ASYMPTOTIC BEHAVIOR OF TREMBLING FLUIDS

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Abstract. A general problem modeling trembling fluids with applications in electro, magneto or thermo-rheological fluids is consider in this work. The fluid flow is governed by the generalized Navier-Stokes equations with a variable q–structure. We prove that the solutions of the associated initial and boundary-value problem extinct in a finite time as long as the trembling fluid remains in the pseudo-plastic zone. For trembling fluids strictly confined to the dilatant zone or that can cross the Newtonian barrier and eventually go back, we study the large time behavior of the solutions. Perturbations of the asymptotically stable equilibrium are analyzed as well.

Keywords and phrases: generalized Navier-Stokes; variable exponent; extinction in a finite time; large time behavior; asymptotically stable equilibrium.

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1. Introduction

Let us consider a general cylinder

$$Q_T := \Omega \times [0,T], \quad \text{with} \quad \Gamma_T := \partial\Omega \times [0,T],$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with a compact boundary $\partial\Omega$, and $0 < T < \infty$. In this cylinder, we consider the generalized Navier-Stokes equations:

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad Q_T, \tag{1.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{div}(\mathbf{u} \otimes \mathbf{u}) = \mathbf{f} - \nabla p + \mathbf{div}\, \mathbf{S} \quad \text{in} \quad Q_T, \tag{1.2}$$

supplemented with the following initial and boundary conditions

$$\mathbf{u} = \mathbf{u}_0 \qquad \text{in} \quad \Omega \quad \text{for} \quad t = 0, \tag{1.3}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on} \quad \Gamma_T. \tag{1.4}$$

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Here, $\mathbf{u}$ is the velocity field, p stands for the pressure divided by the constant density and $\mathbf{f}$ is the external forces field. $\mathbf{S}$ is the deviatoric part of the Cauchy stress tensor, which depends on $(\mathbf{x},t)$ and on the strain rate tensor $\mathbf{D}$:

$$\mathbf{S} = \nu(D_{II})\mathbf{D}, \quad \mathbf{D} \equiv \mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right), \quad D_{II} = \frac{1}{2}|\mathbf{D}|^2, \tag{1.5}$$

where $|\mathbf{D}|$ accounts for the shear rate and D_{II} is the second invariant of $\mathbf{D}$. We prescribe the velocity $\mathbf{u}_0$ at the initial time and we assume the fluid adheres to the fixed boundary for all time. Many constitutive laws have been proposed in the rheological literature to model diverse non-Newtonian fluids in different flow conditions (see *e.g.* Barnes *et al.* [7]), but here we consider a general deviatoric stress tensor $\mathbf{S}$ having a variable q–structure in the following sense:

(A) $\mathbf{S} : Q_T \times \mathbb{M}^N_{\text{sym}} \to \mathbb{M}^N_{\text{sym}}$ is a Charathéodory function;
(B) growth condition: $|\mathbf{S}(\mathbf{x},t,\mathbf{A})| \leq C|\mathbf{A}|^{q(\mathbf{x},t)-1}$ for all $\mathbf{A}$ in $\mathbb{M}^N_{\text{sym}}$ and for a.a. $(\mathbf{x},t)$ in Q_T;
(C) coercivity condition: $\mathbf{S}(\mathbf{x},t,\mathbf{A}) : \mathbf{A} \geq C|\mathbf{A}|^{q(\mathbf{x},t)}$ for all $\mathbf{A}$ in $\mathbb{M}^N_{\text{sym}}$ and for a.a. $(\mathbf{x},t)$ in Q_T;
(D) monotonicity condition: $(\mathbf{S}(\mathbf{x},t,\mathbf{A}) - \mathbf{S}(\mathbf{x},t,\mathbf{B})) : (\mathbf{A} - \mathbf{B}) > 0$ for all $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{M}^N_{\text{sym}}$, with $\mathbf{A} \neq \mathbf{B}$, and for a.a. $(\mathbf{x},t)$ in Q_T.

Here C denotes a positive constant which may vary from line to line and $\mathbb{M}^N_{\text{sym}}$ is the vector space of all symmetric $N \times N$ matrices, which is equipped with the scalar product $\mathbf{A} : \mathbf{B}$ and norm $|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}}$. Throughout this text we shall drop the dependence of $\mathbf{S}$ on $(\mathbf{x},t,\mathbf{A})$, writing only the different dependencies on the tensor $\mathbf{A}$.

The best example of a fluid model obeying to conditions (A)-(D), is a generalization of the well-known power-law model, widely used during the last century to describe diverse non-Newtonian fluids:

$$\mathbf{S} = \tau_0|\mathbf{D}|^{q(\cdot)-2}\mathbf{D}\,, \tag{1.6}$$

where q is the variable power-law index that characterizes the flow and τ_0 is the consistency factor, *i.e.* a positive constant that corresponds to the fluid viscosity when $q \equiv 2$. If q is constant and $1 < q < 2$, the fluid is pseudoplastic and it thins, *i.e.* it deforms more rapidly with an increase in shear stress, and therefore it is also called shear-thinning fluid. When $q = 2$, we recover the Stokes law and consequently the fluid is Newtonian. If $q > 2$, the fluid is dilatant and it thickens with an increase in shear stress, and due to that such fluids are also called shear-thickening. With slight modifications on the conditions (B)-(C), we can see that many other fluid models obey to (A)-(D) as the Sisko and Carreau models (see *e.g.* [7]). Fluids modeled by the power-law (1.6) cannot be cataloged into a single class of non-Newtonian fluids as pseudo-plastic or dilatant. The problems we have in mind in this work are the cases when q depends on an electric field or on a magnetic field, or still when q is a temperature-dependent function. Therefore the incompressible generalized Navier-Stokes equations (1.1)-(1.2) have to be solved with the Maxwell equations, in the cases of q depending on an electric or on a magnetic field, or must be supplemented with the equation for the transport of the temperature, when q is temperature-dependent. Nevertheless, the resulting governing equations are essentially uncoupled, hence the Maxwell equations, or the temperature equation, can be solved first. The solution of electric or magnetic field, or the temperature, can be then considered as a known function, resulting that the original problem reduces to the problem of the incompressible generalized Navier-Stokes equations (1.1)-(1.2) with the deviatoric part of the Cauchy stress tensor given by (1.6), with

$$q(\cdot) = q(\mathbf{x}, t) \quad \text{for} \quad (\mathbf{x}, t) \in Q_T. \tag{1.7}$$

In these cases are the electro, magneto or thermo-rheological fluids whose rheological properties are controllable through the application of an electric or magnetic field or by changes in the temperature, showing useful and special function with the effect of reversibility. Now we are in the presence of fluids that can go, for instance, from the consistency of a liquid (Newtonian) to that of a gel (non-Newtonian), and back, with response times on the order of milliseconds. Examples of electro-rheological fluids are suspensions dispersed with some polymeric colloids and typical magneto-rheological fluids are made of very small solid particles that are suspended in a Newtonian fluid. Applications of electro and magneto-rheological fluids in the automotive industry such as dampers, clutches, brakes and active bearings have already came to the market. Other application is seismic dampers, which are used in buildings in seismically-active zones to damp the oscillations (see *e.g.* Hao [12] and Henrie and Carlson [11]). Because electro and magneto-rheological fluids have the ability to change their flow characteristics according to the practical needs, they are often referred to as smart fluids. On the other hand, thermo-rheological fluids are made of nanometer-sized particles dispersed in a Newtonian fluid and are being used, for instance, to model certain cooling processes of volcano lava flows (see *e.g.* Das *et al.* [8]). Due to their trembling shear behavior, in the sequel, we shall call fluids with the deviatoric part of the stress tensor satisfying to conditions (A)-(D) above by *trembling fluids*.

