THE TRANSIENT NAVIER-STOKES PROBLEM WITH ANISOTROPIC DIFFUSION

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Abstract. In this work, we consider the problem for the incompressible Navier-Stokes equations with a general diffusion which can be fully anisotropic. The existence of weak solutions is proved for the associated initial problem supplemented with no-slip boundary conditions. We prove also the properties of extinction in a finite time, exponential time decay and power time decay. Perturbations of the asymptotically stable equilibrium are established as well. By exploiting several examples, we show that the results established here improve its isotropic versions in almost all directions or for particular choices of all the diffusion coefficients.

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1. Introduction

In this work is considered the initial-boundary value problem posed by the generalized Navier-Stokes equations with anisotropic diffusion. The problem is considered in a general cylinder $Q_T := \Omega \times [0, T]$ bounded by $\Gamma_T := \partial\Omega \times [0, T]$, where Ω is a bounded subdomain of $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$, and $T > 0$. The governing equations are:

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad Q_T; \tag{1.1}$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f} - \nabla p + \sum_{i=1}^{N} D_i \left(|D_i \mathbf{u}|^{q_i-2} D_i \mathbf{u}\right) \quad \text{in} \quad Q_T; \tag{1.2}$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{in} \quad \Omega \quad \text{for} \quad t = 0; \tag{1.3}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_T. \tag{1.4}$$

Here, the vector function $\mathbf{u} = (u_1, \cdots, u_N)$ and the scalar function p are the unknowns of the problem. The vector function $\mathbf{f} = (f_1, \cdots, f_N)$ is a given problem data, $D_i\mathbf{u} = (\partial_i u_1, \ldots, \partial_i u_N)$ and $\partial_i u_j = \frac{\partial u_j}{\partial x_i}$. The exponents q_i are assumed to be constant with possible distinct values but such that $1 < q_i < \infty$ for any $i = 1, \ldots, N$.

The isotropic version of the problem (1.1)-(1.4) has been studied in the context of Mathematical Fluid Mechanics during the last 50 years and the simplest case corresponds to the following (isotropic) diffusion term (see *e.g.* [14, 15, 16, 23, 22, 8]):

$$\mathcal{A}(\mathbf{u}) = \mathbf{div}\left(|\mathbf{D}(\mathbf{u})|^{q-2}\mathbf{D}(\mathbf{u})\right), \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)_{i,\, j=1,\ldots,N}, \quad 1 < q < \infty. \tag{1.5}$$

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In this scope, $\mathbf{u}$ is the velocity field, p stands for the pressure divided by the constant density, $\mathbf{f}$ is the external forces field and q is the power-law index that characterizes the flow. Using the same notation of (1.5), we can write the anisotropic diffusion term present in (1.2) as

$$\mathbf{A}(\mathbf{u})=\sum_{i=1}^{N}\mathcal{A}_i(\mathbf{u}),\quad \text{where}\quad \mathcal{A}_i(\mathbf{u})=D_i\left(|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right). \tag{1.6}$$

Letting $\{\mathbf{e}_1,\ldots,\mathbf{e}_N\}$ be the canonical basis of $\mathbb{R}^N$, we can write the anisotropic diffusion in the following meaningful formulation

$$\mathcal{A}_i(\mathbf{u})=\mathbf{div}\left(|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\otimes\mathbf{e}_i\right)\Rightarrow\mathbf{A}(\mathbf{u})=\mathbf{div}\left(\sum_{i=1}^{N}|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\otimes\mathbf{e}_i\right).$$

In a celebrating work dating back to 1967, Ladyzhenskaya [13] started the study of the existence and uniqueness of weak solutions to an isotropic version of the problem (1.1)-(1.4) for $N=3$. A little bit later, Lions [15, Section 2.5] have improved and extended, to a general dimension $N\geq 2$, the existence and uniqueness results [13]. Under the assumptions that $\mathbf{u}_0\in\mathbf{H}$ and $\mathbf{f}\in\mathrm{L}^{q'}(0,T;\mathbf{V}_q')$, [15, Théorème 2.5.1] asserts the existence of weak solutions $\mathbf{u}\in\mathrm{L}^q(0,T;\mathbf{V}_q)\cap\mathrm{L}^\infty(0,T;\mathbf{H})$, where, given q such that $1<q<\infty$, we set

$$\mathcal{V}:=\{\mathbf{v}\in\mathbf{C}_0^\infty(\Omega):\operatorname{div}\mathbf{v}=0\},$$
$$\mathbf{H}:=\text{closure of }\mathcal{V}\text{ in the norm }\|\cdot\|_{\mathbf{L}^2(\Omega)},$$
$$\mathbf{V}_q:=\text{closure of }\mathcal{V}\text{ in the norm }\|\cdot\|_{\mathbf{W}^{1,q}(\Omega)}.$$

Both proofs in [13] and [15] use Galerkin approximations and compactness arguments together with the theory of monotone operators. The improvement of [15, Théorème 2.5.1] relies in the fact that the continuous imbedding

$$\mathrm{L}^q(0,T;\mathbf{V}_q)\cap\mathrm{L}^\infty(0,T;\mathbf{H})\hookrightarrow\mathrm{L}^{q\frac{N+2}{N}}(Q_T) \tag{1.7}$$

implies the boundedness of $\mathbf{u}\otimes\mathbf{u}:\mathbf{D}(\mathbf{v})$ in $\mathbf{L}^1(Q_T)$ for all $\mathbf{u},\ \mathbf{v}\in\mathrm{L}^q(0,T;\mathbf{V}_q)\cap\mathrm{L}^\infty(0,T;\mathbf{H})$, provided that

$$q\geq\frac{3N+2}{N+2}.$$

A demand in the search of results of existence for lower values of q was subsequently carried out by several authors. However, the improvements on this direction were obtained for a notion of weak solution with test functions much more regular (see *e.g.* [8, 22, 23] and the references cited therein). In this article we shall be confined to weak solutions in the spirit of the works developed in [13, 15].

Differential equations with anisotropic diffusion have been studied intensively as witness the large number of works published in recent years (see *e.g.* [2, 5, 19, 21] and the references cited therein). In the literature of classical Navier-Stokes equations, anisotropic diffusion is used in the context of geophysical flows with the meaning that the vertical viscosity is distinct from the horizontal one (see *e.g.* [6, 7]). The resolution of the Navier-Stokes with small initial data in suitable anisotropic spaces is also considered in some literature (see [12] and some of the several works where it is cited). There are also some works in which are studied some anisotropic regularity criteria for the Navier-Stokes equations that extend the well-known Serrin condition (see [17]). There is also in the literature other types of Navier-Stokes equations where anisotropy is considered in other terms (see *e.g.* [3]). In the present work, although we can find some features of our problem on the applications, our motivation is purely mathematical. In particular, we shall see that, by considering an anisotropic diffusion, we can considerably decrease, in several directions, the lower bound for the existence result. We will see also that for the phenomenons of fast or slow diffusion, related with the properties of extinction in finite time or (infinity) decay in time, respectively, it will be important, in most situations, the behavior in only two distinct directions. The steady version of the problem (1.1)-(1.4) was analyzed by the authors in the work [4].

The article is organized as follows. Section 2 is dedicated to introduce the main concepts of the anisotropic function spaces we are going to work with, as well to define the notion of weak solution to our problem we shall consider. The existence of weak solutions to the problem (1.1)-(1.5) is established in Section 3. Still in this section, we will provide a list of examples, in the case of $N = 3$ and sometimes $N = 4$, for several situations where the anisotropic existence result improve the isotropic one. An energy inequality fundamental for the properties that we will establish in the sequel will be proved in Section 4. Properties of extinction in a finite time are studied in Section 5. The large time behavior of the solutions is analyzed in Sections 6–7. Several examples showing the improvements of the anisotropic results established in Sections 5–7, with respect to its isotropic versions, will be provided throughout these sections. Finally, in Section 8, are studied some perturbations of the asymptotically stable equilibrium.

The notation used throughout this article and the main notions of the considered (isotropic) function spaces are largely standard in the literature of Partial Differential Equations and in Mathematical Fluid Mechanics as well. We address the reader to the monographs [14, 15, 10, 16] for any question related to that matter.

2. Weak formulation

Due to the presence of possibly different exponents q_i for distinct directions $\mathbf{e}_i$, we need to consider the solutions to the problem (1.1)-(1.4) in some anisotropic Sobolev space. We define the vector $\mathbf{q}$ in $\mathbb{R}^N$, whose components are the exponents of the anisotropic diffusion term considered in (1.2), by

$$\mathbf{q} := (q_1, \cdots, q_N), \quad 1 < q_i < \infty \quad \forall\, i \in \{1, \ldots, N\},$$

and let us set

$$\beta := \max_{i=1,\ldots,N} q_i \quad \text{and} \quad \alpha := \min_{i=1,\ldots,N} q_i\,.$$

To avoid any confusion that $\mathbf{q}$ is in fact multi-component, in the rest of our work we will emphasize this meaning by writing an arrow over $\mathbf{q}$: $\overrightarrow{\mathbf{q}}$. For simplicity, we assume throughout the text that the components of $\overrightarrow{\mathbf{q}} = (q_1, q_2, \ldots, q_N)$ satisfy to

$$\alpha = q_1 \leq q_2 \leq \cdots \leq q_N = \beta\,.$$

We define the following anisotropic Lebesgue and Sobolev spaces

$$\mathbf{L}^{\overrightarrow{\mathbf{q}}}(\Omega) = \{\mathbf{v} : \mathbf{v} \in \mathbf{L}^{q_i}(\Omega) \quad \forall\, i = 1, \ldots, N\}\,,$$
$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) = \left\{\mathbf{v} \in \mathbf{W}^{1,1}(\Omega) : \quad D_i\mathbf{v} \in \mathbf{L}^{q_i}(\Omega) \quad \forall\, i = 1, \ldots, N\right\}\,,$$

which are Banach spaces for the norms

$$\|\mathbf{v}\|_{\mathbf{L}^{\overrightarrow{\mathbf{q}}}(\Omega)} = \sum_{i=1}^{N} \|\mathbf{v}\|_{\mathbf{L}^{q_i}(\Omega)}\,, \quad \|\mathbf{v}\|_{\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega)} = \|\mathbf{v}\|_{\mathbf{L}^{1}(\Omega)} + \sum_{i=1}^{N} \|D_i\mathbf{v}\|_{\mathbf{L}^{q_i}(\Omega)}\,.$$

An equivalent approach to introduce $\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega)$ is to firstly consider the unidirectional Sobolev spaces

$$\mathbf{W}_i^{1,q_i}(\Omega) = \left\{\mathbf{v} \in \mathbf{W}^{1,1}(\Omega) : \quad D_i\mathbf{v} \in \mathbf{L}^{q_i}(\Omega)\right\}\,, \quad i = 1, \ldots, N\,,$$

which are Banach spaces for the norm

$$\|\mathbf{v}\|_{\mathbf{W}_i^{1,q_i}(\Omega)} = \|\mathbf{v}\|_{\mathbf{L}^{1}(\Omega)} + \|D_i\mathbf{v}\|_{\mathbf{L}^{q_i}(\Omega)}\,, \quad i = 1, \ldots, N\,,$$

and then to consider

$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) = \bigcap_{i=1}^{N} \mathbf{W}_i^{1,q_i}(\Omega)\,.$$

An important limitation of the anisotropic Sobolev space $\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega)$, is that, for bounded domains Ω, the validity of Sobolev imbeddings is restricted to rectangular domains (see *e.g.* [11]). In fact, for rectangular domains Ω, the following imbedding is continuous (*cf.* [18, Theorem 1])

