The Cauchy Problem for a Coupled Nonlinear Schrödinger Equations with Linear Damping: Local and Global Existence and Blowup of Solutions

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Abstract. We study, by applying and extending the methods developed in [8], [2], [6], [10], [14], [4] and [5], the Cauchy problem for a damped coupled system of nonlinear Schrödinger equations and we obtain new results on the local and global existence of H^1-strong solutions and on their possible blowup in the supercritical case and, in a special situation, in the critical or supercritical cases.

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1. Introduction

The study of blowup of solutions for a damped nonlinear Schrödinger equation has been developed in the papers by M. Tsutsumi [14] and by M. Ohta and G. Todorova [10]. More recently the problem was addressed by [4], for the case of inhomogeneous damping. Stimulated by the relevance for physical applications there was also significant interest in exploring the blowup phenomenon in a system of coupled nonlinear Schrödinger equations with cubic nonlinearity, without [13, 12] and with [7] the linear coupling. Rather complete list of the available results can be found in [7]. Two sufficient condition for the finite-time blowup have also been established for the supercritical case of the coupled nonlinear Schrödinger equations one of which has gain and another has dissipation, both balanced with each other [5].

In this paper we consider the system

$$\begin{cases} i\,u_t \;=\; -\Delta u + i\,\gamma_1\,u + k\,v - \left(g_1|u|^{p-1} + g|v|^2\right) u\ , \\ i\,v_t \;=\; -\Delta v + i\,\gamma_2\,v + k\,u - \left(g|u|^2 + g_2|v|^{p-1}\right) v\ , \end{cases} \tag{1.1}$$

with initial data $u_0, v_0 \in H^1(\mathbb{R}^N)$, $1 \leq N \leq 3$, and $\frac{4}{N} \leq p-1$, with $p-1 < \frac{4}{N-2}$ if $N = 3$, $\gamma_1, \gamma_2 \in \mathbb{R}$ describe gain ($\gamma_{1,2} > 0$) or dissipation ($\gamma_{1,2} < 0$), $k \in \mathbb{R}$ is the linear coupling, $g_1, g_2 > 0$, $g \in \mathbb{R}$, $u(x,t), v(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^N$, $t \geq 0$. The particular case $p-1 = 2$, $N \geq 3$ and $\gamma_1 = -\gamma_2$, was considered in [5] for the study of the possible blowup of H^1-strong solutions.

System (1.1) may appear in various physical contexts. As a few examples we mention an optical coupler ($N = 1$) with passive and active arms [1] and the self-phase modulation (described by g_1 and g_2) stronger than the Kerr nonlinearity ($p > 3$). Alternatively the model describes propagation of a pulse in an elliptically polarized medium [9] with dissipation where the two polarization are linearly coupled. In two- and three-dimensional settings the model can describe diffraction, focusing and filamentation of a transversely polarized electromagnetic wave (see [3]) where the orthogonally polarized components (they are described by u and v) are linearly coupled (or alternatively two beams are linearly coupled) and are subject to absorption or gain (described by $\gamma_{1,2}$). In these cases the evolutional variable t describes distance along the propagation direction of the beam. Further, at $N = 3$ the model describes a collapse of an unstable binary mixture of Bose–Einstein condensates [11] subjected to the removal and adding atoms.

In this paper we first study the existence and uniqueness of H^1-strong solutions of the system (1.1) in the sense of T. Kato [8] (see also [2]) by applying some variants of Strichartz's inequalities (cf. [10]) and some convenient *a priori* estimates (Theorems 2.1 and 3.1). In the second part of the paper, we extend the main result of [5] in the supercritical case (Theorem 4.1) and give a new result in the critical case (Theorem 4.2).

2. Local existence in $H^1(\mathbb{R}^N)$

In this section we will study the local existence in $H^1(\mathbb{R}^N)$ to the Cauchy problem for the system (1.1) with initial data $(u_0, v_0) \in (H^1(\mathbb{R}^N))^2$. Recall that we have $\frac{4}{N} \leq p-1 < \frac{4}{N-2}$ ($< +\infty$ if $N = 1, 2$) and $1 \leq N \leq 3$. The case $p-1 = \frac{4}{N}$ is called the critical case.

To prove the local existence of solution we apply Kato's method (cf. [8], [2]) by adapting the proof of theorem 4.4.6 in [2].

We start by writing system (1.1) in the form

$$\begin{cases} i\,\varphi_t \;=\; H\varphi + \mathcal{G}(\varphi) + B\varphi\ , \\ \varphi(0) \;=\; \varphi_0\ , \end{cases} \tag{2.1}$$

where $\varphi = \begin{pmatrix} u \\ v \end{pmatrix}$, $\varphi_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in (H^1(\mathbb{R}^N))^2$,

$$H\varphi \;=\; \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} \varphi\ ,$$

$$\mathcal{G}(\varphi) = \begin{pmatrix} \mathcal{G}_1(\varphi) \\ \mathcal{G}_2(\varphi) \end{pmatrix} = -\begin{pmatrix} g_1|u|^{p-1} + g|v|^2 & 0 \\ 0 & g|u|^2 + g_2|v|^{p-1} \end{pmatrix} \varphi$$

and

$$B\varphi = \begin{pmatrix} i\,\gamma_1 & k \\ k & i\,\gamma_2 \end{pmatrix} \varphi = \begin{pmatrix} B_1(u,v) \\ B_2(u,v) \end{pmatrix},$$

i.e.

$$B_1(u,v) = i\,\gamma_1 u + k\,v\,, \qquad B_2(u,v) = k\,u + i\,\gamma_2\,v\ .$$

We decompose $\mathcal{G}_{1,2}$ as follows:

$$\mathcal{G}_1(\varphi) = g_1 G(u) + \tilde{G}(v,u)\,, \qquad \mathcal{G}_2(\varphi) = g_2 G(v) + \tilde{G}(u,v)\ ,$$

with the functions $G(\cdot)$ and $\tilde{G}(\cdot,\cdot)$ defined as follows

$$G(u) = -|u|^{p-1}\,u\,, \qquad \tilde{G}(u,v) = -g|u|^2\,v\ .$$

Now we easily derive

$$\big||v_1|^2\,u_1 - |v_2|^2\,u_2\big| \le c\left(|v_1|^2 + |v_2|^2 + |u_1|^2 + |u_2|^2\right)\left(|u_1 - u_2| + |v_1 - v_2|\right)$$

and the same estimate for $\big||u_1|^2\,v_1 - |u_2|^2\,v_2\big|$. Moreover,

$$\big||u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\big| \le c\left(|u_1|^{p-1} + |u_2|^{p-1}\right)|u_1 - u_2|\ ,$$

and similar estimate for $\big||v_1|^{p-1}v_1 - |v_2|^{p-1}v_2\big|$.

With $r = (p-1)+2 = p+1$, we derive, for r' such that $\frac{1}{r} + \frac{1}{r'} = 1$, $(u_{1,2}, v_{1,2}) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$,

$$\big\|G(u_2) - G(u_1)\big\|_{L^{r'}} \le c\left(\|u_1\|_{L^r}^{p-1} + \|u_2\|_{L^r}^{p-1}\right)\|u_2 - u_1\|_{L^r}\ ,$$

$$\begin{aligned}\big\|\tilde{G}(u_2,v_2) - \tilde{G}(u_1,v_1)\big\|_{L^{4/3}} &\le c\left(\|u_1\|_{L^4}^2 + \|u_2\|_{L^4}^2 + \|v_1\|_{L^4}^2 + \|v_2\|_{L^4}^2\right) \\ &\quad \times \left(\|u_1 - u_2\|_{L^4} + \|v_1 - v_2\|_{L^4}\right)\ ,\end{aligned}$$

and similar estimates for $G(v)$ and $\|\tilde{G}(v_2,u_2) - \tilde{G}(v_1,u_1)\|_{L^{4/3}}$,

$$\big\|\nabla G(u)\big\|_{L^{r'}} \le c\,\|u\|_{L^r}^{p-1}\,\|\nabla u\|_{L^r}\ ,$$

$$\begin{aligned}\big\|\nabla\tilde{G}(u,v)\big\|_{L^{4/3}} &\le c\left[\|v\|_{L^4}^2\,\|\nabla u\|_{L^4} + \|\nabla v\|_{L^4}\,\|v\|_{L^4}\,\|u\|_{L^4}\right] \\ &\le c\left[\left(\|v\|_{L^4}^2 + \|u\|_{L^4}^2\right)\left(\|\nabla u\|_{L^4} + \|\nabla v\|_{L^4}\right)\right]\ ,\end{aligned}$$

and similar estimates for $\big\|\nabla G(v)\big\|_{L^{r'}}$ and $\|\nabla\tilde{G}(v,u)\|_{L^{4/3}}$.