The mathematical analysis of non-Newtonian fluid models, with constant power-law indexes q, started with the works of Ladyzhenskaya [13, 14] and then followed by Lions [15]. A few years ago the existence results established in [13, 14, 15] were improved by Zhikov [20], Wolf [19] and Diening *et al.* [10] for values of q that reach the pseudo-plastic zone. The asymptotic behavior of the solutions to these problems, with constant power-law indexes q, have been performed, among others, by Antontsev *et al.* [1] and Bae [6]. In recent years there has been a demand looking for similar results for the same problems, but considering variable power-law indexes q. The aim of this work is to study the asymptotic behavior of the weak solutions to the problem (1.1)-(1.4), with the deviatoric tensor $\mathbf{S}$ obeying to conditions (A)-(D). We shall study the extinction in a finite time property and the large time behavior of the weak solutions. Perturbations of the asymptotically stable equilibrium will be analyzed as well.

The notation used throughout this article is largely standard in Mathematical Analysis and in particular in Mathematical Fluid Mechanics (see *e.g.* Lions [15]). We distinguish tensors and vectors from scalars by using boldface letters. For functions and function spaces we will use this distinction as well. The symbol C, with or without subscripts, will denote a generic positive constant, whose value will not be specified – it can change from one inequality to another. The dependence of C on other constants or parameters will always be clear from the exposition. Given $k \in \mathbb{N}$, we denote by $\mathrm{C}^k(\Omega)$ the space of all k-differentiable functions in Ω. By $\mathrm{C}_0^\infty(\Omega)$, we denote the space of all infinity-differentiable functions with compact support in Ω. If X is a generic Banach space, its dual space is denoted by X'. Let $1 \leq q \leq \infty$ and $\Omega \subset \mathbb{R}^N$, with $N \geq 1$, be a domain. We will use the classical Lebesgue spaces $\mathrm{L}^q(\Omega)$, whose norm is denoted by $\|\cdot\|_{\mathrm{L}^q(\Omega)}$. $\mathrm{W}^{1,q}(\Omega)$ denotes the Sobolev space of all functions $u \in \mathrm{L}^q(\Omega)$ such that the weak derivatives $\mathrm{D}^\gamma u$ exist, in the generalized sense, and are in $\mathrm{L}^q(\Omega)$ for any multi-index γ such that $0 \leq |\gamma| \leq 1$. The norm in $\mathrm{W}^{1,q}(\Omega)$ is denoted by $\|\cdot\|_{\mathrm{W}^{1,q}(\Omega)}$. The corresponding spaces of vector-valued or tensor-valued functions are denoted by boldface letters. All these spaces are Banach spaces and the Hilbert framework corresponds to $q = 2$. Given $T > 0$ and a Banach space X, $\mathrm{L}^q(0,T;X)$ denotes the usual Bochner space used in evolutive problems, with the norm denoted by $\|\cdot\|_{\mathrm{L}^q(0,T;X)}$.

2. Weak formulation

The analysis of our problem must be done in the context of Orlicz spaces. These spaces resemble in many aspects to the classical Lebesgue and Sobolev spaces, but there are some important differences which must be pointed out. We denote by $\mathcal{P}(\Omega)$ the set of all measurable functions $q : \Omega \to [1,\infty]$ and define

$$q^- := \operatorname*{ess\,inf}_{x\in\Omega} q(x), \quad q^+ := \operatorname*{ess\,sup}_{x\in\Omega} q(x).$$

Given $q \in \mathcal{P}(\Omega)$, we denote by $\mathrm{L}^{q(\cdot)}(\Omega)$ the space of all measurable functions f in Ω such that its semimodular is finite:

$$A_{q(\cdot)}(f) := \int_\Omega |f(x)|^{q(x)} d\,x < \infty. \tag{2.1}$$

The space $\mathrm{L}^{q(\cdot)}(\Omega)$ is called Lebesgue space with variable exponent and is also known by generalized Lebesgue space. Equipped with the norm

$$\|f\|_{\mathrm{L}^{q(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : A_{q(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\}, \tag{2.2}$$

$\mathrm{L}^{q(\cdot)}(\Omega)$ becomes a Banach space. Note that the infimum in (2.2) is attained if $A_{q(\cdot)}(f) > 0$. If $q^+ < \infty$, $\mathrm{L}^{q(\cdot)}(\Omega)$ is separable and the space $\mathrm{C}_0^\infty(\Omega)$ is dense in $\mathrm{L}^{q(\cdot)}(\Omega)$. Moreover, if $1 < q^- \leq q^+ < \infty$, $\mathrm{L}^{q(\cdot)}(\Omega)$ is reflexive. One problem in these Orlicz spaces, is the relation between the semimodular (2.1) and the norm (2.2). If $1 < q^- \leq q^+ < \infty$, one can shows that

$$\min\left(\|f\|^{q^-}_{\mathrm{L}^{q(\cdot)}(\Omega)}, \|f\|^{q^+}_{\mathrm{L}^{q(\cdot)}(\Omega)}\right) \leq A_{q(\cdot)}(f) \leq \max\left(\|f\|^{q^-}_{\mathrm{L}^{q(\cdot)}(\Omega)}, \|f\|^{q^+}_{\mathrm{L}^{q(\cdot)}(\Omega)}\right). \tag{2.3}$$

In Lebesgue spaces with variable exponents, there holds a version of Hölder's inequality, called generalized Hölder's inequality.

Given $q \in \mathcal{P}(\Omega)$, the Orlicz-Sobolev space $W^{1,q(\cdot)}(\Omega)$ is defined as:

$$W^{1,q(\cdot)}(\Omega) := \left\{f \in \mathrm{L}^{q(\cdot)}(\Omega) : \mathrm{D}^\xi f \in \mathrm{L}^{q(\cdot)}(\Omega),\ 0 \leq |\xi| \leq 1\right\}.$$

This space is also known by Sobolev space with variable exponent or generalized Sobolev space. In $W^{1,q(\cdot)}(\Omega)$ is defined a semimodular and the correspondent induced norm analogously as in (2.1)-(2.2), which are equivalent, respectively, to

$$A_{1,q(\cdot)}(f) := A_{q(\cdot)}(f) + A_{q(\cdot)}(|\nabla f|)$$

and

$$\|f\|_{W^{1,q(\cdot)}(\Omega)} := \|f\|_{\mathrm{L}^{q(\cdot)}(\Omega)} + \|\nabla f\|_{\mathrm{L}^{q(\cdot)}(\Omega)}.$$

For this norm, $W^{1,q(\cdot)}(\Omega)$ is a Banach space, which becomes separable and reflexive in the same conditions as $\mathrm{L}^{q(\cdot)}(\Omega)$. The Orlicz-Sobolev space with zero boundary values is defined by:

$$W_0^{1,q(\cdot)}(\Omega) := \overline{\left\{f \in W^{1,q(\cdot)}(\Omega) : \operatorname{supp} f \subset\subset \Omega\right\}}^{\;\|\cdot\|_{W^{1,q(\cdot)}(\Omega)}}.$$

In contrast to the case of classical Sobolev spaces, the set $C_0^\infty(\Omega)$ is not necessarily dense in $W_0^{1,q(\cdot)}(\Omega)$ – the closure of $C_0^\infty(\Omega)$ in $W^{1,q(\cdot)}(\Omega)$ is strictly contained in $W_0^{1,q(\cdot)}(\Omega)$. A necessary condition for the equality is the globally log-Hölder continuity of q: there are positive constants C_1, C_2 and q_∞ such that

$$\text{(2.4)}\quad |q(\mathbf{x}) - q(\mathbf{y})| \leq \frac{C_1}{\ln(e + 1/|\mathbf{x}-\mathbf{y}|)}, \quad |q(\mathbf{x}) - q_\infty| \leq \frac{C_2}{\ln(e + |\mathbf{x}|)} \quad \forall\ \mathbf{x},\ \mathbf{y} \in \Omega.$$

In the sequel, we will denote by $\mathcal{P}^{log}(\Omega)$ the set of functions defined in $\mathcal{P}(\Omega)$ which satisfy to (2.4) in the domain Ω. For a thorough analysis of Lebesgue and Sobolev spaces with variable exponents, we address the reader to the monograph by Diening *et al.* [9].