$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow \mathbf{L}^{s}(\Omega) \quad \text{for any } s : \begin{cases} 1 \leq s \leq \overline{q}^{\,*}\,, & \sum_{j=1}^{N} \frac{1}{q_j} > 1 \\ 1 \leq s < \infty\,, & \sum_{j=1}^{N} \frac{1}{q_j} \leq 1\,, \end{cases} \tag{2.1}$$

where $\overline{q}^*$ denotes the Sobolev conjugate of $\overline{q}$, the harmonic mean of $q_1,\dots,q_N$:

$$\overline{q}^* := \frac{N\overline{q}}{N-\overline{q}}, \quad \overline{q} := \frac{N}{\sum_{j=1}^N \frac{1}{q_j}}, \quad \sum_{j=1}^N \frac{1}{q_j} > 1 \quad \Rightarrow \quad \overline{q}^* = \frac{N}{\sum_{j=1}^N \frac{1}{q_j} - 1}. \tag{2.2}$$

Moreover, the imbedding (2.1) is compact (*cf.* [18, Theorem 2]), and we denote this fact by writing

$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^s(\Omega) \quad \text{for any } q: 1 \le s < \overline{q}^*. \tag{2.3}$$

In some situations it is possible to remove the restrictions on the shape's domain and to enlarge the interval of s for the validity of (2.1) and (2.3). Let us see this fact by defining

$$\mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) := \text{closure of } \mathbf{C}_0^\infty(\Omega) \text{ in the anisotropic norm of } \mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega).$$

In this case, we have (*cf.* [9, Theorem 1]), in the interesting case of $\sum_{j=1}^N \frac{1}{q_j} > 1$, that

$$\mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow \mathbf{L}^s(\Omega) \quad \text{for any } s: 1 \le s \le q_a^*, \tag{2.4}$$

$$\mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^s(\Omega), \quad \text{for any } q: 1 \le s < q_a^*, \tag{2.5}$$

where q_a^* is the critical exponent defined by

$$q_a^* := \max\left\{\overline{q}^*, \beta\right\}. \tag{2.6}$$

Remark 2.1. Note that, for $N=2$, $\overline{q}^* > \beta$ and therefore $q_a^* = \overline{q}^*$. But, if $N>2$, it may well happen that $\beta > \overline{q}^*$. In fact, for $N>2$,

$$\beta > \overline{q}^* \Leftrightarrow \overline{q} < \frac{\beta N}{\beta + N} \Leftrightarrow \sum_{j=1,\ q_j \neq \beta}^N \frac{1}{q_j} > 1 + \frac{N-1}{\beta}.$$

This means that, in typical situations when $N>2$, $q_a^* = \beta$ if, by one hand, all the components q_i of $\overrightarrow{\mathbf{q}}$ but β are not too far apart and must belong to the lowest regions of the interval $(1,\infty)$. On the other hand, β must be too far apart from the other components (see details in [4, Remark 2.1]).

As a particular case of (2.4), it can be derived the following inequality

$$\|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} \le C \left(\prod_{i=1}^N \|D_i \mathbf{u}\|_{\mathbf{L}^{q_i}(\Omega)} \right)^{\frac{1}{N}} \qquad \forall\ \mathbf{u} \in \mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) \tag{2.7}$$

for $s \le q_a^*$ if $\sum_{i=1}^N \frac{1}{q_i} > 1$, or $s \in [1,\infty)$ otherwise (see [20, Theorem 1.2]).

Let us now define the anisotropic analogue of $\mathbf{V}_q$ by

$$\mathbf{V}_{\overrightarrow{\mathbf{q}}} := \text{closure of } \mathcal{V} \text{ in the anisotropic norm of } \mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega). \tag{2.8}$$

The natural parabolic anisotropic space considered here is

$$\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}}) := \left\{ \mathbf{v} : [0,T] \to \mathbf{V}_{\overrightarrow{\mathbf{q}}} \quad : \quad \mathbf{v},\ |D_i \mathbf{v}|^{q_i} \in \mathbf{L}^1(Q_T)\ \forall\ i=1,\dots,N \right\}, \tag{2.9}$$

whose norm is defined by

$$\|\mathbf{v}\|_{L^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})} = \|\mathbf{v}\|_{\mathbf{L}^1(Q_T)} + \sum_{i=1}^N \|D_i \mathbf{v}\|_{\mathbf{L}^{q_i}(Q_T)}. \tag{2.10}$$

A different approach to introduce this space, is to firstly define the unidirectional analogue of $\mathbf{V}_q$

$$\mathbf{V}_{q_i} := \text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{W}_i^{1,q_i}(\Omega)$$

and to consider the unidirectional Bochner space $\mathrm{L}^{q_i}(0,T;\mathbf{V}_{q_i})$, which is a Banach space for the norm

$$\|\mathbf{v}\|_{L^{q_i}(0,T;\mathbf{V}_{q_i})} = \|\mathbf{v}\|_{\mathbf{L}^1(Q_T)} + \|D_i \mathbf{v}\|_{\mathbf{L}^{q_i}(Q_T)}.$$

Then we define

$$\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})=\bigcap_{i=1}^{N}\mathrm{L}^{q_i}(0,T;\mathbf{V}_{q_i})\,.$$

Note that, for a bounded domain Ω and for a finite T, the following continuous imbeddings hold

$$\mathrm{L}^{\beta}(0,T;\mathbf{V}_{\beta})\hookrightarrow\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})\hookrightarrow\mathrm{L}^{\alpha}(0,T;\mathbf{V}_{\alpha})\,.$$

Therefore, as a closed subspace of $\mathrm{L}^{\alpha}(0,T;\mathbf{V}_{\alpha})$, the anisotropic parabolic space $\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$ is separable and reflexive. By $\mathrm{L}^{q_i'}(0,T;\mathbf{V}'_{q_i})$ and $\mathrm{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}'_{\overrightarrow{\mathbf{q}}})$, we denote the dual spaces of $\mathrm{L}^{q_i}(0,T;\mathbf{V}_{q_i})$ and $\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$, respectively, and where $\mathbf{V}'_{q_i}$ and $\mathbf{V}'_{\overrightarrow{\mathbf{q}}}$ stay for the dual spaces of $\mathbf{V}_{q_i}$ and $\mathbf{V}_{\overrightarrow{\mathbf{q}}}$.

We are now in conditions to define the notion of weak solution we shall consider. But first, we observe that, according to what is customary in Mathematical Fluid Mechanics, the determination of the pressure p is not a problem. In fact, after we determine $\mathbf{u}$, we can recover p by applying de Rham's theorem (see *e.g.* [14, 15, 10, 16]).

Definition 2.1. Let $N\geq 2$ and $\overrightarrow{\mathbf{q}}=(q_1,\dots,q_N)$, with $1<q_i<\infty$ for any $i=1,\dots,N$. Assume that $\mathbf{f}\in\mathbf{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}'_{\overrightarrow{\mathbf{q}}})$ and $q_a^*\geq 2$. A vector field $\mathbf{u}$ is a weak solution to the problem (1.1)-(1.4), if:

(1) $\mathbf{u}\in\mathrm{L}^{\infty}(0,T;\mathbf{H})\cap\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$;
(2) $\mathbf{u}(0)=\mathbf{u}_0$;
(3) For every $\mathbf{v}\in\mathbf{V}_{\overrightarrow{\mathbf{q}}}\cap\mathbf{L}^{\theta}(\Omega)$ and for a.a. $t\in[0,T]$

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega}\mathbf{u}(t)\cdot\mathbf{v}\,d\mathbf{x}+\sum_{i=1}^{N}\int_{\Omega}|D_i\mathbf{u}(t)|^{q_i-2}D_i\mathbf{u}(t)\cdot D_i\mathbf{v}\,d\mathbf{x}+\int_{\Omega}[(\mathbf{u}(t)\cdot\nabla)\,\mathbf{u}(t)]\cdot\mathbf{v}\,d\mathbf{x}=\\&\int_{\Omega}\mathbf{f}(t)\cdot\mathbf{v}\,d\mathbf{x},\end{aligned}\tag{2.11}$$

where θ is defined by

$$\frac{1}{q_a^*}+\frac{1}{\alpha}+\frac{1}{\theta}=1\,.\tag{2.12}$$

Remark 2.2. As usual, the condition $\mathbf{u}(0)=\mathbf{u}_0$ is interpreted in the following sense:

$$\lim_{t\to 0^+}\int_{\Omega}\mathbf{u}(t)\cdot\mathbf{v}\,d\mathbf{x}=\int_{\Omega}\mathbf{u}_0\cdot\mathbf{v}\,d\mathbf{x}\quad\forall\,\mathbf{v}\in\mathbf{V}_{\overrightarrow{\mathbf{q}}}\cap\mathbf{L}^{\theta}(\Omega)\,.$$

Remark 2.3. Note that $\mathbf{v}\in\mathbf{L}^{\theta}(\Omega)$ is necessary to control the boundedness of the convective integral term when $\mathbf{u}$ merely belongs to $\mathrm{L}^{\infty}(0,T;\mathbf{H})\cap\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$. But, if $\theta\leq q_a^*$ we only need to require that $\mathbf{v}\in\mathbf{V}_{\overrightarrow{\mathbf{q}}}$, since, in this case, the anisotropic Sobolev imbedding $\mathbf{V}_{\overrightarrow{\mathbf{q}}}\hookrightarrow\mathbf{L}^{\theta}(\Omega)$ holds. On the other hand, the assumption $q_a^*\geq 2$ is needed to control the first integral when $\mathbf{u}$ and $\mathbf{v}$ solely belong to the mentioned spaces.

3. Existence of weak solutions

The main result of this section is written in the following theorem. As it was already mentioned in the previous section and in what regards to the existence, in this work we shall only be concerned with the existence of the unknown field $\mathbf{u}$.

Theorem 3.1. Let Ω be a bounded domain in $\mathbb{R}^N$, $N\geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume that

$$\mathbf{u}_0\in\mathbf{H}\,,\tag{3.1}$$

$$\mathbf{f}\in\mathbf{L}^{\overrightarrow{\mathbf{q}}'}(Q_T),\quad\text{where }\overrightarrow{\mathbf{q}}'=(q_1',\dots,q_N').\tag{3.2}$$

If $q_a^* \geq q_*$, where

$$q_* := \begin{cases} \dfrac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)}\,, & 2<\alpha<3 \\ \dfrac{2\alpha}{\alpha-1}\,, & \alpha \geq 3\,, \end{cases} \tag{3.3}$$

then there exists, at least, a weak solution to the problem (1.1)-(1.4) in the sense of Definition 2.1.

Remark 3.1. If $q_i = q$ for all $i \in \{1,\ldots,N\}$, then, from (3.3) together with (2.2) and (2.6), we recover the well known restrictions on q of the isotropic problem (see *e.g.* [15, pp. 209-217]):

$$q_i = q \;\forall\, i \;\wedge\; q_a^* \geq \frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)} \Leftrightarrow q \geq \frac{3N+2}{N+2}\,;$$
$$q_i = q \;\forall\, i \;\wedge\; q_a^* \geq \frac{2\alpha}{\alpha-1} \Leftrightarrow q \geq \frac{3N}{N+2}\,.$$

But in this case, and contrary to the anisotropic situation, the intersection of the two restrictions obviously gives $q \geq \frac{3N+2}{N+2}$.

Proof. The proof of Theorem 3.1 will be split into several subsections.