Moreover, we have

$$\big\|B_1(u_2,v_2) - B_2(u_1,v_1)\big\|_{L^2} \le c\left(\|u_2 - u_1\|_{L^2} + \|v_2 - v_1\|_{L^2}\right)\ ,$$

$$\big\|\nabla B_1(u,v)\big\|_{L^2} \le c\left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right)\ ,$$

and similar estimates for B_2.

Now we fix $M, T > 0$ to be chosen latter as in the proof of theorem 4.4.6 in [2] and, with $r = \max(2, p+1, 4) = \max(p+1, 4)$, we consider the admissible pair (in the Strichartz's sense, cf. [2, section 2.3])

$$(q,r)\,, \qquad \frac{2}{q} = N\left(\frac{1}{2} - \frac{1}{r}\right)\ .$$

We introduce the space

$$X = \Big\{\varphi \in \big(L^\infty(-T,T;H^1)\cap L^q(-T,T;W^{1,r})\big)^2\Big\}$$

with the distance

$$\begin{aligned} d\big(\varphi_1,\varphi_2\big) &= \|u_1-u_2\|_{L^q(-T,T;L^r)} + \|u_1-u_2\|_{L^\infty(-T,T,L^2)} \\ &\quad + \|v_1-v_2\|_{L^q(-T,T;L^r)} + \|v_1-v_2\|_{L^\infty(-T,T;L^2)} \ , \end{aligned}$$

where $\varphi_{1,2} = \begin{pmatrix} u_{1,2} \\ v_{1,2} \end{pmatrix}$ and the subset

$$\begin{aligned} E = \Big\{\varphi \in X \mid \ & \|u\|_{L^q(-T,T;W^{1,r})} + \|u\|_{L^\infty(-T,T;H^1)} \\ & + \|v\|_{L^q(-T,T;W^{1,r})} + \|v\|_{L^\infty(-T,T;H^1)} \leq M\Big\} \ , \end{aligned}$$

which is a complete metric space with distance d.

Now, with $S(t) = e^{i\Delta t}$ and $t \in \mathbb{R}$ denoting the Schrödinger group in L^2, we introduce, for $\varphi \in E$, $(u_0, v_0) \in (H^1(\mathbb{R}^N))^2$,

$$\mathcal{H}(\varphi) = \begin{pmatrix} \mathcal{H}_1(\varphi) \\ \mathcal{H}_2(\varphi) \end{pmatrix}, \tag{2.2}$$

with the entries

$$\begin{aligned} \mathcal{H}_1(\varphi) &= S(t)u_0 + i\int_0^t S(t-s)\ \mathcal{G}_1\big(u(s),v(s)\big)\ ds \\ &\quad + i\int_0^t S(t-s)\ B_1\big(u(s),v(s)\big)\ ds\ , \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathcal{H}_2(\varphi) &= S(t)\,v_0 + i\int_0^t S(t-s)\ \mathcal{G}_2\big(u(s),v(s)\big)\ ds \\ &\quad + i\int_0^t S(t-s)\,B_2(u(s),v(s))\ ds\ . \end{aligned} \tag{2.4}$$

Now, reasoning as in the proof of theorem 4.4.6, we can prove, by the previous estimates and applying Strichartz's inequalities, that

$$\mathcal{H}(\varphi) \in C\big([-T,T];H^1\big)\cap L^q\big(-T,T;W^{1,r}\big)\ ,$$

and, for a convenient M and a sufficient small $T>0$, $\mathcal{H}(u,v)\in E$ and

$$d\big(\mathcal{H}(\varphi_1),\,\mathcal{H}(\varphi_2)\big) \leq \frac{1}{2}\,d\big(\varphi_1,\varphi_2\big)\,, \quad \text{for } \ \varphi_1,\varphi_2\in E\ .$$

The uniqueness in $C([-T,T];H^1)$ and the blowup alternative follows as in theorem 4.4.6 (cf. [2]). We have:

Theorem 2.1. *Let $(u_0,v_0)\in(H^1(\mathbb{R}^N))^2$. Then, the Cauchy problem for system* (1.1) *has a unique strong solution $(u,v)\in C\big([0,T_{\max});(H^1)^2(\mathbb{R}^N)\big)$ with initial data (u_0,v_0), defined on a maximal time interval $[0,T_{\max})$.*

3. Global existence for $k = 0$

In this section we prove the global existence of the particular case when the linear coupling is absent, and the system obey sufficiently strong dissipation.

Given $\gamma \in \mathbb{R}$ let us consider the semigroup $(S_\gamma(t))_{t\geq 0}$ in $L^2(\mathbb{R}^N)$ defined by

$$S_\gamma(t) = e^{\gamma t} S(t) \ .$$

We need to apply Strichartz's estimates (cf. [2]). We recall that a pair (q,r) is admissible if

$$\frac{2}{q} = N\left(\frac{1}{2} - \frac{1}{r}\right)$$

with $2 \leq r \leq \frac{2N}{N-2}$ ($2 \leq r \leq \infty$ if $N = 1$, $2 \leq r < \infty$ if $N = 2$).

Using the same notation as in [2], [10], we define

$$\Phi_f(t) = \int_0^t S(t-s)\, f(s)\, ds$$

and

$$\Phi_f^\gamma(t) = \int_0^t S_\gamma(t-s)\, f(s)\, ds = e^{\gamma t}\int_0^t S(t-s)\, e^{-\gamma s}\, f(s)\, ds \ .$$

We have the following estimates (cf. [2], [10]):

For every admissible pair (q,r) and $\forall\, \varphi \in L^2$ there exists a constant $c > 0$ such that, with $L^p = L^p(\mathbb{R}^N)$ and $T > 0$,

$$\|S(\cdot)\, f\|_{L^q(0,T;L^r)} \leq c\, \|f\|_{L^2} \,, \qquad \forall\, \varphi \in L^2 \ , \tag{3.1}$$

with c independent of T,

$$\left\|\Phi_f^\gamma\right\|_{L^q(0,T;L^r)} \leq c\, \|f\|_{L^{q'}(0,T;L^{r'})} \ , \tag{3.2}$$

$$\left\|\Phi_f^\gamma\right\|_{L^\infty(0,T;L^r)} \leq c\, \|f\|_{L^{q'}(0,T;L^{r'})} \ , \tag{3.3}$$

with c independent of T, $\frac{1}{r} + \frac{1}{r'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Moreover, if $2 < r < \frac{2N}{N-2}$ (or $2 < r < +\infty$ if $N = 1$) and $\theta, \widetilde{\theta} \in \,]1, +\infty[$ are such that $\frac{1}{\theta} + \frac{1}{\widetilde{\theta}} = N(\frac{1}{2} - \frac{1}{r})$, then

$$\left\|\Phi_f^\gamma\right\|_{L^\theta(0,T;L^r)} \leq c\, \|f\|_{L^{\widetilde{\theta}'}(0,T;L^{r'})} \tag{3.4}$$

with c independent of T and $\frac{1}{\widetilde{\theta}} + \frac{1}{\widetilde{\theta}'} = 1$.