In order to define the notion of solutions we shall look for, let us introduce the usual functional setting of Mathematical Fluid Mechanics:

$$\text{(2.5)}\qquad \mathcal{V} := \{\mathbf{v} \in \mathbf{C}_0^\infty(\Omega) : \operatorname{div} \mathbf{v} = 0\};$$

$$\text{(2.6)}\qquad \mathbf{H} := \text{closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega);$$

$$\text{(2.7)}\qquad \mathbf{V}_q := \text{closure of } \mathcal{V} \text{ in } \mathbf{W}^{1,q}(\Omega).$$

In the rest of this work, and in order to simplify the notation, given $q \in \mathcal{P}(Q_T)$, we set

$$\text{(2.8)}\qquad \alpha := \operatorname*{ess\ inf}_{(\mathbf{x},t)\in Q_T} q(\mathbf{x},t), \qquad \beta := \operatorname*{ess\ sup}_{(\mathbf{x},t)\in Q_T} q(\mathbf{x},t)$$

and let us assume that

$$\text{(2.9)}\qquad 1 < \alpha \leq q(\mathbf{x},t) \leq \beta < \infty \qquad \text{for all } (\mathbf{x},t) \in Q_T.$$

We recall that, for a fixed $t \in [0,T]$ and without any extra condition on q, the set $\mathbf{C}_0^\infty(\Omega)$ is not necessarily dense in $\mathbf{W}_0^{1,q(\cdot,t)}(\Omega)$. For this reason, we define the analogue of $\mathrm{L}^q(0,T;\mathbf{V}_q)$ by

$$\text{(2.10)}\qquad \mathbf{W}_q(Q_T) := \overline{\{\mathbf{v} \in \mathbf{C}^\infty(\overline{Q}_T) : \operatorname{div}\mathbf{v} = 0,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_T\}}^{\|\mathbf{D}(\mathbf{v})\|_{\mathbf{L}^{q(\cdot,\cdot)}(Q_T)}}.$$

It is a easy task to verify that the space $\mathbf{W}_q(Q_T)$ satisfies to the following imbeddings:

$$\mathrm{L}^\beta(0,T;\mathbf{V}_\beta) \hookrightarrow \mathbf{W}_q(Q_T) \hookrightarrow \mathrm{L}^\alpha(0,T;\mathbf{V}_\alpha).$$

Moreover, $\mathbf{W}_q(Q_T)$ is a closed subspace of $\mathrm{L}^\alpha(0,T;\mathbf{W}_0^{1,\alpha}(\Omega))$ and therefore it is a reflexive and separable Banach space for the norm

$$\|\mathbf{v}\|_{\mathbf{W}_q(Q_T)} := \|\mathbf{D}(\mathbf{v})\|_{\mathbf{L}^{q(\cdot,\cdot)}(Q_T)}.$$

Definition 2.1. *Let $N \geq 2$ and $q \in \mathcal{P}(Q_T)$ be a variable exponent in the conditions of (2.8)-(2.9). Let also $\mathbf{u}_0 \in \mathbf{H}$ and $\mathbf{f} \in \mathbf{L}^1(Q_T)$, and assume conditions (A)-(D) are fulfilled. A vector field $\mathbf{u}$ is a weak solution to the problem (1.1)-(1.5), if:*

(1) $\mathbf{u} \in \mathbf{W}_q(Q_T) \cap \mathrm{L}^\infty(0,T;\mathbf{H})$;

(2) *For every* $\varphi \in \mathbf{C}^\infty(Q_T)$, *with* $\operatorname{div}\varphi = 0$ *and* $\operatorname{supp}\varphi \subset\subset \Omega \times [0,T)$,

$$\begin{aligned} & -\int_{Q_T} \mathbf{u}\cdot\varphi_t \, d\mathbf{x}dt + \int_{Q_T} [\mathbf{S}(\mathbf{D}(\mathbf{u})) - \mathbf{u}\otimes\mathbf{u}] : \mathbf{D}(\varphi)\, d\mathbf{x}dt \\ & = \int_{Q_T} \mathbf{f}\cdot\varphi \, d\,\mathbf{x}dt + \int_{\Omega} \mathbf{u}_0\cdot\varphi(0)\, d\mathbf{x}. \end{aligned} \tag{2.11}$$

We observe that (2.11) is equivalent to assert that for every $\eta \in \mathbf{C}^\infty(\overline{Q_T})$, with $\operatorname{div}\eta = 0$ in Q_T and $\eta = 0$ on Γ_T,

$$\begin{aligned} & -\int_{Q_{t_1}\setminus Q_{t_0}} \mathbf{u}\cdot\eta_t \, d\mathbf{x}ds + \int_{Q_{t_1}\setminus Q_{t_0}} [\mathbf{S}(\mathbf{D}(\mathbf{u})) - \mathbf{u}\otimes\mathbf{u}] : \mathbf{D}(\eta)\, d\mathbf{x}ds \\ & = \int_{Q_{t_1}\setminus Q_{t_0}} \mathbf{F} : \mathbf{D}(\eta)\, d\mathbf{x}ds - \int_{\Omega} \mathbf{u}(t)\cdot\eta(t)\, d\mathbf{x}\Big|_{t=t_0}^{t=t_1} \end{aligned}$$

for all $t_0,\ t_1 \in [0,T]$ and $\mathbf{f} = -\mathbf{div}\,\mathbf{F}$, with $\mathbf{F} \in \mathbb{M}^N_{\mathrm{sym}}$ (see *e.g.* [18, Proposition 4.2]). To the best of our knowledge, we do not know any result that assures the existence of weak solutions to the problem (1.1)-(1.5) in the sense of Definition 2.1 for all $\alpha > 1$, nor even for all

$$\alpha > \frac{2N}{N+2} \tag{2.12}$$

as we know in the case of a constant q-structure (*cf.* [10]). For the problem (1.1)-(1.5) we only have some partial results (see Antontsev *et al.* [4], Pastukhova [18] and Zhikov [20]). In [20] is proved the existence of weak solutions to the problem (1.1)-(1.5) for q depending only on the space variable (and considering the parabolic case) and such that

$$\alpha \geq \max\left\{\frac{3N}{N+2}, \frac{N+\sqrt{3N^2+4N}}{N+2}\right\}. \tag{2.13}$$

Considering a space and time dependent exponent q and assuming that q is globally log-Hölder continuous on the space variable (see (2.4)), it was proved in [18] the existence of weak solutions for

$$\frac{3N}{N+2} < \alpha \leq q(\mathbf{x},t) \leq \beta < \alpha_* := \begin{cases} \alpha\frac{N+2}{N}, & \alpha < N \\ N+2, & \alpha \geq N. \end{cases} \tag{2.14}$$

A better improvement of the range of q is provided in [4], where the existence result is established for

$$\frac{2N}{N+2} < \alpha \leq q(\mathbf{x},t) \leq \beta < \alpha^* := \begin{cases} \frac{\alpha N}{N-\alpha}, & \alpha < N \\ \infty, & \alpha \geq N\,, \end{cases} \tag{2.15}$$

where α^* is the Sobolev conjugate of α. This result does not require the log-Hölder continuity property (2.4), but holds only for solutions whose gradients *a priori* satisfy a condition of local higher-integrability.

In the sequel we shall assume the existence of weak solutions to the problem (1.1)-(1.5), with a variable q–structure, for $\alpha > 1$. The case $1 < \alpha \leq 2N/(N+2)$ and $N > 2$ seems to be very difficult to achieve, at least if we want to use the same techniques used for the case $\alpha > 2N/(N+2)$ (see the introductory section of [10]). However, we strongly believe that an existence result for the problem (1.1)-(1.5) with a variable q–structure will be proved for α satisfying to (2.12), at least for log-Hölder continuous exponents q and without requiring extra regularity on the solutions.