3.1. Step 1: Existence of approximative solutions.

For the smallest integer $s > 1 + \frac{N}{2}$, we define

$$\mathbf{V}^s := \text{closure of } \mathcal{V} \text{ in } \mathbf{W}^{s,2}(\Omega)\,. \tag{3.4}$$

Let $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ be a set of non-trivial solutions $\mathbf{v}_j$ of the following spectral problem associated to the eigenvalues $\lambda_j > 0$:

$$\sum_{|\gamma|=s} \int_\Omega \mathrm{D}^\gamma \mathbf{v}_j \cdot \mathrm{D}^\gamma \varphi\, d\mathbf{x} = \lambda_j \int_\Omega \mathbf{v}_j \cdot \varphi\, d\mathbf{x} \quad \forall\, \varphi \in \mathbf{V}^s\,.$$

The family $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ is orthogonal in $\mathbf{V}^s$ and can be chosen as being orthonormal in $\mathbf{H}$ (see *e.g.* [16, p. 290]). Given $m \in \mathbb{N}$, let us also consider the correspondingly m-dimensional space, say $\mathbf{V}^m$, spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$. For each $m \in \mathbb{N}$, we search for an approximative solution $\mathbf{u}^m(t)$ of (2.11) in the form

$$\mathbf{u}^m(t) = \sum_{k=1}^m c_k^m(t)\mathbf{v}_k\,, \tag{3.5}$$

where $\mathbf{v}_k \in \mathbf{V}^m$. This function is found by solving the following system of m nonlinear ordinary differential equations, with respect to the m unknowns $c_1^m, \ldots, c_m^m$, obtained from (2.11):

$$\begin{aligned} &\int_\Omega \frac{\partial\, \mathbf{u}^m(t)}{\partial\, t} \cdot \mathbf{v}_k\, d\mathbf{x} + \sum_{i=1}^N \int_\Omega |D_i \mathbf{u}^m(t)|^{q_i-2} D_i \mathbf{u}^m(t) \cdot D_i \mathbf{v}_k\, d\mathbf{x} + \int_\Omega \left[(\mathbf{u}^m(t)\cdot\nabla)\,\mathbf{u}^m(t)\right]\cdot \mathbf{v}_k\, d\mathbf{x} \\ &= \int_\Omega \mathbf{f}(t)\cdot \mathbf{v}_k\, d\,\mathbf{x}, \qquad \text{for } k=1,\ldots,m, \end{aligned} \tag{3.6}$$

and with

$$\mathbf{u}^m(0) = \mathbf{u}_0^m \qquad \text{and} \qquad \mathbf{u}_0^m \to \mathbf{u}_0\,, \quad \text{as } m\to\infty, \text{ in } \mathbf{H}. \tag{3.7}$$

From the Cauchy-Kowalevskaya theorem there exist $t^m \in (0,T)$ and $\mathbf{c}^m(t) \equiv (c_1^m(t), \ldots, c_N^m(t))$ such that $\mathbf{c}^m(t)$ is a solution to the system (3.6)-(3.7) in the interval $[0,t^m]$. To show that this solution holds for all the interval $[0,T]$, we shall establish an *a priori* estimate. To do it so, we multiply (3.6) by c_k^m, we add up from $k=1$ until $k=m$ and we integrate the resulting equation between 0 and t, with $t \in (0,t^m)$. After all, we obtain

$$\frac{1}{2}\|\mathbf{u}^m(t)\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^N \|D_i\mathbf{u}^m(t)\|^{q_i}_{\mathbf{L}^{q_i}(Q_t)} = \frac{1}{2}\|\mathbf{u}_0^m\|^2_{\mathbf{L}^2(\Omega)} + \int_{Q_t} \mathbf{f}\cdot\mathbf{u}^m(t)\, d\,\mathbf{x}dt\,, \tag{3.8}$$

where $Q_t := \Omega \times [0,t]$. Taking the essential supreme in $[0,T]$, using the assumptions (3.1)-(3.2) and (3.7), and applying Cauchy's inequality with a suitable ϵ, one obtains from (3.8)

$$\|\mathbf{u}^m\|_{\mathrm{L}^\infty(0,T;\mathbf{H})} + \sum_{i=1}^{N} \|D_i \mathbf{u}^m\|^{q_i}_{\mathbf{L}^{q_i}(Q_T)} \leq C, \tag{3.9}$$

where C is an independent of m positive constant. Thus, from the Theory of the ODEs, we can take $t^m = T$.

3.2. **Convergence of the approximative solutions.** Due to (3.9) and by means of separability and reflexivity, there exists a subsequence (still denoted by) $\mathbf{u}^m$ and $\mathbf{u} \in \mathrm{L}^\infty(0,T;\mathbf{H}) \cap \mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$ such that

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{weakly-}* \text{ in } \mathrm{L}^\infty(0,T;\mathbf{H}), \quad \text{as } m \to \infty, \tag{3.10}$$

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{weakly in } \mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}}), \quad \text{as } m \to \infty. \tag{3.11}$$

On the other hand, using the notations of (1.6), it can be proved that

$$\|\mathcal{A}_i(\mathbf{u}^m)\|_{\mathrm{L}^{q_i'}(0,T;\mathbf{V}'_{q_i})} \leq C, \tag{3.12}$$

$$\|\mathbf{A}(\mathbf{u}^m)\|_{\mathrm{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}'_{\overrightarrow{\mathbf{q}}})} \leq C\,, \tag{3.13}$$

for possibly distinct positive constants C. Then, owe to (3.12)-(3.13) and by means of reflexivity, there exist $\mathbf{S}_i \in \mathrm{L}^{q_i'}(0,T;\mathbf{V}'_{q_i})$, with $i = 1,\dots,N$, and $\mathbf{S} \in \mathrm{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}'_{\overrightarrow{\mathbf{q}}})$ such that

$$\mathcal{A}_i(\mathbf{u}^m) \longrightarrow \mathbf{S}_i \quad \text{weakly in } \mathrm{L}^{q_i'}(0,T;\mathbf{V}'_{q_i}), \quad \text{as } m \to \infty, \quad \text{for all } i = 1,\dots,N, \tag{3.14}$$

$$\mathbf{A}(\mathbf{u}^m) \longrightarrow \mathbf{S} \quad \text{weakly in } \mathrm{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}'_{\overrightarrow{\mathbf{q}}}), \quad \text{as } m \to \infty. \tag{3.15}$$

Obviously, $\mathbf{S} = \mathbf{S}_1 + \cdots + \mathbf{S}_N$.

Next, using the orthogonal projection operator $P^m : \mathbf{H} \to \mathbf{V}^m$ in the spirit of [15, p. 76] and owing to (3.6), (3.9) and (3.2), it can be proved that

$$\frac{\partial \mathbf{u}^m}{\partial t} \in \mathrm{L}^\infty(0,T;\mathbf{V}'_s) \cap \mathrm{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}'_s)\,,$$

where $\mathbf{V}'_s$ is the dual of the space $\mathbf{V}_s$ defined at (3.4) and $\mathrm{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}'_s)$ is the dual space of $\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_s)$. In particular, for finite T,

$$\mathrm{L}^\infty(0,T;\mathbf{V}'_s) \hookrightarrow \mathrm{L}^{\overrightarrow{\mathbf{q}}'}(0,T;\mathbf{V}'_s) \hookrightarrow \mathrm{L}^{\beta'}(0,T;\mathbf{V}'_s)$$

and therefore

$$\frac{\partial \mathbf{u}^m}{\partial t} \in \mathrm{L}^{\beta'}(0,T;\mathbf{V}'_s)\,. \tag{3.16}$$

By means of reflexivity, we have

$$\frac{\partial \mathbf{u}^m}{\partial t} \longrightarrow \frac{\partial \mathbf{u}}{\partial t} \quad \text{weakly in } \mathrm{L}^{\beta'}(0,T;\mathbf{V}'_s)\,. \tag{3.17}$$

On the other hand, considering a bounded domain Ω and also a finite T, there holds $\mathrm{L}^{\overrightarrow{\mathbf{q}}}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}}) \hookrightarrow \mathrm{L}^{\alpha}(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})$ and, due to (3.9),

$$\mathbf{u}^m \in \mathrm{L}^\alpha(0,T;\mathbf{V}_{\overrightarrow{\mathbf{q}}})\,. \tag{3.18}$$

Now, we observe that by the choice of s in (3.4), the continuous imbedding holds

$$\mathbf{H} \hookrightarrow \mathbf{V}'_s \tag{3.19}$$

and by the anisotropic compactness imbedding (2.5), there holds

$$\mathbf{V}_{\overrightarrow{\mathbf{q}}} \hookrightarrow\hookrightarrow \mathbf{H} \quad \text{for } 2 < q_a^*. \tag{3.20}$$

Then, due to (3.16)-(3.20), we can apply Aubin-Lions compactness Lemma (see [15, p. 58]) to deduce that

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{strongly in } \mathrm{L}^\alpha(0,T;\mathbf{H}), \quad \text{as } m \to \infty. \tag{3.21}$$

Using parabolic interpolation, we obtain from (3.9) and (3.21) that

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{strongly in } \mathrm{L}^s(0,T;\mathbf{H}),\ \text{as } m\to\infty, \text{ for any } s\geq 1 \tag{3.22}$$

3.3. **Passing to the limit.** Fixing k, we pass the equation (3.6) to the limit $m\to\infty$ by using (3.11), (3.14), (3.17) and (3.22), and yet observing the definition of the space $\mathbf{V}^s$ given at (3.4), we obtain

$$\begin{aligned}&\int_\Omega \frac{\partial\,\mathbf{u}(t)}{\partial\,t}\cdot\mathbf{v}_k\,d\mathbf{x}+\sum_{i=1}^N\int_\Omega \mathbf{S}_i\cdot\mathbf{v}_k\,d\mathbf{x}+\int_\Omega[(\mathbf{u}(t)\cdot\nabla)\,\mathbf{u}(t)]\cdot\mathbf{v}_k\,d\mathbf{x}\\&=\int_\Omega \mathbf{f}(t)\cdot\mathbf{v}_k\,d\mathbf{x}\qquad\forall\ \mathbf{v}_k\in\mathbf{V}^m\quad\text{for a.e. } t\in[0,T].\end{aligned}\tag{3.23}$$

Since $\mathbf{V}^s=\bigcup_{m=1}^\infty\mathbf{V}^m$, this equation holds for all $\mathbf{v}\in\mathbf{V}^s$. By a continuity argument, the equation (3.23) holds true for all $\mathbf{v}\in\mathbf{V}_{\vec{\mathbf{q}}}$ as long as the integrals there remain bounded for $\mathbf{u}\in\mathrm{L}^\infty(0,T;\mathbf{H})\cap\mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$. The only difficulty here is to show the boundedness of the convective integral term. For this, we use Hölder's inequality, to show that for a.e. $t\in[0,T]$

$$\int_\Omega[(\mathbf{u}(t)\cdot\nabla)\,\mathbf{u}(t)]\cdot\mathbf{v}\,d\mathbf{x}=-\int_\Omega\mathbf{u}(t)\otimes\mathbf{u}(t):\nabla\mathbf{v}\,d\mathbf{x}\leq\|\mathbf{u}(t)\|^2_{\mathbf{L}^{2\alpha'}(\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^\alpha(\Omega)}\,.$$