Now, by using the Duhamel formula, we write the system (1.1), for the local solution, in the integral form. In the case $k = 0$ for $t \in [0, T_{\max})$ we have

$$\begin{cases} u(t) = S_{\gamma_1}(t)\, u_0 + i\displaystyle\int_0^t S_{\gamma_1}(t-s)\left(-g_1|u(s)|^{p-1}u(s) - g|v(s)|^2u(s)\right) ds\,, \\ v(t) = S_{\gamma_2}(t)\, v_0 + i\displaystyle\int_0^t S_{\gamma_2}(t-s)\left(-g|u(s)|^2v(s) - g_2|v(s)|^{p-1}v(s)\right) ds\,. \end{cases} \tag{3.5}$$

Next we state a global existence result of the Cauchy problem for the system (1.1) with $k = 0$:

Theorem 3.1. *Assume* $\gamma_1, \gamma_2 < 0$ *and* $k = 0$. *Then, for any* $(u_0, v_0) \in (H^1(\mathbb{R}^N))^2$ *there exists* $\gamma^*(\|u_0\|_{H^1}, \|v_0\|_{H^1}) > 0$ *such that, for all* $\gamma_1, \gamma_2 < -\gamma^*$, $T_{\max} = +\infty$.

First we prove the following important result:

Lemma 3.1. *Under the conditions of the Theorem* 3.1, *assume that there exist constants* $\varepsilon > 0$ *and* $\overline{\gamma} < 0$ *such that for* $\gamma_i \le \overline{\gamma}$, $i = 1, 2$, *we have, with* $w_0^1 = u_0$, $w_0^2 = v_0$,

$$\left\|S_{\gamma_i}(\cdot)\, w_0^i\right\|_{L^\theta(0,+\infty;L^{p+1})} \le \varepsilon\,, \qquad \text{if } \; p-1 \ge 2\;, \tag{3.6}$$

where $\theta = \frac{2(p-1)(p+1)}{4-(N-2)(p-1)}$,

$$\left\|S_{\gamma_i}(\cdot)\, w_0^i\right\|_{L^8(0,+\infty;L^4)} \le \varepsilon\,, \qquad \text{if } \; p-1 < 2\;. \tag{3.7}$$

Then $T_{\max} = +\infty$.

Proof. Let $\widetilde{\theta}$ be defined by $\frac{1}{\theta} + \frac{1}{\widetilde{\theta}} = N(\frac{1}{2} - \frac{1}{r})$ for $2 < r < \frac{2N}{N-2}$ (or $2 < r < +\infty$ if $N = 1$) (cf. (3.4)). We have

$$p\,r' = r\,, \qquad p\,\widetilde{\theta}' = \theta\,, \qquad \frac{1}{q'} = \frac{1}{q} + \frac{p-1}{\theta}\;. \tag{3.8}$$

Applying (3.5) we derive for $t \in [0, T[$, $T < T_{\max}$,

$$\begin{aligned} \|u(t)\|_{H^1} + \|v(t)\|_{H^1} \;\le& \\ \le\; & \|u_0\|_{H^1} + \|v_0\|_{H^1} \\ &+ \left\|\Phi^{\gamma_1}_{|u|^{p-1}u}\right\|_{L^\infty(0,T;H^1)} + \left\|\Phi^{\gamma_2}_{|v|^{p-1}v}\right\|_{L^\infty(0,T;H^1)} \\ &+ \left\|\Phi^{\gamma_1}_{(|u|^2+|v|^2)u}\right\|_{L^\infty(0,T;H^1)} + \left\|\Phi^{\gamma_2}_{(|u|^2+|v|^2)v}\right\|_{L^\infty(0,T;H^1)}\;. \end{aligned} \tag{3.9}$$

Now we start with the case

I) $p - 1 \ge 2$.

In this case we have

$$|u|^2 \le 1 + |u|^{p-1}\,, \qquad |v|^2 \le 1 + |v|^{p-1}\;,$$

and so we estimate

$$\begin{aligned} \left\|\Phi^{\gamma_1}_u\right\|_{L^\infty(0,T;H^1)} &\le \sup_{t\in(0,T)} e^{\gamma_1 t} \left\|\int_0^t S(t-s)\, e^{-\gamma_1 s}\, u(s)\; ds\right\|_{H^1} \\ &\le \sup_{t\in(0,T)} \left(e^{\gamma_1 t}\, \frac{e^{-\gamma_1 t} - 1}{|\gamma_1|}\right) \|u\|_{L^\infty(0,T;H^1)} \\ &\le \frac{1}{|\gamma_1|}\, \|u\|_{L^\infty(0,T;H^1)} \end{aligned} \tag{3.10}$$

and similarly

$$\left\|\Phi^{\gamma_2}_v\right\|_{L^\infty(0,T;H^1)} \le \frac{1}{|\gamma_2|}\, \|v\|_{L^\infty(0,T;H^1)}\;. \tag{3.11}$$

Hence, for $|\gamma_i|$ large enough, it follows from (3.3) and (3.9)

$$\begin{aligned} \|u\|_{L^\infty(0,T;H^1)} + \|v\|_{L^\infty(0,T;H^1)} \le \\ \le\ & 2\left(\|u_0\|_{H^1} + \|v_0\|_{H^1}\right) \\ &+ c\left\||u|^{p-1}u\right\|_{L^{q'}(0,T;W^{1,r'})} + c\left\||v|^{p-1}v\right\|_{L^{q'}(0,T;L^{r'})} \\ &+ c\left\||u|^{p-1}(v+|\nabla v|)\right\|_{L^{q'}(0,T;L^{r'})} \\ &+ c\left\||v|^{p-1}(u+|\nabla u|)\right\|_{L^{q'}(0,T;L^{r'})}, \end{aligned} \tag{3.12}$$

with c a constant independent of T.

Now we estimate the typical term $\||u|^p\, Dv\|_{L^{q'}(0,T;L^{r'})}$ ($D = \frac{\partial}{\partial x_j}$) in the right-hand side of (3.12).

From (3.8) we derive

$$\left\||u|^{p-1}Dv\right\|_{L^{q'}(0,T;L^{r'})} \le c\,\|u\|^{p-1}_{L^\theta(0,T;L^r)}\,\|Dv\|_{L^q(0,T,W^{1,r})}\,. \tag{3.13}$$

Next, we fix $0 < t \le T$. By using the estimate (3.4), from the Duhamel formula (3.5) we deduce by (3.6)

$$\begin{aligned} \|u\|_{L^\theta(0,t;L^r)} \le\ & \|S_{\gamma_1}(\cdot)\,u_0\|_{L^\theta(0,t;L^r)} \\ &+ \left\|\Phi^{\gamma_1}_{|u|^{p-1}u}\right\|_{L^\theta(0,t;L^r)} + \left\|\Phi^{\gamma_1}_{|u|^2u+|v|^2u}\right\|_{L^\theta(0,t;L^r)} \\ \le\ & \varepsilon + c\left\||u|^{p-1}u\right\|_{L^{\tilde\theta'}(0,t;L^{r'})} \\ &+ c\left\||u|^2\,u\right\|_{L^{\tilde\theta'}(0,t;L^{r'})} + c\left\||v|^2\,v\right\|_{L^{\tilde\theta'}(0,t;L^{r'})}\,. \end{aligned} \tag{3.14}$$

Now, we remark that, for $D_u = \left\{x \in \mathbb{R}^N \mid |u(x)| \ge 1 \text{ a.e.}\right\}$, with χ_{D_u} the characteristic function of D_u, and for each t,

$$\begin{aligned} \left\||u|^2\,u\right\|_{L^{r'}} &= \left\|\chi_{D_u}|u|^2\,u + (1-\chi_{D_u})\,|u|^2\,u\right\|_{L^{r'}} \\ &\le \left\||u|^{p-1}\,u\right\|_{L^{r'}} + \left\|(1-\chi_{D_u})\,|u|^2\,u\right\|_{L^{r'}} \end{aligned}$$

and

$$\begin{aligned} \left\|(1-\chi_{D_u})\,|u|^2\,u\right\|_{L^{r'}} &\le \left\|(1-\chi_{D_u})\,u\right\|^2_{L^{2\frac{p+1}{p-1}}}\,\|u\|_{L^r} \\ &\le \|u\|^{2\frac{p-1}{p+1}}\,\|u\|_{L^r} \\ &\le e^{2\gamma_1\frac{p-1}{p+1}t}\,\|u_0\|_{L^2}\,\|u\|_{L^r}\,. \end{aligned}$$

Since $\widetilde{\theta}' = \frac{\theta}{p} < \theta$, it follows that

$$\begin{aligned} \left\||u|^2\,u\right\|_{L^{\tilde\theta'}(0,t;L^{r'})} &\le c\left\||u|^{p-1}\,u\right\|_{L^{\tilde\theta'}(0,t;L^{r'})} + \frac{c}{|\gamma_1|}\,\|u\|_{L^\theta(0,t;L^r)} \\ &\le c\,\|u\|^p_{L^\theta(0,t;L^r)} + \frac{c}{|\gamma_1|}\,\|u\|_{L^\theta(0,t;L^r)}\,, \end{aligned}$$

with c a constant independent of t.