3. Energy inequality

Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.5) and let us denote the kinetics energy associated with this problem by

$$e(t) := \frac{1}{2}\int_\Omega |\mathbf{u}(t)|^2\, d\mathbf{x}\,. \tag{3.1}$$

In this section, we are interested in solutions to the problem (1.1)-(1.5) with a finite energy e. Therefore, we may assume that

$$|e(t)| \leq M \quad \text{for all } t \in [0,T], \quad \text{for some constant } M > 0\,. \tag{3.2}$$

Normalizing (3.2), we obtain

$$|E(t)| \leq 1 \quad \text{for all } t \in [0,T], \quad E(t) := \frac{e(t)}{M}\,. \tag{3.3}$$

In order to establish the main result of this section, let us now introduce the following notations for any fixed $t \in [0,T]$:

$$q(t) := q(\cdot,t) \quad \text{and} \quad q^-(t) := \inf_{\mathbf{x}\in\Omega} q(\mathbf{x},t), \quad q^+(t) := \sup_{\mathbf{x}\in\Omega} q(\mathbf{x},t). \tag{3.4}$$

Only in the norms and semimodulars, we still shall use the notation $q(\cdot,t)$, instead of $q(t)$, to emphasize the dependence of q on the space variable. According to (2.8) and (2.9), we can readily see that, for any fixed $t \in [0,T]$,

$$\alpha \leq \ q^-(t) \leq q(\mathbf{x},t) \leq q^+(t) \leq \beta \quad \text{for all } \mathbf{x} \in \Omega.$$

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume that conditions (A)-(D) are fulfilled with a variable $q \in \mathcal{P}(Q_T)$ and*

$$\mathbf{f} = -\mathbf{div}\,\mathbf{F}, \quad \mathbf{F} \in \mathbb{M}^N_{\text{sym}}, \quad \mathbf{F} \in \mathbf{L}^{q'(\cdot,\cdot)}(Q_T), \tag{3.5}$$

$$\mathbf{u}_0 \in \mathbf{H}. \tag{3.6}$$

If $q(\cdot,t) \in \mathcal{P}^{log}(\Omega)$ for all $t \in [0,T]$,

$$\alpha \geq \frac{2N}{N+2} \tag{3.7}$$

and $\mathbf{u}$ is a weak solution to the problem (1.1)-(1.5) such that the energy inequality

$$\frac{1}{2}\|\mathbf{u}(t)\|^2_{\mathbf{H}}\Big|_{t=t_0}^{t=t_1} + \int_{Q_{t_1}\setminus Q_{t_0}} \mathbf{S}(\mathbf{D}(\mathbf{u})) : \mathbf{D}(\mathbf{u}) d\mathbf{x}dt \leq \int_{Q_{t_1}\setminus Q_{t_0}} \mathbf{F} : \mathbf{D}(\mathbf{u}) d\mathbf{x}dt\,, \tag{3.8}$$

is valid for all $t_0,\ t_1 \in [0,T]$, with $t_0 < t_1$, then there exist two independent of t constants C_1 and C_2 such that

$$\frac{d}{dt}E(t) + C_1 E(t)^{\frac{q^+(t)}{2}} \leq C_2 \int_\Omega |\mathbf{F}(t)|^{q'(\cdot,t)}\, d\mathbf{x} \quad \textit{for a.a. } t \in [0,T]. \tag{3.9}$$

Note that the assumption (3.5) is made only to simplify the exposition. But, if we assumed only $q(\cdot,t)\in\mathcal{P}(\Omega)$, instead of $q(\cdot,t)\in\mathcal{P}^{log}(\Omega)$, for all $t\in[0,T]$, then the symmetric aspect of the forces field would be crucial, since, in that case, we could not use Korn's inequality in the scope of Orlicz-Sobolev spaces. On the other hand, the hypothesis on the existence of a weak solution which fulfills the energy relation (3.8) is perfectly reasonable. In fact, it was proved in [20, 18, 4] the existence of weak solutions $\mathbf{u}\in\mathbf{W}_q(Q_T)\cap\mathrm{L}^\infty(0,T;\mathbf{H})$ satisfying to (3.8) under all assumptions of Theorem 3.1, except that (3.7) was replaced by (2.13), (2.14) or (2.15) in each corresponding reference. Moreover, for all initial-boundary problems of Fluid Mechanics analogous energy relations follow from the existence results. Since important zones of the pseudo-plastic and of dilatant zones are not covered by (2.14) and once that we shall study the asymptotic behavior of the solutions to the problem (1.1)-(1.5) in the most extensible range of q, we shall need to assume the existence of a weak solution to the problem (1.1)-(1.5) such that the energy inequality (3.8) holds for all α obeying to (3.7). We cannot go below of (3.7) due to the imperial need to use, in our reasoning, Sobolev type inequalities which only hold for values of α complied with (3.7).

Proof. (Theorem 3.1) Due to (3.8), we can write for every $t,\ t+\Delta t\in[0,T]$, with $\triangle t>0$, the following inequality

(3.10)
$$\frac{1}{2}\|\mathbf{u}(s)\|^2_{\mathbf{H}}\Big|_{s=t}^{s=t+\Delta t}+\int_{Q_{t+\Delta t}\setminus Q_t}\mathbf{S}(\mathbf{D}(\mathbf{u})):\mathbf{D}(\mathbf{u})d\mathbf{x}ds\leq\int_{Q_{t+\Delta t}\setminus Q_t}\mathbf{F}:\mathbf{D}(\mathbf{u})d\mathbf{x}ds\,.$$

Using the coercivity condition (C) and the notations (3.1)-(3.2), we obtain from (3.10) that

(3.11)
$$\frac{M}{|\Delta t|}\,E(s)|_{s=t}^{s=t+\Delta t}\leq-\frac{C}{|\Delta t|}\int_{Q_{t+\Delta t}\setminus Q_t}|\mathbf{D}(\mathbf{u})|^{q(\cdot,\cdot)}d\mathbf{x}ds+\frac{1}{|\Delta t|}\int_{Q_{t+\Delta t}\setminus Q_t}|\mathbf{F}:\mathbf{D}(\mathbf{u})|d\mathbf{x}ds\,.$$

Since $\mathbf{u}\in\mathbf{W}_q(Q_T)$ and $\mathbf{F}\in\mathbf{L}^{q'(\cdot,\cdot)}(Q_T)$, then

$$\int_\Omega|\mathbf{D}(\mathbf{u})|^{q(\mathbf{x},\cdot)}d\mathbf{x}\quad\text{and}\quad\int_\Omega|\mathbf{F}:\mathbf{D}(\mathbf{u})|d\mathbf{x}\quad\text{are in}\quad\mathrm{L}^1[0,T].$$

In consequence, every term on the right-hand side of (3.11) has a limit, for almost all $t\in[0,T]$, as $\Delta t\to0$. This in turn yields the existence of a limit of the left-hand side of (3.11), for almost all $t\in[0,T]$, as $\Delta t\to0$. Whence we can write for almost all $t\in[0,T]$

$$\frac{d}{d\,t}E(t)+\frac{C}{M}\int_\Omega|\mathbf{D}(\mathbf{u}(t))|^{q(\cdot,t)}d\mathbf{x}\leq\frac{1}{M}\int_\Omega|\mathbf{F}(t):\mathbf{D}(\mathbf{u}(t))|d\mathbf{x}\,. \tag{3.12}$$

Then making use of the generalized Young's inequality, we obtain for almost all $t\in[0,T]$

$$\frac{d}{dt}E(t)+C_1\int_\Omega|\mathbf{D}(\mathbf{u}(t))|^{q(\cdot,t)}\,d\mathbf{x}\leq C_2\int_\Omega|\mathbf{F}(t)|^{q'(\cdot,t)}\,d\mathbf{x}\,. \tag{3.13}$$