The boundedness follows by using the imbedding $\mathbf{V}_{\vec{\mathbf{q}}}\hookrightarrow\mathbf{V}_\alpha$ and the anisotropic Sobolev imbedding (2.1) which holds for

$$q_a^*\geq\frac{2\alpha}{\alpha-1}\,. \tag{3.24}$$

3.4. **Use of the monotonicity.** Since the diffusion term is the sum of N possible different diffusion terms which are strictly monotonous, the anisotropic diffusion term is strictly monotonous:

$$\sum_{i=1}^N\langle|D_i\xi|^{q_i-2}D_i\xi-|D_i\eta|^{q_i-2}D_i\eta,D_i\xi-D_i\eta\rangle_{\mathbf{L}^{q_i'}(Q_T)\times\mathbf{L}^{q_i}(Q_T)}>0$$

for every $\xi,\ \eta\in\mathbf{L}^{\vec{\mathbf{q}}}(0,T;\in\mathbf{V}_{\vec{\mathbf{q}}})$, with $\xi\neq\eta$. Consequently, appealing to this strict monotonicity property, we can construct a suitable test function $\mathbf{v}\in\mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$ to use the Minty trick in the spirit of Lions [15, pp. 212-215]) and to identify

$$\mathbf{S}_i=D_i\left(|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right)$$

for all $i=1,\ldots,N$. For the application of the this reasoning, we need to show in what conditions we have

$$\int_\Omega[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\,d\mathbf{x}=-\int_\Omega\mathbf{u}\otimes\mathbf{u}:\nabla\,\mathbf{v}\,d\mathbf{x}\ \in\mathrm{L}^1(0,T) \tag{3.25}$$

for $\mathbf{u}\in\mathrm{L}^\infty(0,T;\mathbf{H})\cap\mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$ and $\mathbf{v}\in\mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$. To show (3.25), we first observe that by the imbedding $\mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})\hookrightarrow\mathrm{L}^\alpha(0,T;\mathbf{V}_{\vec{\mathbf{q}}})$ and by the anisotropic imbedding (2.1), we can use parabolic interpolation to prove that

$$\mathrm{L}^\infty(0,T;\mathbf{H})\cap\mathrm{L}^{\vec{\mathbf{q}}}(0,T;\mathbf{V}_{\vec{\mathbf{q}}})\hookrightarrow\mathbf{L}^\rho(Q_T)\quad\text{for}\quad\rho:=2+\alpha-\frac{2\alpha}{q_a^*}\,. \tag{3.26}$$

Observe that (3.26) is the anisotropic analogue of the well known imbedding (1.7) used to prove the existence results for the isotropic problem. Then, by Hölder's inequality, we have

$$\int_{Q_T}\mathbf{u}\otimes\mathbf{u}:\nabla\,\mathbf{v}\,d\mathbf{x}dt\leq\|\mathbf{u}\|^2_{\mathbf{L}^\rho(Q_T)}\|\nabla\,\mathbf{v}\|_{\mathbf{L}^\alpha(Q_T)}\quad\text{for}\quad\frac{2}{\rho}+\frac{1}{\alpha}\leq 1\,. \tag{3.27}$$

Due to (3.26) and (3.27), (3.25) holds provided

$$q_a^*\geq\frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)}\quad\text{and } \alpha>2\,. \tag{3.28}$$

Finally, as a consequence of (3.20), (3.24) and (3.28), the result of this theorem holds provided the following conditions are mutually satisfied:

$$\textbf{Case 1:} \qquad q_a^* \geq \frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)} \quad \text{and} \quad \alpha > 2\,, \tag{3.29}$$

$$\textbf{Case 2:} \qquad q_a^* \geq \frac{2\alpha}{\alpha-1} \quad \text{and} \quad \alpha > 1\,. \tag{3.30}$$

But, since $\frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)} \geq \frac{2\alpha}{\alpha-1} \Leftrightarrow \alpha \leq 3$, (3.29)-(3.30) is a consequence of the assumption (3.3). □

Remark 3.2. Observe that the restriction (3.24) implies that $q_a^* > 2$ for all $\alpha > 1$. In consequence, this last condition, essential for the compactness imbedding (3.20), is trivially fulfilled.

Remark 3.3. We observe that in the case of $q_i = q$ for all $i \in \{1, \dots, N\}$, we obtain, in view of (2.2), the well-known bounds of the isotropic problem (see the proofs of [15, Théorème 2.5.1] and of [16, Lemma 5.2.44]):

$$\begin{aligned} &\text{Condition (3.20):} \quad q_a^* > 2 \Leftrightarrow q > \frac{2N}{N+2}\,, \\ &\text{Condition (3.24):} \quad q_a^* \geq \frac{2\alpha}{\alpha-1} \Leftrightarrow q \geq \frac{3N}{N+2}\,, \\ &\text{Condition (3.28):} \quad q_a^* \geq \frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)} \Leftrightarrow q \geq \frac{3N+2}{N+2}\,. \end{aligned}$$

In the rest of this section, we will be concerned in analyzing in what extent Theorem 3.1 improves the analogue existence result of the correspondingly isotropic problem. According to [15, Théorème 2.5.1], the existence of a weak solution to the isotropic version of the problem (1.1)-(1.4) can be proved as long as

$$q \geq \frac{3N+2}{N+2} = \begin{cases} 2 & \text{if } N = 2\,, \\ \frac{11}{5} & \text{if } N = 3\,. \end{cases} \tag{3.31}$$

And to the best of our knowledge, there is no better result for the existence of a weak solution in the sense of Definition 2.1. Now we are going to compare the lower bounds for the anisotropic exponents given by (3.29)-(3.30) with the lower bound for the isotropic exponent given by (3.31), in the particular cases of $N = 2$ or $N = 3$. We first observe that Case 2 is useful for values of $\alpha \geq 3$. But, in this case, the isotropic restriction (3.31) always provides us with lower values, since $\frac{3N+2}{N+2} < 3$ for all $N \in \mathbb{N}$.

Case-1: We assume that (3.29) is verified for $2 < \alpha < 3$.

Case-1.1: If $q_a^* = \overline{q}^*$, then $N = 2$ or $\beta \leq \frac{N-1}{\sum_{i\neq N}\frac{1}{q_i}-1}$ (see Remark 2.1 and (2.2)).

Case-1.1.1: If $N = 2$, from (3.29), we have for $\alpha > 2$ that

$$q_a^* \geq \frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)} \Leftrightarrow \beta \geq \frac{\alpha(\alpha-1)}{\alpha(2\alpha-3)-1}\,.$$

But since $\frac{3}{4} < \frac{\alpha(\alpha-1)}{\alpha(2\alpha-3)-1} < 2$ for $2 < \alpha < 3$, the restriction on β reduces to $\beta \geq \alpha > 2$. Therefore α and β can be infinitely close to 2 with the only restriction that $\beta \geq \alpha > 2$. Note that if $\beta = \alpha$, we can use the isotropic result to show that it holds for $\beta = \alpha \geq 2$.

Case-1.1.2: If $\beta \leq \frac{N-1}{\sum_{i\neq N}\frac{1}{q_i}-1}$ (and $N \neq 2$), then conjugating this with (3.29), we have

$$\frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)} \leq \beta \leq \frac{N-1}{\sum_{i\neq N}\frac{1}{q_i}-1} \quad \text{if} \quad \sum_{i\neq N}\frac{1}{q_i} - 1\,.$$

Observe that β takes solely finite values if and only if $N - 1 > \alpha$, which in turn implies that it must be $N > 3$ for $\alpha > 2$. On the other hand, we have $3 \leq \frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)} < \infty$ for $2 < \alpha < 3$, which in turn implies that $\beta \geq 3$.

Example 3.1. If $N=4$ and $q_i=\alpha$ for all $i\neq 4$, then $\frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)}\leq\beta\leq\frac{3\alpha}{3-\alpha}$ and the range of β is non-empty only if $\frac{11}{5}\leq\alpha<3$. For instance, if $\alpha=q_2=q_3=\frac{23}{10}$, then $6.040\simeq\frac{598}{99}\leq\beta\leq\frac{69}{7}\simeq 9.857$.

In this case, the previous example shows us that we have lower values for the first $N-1$ components of $\overrightarrow{\mathbf{q}}$ than in the isotropic case that gave us $q=\frac{7}{3}>\frac{23}{10}$. In particular, if $\alpha=q_2=q_3=\frac{11}{5}$, then $\beta=\frac{33}{4}=8.25$.

Case-1.2: If $q_a^*=\beta$, then $N\neq 2$ and $\beta\geq\frac{N-1}{\sum_{i\neq N}\frac{1}{q_i}-1}$. By Remark 2.1, we can see that $q_a^*=\beta$ if $N>2$ and all the components q_i of $\overrightarrow{\mathbf{q}}$ but β belong to the lowest regions of the interval $(1,\infty)$. This consideration and the restriction $\alpha>2$ at (3.29) considerably reduces the possibilities of this case. In fact, we can see that

$$q_i\geq\alpha\quad\forall\ i\quad\Rightarrow\quad\beta\geq\frac{\alpha(N-1)}{N-1-\alpha}.$$

Thus, β does not blow up to infinity only if $N-1>\alpha$, and, since we are considering $\alpha>2$, then it must be $N>3$.

Example 3.2. If $N=4$ and $q_i=\alpha$ for all $i\neq 4$, then $\beta\geq\max\left\{\frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)},\frac{3\alpha}{3-\alpha}\right\}$. Observing that $2<\alpha<3$, we have $\max\left\{\frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)},\frac{3\alpha}{3-\alpha}\right\}=\frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)}$ if $2<\alpha<\frac{11}{5}$ and $\max\left\{\frac{2\alpha(\alpha-1)}{(\alpha+1)(\alpha-2)},\frac{3\alpha}{3-\alpha}\right\}=\frac{3\alpha}{3-\alpha}$ if $\frac{11}{5}\leq\alpha<3$. In particular, for $\alpha=q_2=q_3=\frac{21}{10}<\frac{11}{5}$, then $\beta\geq\frac{462}{31}\simeq 14.903$. But if $\alpha=q_2=q_3=\frac{23}{10}>\frac{11}{5}$, then $\beta\geq\frac{69}{7}\simeq 9.857$.

In this case, we have even lower values for the first $N-1$ components of $\overrightarrow{\mathbf{q}}$ than in the previous case, in particular when $2<\alpha=q_2=q_3<\frac{11}{5}$.

4. Energy inequality

Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.4) and let us denote the kinetics energy associated with this problem by

$$e(t):=\frac{1}{2}\int_\Omega|\mathbf{u}(t)|^2\,d\mathbf{x}\,. \tag{4.1}$$

In this section, we are interested in solutions to the problem (1.1)-(1.4) with a finite energy e. Therefore, we may assume that

$$|e(t)|<M\quad\text{for all } t\in[0,T],\quad\text{for some constant } M>0\,. \tag{4.2}$$

Normalizing (4.2), we obtain

$$E(t):=\frac{e(t)}{M}<1\quad\text{for all } t\in[0,T]\,. \tag{4.3}$$

Theorem 4.1. Let Ω be a bounded domain in $\mathbb{R}^N$, $N\geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume that

$$\mathbf{f}=-\mathbf{div}\,\mathbf{F},\quad\mathbf{F}\in\mathbf{L}^{\overrightarrow{\mathbf{q}}'}(Q_T), \tag{4.4}$$

$$\mathbf{u}_0\in\mathbf{H}. \tag{4.5}$$

If $\mathbf{u}$ is a weak solution to the problem (1.1)-(1.4) in the sense of Definition 2.1, then the following energy inequality

$$\frac{1}{2}\|\mathbf{u}(t)\|^2_{\mathbf{H}}\Big|_{t=t_0}^{t=t_1}+\sum_{i=1}^N\int_{t_0}^{t_1}\|D_i\mathbf{u}(t)\|^{q_i}_{\mathbf{L}^{q_i}(\Omega)}dt\leq\int_{Q_{t_1}\setminus Q_{t_0}}\mathbf{F}:\nabla\mathbf{u}\,d\mathbf{x}dt\,, \tag{4.6}$$

holds for all $t_0,\ t_1\in[0,T]$, with $t_0<t_1$.

Note that the assumption (4.4) is made only to simplify the exposition.