The same conclusion can be obtained for the term $\||v|^2\,v\|_{L^{\tilde\theta'}(0,t;L^{r'})}$. Therefore, putting together all the terms, we obtain, for $|\gamma_1|$ and $|\gamma_2|$ big enough,

$$\|u\|_{L^\theta(0,t;L^r)} + \|v\|_{L^\theta(0,t;L^r)} \le \le 4\,\varepsilon + c_1\big(\|u\|_{L^\theta(0,t;L^r)} + \|v\|_{L^\theta(0,t;L^r)}\big)^p , \tag{3.15}$$

with c_1 a constant independent of t.

On the other hand, and using again the Duhamel formula and the Strichartz's estimates we derive

$$\begin{aligned} \|u\|_{L^q(0,t;W^{1,r})} + \|v\|_{L^q(0,t;W^{1,r})} \le \\ &\le \big\|S^{\gamma_1}(\cdot)\,u_0\big\|_{L^q(0,t;W^{1,r})} + \big\|S^{\gamma_2}(\cdot)\,v_0\big\|_{L^q(0,t;W^{1,r})} \\ &\quad + \big\|\Phi^{\gamma_1}_{|u|^{p-1}u}\big\|_{L^q(0,t;W^{1,r})} + \big\|\Phi^{\gamma_2}_{|v|^{p-1}v}\big\|_{L^q(0,t;W^{1,r})} \\ &\le c\,\big(\|u_0\|_{H^1} + \|v_0\|_{H^1}\big) \\ &\quad + c\,\big\||u|^{p-1}u\big\|_{L^{q'}(0,t;W^{1,r'})} + c\,\big\||v|^{p-1}v\big\|_{L^{q'}(0,t;W^{1,r'})} \\ &\quad + c\,\big\|\big(|u|^2+|v|^2\big)\,u\big\|_{L^{q'}(0,t;W^{1,r'})} \\ &\quad + c\,\big\|\big(|v|^2+|u|^2\big)\,v\big\|_{L^{q'}(0,t;W^{1,r'})} . \end{aligned} \tag{3.16}$$

Next we proceed as before to estimate the last two terms on the right-hand side, more precisely (with $D_v = \big\{x \in \mathbb{R}^N \mid |v(x)| \ge 1 \text{ a.e.}\big\}$):

$$\begin{aligned} \big\||v|^2\,Du\big\|_{L^{q'}(0,t;L^{r'})} &= \big\|\chi_{D_v}|v|^2 Du + (1-\chi_{D_v})\,|v|^2 Du\big\|_{L^{q'}(0,t;L^{r'})} \\ &\le \big\||v|^{p-1}Du\big\|_{L^{q'}(0,t;L^{r'})} + \big\|(1-\chi_{D_v})\,|v|^2 Du\big\|_{L^{q'}(0,t;L^{r'})} \\ &\le \|v\|^{p-1}_{L^\theta(0,t;L^r)}\,\|u\|_{L^q(0,t;W^{1,r})} + \frac{c}{|\gamma_2|}\,\|u\|_{L^q(0,t;W^{1,r})} . \end{aligned}$$

Then, for $|\gamma_1|$ and $|\gamma_2|$ large enough, it follows from (3.16)

$$\begin{aligned} \|u\|_{L^q(0,t;W^{1,r})} + \|v\|_{L^q(0,t;W^{1,r})} \le \\ &\le c_2\,\big(\|u_0\|_{H^1} + \|v_0\|_{H^1}\big) \\ &\quad + c_2\,\big(\|u\|_{L^\theta(0,t;L^r)} + \|v\|_{L^\theta(0,t;L^r)}\big)^{p-1} \\ &\quad \cdot\,\big(\|u\|_{L^q(0,t;W^{1,r})} + \|v\|_{L^q(0,t;W^{1,r})}\big) , \end{aligned} \tag{3.17}$$

with c_2 a constant independent of t.

Now, let $c_0 = \max(c_1, c_2)$ and choose ε such that $2^{3p-2}\,c_0\,\varepsilon^{p-1} \le 1$. By the continuity of the functions $t \to \|u\|_{L^\theta(0,t;L^r)} + \|v\|_{L^\theta(0,t;L^r)}$ and $t \to \|u\|_{L^q(0,t;W^{1,r})} + \|v\|_{L^\theta(0,t;W^{1,r})}$ it follows from (3.15) and (3.17) that

$$\|u\|_{L^\theta(0,T;L^r)} + \|v\|_{L^\theta(0,T;L^r)} \le 8\,\varepsilon$$

and

$$\|u\|_{L^q(0,T;W^{1,r})} + \|v\|_{L^q(0,T;W^{1,r})} \le 2\,c_0\,\big(\|u_0\|_{H^1} + \|v_0\|_{H^1}\big) .$$

The conclusion follows now from (3.12) and (3.13).

II) $p-1<2$.

Notice that, since $\frac{4}{N} \le p-1$, the condition $p-1<2$ implies $N>2$, which means in our case $N=3$. The proof follows the same steps used in the previous case $p-1 \ge 2$. The first estimate (3.12) remains true with $p-1$ replaced by 2 and when the admissible pair $(\overline{q},\overline{r})$ corresponds now to $\overline{r}=4$. The estimate (3.13) is now

$$\left\| |u|^2 Dv \right\|_{L^{\overline{q}'}(0,T;L^{4/3})} \le c\, \|u\|^2_{L^8(0,T;L^4)} \, \|Dv\|_{L^{\overline{q}}(0,T;W^{1,4})}$$

and the estimates (3.15) and (3.17) are obtained following the same scheme. For example, to estimate $\|u\|_{L^8(0,t;L^4)}$, just like in (3.14), we use the assumption $\|S_{\gamma_1}(\cdot)\, u_0\|_{L^8(0,+\infty,L^4)} \le \varepsilon$ and we must only estimate

$$\left\| |u|^{p-1} u \right\|_{L^{\tilde{8}}(0,t;L^{4/3})}\,, \qquad \frac{1}{8}+\frac{1}{\tilde{8}} = 3\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{3}{4}\,,$$

using the decomposition $|u|^{p-1}u = \chi_{D_u} |u|^{p-1}u + (1-\chi_{D_u})\, |u|^{p-1}u$. With the corresponding estimated (3.15) and (3.17) we conclude in the same way. □

Now we can pass to the

Proof of Theorem 3.1. Assume first $p-1 \ge 2$. We will prove that (3.6) holds. Since we have $\|S(t)\, u_0\|_{L^{p+1}} \le c\, \|u_0\|_{H^1}$, $t \ge 0$, we derive

$$\|S_{\gamma_1} u_0\|^{\theta}_{L^{\theta}(0,+\infty;L^{p+1})} = \int_0^{+\infty} e^{\theta\gamma_1 t}\, \|S(t)\, u_0\|^{\theta}_{L^{p+1}}\; dt \le -\frac{c}{\theta\gamma_1}\, \|u_0\|^{\theta}_{H^1}\,.$$

Therefore

$$\|S_{\gamma_1} u_0\|_{L^{\theta}(0,+\infty;L^{p+1})} \le c\, \|u_0\|_{H^1} \left(\frac{1}{|\gamma_1|\,\theta}\right)^{\frac{1}{\theta}} \xrightarrow[|\gamma_1|\to+\infty]{} 0$$

and the same conclusion holds for $S_{\gamma_2} v_0$. Similar estimates prove (3.7) in the case $p-1<2$. Hence, the assumptions in Lemma 3.1 are satisfied and so Theorem 3.1 is proved. □

4. Blowup results

In this section we will study the possible blowup of the local in time H^1-strong solutions (u,v) of the Cauchy problem for system (1.1) with initial data $(u_0,v_0) \in (H^1(\mathbb{R}^N))^2$ such that

$$|x|u_0,\ |x|v_0 \in L^2(\mathbb{R}^N)\,.$$

In the following we perform formal calculations which can be justified by suitable regularisations that allow us to prove that

$$|x|\, u(x,\cdot),\ |x|\, v(,\cdot) \in C\big([0,T_{\max});L^2\big)\,.$$

The main ideas are based in the seminal work of R.T. Glassey [6], in [10], [7] and in the previous paper [5] when the case $p-1=2$ is studied.