Now, observing that $q(\cdot,t)\in\mathcal{P}^{log}(\Omega)$ for all $t\in[0,T]$ and, from (3.7), $\alpha\geq\frac{2N}{N+2}$, we can use Sobolev's and Korn's inequalities, in the scope of Orlicz-Sobolev spaces, to obtain

$$\frac{1}{C}\left\|\frac{\mathbf{u}(t)}{\sqrt{2M}}\right\|_{\mathbf{L}^2(\Omega)} \leq \frac{1}{\sqrt{2M}}\left\|\nabla\mathbf{u}(t)\right\|_{\mathbf{L}^{q(\cdot,t)}(\Omega)} \leq \frac{C}{\sqrt{2M}}\left\|\mathbf{D}(\mathbf{u}(t))\right\|_{\mathbf{L}^{q(\cdot,t)}(\Omega)} \tag{3.14}$$

for all $t\in[0,T]$, with an independent of t constant C, and where M is the constant defined at (3.2). On the other hand, from assumption (3.5), the right-hand side of (3.13) is finite and, consequently,

$$|\mathbf{D}(\mathbf{u}(t))|^{q(\cdot,t)} \in \mathbf{L}^1(\Omega) \text{ for a.a. } t\in[0,T].$$

According to the relations between the norms and the semimodulars in Orlicz-Sobolev spaces (see (2.3)), and attending to the notations (3.4), we know that

$$\|\mathbf{D}(\mathbf{u}(t))\|_{\mathbf{L}^{q(\cdot,t)}(\Omega)} \leq \max\left\{\left(\int_\Omega |\mathbf{D}(\mathbf{u}(t))|^{q(\cdot,t)}\,d\mathbf{x}\right)^{\frac{1}{q^+(t)}}, \left(\int_\Omega |\mathbf{D}(\mathbf{u}(t))|^{q(\cdot,t)}\,d\mathbf{x}\right)^{\frac{1}{q^-(t)}}\right\}$$

for all $t\in[0,T]$, and, consequently,

$$\min\left\{\|\mathbf{D}(\mathbf{u}(t))\|^{q^+(t)}_{q(\cdot,t),\ \Omega}, \|\mathbf{D}(\mathbf{u}(t))\|^{q^-(t)}_{\mathbf{L}^{q(\cdot,t)}(\Omega)}\right\} \leq \int_\Omega |\mathbf{D}(\mathbf{u}(t))|^{q(\cdot,t)}\,d\mathbf{x}\,. \tag{3.15}$$

Gathering (3.14) and (3.15), we obtain

$$C\min\left\{\left\|\frac{\mathbf{u}(t)}{\sqrt{2M}}\right\|^{q^-(t)}_{\mathbf{L}^2(\Omega)}, \left\|\frac{\mathbf{u}(t)}{\sqrt{2M}}\right\|^{q^+(t)}_{\mathbf{L}^2(\Omega)}\right\} \leq \int_\Omega |\mathbf{D}(\mathbf{u}(t))|^{q(\cdot,t)}\,d\mathbf{x} \quad \forall\ t\in[0,T]. \tag{3.16}$$

Now we observe that (3.2) yields

$$E(t) = \left\|\frac{\mathbf{u}(t)}{\sqrt{2M}}\right\|^2_{\mathbf{L}^2(\Omega)} \leq 1 \quad \text{for all } t\in[0,T]. \tag{3.17}$$

In consequence (3.17) allows us to write the inequality (3.16) in the form

$$CE(t)^{\frac{q^+(t)}{2}} \leq \int_\Omega |\mathbf{D}(\mathbf{u}(t))|^{q(\cdot,t)}\,d\mathbf{x} \quad \forall\ t\in[0,T]\,. \tag{3.18}$$

Finally, using (3.18) into (3.13), we obtain (3.9). □

4. Extinction in a finite time

In this section we are interested in trembling fluids that are confined to the pseudo-plastic zone. We shall show that a well known effect characterizing classical pseudo-plastic fluids is inherited by trembling fluids if they are confined to this zone. Here we shall assume that the upper bound of the time interval is large enough in order that the properties established in this section may occur inside our time interval.

Theorem 4.1. *Let* $\mathbf{u}$ *be a weak solution to the problem (1.1)-(1.5) in the conditions of Theorem 3.1 and assume that*

$$\frac{2N}{N+2} \leq \alpha \leq \beta < 2\,. \tag{4.1}$$

(1) *If* $\mathbf{F}=\mathbf{0}$ *a.e. in* Q_T*, then there exists* $t^*>0$ *such that* $\mathbf{u}=\mathbf{0}$ *a.e. in* Ω *and for almost all* $t\geq t^*$*.*

(2) *Let $\mathbf{F} \neq \mathbf{0}$ and assume that exist positive constants δ and γ and a positive time $t_{\mathbf{f}}$ such that, for almost all $t \in [0,T]$,*

$$\int_\Omega |\mathbf{F}(t)|^{q'(\cdot,t)}\,d\mathbf{x} \leq \delta\left(1-\frac{t}{t_{\mathbf{F}}}\right)_+^{\gamma}, \quad \textit{where} \quad \gamma := \frac{\beta}{2-\beta}. \tag{4.2}$$

Then there exists a constant $\delta_0 > 0$ such that $\mathbf{u} = \mathbf{0}$ a.e. in Ω and for almost all $t \geq t_{\mathbf{F}}$, provided that $0 < \delta \leq \delta_0$.

The notation u_+ means the positive part of u, *i.e.* $u_+ = \max\{u, 0\}$.

Proof. Firstly, we observe that (3.17) implies that (3.9) can be written in the form

$$\frac{d}{dt}E(t) + C_1 E(t)^{\frac{\beta}{2}} \leq C_2 \int_\Omega |\mathbf{F}(t)|^{q'(\cdot,t)}\,d\mathbf{x} \quad \text{for a.a. } t \in [0,T]. \tag{4.3}$$

1. If $\mathbf{F} = \mathbf{0}$ a.e. in Q_T, then (4.3) comes in the form

$$\frac{d}{dt}E(t) + C_1 E(t)^{\frac{\beta}{2}} \leq 0 \quad \text{for a.a. } t \in [0,T]. \tag{4.4}$$

Knowing that, from (4.1), $\beta < 2$, an explicit integration of (4.4) between $t = 0$ and t leads us to

$$E(t) \leq \left(E(0)^{\frac{2-\beta}{2}} - \frac{2-\beta}{2}C_1 t\right)^{\frac{2}{2-\beta}} \quad \text{for a.a. } t \in [0,T]. \tag{4.5}$$

The right-hand side of (4.5) vanishes for

$$t \geq t^* := C_2\|\mathbf{u}_0\|_{\mathbf{H}}^{2-\beta}, \quad C_2 := \frac{2^{\frac{\beta}{2}}}{(2-\beta)M^{\frac{2-\beta}{2}}C_1},$$

and the first assertion is then proved.

2. If $\mathbf{F}$ satisfies to (4.2), then from (4.3), we obtain the following non-homogeneous ordinary differential inequality

$$\frac{d}{d\,t}E(t) + C_1 E(t)^{\frac{\beta}{2}} \leq C_2\,\delta\left(1-\frac{t}{t_{\mathbf{F}}}\right)_+^{\gamma}, \tag{4.6}$$

with the constant exponent γ to be defined later on. To analyze (4.6), we need the following result.