Proof. We proceed similarly as in the proof of Theorem 3.1, but by taking $\mathbf{f} = -\mathbf{div}\,\mathbf{F}$ in the equation (3.6). We multiply this equation by c_k^m, we add up from $k = 1$ until $k = m$ and we integrate the resulting equation between t_0 and t_1, with $t_0 < t_1$ and $t_0,\ t_1 \in [0, T)$. After all, we obtain the following energy inequality

$$\frac{1}{2}\|\mathbf{u}^m(t)\|_{\mathbf{H}}^2\Big|_{t=t_0}^{t=t_1} + \sum_{i=1}^{N}\int_{t_0}^{t_1} \|D_i\mathbf{u}^m(t)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i} dt = \int_{Q_{t_1}\setminus Q_{t_0}} \mathbf{F} : \nabla\mathbf{u}^m \, d\mathbf{x}dt\,, \tag{4.7}$$

Then by (3.7) and (3.10)-(3.11), and by a well known classical property of weak limits, we obtain (4.6). □

Theorem 4.2. Assume that all the conditions of Theorem 4.1 are fulfilled. If

$$q_a^* \geq 2\,, \tag{4.8}$$

then there exist two independent of t constants C_1 and C_2 such that

$$\frac{d}{dt}E(t) + C_1 E(t)^{\mu} \leq C_2 \sum_{i=1}^{N}\int_{\Omega} |\mathbf{F}(t)|^{q_i'}\, d\mathbf{x} \quad \text{for all } t \in [0, T], \tag{4.9}$$

where

$$\mu := \left(\frac{2}{N}\sum_{i=1}^{N}\frac{1}{q_i}\right)^{-1}\,. \tag{4.10}$$

Remark 4.1. Recalling the observations of Remark 2.1, it should be noted that (4.8) is equivalent to

$$\begin{cases} 1 < \displaystyle\sum_{i=1}^{N}\frac{1}{q_i} \leq 1 + \frac{N}{2} & \text{if} \quad \overline{q}^* \geq \beta \Leftrightarrow \displaystyle\sum_{i=1}^{N}\frac{1}{q_i} \leq 1 + \frac{N}{\beta} \quad \text{or} \quad N = 2\,, \\ \text{or} \quad \beta \geq 2 & \text{if} \quad \beta > \overline{q}^* \Leftrightarrow \displaystyle\sum_{i=1}^{N}\frac{1}{q_i} > 1 + \frac{N}{\beta} \quad \text{and} \quad N \neq 2\,. \end{cases}$$

Remark 4.2. In the isotropic case, *i.e.* when $q_i = q$ for all $i \in \{1, \ldots, N\}$, relation (4.10) resumes to $\mu = \frac{q}{2}$.

Proof. (Theorem 4.2) Due to (4.6), we can write for every $t,\ t + \Delta t \in [0, T]$, with $\triangle t > 0$, the following inequality

$$\frac{1}{2}\|\mathbf{u}(s)\|_{\mathbf{H}}^2\Big|_{s=t}^{s=t+\Delta t} + \sum_{i=1}^{N}\int_{t}^{t+\Delta t} \|D_i\mathbf{u}(s)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i} ds \leq \int_{Q_{t+\Delta t}\setminus Q_t} \mathbf{F} : \nabla\mathbf{u}\, d\mathbf{x}ds\,. \tag{4.11}$$

Using the notations (4.1)-(4.2), we obtain from (4.11) that

$$\frac{M}{|\Delta t|}\ E(s)|_{s=t}^{s=t+\Delta t} \leq -\frac{1}{|\Delta t|}\sum_{i=1}^{N}\int_{t}^{t+\Delta t} \|D_i\mathbf{u}(s)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i} ds + \frac{1}{|\Delta t|}\int_{Q_{t+\Delta t}\setminus Q_t} \mathbf{F} : \nabla\mathbf{u}\, d\mathbf{x}ds\,. \tag{4.12}$$

Since $\mathbf{u} \in \mathrm{L}^{\overrightarrow{\mathbf{q}}}(0, T; \mathbf{V}_{\overrightarrow{\mathbf{q}}})$ and $\mathbf{F} \in \mathbf{L}^{\overrightarrow{\mathbf{q}}'}(Q_T)$, then, for any $i = 1, \ldots, N$,

$$\|D_i\mathbf{u}\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i} \qquad \int_{\Omega} |\mathbf{F} : \nabla\mathbf{u}| d\mathbf{x} \quad \text{are in} \quad \mathrm{L}^1[0, T].$$

In consequence, every term on the right-hand side of (4.12) has a limit, for all $t \in [0, T]$, as $\Delta t \to 0$. This in turn yields the existence of a limit of the left-hand side of (4.12), for all $t \in [0, T]$, as $\Delta t \to 0$. Whence we can write for all $t \in [0, T]$

$$\frac{d}{d\,t}E(t) + \frac{1}{M}\sum_{i=1}^{N}\|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i} \leq \frac{1}{M}\int_{\Omega} |\mathbf{F}(t) : \nabla\mathbf{u}(t)| d\mathbf{x}\,. \tag{4.13}$$

Then making use of Young's inequality, we obtain for all $t \in [0, T]$

$$\frac{d}{dt}E(t) + C_1\sum_{i=1}^{N}\|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}^{q_i} \leq C_2\sum_{i=1}^{N}\int_{\Omega} |\mathbf{F}(t)|^{q_i'}\, d\mathbf{x}\,. \tag{4.14}$$

Observing that the assumption (4.8) holds, we can use the anisotropic inequality (2.7) together with the algebraic inequality $A^a \times B^b \leq (A+B)^{a+b}$, valid for any real numbers $A,\ B,\ a,\ b$, with $A,\ B \geq 0$, to prove that

$$E(t) = \frac{1}{2M}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{C}{2M}\left(\prod_{i=1}^{N}\|D_i\mathbf{u}(t)\|_{\mathbf{L}^{q_i}(\Omega)}\right)^{\frac{2}{N}} \leq \frac{C}{2M}\left(\sum_{i=1}^{N}\|D_i\mathbf{u}(t)\|^{q_i}_{\mathbf{L}^{q_i}(\Omega)}\right)^{\frac{2}{N}\sum_{i=1}^{N}\frac{1}{q_i}}. \tag{4.15}$$

The inequality (4.9) follows by plugging (4.15), with the notation given by (4.10), into the differential inequality (4.14). □

Remark 4.3. If instead of (4.4), we assume that

$$\int_0^t \|\mathbf{f}(s)\|_{\mathbf{L}^2(\Omega)} ds < \infty \quad \forall\, t \in [0,T]$$

and the energy inequality (4.6) is replaced by

$$\frac{1}{2}\|\mathbf{u}(t)\|^2_{\mathbf{H}}\Big|_{t=t_0}^{t=t_1} + \sum_{i=1}^{N}\int_{t_0}^{t_1}\|D_i\mathbf{u}(s)\|^{q_i}_{\mathbf{L}^{q_i}(\Omega)}ds \leq \int_{Q_{t_1}\setminus Q_{t_0}} \mathbf{f}\cdot\mathbf{u}\, d\mathbf{x}ds\,, \tag{4.16}$$

then we obtain the following differential inequality

$$\frac{d}{dt}E(t) + C_1 E(t)^{\mu} \leq C_2\|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)} \quad \text{for all } t \in [0,T]\,. \tag{4.17}$$

In fact, by using Hölder's inequality together with (4.3) and (4.15), we obtain from (4.16) that (4.17) holds with C_1 deduced from (4.15) and $C_2 = \sqrt{2M}$.

5. Extinction in a finite time

In this section, we assume that the time interval $[0,T]$ is so large that the properties established here occur inside our considered domain. For our purposes it is enough that the vanishing moments belong to the interior of $[0,T]$.

Theorem 5.1. Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.4) in the sense of Definition 2.1 and such that (3.1), (4.2) and (4.8) hold. In addition, assume that

$$0 < \mu < 1\,. \tag{5.1}$$

(1) If $\mathbf{F} = \mathbf{0}$ a.e. in Q_T, then there exists a (positive) time

$$t^* := \frac{E(0)^{1-\mu}}{C_1(1-\mu)} = \frac{\|\mathbf{u}_0\|^{2(1-\mu)}_{\mathbf{H}}}{C_1(1-\mu)(2M)^{1-\mu}}, \tag{5.2}$$

such that $\mathbf{u} = \mathbf{0}$ a.e. in Ω and for all $t \geq t^*$.

(2) Let $\mathbf{F} \neq \mathbf{0}$ and assume that exist positive constants $\delta_1, \dots, \delta_N$, $\gamma_1, \dots, \gamma_N$ and γ, and positive times $t^1_{\mathbf{F}}, \dots, t^N_{\mathbf{F}}$ and $t_{\mathbf{F}}$ such that, for any $i \in \{1,\dots,N\}$ and for all $t \in [0,T]$,

$$\int_\Omega |\mathbf{F}(t)|^{q_i'}\, d\mathbf{x} \leq \delta_i\left(1 - \frac{t}{t^i_{\mathbf{F}}}\right)_+^{\gamma_i} \quad \text{and} \quad \frac{\gamma_i}{\gamma} \geq \frac{t^i_{\mathbf{F}}}{t_{\mathbf{F}}}\,, \tag{5.3}$$

where

$$\gamma \geq \gamma^* := \frac{\mu}{1-\mu} \qquad \text{and} \qquad t_{\mathbf{F}} > t^* \quad (\text{defined in (5.2)})\,. \tag{5.4}$$

Then there exists a constant $\delta_0 > 0$ such that $\mathbf{u} = \mathbf{0}$ a.e. in Ω and for all $t \geq t_{\mathbf{F}}$, provided that

$$0 < \max_{i\in\{1,\dots,N\}} \delta_i \leq \delta_0\,. \tag{5.5}$$

The notation u_+ means the positive part of u, *i.e.* $u_+ = \max\{u, 0\}$.

Remark 5.1. Observe that typical situations in which $(5.3)_2$ occurs are when $\gamma_i \geq \gamma$ and $t^i_{\mathbf{F}} \leq t_{\mathbf{F}}$ for all $i \in \{1, \ldots, N\}$. In particular, when $\gamma_i = \gamma$ for all $i \in \{1, \ldots, N\}$, $(5.3)_2$ reduces to assume that $t^i_{\mathbf{F}} \leq t_{\mathbf{F}}$ for all $i \in \{1, \ldots, N\}$. If $t^i_{\mathbf{F}} = t_{\mathbf{F}}$ for all $i \in \{1, \ldots, N\}$, then $(5.3)_2$ is equivalent to assume that $\gamma_i \geq \gamma$ for all $i \in \{1, \ldots, N\}$. However, it should be stressed that $(5.3)_2$ is a much more general condition, because it may well happen that $\gamma_i < \gamma$ and $t^i_{\mathbf{F}} < t_{\mathbf{F}}$ for all $i \in \{1, \ldots, N\}$, or that $\gamma_i > \gamma$ and $t^i_{\mathbf{F}} > t_{\mathbf{F}}$ for all $i \in \{1, \ldots, N\}$.

Remark 5.2. Observe that if $q_i = q$ for all $i \in \{1, \ldots, N\}$, then we fall in the isotropic case and, in this case, (5.1) reduces to $q < 2$. This means that the structure of the stress tensor alone is able to stop shear-thinning fluids in a finite time. It is well known that this phenomenon does not happen if $q \geq 2$ (see *e.g.* [1]).

Proof. (Theorem 5.1) 1. If $\mathbf{F} = \mathbf{0}$ a.e. in Q_T, then (4.9) is reduced to

$$\frac{d}{dt}E(t) + C_1 E(t)^{\mu} \leq 0 \quad \text{for all } t \in [0, T]. \tag{5.6}$$

Knowing that, from (5.1), $\mu < 1$, an explicit integration of (5.6) between $t = 0$ and t leads us to

$$E(t) \leq \left[E(0)^{1-\mu} - C_1(1-\mu)t\right]^{\frac{1}{1-\mu}} \quad \text{for all } t \in [0, T]. \tag{5.7}$$

The right-hand side of (5.7) vanishes for $t \geq t^*$ and the first assertion is then proved.