We start by proving some preliminar estimates to the local solution $(u,v)\in C([0,T_{\max});(H^1)^2)$.

It is easy to derive, for $t\in[0,T_{\max})$ and with $\int\cdot\,dx=\int_{\mathbb{R}^N}\cdot\,dx$,

$$\frac{1}{2}\frac{d}{dt}\int|u|^2\,dx = \gamma_1\int|u|^2\,dx + k\ \mathrm{Im}\int v\,\overline{u}\,dx\ ,$$
$$\frac{1}{2}\frac{d}{dt}\int|v|^2\,dx = \gamma_2\int|v|^2\,dx + k\ \mathrm{Im}\int u\,\overline{v}\,dx\ ,$$

and so,

$$\begin{aligned}\frac{d}{dt}\int\big(|u|^2+|v|^2\big)\,dx &= 2\,\gamma_1\int|u|^2\,dx + 2\,\gamma_2\int|v|^2\,dx\\ &\le 2\,\gamma\int\big(|u|^2+|v|^2\big)\,dx\ ,\\ &\qquad\text{with } \gamma=\max\big(|\gamma_1|,|\gamma_2|\big)\ .\end{aligned}\tag{4.1}$$

Then we obtain

$$Q(t) = \|u(t)\|^2_{L^2}+\|v(t)\|^2_{L^2} \le e^{\gamma t}\big(\|u_0\|^2_{L^2}+\|v_0\|^2_{L^2}\big)\ .\tag{4.2}$$

We define the energy

$$\begin{aligned}E(t) &= \frac{1}{2}\int|\nabla u(t)|^2\,dx + \frac{1}{2}\int|\nabla v(t)|^2\,dx + k\ \mathrm{Re}\int u(t)\,\overline{u}(t)\,dx\\ &\quad - \frac{g_1}{p+1}\int|u(t)|^{p+1}dx - \frac{g_2}{p+1}\int|v(t)|^{p+1}\,dx\\ &\quad - \frac{g}{2}\int|u(t)|^2\,|v(t)|^2\,dx\ .\end{aligned}\tag{4.3}$$

From the system (1.1) we deduce

$$\begin{aligned}\frac{dE}{dt} &= \gamma_1\int|\nabla u|^2\,dx + \gamma_2\int|\nabla v|^2\,dx\\ &\quad - \gamma_1 g_1\int|u|^{p+1}dx - \gamma_2\,g_2\int|v|^{p+1}dx\\ &\quad - (\gamma_1+\gamma_2)\,g\int|u|^2\,|v|^2\,dx + k\,(\gamma_1+\gamma_2)\ \mathrm{Re}\int u\,\overline{v}\,dx\ .\end{aligned}\tag{4.4}$$

We need the following result:

Lemma 4.1. *Assume $p-1>\frac{4}{N}$. Then the solution $(u,v)\in C([0,T_{\max});H^1)$ of system (1.1) with initial data $(u_0,v_0)\in(H^1)^2$ verifies the inequality*

$$\begin{aligned}E(t) &\le E(0) + \widetilde{c}_3\,\gamma\int_0^t\big(\|\nabla u\|^2_{L^2}+\|\nabla v\|^2_{L^2}+\|u\|^{p+1}_{L^{p+1}}+\|v\|^{p+1}_{L^{p+1}}\big)\,d\tau\\ &\quad + \widetilde{\widetilde{c}}_3\,Q(0)\,e^{2\gamma t}\ ,\end{aligned}$$

with

$$\widetilde{c}_3 = \begin{cases}\max\{1,g_1,g_2\}\,, & \textit{if } (\gamma_1+\gamma_2)\,g\ge 0\ ,\\ \max\{1,|g|+g_1,|g|+g_2\}\,, & \textit{if } (\gamma_1+\gamma_2)\,g<0,\quad N=1,2\ ,\\ & \textit{or if } (\gamma_1+\gamma_2)\,g<0,\quad p-1\ge 2,\quad N=3\ ,\end{cases}$$

and

$$\widetilde{\widetilde{c}}_3 \;=\; \begin{cases} \dfrac{|k|}{2}\,, & \textit{if}\;\; (\gamma_1{+}\gamma_2)\, g \ge 0\;, \\ \dfrac{|k|+|g|}{2}\,, & \textit{if}\;\; (\gamma_1{+}\gamma_2)\, g < 0, \quad N{=}1,2\;, \\ & \textit{or if}\;\; (\gamma_1{+}\gamma_2)\, g < 0, \quad p{-}1 \ge 2, \quad N{=}3\;. \end{cases}$$

Proof. If $(\gamma_1{+}\,\gamma_2)\, g \ge 0$ it follows from (4.4) that

$$\begin{aligned} E(t) \;\le\; E(0) \,+\, \widetilde{c}_3\, \gamma \int_0^t \big(\|\nabla u\|^2_{L^2} + \|\nabla v\|^2_{L^2} + \|u\|^{p+1}_{L^{p+1}} + \|v\|^{p+1}_{L^{p+1}} \big)\, d\tau \\ +\, |k|\, \gamma \int_0^t Q(\tau)\, d\tau\;, \end{aligned}$$

$\widetilde{c}_3 = \max\{1, g_1, g_2\}$, and, since $Q(\tau) \le e^{2\gamma\tau} Q(0)$, we obtain the result with $\widetilde{\widetilde{c}}_3 = \frac{|k|}{2}$. If $(\gamma_1{+}\,\gamma_2)\, g < 0$, we remark that (recall that $p-1 \ge \frac{4}{N}$)

$$\begin{aligned} \int |u|^2\, |v|^2\, dx \;&\le\; \frac{1}{2} \int |u|^4\, dx \,+\, \frac{1}{2} \int |v|^4\, dx \\ &\le\; \frac{1}{2} \int |u|^{p+1} dx \,+\, \frac{1}{2} \int |v|^{p+1} dx \,+\, \frac{1}{2} \int \big(|u|^2{+}|v|^2 \big)\, dx \end{aligned}$$

for $N = 1, 2$ and assuming $p - 1 \ge 2$ for $N = 3$. The result now follows from (4.4) as before. □

Now we define the variance

$$Y(t) \;=\; Y_1(t) + Y_2(t)\;,$$

with

$$Y_1(t) \;=\; \int |x|^2\, |u|^2\, dx\,, \qquad Y_2(t) \;=\; \int |x|^2\, |v|^2\, dx\;,$$

and let

$$V_1(t) \;=\; 4\, \mathrm{Im} \int (\nabla u \cdot x)\, \overline{u}\; dx\,, \qquad V_2(t) \;=\; 4\, \mathrm{Im} \int (\nabla v \cdot x)\, \overline{v}\; dx\;.$$

We derive from (1.1)

$$\begin{aligned} Y_1'(t) \;&=\; V_1(t) + 2\,\gamma_1 Y_1(t) + 2k\; \mathrm{Im} \int |x|^2\, v\, \overline{u}\; dx\;, \\ Y_2'(t) \;&=\; V_2(t) + 2\,\gamma_2 Y_2(t) + 2k\; \mathrm{Im} \int |x|^2\, u\, \overline{v}\; dx\;, \end{aligned}$$

and so

$$Y'(t) \;=\; V_1(t) + V_2(t) + 2\,\gamma_1 Y_1(t) + 2\,\gamma_2 Y_2(t)\;. \tag{4.5}$$