Lemma 4.1. [1, Lemma 1.2.3] *Let $\rho > 0$ be such that $(t_{\mathbf{F}} - \rho, t_{\mathbf{F}} + \rho) \subset [0,T]$ and assume $E \in \mathrm{W}^{1,1}(t_{\mathbf{F}} - \rho,\ t_{\mathbf{F}} + \rho)$ satisfies to the differential inequality*

$$\frac{d}{dt}E(t) + \varphi(E(t)) \leq F\left(\left(1-\frac{t}{t_{\mathbf{F}}}\right)_+\right) \quad \textit{a.e. in } (t_{\mathbf{F}} - \rho, t_{\mathbf{F}} + \rho)\,, \tag{4.7}$$

where φ is a continuous and non-decreasing function such that

$$\varphi(0) = 0 \quad \textit{and} \quad \int_{0^+} \frac{ds}{\varphi(s)} < \infty\,, \tag{4.8}$$

and the function F fulfills, for some $\overline{k} \in (0,1)$, to

$$F(s) \leq (1-\overline{k})\varphi(\eta_{\overline{k}}(s)) \quad \textit{in } (0, t_{\mathbf{F}})\,, \tag{4.9}$$

where

$$\eta_k(s) = \theta_k^{-1}(s) \quad \textit{and} \quad \theta_k(s) = \int_0^s \frac{d\tau}{k\varphi(\tau)}\,.$$

Then $E(t) = 0$ *for all* $t \geq t_{\mathbf{F}}$.

In order to read (4.6) in the form (4.7), we define

$$\varphi(s) := C_1 s^{\frac{\beta}{2}} \quad \text{and} \quad F(s) := C_2\, \delta s^{\gamma}\,.$$

Then clearly (4.8) is satisfied and we have

$$\theta_k(s) = \frac{2}{kC_1(2-\beta)} s^{\frac{2-\beta}{2}} \quad \text{and} \quad \eta_k(s) = \left(\frac{kC_1(2-\beta)}{2} s\right)^{\frac{2}{2-\beta}}\,.$$

Moreover, (4.9) is verified if

$$\gamma := \frac{\beta}{2-\beta} \quad \text{and} \quad \delta \leq \delta_0 := \frac{(1-k)C_1}{C_2}\left(\frac{kC_1(2-\beta)}{2}\right)^{\frac{\beta}{2-\beta}}$$

for a certain $k \in (0,1)$, for instance $k = \beta/2$. Then Lemma 4.1 proves the second assertion. □

Remark 4.1. *We recall that, in the scope of Fluid Mechanics , the physical meaning (see* [1]*) of the properties established in Theorem 4.1 are the following. The first assertion, states that if the trembling flow is generated only by the initial velocity, then in a finite time the fluid becomes immobile. The second, states that if the flow is stirred by a forces field that stops to act at some moment, then the fluid is motionless for all posterior times, provided that the force is suitably small.*

Remark 4.2. *The extension of the results of Theorem 4.1 to the case when* $\beta > 2$ *are not known as we will see in the next section. But the results of the present section hold if we modify the problem by introducing, in the momentum equation (1.2), a damping term like* $|\mathbf{u}|^{\sigma(\cdot,\cdot)}\mathbf{u}$*, where* σ *is a new variable exponent (see* [3, 17]*). Another possibility, which seems to be more application oriented in certain branches of Fluid Mechanics, is to consider such damping term as being part of a feedback forces field (see* [2]*).*

5. Large time behavior

In this section we are interested in trembling fluids that are not strictly confined to the pseudo-plastic zone. We shall consider the other two possibilities: trembling fluids strictly confined to the dilatant zone and trembling fluids that can cross the barrier between pseudo-plastic and dilatant zones and that eventually can go back. For these fluids, we are not able to prove a property of extinction in a finite time, as we did for trembling fluids which are confined to the pseudo-plastic zone. Many arguments can be used to claim the impossibility of establishing this property, but, in our opinion, the main one comes from the own definition of a Newtonian fluid in Physics: a fluid that continues to flow regardless the forces acting on it. Nonetheless we shall study the asymptotic behavior of these flows in the presence of forces fields with a specific decay. Again, we shall assume that the upper bound of the time interval is large enough in order that the property established in the next theorem may occur inside our time interval.

Theorem 5.1. *Let* $\mathbf{u}$ *be a weak solution to the problem (1.1)-(1.5) in the conditions of Theorem 3.1 and assume that*

$$\alpha \geq \frac{2N}{N+2} \qquad \text{and} \qquad \beta > 2. \tag{5.1}$$

(1) *If* $\mathbf{F}=\mathbf{0}$ *a.e. in* Q_T*, then there exists a positive constant* C *such that*

$$\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq C(1+t)^{-\frac{2}{\beta-2}} \quad \textit{for a.a. } t\geq 0. \tag{5.2}$$

(2) *More generally, if* $\mathbf{F}\neq\mathbf{0}$ *but exists a positive constant* $C_{\mathbf{F}}$ *such that*

$$\int_\Omega |\mathbf{F}(t)|^{q'(\cdot,t)}\,d\mathbf{x} \leq C_{\mathbf{F}}(1+t)^{-\frac{\beta}{\beta-2}} \quad \textit{for a.a. } t\in[0,T], \tag{5.3}$$

then there exists a positive constant C *such that*

$$\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq C(1+t)^{-\frac{\beta}{\beta-2}} \quad \textit{for all } t\geq 0. \tag{5.4}$$

Proof. 1. If $\mathbf{F}=\mathbf{0}$ a.e. in Q_T, we proceed analogously as we did for the proof of Theorem 4.1 to obtain (4.4). Knowing that, from (5.1), $\beta>2$, an explicit integration of (4.4) between $t=0$ and $t\in(0,T]$ leads us to

$$E(t)\leq\left(E(0)^{\frac{2-\beta}{2}}+\frac{\beta-2}{2}C_1 t\right)^{-\frac{2}{\beta-2}} \quad \text{for a.a. } t\in[0,T]. \tag{5.5}$$

Thus, (5.2) is an immediate consequence of (5.5), with

$$C:=\left[\min\left\{\frac{\beta-2}{2}C_1,\left(\frac{\|\mathbf{u}_0\|_{\mathbf{H}}}{\sqrt{2M}}\right)^{2-\beta}\right\}\right]^{-\frac{2}{\beta-2}}.$$

2. Now, we assume that $\mathbf{F}\neq\mathbf{0}$ and satisfies to (5.3). Recalling the derivation of (4.3) and using the assumption (5.3), we obtain

$$\frac{d}{dt}E(t)+C_1E(t)^{\frac{\beta}{2}}\leq C_{\mathbf{F}}(1+t)^{-\frac{\beta}{\beta-2}} \quad \text{for a.a. } t\in[0,T]. \tag{5.6}$$

Let us look know for solutions of the form $y(t)=C(1+t)^{\gamma}$ to the ordinary differential equation

$$\frac{d}{dt}E(t)+C_1E(t)^{\frac{\beta}{2}}=C_*(1+t)^{-\gamma_*}, \tag{5.7}$$

where γ_* and C_* are positive constants to be found. After some algebraic manipulations, we can readily see that

$$\gamma_*=\frac{\beta}{\beta-2}$$

and C is a solution to the equation

$$f(C)=0,\quad f(C):=-\frac{2}{\beta-2}C+C_1C^{\frac{\beta}{2}}-C_*\,.$$

Since $f(0)=-C_*$, $C_*>0$ and $\lim_{C\to+\infty}f(C)=+\infty$, the equation $f(C)=0$ has a solution $\overline{C}>0$. On the other hand, $\overline{C}$ can be estimated from below as follows

$$\overline{C}=\frac{1}{C_1}\left(C_*+\frac{2}{\beta-2}\overline{C}\right)^{\frac{2}{\beta}}>\frac{1}{C_1}\left(\frac{2}{\beta-2}\overline{C}\right)^{\frac{2}{\beta}}\Leftrightarrow\overline{C}>\frac{1}{C_1^{\frac{\beta}{\beta-2}}}\left(\frac{2}{\beta-2}\right)^{\frac{2}{\beta-2}}.$$