2. If $\mathbf{F}$ satisfies to $(5.3)_1$, then from (4.9), we obtain the following non-homogeneous ordinary differential inequality

$$\frac{d}{d\,t}E(t) + C_1 E(t)^{\mu} \leq C_2 \sum_{i=1}^{N} \delta_i \left(1 - \frac{t}{t^i_{\mathbf{F}}}\right)_+^{\gamma_i}. \tag{5.8}$$

Using Lagrange Theorem, we obtain

$$\frac{t^i_{\mathbf{F}}}{t_{\mathbf{F}}} = \frac{\frac{1}{t_{\mathbf{F}}}}{\frac{1}{t^i_{\mathbf{F}}}} = \frac{\dfrac{\ln(t_{\mathbf{F}}) - \ln(t_{\mathbf{F}} - t)}{t} + o(t)}{\dfrac{\ln(t^i_{\mathbf{F}}) - \ln(t^i_{\mathbf{F}} - t)}{t} + o(t)}, \quad \text{where} \quad \lim_{t\to 0^+} \frac{o(t)}{t} = 0$$

$$= \frac{\ln\left[\left(\frac{t_{\mathbf{F}} - t}{t_{\mathbf{F}}}\right)^{-\frac{1}{t}} \omega(t)\right]}{\ln\left[\left(\frac{t^i_{\mathbf{F}} - t}{t^i_{\mathbf{F}}}\right)^{-\frac{1}{t}} \omega(t)\right]}, \quad \text{where} \quad \omega(t) = e^{o(t)}.$$

Then by assumption $(5.3)_2$, we have for each $i \in \{1, \ldots, N\}$ that

$$\frac{\gamma_i}{\gamma} \geq \frac{t^i_{\mathbf{F}}}{t_{\mathbf{F}}} \Leftrightarrow \ln\left[\left(1 - \frac{t}{t^i_{\mathbf{F}}}\right)^{-\frac{\gamma_i}{t}} \omega(t)^{\gamma_i}\right] \geq \ln\left[\left(1 - \frac{t}{t_{\mathbf{F}}}\right)^{-\frac{\gamma}{t}} \omega(t)^{\gamma}\right]$$

$$\Leftrightarrow \left(1 - \frac{t}{t^i_{\mathbf{F}}}\right)_+^{-\frac{\gamma_i}{t}} \omega(t)^{\gamma_i} \geq \left(1 - \frac{t}{t_{\mathbf{F}}}\right)_+^{-\frac{\gamma}{t}} \omega(t)^{\gamma}$$

$$\Leftrightarrow \left(1 - \frac{t}{t^i_{\mathbf{F}}}\right)_+^{\gamma_i} \leq e^{-(\gamma-\gamma_i)\frac{o(t)}{t}} \left(1 - \frac{t}{t_{\mathbf{F}}}\right)_+^{\gamma}$$

Since $o(t)$ is very small when compared with t, as $t \to 0^+$, then there exists a positive constant C such that

$$e^{-(\gamma-\gamma_i)\frac{o(t)}{t}} \leq C \quad \forall\, t : 0 < t \leq \max\left\{t_{\mathbf{F}}, t^i_{\mathbf{F}}\right\}.$$

Then from the above calculus, we have that

$$\left(1 - \frac{t}{t^i_{\mathbf{F}}}\right)_+^{\gamma_i} \leq C\left(1 - \frac{t}{t_{\mathbf{F}}}\right)_+^{\gamma} \quad \forall\, i \in \{1, \ldots, N\}. \tag{5.9}$$

Now plugging (5.9) into (5.8), we get

$$\frac{d}{d\,t}E(t) + C_1 E(t)^{\mu} \leq C_2 \overline{\delta}\left(1 - \frac{t}{t_{\mathbf{F}}}\right)_+^{\gamma}, \quad \text{where} \quad \overline{\delta} := \max_{i\in\{1,\ldots,N\}} \delta_i. \tag{5.10}$$

Next we shall split the proof into the two main different cases:

2.1 The case $\gamma = \frac{\mu}{1-\mu}$. In this case, and in order to linearize the equation (5.10), let us consider a new function defined by

$$H(t) := E(t) - G(t) , \qquad G(t) = E(0)\left(1 - \frac{t}{t_{\mathbf{F}}}\right)_+^{\frac{1}{1-\mu}} . \tag{5.11}$$

It is easy to check that the function $G(t)$ is a non-negative solution to the non-homogenous problem

$$\begin{cases} \dfrac{d}{d\,t}G(t) + C_1 G(t)^\mu = C_2\overline{\delta}\left(1 - \dfrac{t}{t_{\mathbf{F}}}\right)_+^{\frac{\mu}{1-\mu}} \\ G(0) = E(0) , \end{cases} \tag{5.12}$$

because, due to $(5.4)_2$, the following condition is fulfilled

$$0 < -E(0)\frac{1}{(1-\mu)t_{\mathbf{F}}} + C_1 E(0)^\mu = C_2\overline{\delta} . \tag{5.13}$$

On the other hand, it can be easily verified that the function $H(t)$ satisfies to the following linear differential inequality

$$\frac{d}{d\,t}H(t) + C_1(t)H(t) \leq 0 , \qquad C_1(t) := C_1\mu\int_0^1 \frac{d\lambda}{\left[\lambda E(t) + (1-\lambda)G(t)\right]^{1-\mu}} . \tag{5.14}$$

Solving the differential inequality (5.14), we obtain

$$H(t) \leq H(0)e^{-\int_0^t C_1(\tau)d\tau} \quad \text{for all } t > 0.$$

Then, by the choice of the initial condition of the problem (5.12), we get

$$E(t) \leq C_2\overline{\delta}\left(1 - \frac{t}{t_{\mathbf{F}}}\right)_+^{\frac{\mu}{1-\mu}} ,$$

and consequently $\mathbf{u} = \mathbf{0}$ a.e. in Ω and for all $t \geq t_{\mathbf{F}}$, provided that

$$0 < \overline{\delta} < \delta_0 := C_1 E(0)^\mu = C_1\frac{\|\mathbf{u}_0\|_{\mathbf{H}}^{2\mu}}{(2M)^\mu} .$$

2.2 The case $\gamma > \frac{\mu}{1-\mu}$. In this case, we consider a new function $H(t) := E(t) - G(t)$, where now $G(t)$ is a non-negative solution to the homogenous problem

$$\begin{cases} \dfrac{d}{d\,t}G(t) + C_1 G(t)^\mu = 0 \\ G(0) > E(0) . \end{cases} \tag{5.15}$$

For instance, we may take $G(0) = 1$, because, according to (4.3), $E(0) < 1$. An explicit formula for $G(t)$ is given by

$$G(t) = \max\left\{0, \left[G(0)^{1-\mu} - C_1(1-\mu)t\right]^{\frac{1}{1-\mu}}\right\} . \tag{5.16}$$

Recalling the definition of $C_1(t)$ given in $(5.14)_2$, it can also be easily verified that the new function $H(t)$ satisfies now to the following linear differential inequality

$$\frac{d}{d\,t}H(t) + C_1(t)H(t) \leq F(t) , \qquad F(t) := C_2\overline{\delta}\left(1 - \frac{t}{t_{\mathbf{F}}}\right)_+^{\gamma} . \tag{5.17}$$

Solving the linear differential inequality (5.17), we obtain

$$H(t) \leq e^{-\int_0^t C_1(\tau)d\tau}\left[H(0) + \int_0^t F(s)e^{\int_0^s C_1(\tau)d\tau}ds\right] \quad \forall\, t > 0 .$$

In particular, for $t = t_{\mathbf{F}}$, we have

$$H(t_{\mathbf{F}}) \leq e^{-\int_0^{t_{\mathbf{F}}} C_1(\tau)\,d\tau} \left[H(0) + \int_0^{t_{\mathbf{F}}} F(s)\, e^{\int_0^s C_1(\tau)\,d\tau} ds \right]. \tag{5.18}$$

Now, we choose a suitable $t_{\mathbf{F}}$ such that in view of (5.16) we have

$$t_{\mathbf{F}} := \frac{G(0)^{1-\mu}}{C_1(1-\mu)} \Leftrightarrow G(t_{\mathbf{F}}) = 0 \Leftrightarrow H(t_{\mathbf{F}}) = E(t_{\mathbf{F}}). \tag{5.19}$$

Note that by the choice of the initial data $G(0)$, condition $(5.4)_2$ still holds. Using this information and observing that $C_1(t) \geq 0$ for all $t > 0$ implies $e^{-\int_0^{t_{\mathbf{F}}} C_1(\tau)\,d\tau} \leq 1$, the inequality (5.18) give rise to

$$E(t_{\mathbf{F}}) \leq E(0) - G(0) + \int_0^{t_{\mathbf{F}}} F(s)\, e^{\int_0^s C_1(\tau)\,d\tau} ds. \tag{5.20}$$

Next, using the change of variables $1 - \lambda = \theta$, we can prove that

$$\begin{aligned} \int_0^s C_1(\tau)\,d\tau &= C_1\mu \int_0^s \int_0^1 \frac{d\theta d\tau}{[(1-\theta)E(\tau) + \theta G(\tau)]^{1-\mu}} \\ &\leq C_1\mu \int_0^s \int_0^1 \frac{d\theta d\tau}{[\theta G(\tau)]^{1-\mu}} = C_1\mu \int_0^s \frac{d\tau}{G(\tau)^{1-\mu}} \int_0^1 \frac{d\theta}{\theta^{1-\mu}} \\ &= -\ln\left|\frac{G(s)}{G(0)}\right| = \ln\left[\left(1 - \frac{s}{t_{\mathbf{F}}}\right)_+^{-\frac{1}{1-\mu}}\right]. \end{aligned} \tag{5.21}$$

In the last two equalities we have used (5.15) to deduce that $\frac{1}{G(s)^{1-\mu}} = -\frac{G'(s)}{C_1 G(s)}$ and the expressions for $G(s)$ and $t_{\mathbf{F}}$ given in (5.16) and (5.19). Then, plugging (5.21) into (5.20), observing the notation $(5.17)_2$ and using the fact that $\gamma > \frac{\mu}{1-\mu}$, we obtain

$$E(t_{\mathbf{F}}) \leq E(0) - G(0) + C_2\,\overline{\delta} \int_0^{t_{\mathbf{F}}} \left(1 - \frac{s}{t_{\mathbf{F}}}\right)_+^{\gamma - \frac{1}{1-\mu}} ds = E(0) - G(0) + \frac{C_2\,\overline{\delta}\,t_{\mathbf{F}}}{\gamma - \frac{\mu}{1-\mu}}.$$

Since $E(t) \geq 0$ for all $t > 0$, we have $E(t_{\mathbf{F}}) = 0$ only if

$$E(0) - G(0) + \frac{C_2\,\overline{\delta}\,t_{\mathbf{F}}}{\gamma - \frac{\mu}{1-\mu}} \leq 0. \tag{5.22}$$

Observe that by the choice of the initial data $G(0)$ and since $\gamma > \frac{\mu}{1-\mu}$, this implies

$$0 < \overline{\delta} \leq \frac{\gamma - \frac{\mu}{1-\mu}}{C_2\,t_{\mathbf{F}}}\,(G(0) - E(0)) := \delta_0.$$

Considering now the problem for the ordinary differential inequality (5.10) on the interval $[t_{\mathbf{F}}, T]$, we have

$$\begin{cases} \dfrac{d}{dt}E(t) + C_1 E(t)^{\mu} \leq 0 \quad \text{for } t \geq t_{\mathbf{F}} \\ E(t_{\mathbf{F}}) = 0. \end{cases}$$

Solving this problem and observing again that $E(t) \geq 0$ for all $t > 0$, we have $E(t) = 0$ for all $t \geq t_{\mathbf{F}}$ and consequently $\mathbf{u} = \mathbf{0}$ for all $t \geq t_{\mathbf{F}}$ and a.e. in Ω. This concludes the proof of the second assertion. □

Remark 5.3. Notice that conditions (5.13) and (5.22) connect the three parameters which characterize the problem: the instant $t_{\mathbf{F}}$ of vanishing of the source, the source intensity $\overline{\delta}$ and the initial value $E(0)$. By this reason, given an arbitrary intensity $\overline{\delta}$ such that $0 < \overline{\delta} \leq \delta_0$ and δ_0 known, the effect of extinction in a finite time holds for a suitable choice of $t_{\mathbf{F}}$ and $E(0)$.