To compute the record derivative we take the derivative of $V_i(t)$, $i = 1, 2$:

$$\begin{aligned} V_1'(t) \;&=\; 4\, \mathrm{Im} \int (\nabla u_t \cdot x)\, \overline{u}\; dx \,+\, 4\, \mathrm{Im} \int (x \cdot \nabla u)\, \overline{u}_t\; dx \\ &=\; -4N\; \mathrm{Im} \int u_t\, \overline{u}\; dx \,-\, 8\, \mathrm{Im} \int (x \cdot \nabla \overline{u})\, u_t\; dx\;, \end{aligned}$$

and

$$
\begin{aligned}
-4N\,\mathrm{Im}\int u_t\,\overline{u}\;dx \;&=\; 4N\Big[\int|\nabla u|^2\,dx \;-\; g_1\int|u|^{p+1}\,dx \\
&\qquad -\; g\int|u|^2\,|v|^2\,dx \;+\; k\;\mathrm{Re}\int v\,\overline{u}\;dx\Big]\,, \\
-8\,\mathrm{Im}\int(x\cdot\nabla\overline{u})\,u_t\;dx \;&=\; 8\,\mathrm{Re}\int(x\cdot\nabla\overline{u})\,\big[-\Delta u - g_1|u|^{p-1}u - g|v|^2\,u\big]\,dx \\
&\quad +\; 8k\;\mathrm{Re}\int(x\cdot\nabla\overline{u})\,v\;dx \;-\; 8\,\gamma_1\,\mathrm{Im}\int(x\cdot\nabla\overline{u})\,u\;dx \\
&=\; 8\int|\nabla u|^2\,dx \;-\; 4N\int|\nabla u|^2\,dx \\
&\quad +\; \frac{8N}{p+1}\,g_1\int|u|^{p+1}dx \;-\; 4\,g\int(x\cdot\nabla|u|^2)\,|v|^2\,dx \\
&\quad +\; 8k\;\mathrm{Re}\int(x\cdot\nabla\overline{u})\,dx \;+\; 8\,\gamma_1\,\mathrm{Im}\int(x\cdot\nabla u)\,\overline{u}\;dx\;.
\end{aligned}
$$

Then

$$
\begin{aligned}
V_1'(t) \;=\; 8\int|\nabla u|^2\,dx \;&+\; 4Ng_1\,\frac{1-p}{p+1}\int|u|^{p+1}dx \;-\; 4Ng\int|u|^2\,|v|^2\,dx \\
&-\; 4g\int(x\cdot\nabla|u|^2)\,|v|^2\,dx \;+\; 4k\,N\,\mathrm{Re}\int v\,\overline{u}\;dx \\
&+\; 8k\;\mathrm{Re}\int(x\cdot\nabla\overline{u})\,v\;dx \;+\; 8\,\gamma_1\,\mathrm{Im}\int(x\cdot\nabla u)\,\overline{u}\;dx\;.
\end{aligned}
\tag{4.6}
$$

Similarly,

$$
\begin{aligned}
V_2'(t) \;=\; 8\int|\nabla v|^2\,dx \;&+\; 4Ng_2\,\frac{1-p}{p+1}\int|v|^{p+1}dx \;-\; 4Ng\int|u|^2\,|v|^2\,dx \\
&-\; 4g\int(x\cdot\nabla|v|^2)\,|u|^2\,dx \;+\; 4k\,N\,\mathrm{Re}\int u\,\overline{v}\;dx \\
&+\; 8k\;\mathrm{Re}\int(x\cdot\nabla\overline{v})\,u\;dx \;+\; 8\,\gamma_2\,\mathrm{Im}\int(x\cdot\nabla v)\,\overline{v}\;dx\;.
\end{aligned}
\tag{4.7}
$$

By (4.5) we derive

$$
\begin{aligned}
Y''(t) \;&=\; V_1'(t)+V_2'(t)+2\,\gamma_1 Y_1'(t)+2\,\gamma_2\,Y_2'(t) \\
&=\; V_1'(t)+V_2'(t)+2\,\gamma_1 V_1(t)+2\,\gamma_2\,V_2(t) \\
&\quad +\; k(\gamma_1-\gamma_2)\;\mathrm{Im}\int|x|^2\,v\,\overline{u}\;dx \;+\; 4\,\gamma_1^2\,Y_1(t) \;+\; 4\,\gamma_2^2\,Y_2(t)
\end{aligned}
$$

and so we obtain, from (4.6), (4.7):

$$
\begin{aligned}
Y''(t) \;=\; & 8\int |\nabla u|^2\,dx \;+\; 8\int |\nabla v|^2\,dx \;+\; 4Ng_1\frac{1-p}{p+1}\int |u|^{p+1}\,dx \\
& +\;4Ng_2\,\frac{1-p}{p+1}\int |v|^{p+1}\,dx \;-\; 4Ng\int |u|^2\,|v|^2\,dx \\
& +\;16\,\gamma_1\,\mathrm{Im}\int (x\cdot\nabla u)\,\overline{u}\;dx \;+\; 16\,\gamma_2\,\mathrm{Im}\int (x\cdot\nabla v)\,\overline{v}\;dx \\
& +\;4k\,(\gamma_1-\gamma_2)\;\mathrm{Im}\int |x|^2\,v\,\overline{u}\;dx \;+\; 4\,\gamma_1^2\,Y_1(t) \;+\; 4\,\gamma_2^2\,Y_2(t)\;.
\end{aligned}
\tag{4.8}
$$

Now, we will assume $p-1>\frac{4}{N}$.

Since $p-1>\frac{4}{N}$ we can choose δ such that

$$
0<\delta<N \qquad \text{and} \qquad p-1>\frac{4}{\delta}\;.
$$

Rearranging the terms in the right-hand side of (4.8), we derive by (4.3)

$$
\begin{aligned}
Y''(t) \;=\; & \frac{16}{\delta}\,NE(t) \;+\; \frac{8}{\delta}\int (\delta-N)\,\left[|\nabla u|^2+|\nabla v|^2\right]dx \\
& +\;\frac{4N}{p+1}\left(\frac{4}{\delta}+1-p\right)\left[g_1\int |u|^{p+1}dx \;+\; g_2\int |v|^{p+1}dx\right] \\
& +\;\left(\frac{8}{\delta}-4\right)Ng\int |u|^2\,|v|^2\,dx \;+\; 16\,\gamma_1\,\mathrm{Im}\int (x\cdot\nabla u)\,\overline{u}\;dx \\
& +\;16\,\gamma_2\,\mathrm{Im}\int (x\cdot\nabla v)\,\overline{v}\;dx \;+\; 4k\,(\gamma_1-\gamma_2)\;\mathrm{Im}\int |x|^2\,v\,\overline{u}\;dx \\
& -\;8Nk\;\mathrm{Re}\int u\,\overline{v} \;+\; 4\,\gamma_1^2\,Y_1(t) \;+\; 4\,\gamma_2^2\,Y_2(t)\;.
\end{aligned}
\tag{4.9}
$$

First we assume $N=3$. If $g>0$, and since $p-1>\frac{4}{N}$ (we keep the notation with N by technical reasons) we choose δ such that

$$
2<\delta<N \qquad \text{and} \qquad p-1>\frac{4}{\delta}\;.
$$

If $g<0$ we must assume $p-1\geq 2$. In the case $p-1>2$ we choose $\delta=2$ and so the term $(\frac{8}{\delta}-4)\,Ng\int |u|^2\,|v|^2\,dx$ in (4.9) can be canceled. If $p-1=2$ we choose $\delta=\frac{N+2}{2}$: we easily check that $\delta<N$, $p-1=2>\frac{4}{\delta}$ and we have

$$
\begin{aligned}
& \frac{4N}{p+1}\left(\frac{4}{\delta}+1-p\right)\left[g_1\int |u|^{p+1}dx+g_2\int |v|^{p+1}dx\right]+\left(\frac{8}{\delta}-4\right)Ng\int |u|^2|v|^2dx \;= \\
& \quad =\;2\,N\left(\frac{N-2}{N+2}\right)\left[g_1\int |u|^4dx+g_2\int |v|^4dx\right]\;+\;4N\left(\frac{N-2}{N+2}\right)g\int |u|^2\,|v|^2dx
\end{aligned}
$$

and

$$\left| 2\,g \int |u|^2\,|v|^2\,dx \right| \le 2\,|g|\,\|u\|_{L^4}^2\,\|v\|_{L^4}^2$$
$$\le \frac{|g|}{\sqrt{g_1 g_2}}\left(g_1\|u\|_{L^4}^2 + g_2\|v\|_{L^4}^4\right) .$$