Next, we introduce the new function

$$G(t):=E(t)-y(t)\equiv E(t)-\overline{C}(1+t)^{-\frac{\beta}{\beta-2}}\,,$$

which in turn obeys to the following linear differential inequality

$$\frac{d}{dt}G(t)+C_GG(t)\leq\left(C_{\mathbf{F}}-C_*\right)(1+t)^{-\frac{\beta}{\beta-2}}\,, \tag{5.8}$$

where

$$C_G := C_1 \frac{\beta}{2} \int_0^1 [\lambda E(t) - (1-\lambda) y(t)]^{\frac{\beta-2}{2}} \, d\lambda \geq 0 \,.$$

Now, we choose the constant C_* stated in (5.7) in such a way that

$$C_* \geq \max \left\{ C_{\mathbf{F}}, \frac{\|\mathbf{u}_0\|_{\mathbf{H}}^2}{2M} \right\}. \tag{5.9}$$

Then in view of (5.9) it follows from (5.8) that $G(t) \leq 0$ for all $t \in (0, T]$ and $G(0) \leq 0$. As a consequence,

$$0 \leq E(t) \leq \overline{C}(1+t)^{-\frac{\beta}{\beta-2}}$$

and, finally, (5.4) follows by taking $C = 2M\overline{C}$. □

6. Limit cases

In order to simplify our arguing in this section, we shall consider here an infinite time interval. We are interested in the cases when $q^+(t) \to 2$, as $t \to \infty$, in the inequality (3.9), or merely when $\beta \to 2$ in the inequality (4.3), and to look if the properties of the previous sections still remain.

First, we observe that if in the assumptions on the forces field $\mathbf{F}$, $\beta \to 2^-$ in (4.2), or $\beta \to 2^+$ in (5.3), then $\mathbf{F}$ would never vanish in any case. In these cases, the arguments based on comparison with solutions of equations with constant exponents of nonlinearity, we have used in the previous sections, are no longer valid, because the nonlinear differential inequality (4.3) transforms into a linear one in any case.

A different case is when $q^+(t) \to 2$, as $t \to \infty$, because it may well happen that the trembling fluid can go beyond the zones where it is confined by Theorem 4.1 or Theorem 5.1 and yet the properties therein still remain valid. In order to analyze this case, let us split our study in the situations when in the inequality (3.9)

$$q^+(t) \searrow 2 \quad \text{as } t \to \infty \,, \tag{6.1}$$

and when

$$q^+(t) \nearrow 2 \quad \text{as } t \to \infty \,. \tag{6.2}$$

If one of the situations (6.1) or (6.2) occurs, the nonlinear differential inequality (3.9) eventually transforms into a linear one in any case again. The convergence (6.1) means that the class of trembling fluids is endeavoring to escape from the dilatant zone. In this case, Theorem 4.1 provides us with better results than Theorem 5.1 as long as the trembling fluids rest in the pseudo-plastic zone. If the fluid is Newtonian, similar results to those provided by Theorem 5.1 are valid (see *e.g.* [16]). The case (6.2) states that are now the trembling fluids which are confined to pseudo-plastic zone who strive to become Newtonian. Here, a completely different situation happens, because we are coming from a range of $q^+(t)$ with results of localized solutions and, in some sense, we try to extent the range of q hopping that these properties still remain valid. Although the effect of extinction in time is never displayed by the solutions of the Navier-Stokes equations, it may happen that the generalized equation (1.2), with a variable exponent in the diffusion term, transforms into the classical Navier-Stokes equation and nonetheless their solutions possess this property. Therefore we are interested in the maximal velocity of such convergence for which the extinction in a finite time property remains valid.

Theorem 6.1. *Let, in the conditions of Theorem 4.1, $\mathbf{F}=\mathbf{0}$ and $\|\mathbf{u}(t)\|_{\mathbf{H}} \le \|\mathbf{u}_0\|_{\mathbf{H}}$ for all $t\in[0,T]$. If $q^+(t)$ is monotone increasing, (6.2) holds and the equation*

$$C\int_0^r \|\mathbf{u}_0\|_{\mathbf{H}}^{q^+(s)-2}\,ds = \int_0^\infty \frac{dz}{\mathrm{e}^{z\left(1-\frac{q^+(z)}{2}\right)}} \tag{6.3}$$

has a root $r=t_*$, then $\mathbf{u}=\mathbf{0}$ a.e. in Ω and for all $t\ge t_*$.

Proof. If $\mathbf{F}=\mathbf{0}$, we obtain from (3.9) that

$$\frac{d}{dt}E(t)+C_1E(t)^{\frac{q^+(t)}{2}} \le 0 \quad \text{for all } t\in[0,T]. \tag{6.4}$$

The analysis of the ordinary differential inequality (6.4) will rely on the following result.

Lemma 6.1. [5, Lemma 9.1] *Let $\mathcal{E}(t)$ be a nonnegative function obeying to the following conditions*

$$\begin{cases} \mathcal{E}'(t)+C\mathcal{E}^{\mu(t)}(t)\le 0 \quad \textit{for a.a. } t\ge 0, \textit{ and with } \mu(t)\in(0,1) \textit{ for all } t\ge 0,\\ \mathcal{E}(t)\le\mathcal{E}(0)<\infty\,. \end{cases}$$

If the exponent $\mu(t)$ is monotone increasing, then $\mathcal{E}(t)\equiv 0$ for all $t\ge t_*$ with t_* defined from the equality

$$C\int_0^{t_*}\mathcal{E}^{\mu(s)-1}(0)\,ds = \int_0^\infty \frac{dz}{\mathrm{e}^{z(1-\mu(z))}}.$$

Then, the conclusion of Theorem 6.1 follows by using Lemma 6.1 with $\mathcal{E}(t)=E(t)$ and $\mu(t)=\frac{q^+(t)}{2}$. □

Remark 6.1. *Let us introduce the notation*

$$\delta(t):=1-\frac{q^+(t)}{2}\searrow 0.$$

In particular, the integral of the right-hand side of (6.3) converges, if

$$\delta(t)\ge(1+C)\frac{\ln t}{t}, \qquad t>>1, \quad C>0,$$

which in turn, written in terms of $q^+(t)$, gives

$$q^+(t)\le 2-2(1+C)\frac{\ln t}{t}, \qquad t>>1, \quad C>0.$$

7. Perturbations of the asymptotically stable equilibrium

In this section we are interested in solutions to the problem (1.1)-(1.5), with $\mathbf{F}\neq\mathbf{0}$ but such that

$$\int_\Omega |\mathbf{F}(t)|^{q'(\cdot,t)}\,d\mathbf{x}\le C_{\mathbf{F}} \quad \text{for a.a. } t\in[0,T]\,, \tag{7.1}$$

where $C_{\mathbf{F}}$ is an independent of t positive constant. Condition (7.1) is naturally fulfilled if $\mathbf{F}\in\mathbf{L}^{q'(\cdot,\cdot)}(Q_T)$, which in turn is assured by hypothesis (3.5). From (3.9) and (7.1) it follows that

$$\frac{d}{dt}E(t)+C_1E(t)^{\frac{q^+(t)}{2}}\le C_2 \quad \text{for a.a. } t\in[0,T], \tag{7.2}$$

where C_2 is now the product of the right-hand side constants of (3.9) and (7.1). Let us set now

$$\frac{d}{d\,t}E(t)+C_1E(t)^{\frac{q^+(t)}{2}}=C_2\Leftrightarrow\frac{d}{d\,t}E(t)=C_2-C_1E(t)^{\frac{q^+(t)}{2}}:=\Lambda(t)\,. \tag{7.3}$$

If $\Lambda(t)<0$ or $\Lambda(t)>0$ at some time t (possibly different), then $E(t)$ is decreasing or increasing, respectively, at that time. In consequence, the asymptotically stable equilibrium of (7.3) is reached when

$$\Lambda(t)=0\Leftrightarrow E(t)=\left(\frac{C_2}{C_1}\right)^{\frac{2}{q^+(t)}}:=E_*(t)\Leftrightarrow\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}=2M\left(\frac{C_2}{C_1}\right)^{\frac{2}{q^+(t)}}:=e_*(t)\,, \tag{7.4}$$

where C_1 and C_2 are given by (7.2). The proof of the following result follows by standard arguments. We write it here for completeness.

