle pour une équation quasi-linéaire du premier ordre, Ann. Fac. Sci. Toulouse Math. 9 (1988) 137–159.

[2] F. Berthelin, Existence and weak stability for a pressureless model with unilateral constraint, Math. Models Methods Appl. Sci. 12 (2002) 249–272.

[3] F. Berthelin, F. Bouchut, Weak solutions for a hyperbolic system with unilateral constraint and mass loss. Ann. I. H. Poincaré – AN 20, 6 (2003) 975–997

[4] L. Caffarelli, F. Lin, Nonlocal heat flows preserving the L^2 energy. Discrete and continuous dynamical systems, 23, n.1&2 (2009).

[5] G. Gagneux, A.M. Lefevère, M. Madaune-Tort, Une approche analytique d'un modèle black-oil des écoulements triphasiques compressibles en ingénierie pétrolière, J. Mécanique Théor. Appl. 6 (1987) 547–569.

[6] E. Godlewski, P.-A. Raviart, *Hyperbolic systems of conservation laws,* Mathématiques & Applications (Paris). Ellipses, Paris, 1991

[7] D. Kinderlehrer, G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, 1980.

[8] L. Lévi, Problèmes unilatéraux pour des équations non linéaires de réaction diffusion. Annales de la Faculté de Sciences de Toulouse, 6 (4), no.3 (1995), 593–631.

[9] L. Lévi, Obstacle problems for scalar conservation laws. M2AN Math. Model. Numer. Anal. 35 (2001), no. 3, 575–593.

[10] L. Lévi, The singular limit of a bilateral obstacle problem for a class of degenerate parabolic–hyperbolic operators. Advances in Applied Mathematics 35 (2005) 34–57

[11] F. Mignot, J.-P. Puel, Inéquations variationnelles et quasivariationnelles hyperboliques du premier ordre, J. Math. Pures Appl. (9) 55 (1976) 353–378.

[12] J.-F. Rodrigues, *Obstacle problems in mathematical physics.* North-Holland mathematics Studies ; 134 (1987)

[13] J.-F. Rodrigues, On the hyperbolic obstacle problem of first order. Chin. Ann. of Math., 23B:2 (2002) 253–266.

[14] J.-F. Rodrigues, On Hyperbolic Variational Inequalities of First Order and Some Applications Monatsh. Math. 142 (1–2) (2004) 157–177.