Remark 5.4. In the follow up of Remark 4.3, we can derive the same conclusions of Theorem 5.1 as long as we replace assumption (5.3) by its L^2 counterpart

$$\|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)} \leq \delta \left(1 - \frac{t}{t_{\mathbf{F}}}\right)_+^{\gamma}, \qquad \gamma \geq \frac{\mu}{1-\mu}.$$

In the rest of this section, we shall analyze the relation (5.1) responsible for the finite time extinction property. First we observe that, in the isotropic case, $q_i = q$ for all $i \in \{1, \dots, N\}$ and from (4.10), we know that (5.1) is equivalent to

$$q < 2,$$

the case of shear-thinning fluids for which it is well-known the extinction in a finite property time (see *e.g.* [1]). On the other hand, observe that (5.1) should be weighted with the condition (4.8), essential for the anisotropic continuous imbedding $\mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ (see 2.4). As it was already observed in Remark 4.1, condition (4.8) must be separated into the two possible cases: $q_a^* = \overline{q}^*$ or $q_a^* = \beta$.

The case of $q_a^* = \overline{q}^*$. In this case, Remark 4.1 shows us that conditions (4.8) and (5.1) imply that

$$\frac{N}{2} < \sum_{i=1}^{N} \frac{1}{q_i} \leq \min\left\{1 + \frac{N}{2}, 1 + \frac{N}{\beta}\right\} = \begin{cases} 1 + \dfrac{N}{2} & \text{if } \beta \leq 2 \quad \text{or} \quad N = 2 \\ 1 + \dfrac{N}{\beta} & \text{if } \beta > 2 \quad \text{and} \quad N \neq 2\,. \end{cases} \tag{5.23}$$

From (4.10) and due to our assumption that $\alpha = q_1 \leq q_2 \leq \cdots \leq q_N = \beta$, we know that (5.1) implies

$$\sum_{i=1}^{N} \frac{1}{q_i} > \frac{N}{2} \Rightarrow \frac{1}{\alpha} > \frac{N}{2} - \frac{N-1}{\alpha} \Leftrightarrow \alpha < 2\,. \tag{5.24}$$

If $\beta \leq 2$, then (4.8) implies that

$$\sum_{i=1}^{N} \frac{1}{q_i} \leq 1 + \frac{N}{2} \quad \Rightarrow \quad \frac{N}{\beta} \leq 1 + \frac{N}{2} \Leftrightarrow \beta \geq \frac{2N}{N+2}\,. \tag{5.25}$$

If $\beta > 2$, then (4.8) implies that

$$\sum_{i=1}^{N} \frac{1}{q_i} \leq 1 + \frac{N}{\beta} < 1 + \frac{N}{2} \quad \Rightarrow \quad \frac{N}{\beta} < 1 + \frac{N}{2} \Leftrightarrow \beta > \frac{2N}{N+2}\,. \tag{5.26}$$

The case of $q_a^* = \beta$. In this case, Remark 2.1 shows that definitely $N \neq 2$. On the other hand, by Remark 4.1, conditions (4.8) and (5.1) imply that

$$1 + \frac{N}{2} \geq \sum_{i=1}^{N} \frac{1}{q_i} > \max\left\{\frac{N}{2}, 1 + \frac{N}{\beta}\right\} = \begin{cases} \dfrac{N}{2} & \text{if } \beta \geq \frac{2N}{N-2} \\ 1 + \dfrac{N}{\beta} & \text{if } \beta < \frac{2N}{N-2}\,. \end{cases} \tag{5.27}$$

Observing that $\alpha = q_1 \leq q_2 \leq \cdots \leq q_N = \beta$, we obtain from (4.8) that

$$\sum_{i=1}^{N} \frac{1}{q_i} \leq 1 + \frac{N}{2} \quad \Rightarrow \quad \frac{N}{\beta} \leq 1 + \frac{N}{2} \Leftrightarrow \beta \geq \frac{2N}{N+2}\,. \tag{5.28}$$

If $\beta \geq \frac{2N}{N-2}$, then (5.1) implies that

$$\sum_{i=1}^{N} \frac{1}{q_i} > \frac{N}{2} \quad \Rightarrow \quad \frac{N}{\alpha} > \frac{N}{2} \Leftrightarrow \alpha < 2\,. \tag{5.29}$$

If $\beta < \frac{2N}{N-2}$, then (5.1) implies that

$$\sum_{i=1}^{N} \frac{1}{q_i} > 1 + \frac{N}{\beta} \quad \Rightarrow \quad \frac{N}{\alpha} > 1 + \frac{N}{\beta} \quad \Rightarrow \quad \frac{N}{\alpha} > \frac{N}{2} \Leftrightarrow \alpha < 2\,. \tag{5.30}$$

From (5.24)–(5.26) and (5.28)–(5.30), we can readily see that in both cases, either for $q_a^* = \overline{q}^*$ or for $q_a^* = \beta$, a necessary condition for (5.23) or for (5.27) is that

$$1 < \alpha < 2 \quad \text{and} \quad \beta \geq \frac{2N}{N+2}\,. \tag{5.31}$$

This means that a necessary condition for the extinction in a finite time property is that $q_i < 2$ for, at least, one $i \in \{1, \dots, N\}$ and that for, at least, a different one, say q_j, with $j \neq i$, there holds the continuous imbedding $\mathbf{W}_0^{1,q_j}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$, with no upper restriction on q_j.

Example 5.1. Let us consider the particular case of $\overrightarrow{\mathbf{q}} = (\alpha, 2, \dots, 2, \beta)$. We get from (5.23) or (5.27) that, at least,

$$1 < \frac{1}{\alpha} + \frac{1}{\beta} < 2\,.$$

Observe that in the case of $q_a^* = \overline{q}^*$ and $\beta \leq 2$ or in the case of $q_a^* = \beta$, we may have the equality on the right-hand side. In particular, we have

$$\begin{aligned} \alpha \to 1^+ \quad &\text{as} \quad \beta \to +\infty\,, \\ \alpha \to 2^- \quad &\text{as} \quad \beta \to 2^+\,. \end{aligned}$$

However, it should be mentioned that the requirement of condition (5.31) is not sufficient for the extinction in a finite time property as it is shown in the Example 6.1 at the next section.

6. Exponential time decay

In this section we will consider the case in which the range of μ is augmented in order to allow μ to eventually touch 1:

$$\mu \leq 1 \Leftrightarrow \sum_{i=1}^{N} \frac{1}{q_i} \leq \frac{N}{2}\,. \tag{6.1}$$

In Theorem 5.1 we have seen that if $\mu < 1$, the solutions to the problem (1.1)-(1.4) even extinct in a finite time. We shall see in the next theorem that when we allow μ to touch 1, we loose this property, but nevertheless we can prove a time decay property of exponential type, as long as $\mu \leq 1$. In this case, we assume that the time interval is so large that the properties established here occur inside our considered domain. For simplicity, we assume that we are working with the infinite time interval $[0, \infty)$.

Theorem 6.1. Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.4) in the sense of Definition 2.1 and such that (3.1), (4.2) and (4.8) hold. If there holds (6.1) and

$$\int_0^\infty \|\mathbf{F}(t)\|_{\mathbf{L}^{q_i'}(\Omega)}^{q_i'} d\,t < \infty \quad \forall\, i \in \{1 \dots, N\}\,, \tag{6.2}$$

then there exist positive constants C_1 and C_2 such that

$$\|\mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 \leq e^{-C_1\,t} \left(\|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^2 + C_2 \sum_{i=1}^{N} \int_0^t e^{C_1\,s} \|\mathbf{F}(s)\|_{\mathbf{L}^{q_i'}(\Omega)}^{q_i'} d\,s \right). \tag{6.3}$$

Proof. We start by observing that, from assumptions (4.2) and (6.1), we have

$$1 \leq E(t)^{\mu-1} \quad \forall\, t \geq 0\,. \tag{6.4}$$

Then using (6.4) in the ordinary differential inequality (4.9), we get the following linear differential inequality

$$\frac{d}{dt}E(t) + C_1 E(t) \leq \frac{d}{dt}E(t) + C_1 E(t)^{\mu-1}E(t) \leq C_2 \sum_{i=1}^{N} \int_\Omega |\mathbf{F}(t)|^{q_i'}\, d\mathbf{x} \quad \text{for all } t \in [0, T]. \tag{6.5}$$

Now by virtue of (3.1) and (4.2), we can apply Gronwall's lemma to (6.5) in order to get (6.3), where C_1 and C_2 are the same constants of (6.5) but multiplied by $2M$ (see (4.2)). □

The interesting case of this section is when

$$\mu = 1 \Leftrightarrow \sum_{i=1}^{N} \frac{1}{q_i} = \frac{N}{2}\,, \tag{6.6}$$

but not all the components of $\overrightarrow{\mathbf{q}}$ are equal (to 2).

Example 6.1. Let us consider again the particular case of $\overrightarrow{\mathbf{q}} = (\alpha, 2, \ldots, 2, \beta)$. We get from (6.6) that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1\,.$$

For instance for $\beta = 3$ it must be $\alpha = \frac{3}{2}$. Observe that, in this case, $\alpha < 2$ and $\beta > \frac{6}{5}$, which make the requirements of (5.31) to be fulfilled, when the dimension is $N = 3$, but the phenomenon of extinction in a finite time is not displayed.

7. Power time decay

In this section, we consider the case when

$$\mu > 1\,, \tag{7.1}$$

where μ is given by (4.10). Here we also assume that we are working with the infinite time interval $[0, \infty)$. In the next theorem is established that, under assumption (7.1), the solutions to the problem (1.1)-(1.4) have a time decay of power type.

Theorem 7.1. Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.4) in the sense of Definition 2.1 and such that (3.1), (4.2) and (4.8) hold. In addition, assume that (7.1) holds.