Collecting all these cases and taking in account that

$$16\,\gamma_1\,\mathrm{Im}\int (x\cdot\nabla u)\,\overline{u}\;dx \;+\; 16\,\gamma_2\,\mathrm{Im}\int (x\cdot\nabla v)\,\overline{v}\;dx \;\le$$
$$\le \frac{4}{\delta}(N-\delta)\left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_2^2\right) + \gamma^2\delta\,\frac{16}{N-\delta}\left(\|x\,u\|_{L^2}^2 + \|x\,v\|_{L^2}^2\right)$$

and

$$4k\,(\gamma_1-\gamma_2)\;\mathrm{Im}\int |x|^2\,v\,\overline{u}\;dx \;\le\; 4|k|\,\gamma\,Y(t)\ ,$$

it follows from (4.9)

$$Y''(t) + c_2\Big(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1}\Big) \le$$
$$\le c_1 Y(t) + \frac{16}{\delta}\,N E(t) + 4N|k|\,Q(t)\ , \tag{4.10}$$

with

$$c_1 \;=\; c_1(\gamma,k,N) \;=\; 4\,\gamma^2 + 4\,\gamma|k| + \gamma^2\delta\,\frac{16}{N-\delta} \tag{4.11}$$

and

$$c_2 \;=\; c_2(\gamma,k,N) \tag{4.12}$$
$$= \begin{cases} \min\Big\{\frac{4}{\delta}(N-\delta),\ \frac{4}{p+1}\ g_1\Big(p-1-\frac{4}{\delta}\Big),\ \frac{4}{p+1}\ g_2\Big(p-1-\frac{4}{\delta}\Big)\Big\} & \text{if } \ g>0\ ,\\ \min\Big\{2(N-2),\ \frac{4}{p+1}\ g_1(p-3),\ \frac{4}{p+1}\ g_2(p-3)\Big\} & \text{if } \ g<0 \ \text{ and } \ p-1>2\ ,\\ \frac{2}{5}\ \min\Big\{1,\ g_1 + \frac{g\sqrt{g_1}}{\sqrt{g_2}},\ g_2 + \frac{g\sqrt{g_2}}{\sqrt{g_1}}\Big\} & \text{if } \ -\sqrt{g_1 g_2}<g<0 \ \text{ and } \ p-1=2\ . \end{cases}$$

By applying Lemma 4.1 we derive, from (4.10), the following inequality

$$Y''(t) + c_2\Big(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1}\Big) \le$$
$$\le c_1 Y(t) + \frac{16}{\delta}\,N E(0) + c_4\,e^{2\gamma t} \tag{4.13}$$
$$+\ c_3\int_0^t \Big(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1}\Big)\,d\tau\ ,$$

with

$$c_3 = \frac{16}{\delta}\,N\gamma\,\widetilde{c}_3\,, \qquad c_4 = 4N\Big(|k| + \frac{4}{\delta}\,\widetilde{\widetilde{c}}_3\Big)Q(0)\ .$$

Next, we show that the inequality (4.13), which holds for $N = 3$, is also verified for $N = 1, 2$, up to some few changes in the constants.

The critical point is to dominate the term $(\frac{8}{\delta} - 4)\, Ng \int |u|^2 \, |v|^2 \, dx$ in (4.9).

Assume now $N = 2$. Since $p - 1 > \frac{4}{N} = 2$, we have

$$\int |u|^2 \, |v|^2 \, dx \;\le\; \frac{1}{2} \int |u|^{p+1} \, dx \;+\; \frac{1}{2} \int |v|^{p+1} dx \;+\; \frac{1}{2} \int \big(|u|^2 + |v|^2\big) \, dx$$

and we choose $\delta = \delta_2 < 2$ such that

$$p - 1 > \frac{4}{\delta_2}$$

and

$$\left(\frac{8}{\sqrt{2}} - 4\right)|g| \;<\; \min\left\{ \frac{4}{p+1} \; g_1\left(p - 1 - \frac{4}{\delta_2}\right), \; \frac{4}{p+1} \; g_2\left(p - 1 - \frac{4}{\delta_2}\right) \right\} .$$

It follows that (4.13) holds with constants c_2 and c_4 given by

$$c_2(\gamma, k, 2) \;=\; \min\left\{ \frac{4}{\delta_2} \, (2 - \delta_2), \; \frac{4}{p+1} \; g_1\left(p - 1 - \frac{4}{\delta_2}\right), \; \frac{4}{p+1} \; g_2\left(p - 1 - \frac{4}{\delta_2}\right) \right\}$$

and

$$c_4(\gamma, k, 2) \;=\; \left[8\left(|k| + \frac{4}{\delta_2} \, \widetilde{c}(2) \right) + 2\left(\frac{4}{\sqrt{\delta_2}} - 2 \right) |g| \right] Q(0) \; .$$

Finally:

Assume $N = 1$. We have $p - 1 > \frac{4}{N} = 4$ and we choose $\delta = \delta_1 < 1$ such that $p - 1 - \frac{4}{\delta_1} > 0$. By the Gagliardo–Nirenberg inequality we derive

$$\|u\|_{L^4}^4 \;\le\; c \, \|\nabla u\|_{L^2} \, \|u\|_{L^2}^3 \;\le\; \varepsilon \, \|\nabla u\|_{L^2}^2 + c(\varepsilon) \, \|u\|_{L^2}^6 \; ,$$

$\varepsilon > 0$ to be chosen.

Thus

$$\left(\frac{8}{\delta_1} - 4 \right) |g| \int |u|^2 \, |v|^2 \, dx \;\le$$

$$\le \; \left(\frac{4}{\delta_1} - 2 \right) |g| \, \varepsilon \left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right) + c(\varepsilon) \left(\|u\|_{L^2}^6 + \|v\|_{L^2}^6 \right)$$

and we choose ε such that

$$\left(\frac{4}{\delta_1} - 2 \right) |g| \, \varepsilon \;<\; \frac{2}{\delta_1} \, (1 - \delta_1) \; .$$

Once again we obtain (4.13) with the constants

$$c_2(\gamma, k, 1) \;=\; \min\left\{ \frac{2}{\delta_1} \, (2 - \delta_1), \; \frac{4}{p+1} \; g_1\left(p - 1 - \frac{4}{\delta_1}\right), \; \frac{4}{p+1} \; g_2\left(p - 1 - \frac{4}{\delta_1}\right) \right\} ,$$

and the term $c_4 \, e^{2\gamma t}$ is now replaced by $c_4(\gamma, k, 1) \, e^{6\gamma t}$ with

$$c_4(\gamma, k, 1) \;=\; 4\left(|k| + \frac{4}{\delta_1} \, \widetilde{c}_3(1) \right) Q(0) + c(\varepsilon) \, Q^3(0) \; .$$

Now, let

$$\rho(t) \;=\; \int_0^t\!\!\int_0^s \Big(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1} \Big)\, d\tau\, ds \; .$$