Theorem 7.1. *Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.5) in the conditions of Theorem 3.1 and such that (7.1) is fulfilled with $\mathbf{F}\neq\mathbf{0}$.*

1. *If exists a positive time $t_0>0$ such that $\|\mathbf{u}(t_0)\|^2_{\mathbf{L}^2(\Omega)}\leq e_*(t_0)$, then $\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}<e_*(t_0)$ for all t such that $0<t<t_0$, and $\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}\nearrow e_*(t_0)$ as $t\to t_0$.*
2. *If exists a positive time $t^0>0$ such that $\|\mathbf{u}(t^0)\|^2_{\mathbf{L}^2(\Omega)}>e_*(t^0)$, then $\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}>e_*(t^0)$ for all t such that $t^0<t\leq T$, and $\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}\searrow e_*(t^0)$ as $t\to t^0$.*

Proof. 1. Writing the first assumption in terms of the functions $E(t)$ and $E_*(t)$, we have

$$E(t_0)<E_*(t_0)=\left(\frac{C_2}{C_1}\right)^{\frac{2}{q^+(t_0)}}\Leftrightarrow C_1E(t_0)^{\frac{q^+(t_0)}{2}}-C_2<0\Leftrightarrow\frac{d}{d\,t}E(t_0)>0 \tag{7.5}$$

and therefore E is increasing at $t=t_0$. On the other hand, for t such that $0<t<t_0$, we can write

$$E(t_0)=E(t)+(t_0-t)E'(\tau)\quad\text{for some }\tau\ :\ t<\tau<t_0. \tag{7.6}$$

Now, we observe that when $t\nearrow t_0$, we have $\tau=t_0$ and, therefore, $E'(\tau)\simeq E'(t_0)$. By (7.5), $E'(t_0)>0$ and, as a consequence of (7.6),

$$E(t)<E(t_0)<E_*(t_0)\quad\text{for all }t\ :\ 0<t<t_0.$$

Moreover, $E(t)\nearrow E_*(t_0)$ as $t\to t_0$.

2. The proof of the second assertion follows by reverting the inequalities of the previous case. □

Remark 7.1. *Theorem 7.1 asserts that, if the kinetics energy is below, or above, the kinetics energy evaluated at the moment t_* when the asymptotically stable equilibrium is reached, E_*, then the kinetics energy will remain bellow of E_* for all previous times to t_*, or it will remain above of E_* for all after times to t_*, respectively.*

Next we consider the case of the ordinary differential inequality with a constant exponent of nonlinearity,

$$\frac{d}{dt}E(t)+C_1E(t)^{\frac{\beta}{2}}\leq C_2\quad\text{for a.a. }t\in[0,T], \tag{7.7}$$

which is obtained by the application of the hypotheses (3.2) and (7.1) to (3.9), and where C_2 is the same constant of (7.2). In this case, the counterparts of (7.3) and (7.4) are

$$\frac{d}{dt}E(t) + C_1 E(t)^{\frac{\beta}{2}} = C_2 \Leftrightarrow \frac{d}{dt}E(t) = C_2 - C_1 E(t)^{\frac{\beta}{2}} := \Lambda(t)\,, \tag{7.8}$$

$$\Lambda(t) = 0 \Leftrightarrow E(t) = \left(\frac{C_2}{C_1}\right)^{\frac{2}{\beta}} := E_* \Leftrightarrow \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} = 2M\left(\frac{C_2}{C_1}\right)^{\frac{2}{\beta}} := e_*\,, \tag{7.9}$$

where C_1 and C_2 are given by (7.7). Note that here the exponent of nonlinearity is constant.

Theorem 7.2. *Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.5) in the conditions of Theorem 3.1 and such that (7.1) is fulfilled with $\mathbf{F} \neq \mathbf{0}$. If exists a positive time t_1 such that*

$$\|\mathbf{u}(t_1)\|^2_{\mathbf{L}^2(\Omega)} > e_*,$$

where e_* is given by (7.9), then

$$\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq \left(\|\mathbf{u}(t_1)\|^2_{\mathbf{L}^2(\Omega)} - e_*\right) e^{-C(t-t_1)} + e_* \quad \textit{for all } t > t_1\,, \tag{7.10}$$

for some independent of t positive constant C.

Proof. First we observe that from the second assertion of Theorem 7.1, we can infer that if there exists a positive time t_1 such that $E(t_1) > E_*$, then $E(t) > E_*$ for all time $t > t_1$ and, consequently, $E(t) \searrow E_*$ as $t \to t_1$. In order to simplify the notations, let us set

$$E(t) = E, \quad \frac{d}{dt}E(t) = E' \quad \text{and} \quad f(E) := C_1 E^{\gamma}, \quad \text{with } \gamma := \frac{\beta}{2}. \tag{7.11}$$

First, we observe that with these notations we have from (7.8) and (7.9),

$$E' + C_1 E^{\gamma} = C_2 \quad \Longleftrightarrow \quad (E - E_*)' + C_1 \frac{E^{\gamma} - E_*^{\gamma}}{E - E_*}(E - E_*) = 0\,. \tag{7.12}$$

Using (7.9), we can prove that

$$C_1 \frac{E^{\gamma} - E_*^{\gamma}}{E - E_*} \equiv \frac{f(E) - f(E_*)}{E - E_*} > f'(E_*) \equiv \gamma C_1^{\frac{1}{\gamma}} C_2^{\frac{\gamma-1}{\gamma}} \quad \text{iff} \quad \gamma > 1\,. \tag{7.13}$$

According to the expression of γ (see (7.11)), we can see that (7.13) holds if and only if $\beta > 2$. Then, from (7.12) and (7.13), we derive the following linear differential inequality

$$(E - E_*)' + C(E - E_*) < 0\,, \quad C = \gamma\, C_1^{\frac{1}{\gamma}} C_2^{\frac{\gamma-1}{\gamma}} \equiv \frac{\beta}{2} C_1^{\frac{2}{\beta}} C_2^{\frac{\beta-2}{\beta}}\,. \tag{7.14}$$

Integrating (7.14) between t_1 and $t > t_1$, we obtain (7.10), with the constant C given in (7.14).

Now we consider the case $1 < \beta < 2$. In addition to the notations introduced at (7.11), let us set $E(t_1) = E_1$. Using the reasoning developed at the very beginning of the proof, we then prove that

$$C_1 \frac{E^{\gamma} - E_*^{\gamma}}{E - E_*} > C_1 \frac{E_1^{\gamma} - E_*^{\gamma}}{E_1 - E_*} \quad \text{iff} \quad 0 < \gamma < 1\,, \tag{7.15}$$

and according to (7.11), we see that this holds if and only if $1 < \beta < 2$. Then from (7.12) and (7.15), we derive the following linear differential inequality

$$(E - E_*)' + C(E - E_*) < 0\,, \quad C = C_1 \frac{E_1^\gamma - E_*^\gamma}{E_1 - E_*} \equiv C_1 \frac{E_1^{\frac{\beta}{2}} - E_*^{\frac{\beta}{2}}}{E_1 - E_*}\,. \tag{7.16}$$

Integrating (7.16) between t_1 and $t > t_1$ lead us to (7.10), with the constant C given in (7.16). □

Remark 7.2. *Theorem 7.2 expresses the fact that if the kinetics energy is greater than the kinetics energy evaluated at the moment t_1 when the asymptotically stable equilibrium is reached, E_*, then for all $t > t_1$ the kinetics energy remains bellow of a small perturbation of E_*.*

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