(1) If $\mathbf{F} = \mathbf{0}$ a.e. in Q_T, then there exists positive constant C such that

$$\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq C(1+t)^{-\frac{1}{\mu-1}} \quad \text{for all } t \geq 0. \tag{7.2}$$

(2) More generally, if $\mathbf{F} \neq \mathbf{0}$ but exist a positive constants $C^1_{\mathbf{F}}, \ldots, C^N_{\mathbf{F}}$ and $\gamma_1, \ldots, \gamma_N$ such that

$$\int_\Omega |\mathbf{F}(t)|^{q'_i}\, d\mathbf{x} \leq C^i_{\mathbf{F}}(1+t)^{-\gamma_i} \ \ \text{for all } t \in [0,T] \quad \text{and} \quad \gamma_i \geq \frac{\mu}{\mu-1}\ \forall\, i \in \{1, \ldots, N\}\,, \tag{7.3}$$

then there exists a positive constant C such that

$$\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq C(1+t)^{-\frac{\mu}{\mu-1}} \quad \text{for all } t \geq 0. \tag{7.4}$$

Proof. 1. If $\mathbf{F} = \mathbf{0}$ a.e. in Q_T, we use (4.9) from Theorem 4.2 and we proceed analogously as we did for the proof of Theorem 5.1 to obtain (5.6). Knowing that, from (7.1), $\mu > 1$, an explicit integration of (5.6) between $t = 0$ and $t \in (0, T]$ leads us to

$$E(t) \leq \left[E(0)^{1-\mu} + (\mu-1)C_1 t\right]^{-\frac{1}{\mu-1}} \quad \text{for all } t \in [0,T]. \tag{7.5}$$

Thus, (7.2) is an immediate consequence of (7.5), with

$$C := 2M \left[\min\left\{(\mu-1)C_1, \left(\frac{\|\mathbf{u}_0\|_{\mathbf{H}}}{\sqrt{2M}}\right)^{2(1-\mu)}\right\}\right]^{-\frac{1}{\mu-1}}.$$

2. Now, we assume that $\mathbf{F} \neq \mathbf{0}$ and satisfies to (7.3). Using the assumption $(7.3)_1$ into the ordinary differential inequality (4.9), we obtain

$$\frac{d}{dt}E(t) + C_1 E(t)^\mu \leq C_2 \sum_{i=1}^{N} C^i_{\mathbf{F}}(1+t)^{-\gamma_i} \quad \text{for all } t \in [0,T]. \tag{7.6}$$

Then, taking $C_{\mathbf{F}} = C_2 \max_{i\in\{1,\dots,N\}} C^i_{\mathbf{F}}$ and using $(7.3)_2$, we obtain from (7.6) that

$$\frac{d}{dt}E(t) + C_1 E(t)^{\mu} \le C_{\mathbf{F}}(1+t)^{-\frac{\mu}{\mu-1}} \quad \text{for all } t\in[0,T]. \tag{7.7}$$

Let us look now for solutions of the form $y(t) = C(1+t)^{\xi}$ to the ordinary differential equation

$$\frac{d}{dt}y(t) + C_1 y(t)^{\mu} = C_*(1+t)^{-\gamma_*}, \tag{7.8}$$

where γ_* and C_* are positive constants to be found. After some algebraic manipulations, we can readily see that

$$\gamma_* = \frac{\mu}{\mu-1}, \qquad \xi = \frac{1}{1-\mu},$$

and C is a solution to the equation

$$f(C) = 0, \quad f(C) := -\frac{1}{\mu-1}C + C_1 C^{\mu} - C_*\,.$$

Since $f(0) = -C_*$, $C_* > 0$ and $\lim_{C\to+\infty} f(C) = +\infty$, by Bolzano's theorem the equation $f(C) = 0$ has a root $\overline{C} > 0$. On the other hand, $\overline{C}$ can be estimated from below as follows

$$\overline{C} = \left[\frac{1}{C_1}\left(C_* + \frac{1}{\mu-1}\overline{C}\right)\right]^{\frac{1}{\mu}} > \left(\frac{C_*}{C_1}\right)^{\frac{1}{\mu}}. \tag{7.9}$$

Next, we introduce the new function

$$G(t) := E(t) - y(t) \equiv E(t) - C_*(1+t)^{-\frac{\mu}{\mu-1}},$$

which in turn satisfies the following linear differential inequality

$$\frac{d}{dt}G(t) + c_1(t)G(t) \le c_2(1+t)^{-\frac{\mu}{\mu-1}}, \tag{7.10}$$

where

$$c_1(t) := C_1\mu\int_0^1 \left[\lambda E(t) + (1-\lambda)y(t)\right]^{\mu-1}\,d\lambda \ge 0 \quad \text{and} \quad c_2 := C_{\mathbf{F}} - C_*\,. \tag{7.11}$$

Now, we choose the constant C_* stated in (7.8) in such a way that $C_* \ge C_{\mathbf{F}} \Leftrightarrow c_2 \le 0$. Using this information, the linear differential inequality (7.10) becomes homogeneous. Solving the resulting differential inequality, we obtain

$$G(t) \le G(0)e^{-\int_0^t c_1(\tau)\,d\tau} \le G(0) = E(0) - C_*\,. \tag{7.12}$$

Imposing also that $G(0) \le 0 \Leftrightarrow C_* \ge E(0) = \frac{\|\mathbf{u}_0\|^2_{\mathbf{H}}}{2M}$, it follows, from (7.12), that $G(t) \le 0$ for all $t\in(0,T]$. As a consequence, we can see that a choice of $C_* \ge \max\left\{C_{\mathbf{F}}, \frac{\|\mathbf{u}_0\|^2_{\mathbf{H}}}{2M}\right\}$ proves that

$$0 \le E(t) \le C_*(1+t)^{-\frac{\mu}{\mu-1}}.$$

Finally, using (7.9), the decay property (7.4) follows by taking $C = 2MC_1\overline{C}^{\,\mu}$. □

In the rest of this section, we analyze the relation (7.1) responsible for the large time behavior. First we observe that, in the isotropic case, $q_i = q$ for all $i\in\{1,\dots,N\}$ and from (4.10), we know that (7.1) is equivalent to

$$q > 2\,,$$

the case of shear-thickening fluids for which the best we can get is a sort of time decay of power type. As in the previous section, we observe that (7.1) should be weighted with the condition (4.8), essential for the anisotropic continuous imbedding $\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}_0(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ (see 2.4). As it was already observed in Remark 4.1, condition (4.8) must be separated into the two possible cases: $q^*_a = \overline{q}^{\,*}$ or $q^*_a = \beta$.

The case of $q_a^* = \overline{q}^*$. In this case, Remark 4.1 shows us that conditions (4.8) and (7.1) imply that

$$\sum_{i=1}^{N} \frac{1}{q_i} < \frac{N}{2}\,. \tag{7.13}$$

Observing that $\alpha = q_1 \leq q_2 \leq \cdots \leq q_N = \beta$, then we have

$$\sum_{i=1}^{N} \frac{1}{q_i} < \frac{N}{2} \Rightarrow \frac{N}{\beta} < \frac{N}{2} \Leftrightarrow \beta > \frac{2N}{N+2}\,. \tag{7.14}$$

The case of $q_a^* = \beta$. In this case, Remark 2.1 shows that $N \neq 2$. On the other hand, by Remark 4.1, conditions (4.8) and (7.1) imply that

$$1 + \frac{N}{\beta} < \sum_{i=1}^{N} \frac{1}{q_i} < \frac{N}{2}\,. \tag{7.15}$$

This relation makes sense only when $\beta > 2 + \frac{2}{N}$. In this case, (7.14) also holds and, since $\alpha = q_1 \leq q_2 \leq \cdots \leq q_N = \beta$, we have

$$\sum_{i=1}^{N} \frac{1}{q_i} > 1 + \frac{N}{\beta} \quad \Rightarrow \quad \frac{N}{\alpha} > 1 + \frac{N}{\beta} \quad \Rightarrow \quad N > \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^{-1}\,.$$

We can readily see that in both cases, either for $q_a^* = \overline{q}^*$ or for $q_a^* = \beta$, a necessary condition for (7.13) or (7.15) is that

$$\beta > \frac{2N}{N+2}\,.$$

This means that a necessary condition for the asymptotic behavior established in the previous section is that for, at least, one $i \in \{1, \ldots, N\}$ there holds the compact continuous imbedding $\mathbf{W}_0^{1,q_i}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$, with no upper restriction on q_i. In the second case, we have also a lower bound for the dimension N when it happens that $q_a^* = \beta$.

8. Perturbations of the asymptotically stable equilibrium

In this section we are interested in solutions $\mathbf{u}$ to the problem (1.1)-(1.4) in the sense of Definition 2.1, with $\mathbf{F} \neq \mathbf{0}$ but such that

$$\int_\Omega |\mathbf{F}(t)|^{q_i'}\, d\mathbf{x} \leq C_{\mathbf{F}^i} \quad \text{for all } t \in [0,T] \quad \forall\, i \in \{1, \ldots, N\}\,, \tag{8.1}$$

where $C_{\mathbf{F}^i}$, with $i \in \{1, \ldots, N\}$, are positive constants independent of t. Condition (8.1) is naturally fulfilled if $\mathbf{F}(t) \in \mathbf{L}^{\overrightarrow{\mathbf{q}}'}(\Omega)$ for all $t \in [0,T]$, which in turn is assured by hypothesis (4.4). From (4.9) and (8.1) it follows that

$$\frac{d}{dt}E(t) + C_1 E(t)^\mu \leq C_2 \quad \text{for all } t \in [0,T], \tag{8.2}$$

where C_2 groups all the positive constants resulting from (4.9) and (8.1). Let us set now

$$\frac{d}{d\,t}E(t) + C_1 E(t)^\mu = C_2 \Leftrightarrow \frac{d}{d\,t}E(t) = C_2 - C_1 E(t)^\mu := \Lambda(t)\,. \tag{8.3}$$

If $\Lambda(t) < 0$ or $\Lambda(t) > 0$ at some time t (possibly different), then $E(t)$ is decreasing or increasing, respectively, at that time. In consequence, the asymptotically stable equilibrium of (8.3) is reached when

$$\Lambda(t) = 0 \Leftrightarrow E(t) = \left(\frac{C_2}{C_1}\right)^{\frac{1}{\mu}} := E_*(t) \Leftrightarrow \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} = 2M\left(\frac{C_2}{C_1}\right)^{\frac{1}{\mu}} := e_*(t)\,, \tag{8.4}$$

where C_1 and C_2 are given by (8.2). The proof of the following result follows by standard arguments. We write it here for completeness.

Theorem 8.1. Let $\mathbf{u}$ be a weak solution to the problem (1.1)-(1.4) in the sense of Definition 2.1 and such that (4.8) hold and (8.1) is fulfilled with $\mathbf{F} \neq \mathbf{0}$.

(1) If exists a positive time $t_0 > 0$ such that $\|\mathbf{u}(t_0)\|^2_{\mathbf{L}^2(\Omega)} \leq e_*(t_0)$, then $\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} < e_*(t_0)$ for all t such that $0 < t < t_0$, and $\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \nearrow e_*(t_0)$ as $t \to t_0$.

(2) If exists a positive time $t^0 > 0$ such that $\|\mathbf{u}(t^0)\|^2_{\mathbf{L}^2(\Omega)} > e_*(t^0)$, then $\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} > e_*(t^0)$ for all t such that $t^0 < t \leq T$, and $\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \searrow e_*(t^0)$ as $t \to t^0$.

Proof. 1. Writing the first assumption in terms of the functions $E(t)$ and $E_*(t)$, we have

$$E(t_0) < E_*(t_0) = \left(\frac{C_2}{C_1}\right)^{\frac{1}{\mu}} \Leftrightarrow C_1 E(t_0)^\mu - C_2 < 0 \Leftrightarrow \frac{d}{dt}E(t_0) > 0 \tag{8.5}$$

and therefore E is increasing at $t = t_0$. On the other hand, for t such that $0 < t < t_0$, we can write

$$E(t_0) = E(t) + (t_0 - t)E'(\tau) \quad \text{for some } \tau \ : \ t < \tau < t_0. \tag{8.6}$$

Now, we observe that when $t \nearrow t_0$, we have $\tau = t_0$ and, therefore, $E'(\tau) \simeq E'(t_0)$. By (8.5), $E'(t_0) > 0$ and, as a consequence of (8.6),

$$E(t) < E(t_0) < E_*(t_0) \quad \text{for all } t \ : \ 0 < t < t_0.$$

Moreover, $E(t) \nearrow E_*(t_0)$ as $t \to t_0$.

2. The proof of the second assertion follows by reverting the inequalities of the previous case. □

Remark 8.1. Theorem 8.1 asserts that, if the kinetics energy is below, or above, the kinetics energy evaluated at the moment t_* when the asymptotically stable equilibrium is reached, E_*, then the kinetics energy will remain bellow of E_* for all previous times to t_*, or it will remain above of E_* for all after times to t_*, respectively.

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