From (4.13) we obtain

$$Y(t) + c_2\,\rho(t) \;\le\; F(t) \,+\, c_1 \int_0^t\!\!\int_0^s Y(\tau)\, d\tau\, ds \,+\, c_3 \int_0^t \rho(\tau)\, d\tau \qquad (4.14)$$

with

$$F(t) \;=\; \begin{cases} Y(0) + Y'(0)\,t + \dfrac{8}{\delta}\, NE(0)\,t^2 + \dfrac{c_4(N)}{4\,\gamma^2}\,\big(e^{2\gamma t} - 2\,\gamma t - 1\big)\,, & \text{if } \; N > 1\,, \\[2ex] Y(0) + Y'(0)\,t + \dfrac{8}{\delta}\, E(0)\,t^2 + \dfrac{c_4(1)}{36\,\gamma^2}\,\big(e^{6\gamma t} - 6\,\gamma t - 1\big)\,, & \text{if } \; N = 1\;. \end{cases}$$

Next, we introduce the functions

$$M(t) \;=\; \sup_{\tau\in[0,t]} F(\tau) \,+\, 1\,, \qquad t \ge 0\;, \qquad (4.15)$$

$$G(t) \;=\; M(t)\left[c_1 \frac{t^2}{2} + \exp\Big(\frac{c_3}{c_2}\, t\Big) - 1 \right] . \qquad (4.16)$$

We can now state a blowup theorem for the supercritical case:

Theorem 4.1. *Assume* $p - 1 > \frac{4}{N}$ *and let us assume the Cauchy problem* (1.1) *with initial data* $(u_0, v_0) \in (H^1)^2$ *and let be* $(u,v) \in C([0, T_{\max}); (H^1)^2)$ *the corresponding local solution. Assume that* $(|x|u_0, |x|v_0) \in (L^2)^2$ *and, if* $N = 3$*, we have*

$$\begin{cases} p - 1 \ge 2 & \textit{if} \;\; (\gamma_1 + \gamma_2)\, g < 0\;, \\ p - 1 > 2 & \textit{if} \;\; g < 0\;, \\ p - 1 \ge 2 & \textit{if} \;\; -\sqrt{g_1 g_2} < g < 0\;. \end{cases}$$

Assume also that there exists $T_0 > 0$ *such that*

$$F(T_0) + 1 < 0\;, \qquad (4.17)$$

$$G(T_0) < 1\;. \qquad (4.18)$$

Then the solution (u,v) *blows up in finite time with* $T_{\max} \le T_0$.

Proof. Let us define

$$T_1 \;=\; \sup\Big\{ t \in [0, T_0] \,|\; Y(\tau) \le M(T_0),\; \tau \in [0,t] \Big\}\;.$$

It follows, from (4.14), (4.15), (4.17), that, for $t \in [0, T_1]$,

$$\begin{aligned} Y(t) + c_2\,\rho(t) \;&\le\; F(t) + c_1 M(T_0)\frac{T_0^2}{2} + c_3 \int_0^t \rho(\tau)\, d\tau \\ &\le M(T_0) - 1 + c_1 M(T_0)\frac{T_0^2}{2} + c_3 \int_0^t \rho(\tau)\, d\tau \qquad (4.19) \\ &<\; M(T_0) + c_3 \int_0^t \rho(\tau)\, d\tau\;. \end{aligned}$$

Applying Gronwall's inequality we obtain

$$\rho(t) \le \frac{M(T_0)}{c_2} \exp\left(\frac{c_3}{c_2}\, t\right).$$

Using this estimate, back in the right-hand side of (4.15), we derive

$$Y(t) \le F(t) + G(T_0)\,, \qquad t \in [0, T_1]\ ,$$

and by (4.18), $Y(T_1) < M(T_0)$. Then $T_1 = T_0$. Hence, $Y(T_0) < F(T_0)+1 < 0$, which is absurd since $Y \ge 0$. □

We have now two important remarks:

Remark 4.1. As can be seen by an adaptation of the proofs in Lemma 1 and Lemma 2 in [5], where the particular case $p-1=2$ is considered, the blowup assumptions (4.17) and (4.18) are satisfied, for a certain $T_0 > 0$, if the initial energy $E(0)$ (cf. (4.3)) is sufficiently negative or if the initial variance $Y(0)$ is sufficiently negative (depending of the initial data, γ_i, $i = 1, 2$, and k).

Remark 4.2. The blowup result of Theorem 4.1 can be extended to higher dimensions if $p - 1 > \frac{4}{N}$ with the same proof in the case $N = 3$.

Now, we consider a special case of the Cauchy problem for system (1.1), which includes the critical case, although require the absence of the linear coupling:

Theorem 4.2. *Assuming $k = 0$, $\gamma_1 = \gamma_2 = \sigma > 0$ and $g > 0$ and $p - 1 \ge \frac{4}{N}$ (critical and supercritical cases), let $(u, v) \in ([0, T_{\max}); (H^1)^2)$ be the local solution for the Cauchy problem for system (1.1) with initial data $(u_0, v_0) \in (H^1)^2$, $(x\, u_0, x\, v_0) \in (L^2)^2$. Then, if $E(0) < 0$, the solution blows up in finite time, that is $T_{\max} < +\infty$.*

Proof. We have in this case, by (4.4),

$$\begin{aligned}
\frac{dE}{dt} &= \sigma\Big[\int |\nabla u|^2\, dx + \int |\nabla v|^2\, dx - g_1 \int |u|^{p+1} dx - g_2 \int |v|^{p+1} dx \\
&\quad - 2\,\sigma g \int |u|^2\, |v|^2\, dx\Big] \\
&= \sigma\Big[\int |\nabla u|^2\, dx + \int |\nabla v|^2\, dx - \frac{2\, g_1}{p+1} \int |u|^{p+1} dx - \frac{2\, g_2}{p+1} \int |u|^{p+1} dx\Big] \\
&\quad - 2\,\sigma g \int |u|^2\, |v|^2\, dx + \sigma\left(\frac{2\, g_1}{p+1} - g_1\right) \int |u|^{p+1}\, dx \\
&\quad + \sigma\left(\frac{2\, g_2}{p+1} - g_2\right) \int |v|^{p+1}\, dx\ .
\end{aligned}$$

Since $\frac{2}{p+1} - 1 \le 0$, we derive, with $\gamma = |\gamma_1| = |\gamma_2|$, $\frac{dE}{dt} \le 2\,\gamma\, E$ and so $e^{-2\gamma t} E(t) \le E(0) < 0$.

Now, from (4.6) and (4.7) we deduce

$$\begin{aligned}\big(V_1(t)+V_2(t)\big)' - 2\,\sigma\big(V_1(t)+V_2(t)\big) \;=\; & \\ = 8\int |\nabla u|^2\,dx \,+\, 8\int |\nabla v|^2\,dx \,+\, 4Ng_1\frac{1-p}{1+p}\int |u|^{p+1}dx & \\ +\, 4Ng_2\frac{1-p}{1+p}\int |v|^{p+1}dx \,-\, 4Ng\int |u|^2\,|v|^2\,dx & \\ = 16\,E(t) + \frac{4\,g_1}{1+p}\,\big(N(1-p)+4\big)\int |u|^{p+1}dx & \\ +\, \frac{4\,g_2}{1+p}\,\big(N(1-p)+4\big)\int |v|^{p+1}dx \,-\, 4Ng\int |u|^2\,|v|^2\,dx\;. &\end{aligned}$$

Since $g > 0$ and $p - 1 \geq \frac{4}{N}$ we derive

$$\frac{d}{dt}\big[e^{-2\sigma t}(V_1+V_2)\big] \;\leq\; e^{-2\sigma t}E(t)\;.$$

Finally, from (4.5) we have

$$Y_1'(t) - 2\,\sigma\,Y_1(t) = V_1(t)\,, \qquad Y_2'(t) - 2\,\sigma\,Y_2(t) = V_2(t)\;,$$

and so

$$\frac{d}{dt}\big(e^{-2\sigma t}\,Y(t)\big) \;=\; e^{-2\sigma t}(V_1+V_2)\;.$$

Therefore,

$$\frac{d^2}{dt^2}\big(e^{-2\sigma t}\,Y(t)\big) \;=\; \frac{d}{dt}\big[e^{-2\sigma t}(V_1+V_2)\big] \;\leq\; e^{-2\sigma t}E(t) \;\leq\; E_0 \;<\; 0\;,$$

and the conclusion follows. □

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