A global attractor for a nonlocal parabolic problem

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Abstract

In this work we investigate the existence, uniqueness, continuity with respect to the initial values and exponential stability of the weak solutions for a nonlinear parabolic equation of reaction-diffusion nonlocal type by an application of the Faedo-Galerkin approximation and Aubin-Lions compactness result, coupled with some technical ideias. Moreover, we prove the joint continuity of the solution and an important result on the existence of the global attractor is presented.

Keywords: Asymptotic behavior, global attractor, nonlinear parabolic equation, existence and uniqueness of weak solutions.

1. Introduction

For the last several decades, various types of partial differential equations have been employed as some mathematical model describing physical, chem-

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ical, biological and ecological systems. Among them, the most successful and crucial one is the following model of semilinear parabolic partial differential equation, called the reaction-diffusion system

$$\frac{\partial u}{\partial t} - A\Delta u - f(u) = 0,$$

where $f : R^n \to R^n$ is a nonlinear function, and A is an $n \times n$ real matrix of diffusion (see [1, 2, 3, 4, 5, 6]). In 1998, L.A.F. Oliveira [7] considered the reaction-diffusion system where A is an $n \times n$ real matrix and $f : R^n \to R^n$ is a C^2 function. He studied the exponential decay for some cases. Except for some publications on the subject, such as the searching for traveling waves solutions and some problem in ecology and epidemic theory, most authors assume that the diffusion matrix A is diagonal, so that the coupling between the equations are present only on the nonlinearity of the reaction term f . However, cross-diffusion phenomena are not uncommon (see [8] and references therein) and can be treated as equations in which A is not even diagonalizable. In 1997, M. Chipot and B. Lovat [9] studied the existence and uniqueness of the solutions for the following nonlocal problems

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u = f(x,t) \text{ in } \Omega \times (0,T), \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset in R^n, $n \geq 1$ with smooth boundary $\partial\Omega$; T is some arbitrary time, and a is some function from R into $(0, +\infty)$.

This problem arises in various situations, for instance u could describe the density of a population (for instance of bacteria) subject to spreading. The diffusion coeffcient a is then supposed to depend on the entire population in the domain rather than on the local density, i.e., moves are guided by considering the global state of the medium.

In 2004, F.J.S.A. Corrêa, S.D.B. Menezes and J. Ferreira [8], gave a extension of the result obtained by M. Chipot and B. Lovat [9], considering $a = a(l(u))$ and $f = f(x,u)$ continuous functions. Indeed, in [8] the authors improved the results in [9, 10, 11] by considering both stationary and evolution situations where the nonlinearity appears not only in the operator $u \to a(l(u))\Delta u$ but also in the right-hand side in which one has the nonlinear function f.

The problem studied here is nonlocal in the sense that the diffusion coeffcient is determined by a global quantity. This kind of problem, besides its mathematical motivation because of the presence of the nonlocal term $a(l(u))$, arises from physical situations related to migration of a population of bacteria in a container in which the velocity of migration $v = a\nabla u$ depends on the global population in a subdomain $\Omega' \subset \Omega$ given by $a = a\left(\int_{\Omega'} udx\right)$.

In this paper we consider a nonlinear perturbation of the principal operator in (1), i.e., we study the following initial-boundary value problem involving a nonlinear parabolic equation of nonlocal type

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u + |u|^{p-2}\, u = f(u) \ \text{ in } \Omega \times (0,T), \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega, \end{cases} \tag{2}$$

where $u = u(x,t)$ is real valued functions, $\Omega \subset R^n$, $n \geq 1$, is a bounded smooth domain, with regular boundary $\Gamma = \partial\Omega$, $p \geq 2$. In problem (2) a and f are both continuous functions, whose hypotheses will be introduced in the next section, $l : L^2(\Omega) \to R$ is a continuous linear form.

Does the problem (2) have a solution? If afirmative, is the solution stabilized ? Does the problem (2) have a global attractor? Does the solution be continuous with respect to the initial values? We intend to answer these questions.

Until now few results have been obtained concerning the existence of maximal compact invariant global attractors.

M. Anguiano, P.E. Kloeden and T. Lorenz [12] proved the existence of a global attractor for a reaction-diffusion equation on a bounded domain with a nonlocal reaction term.

M. Chipot and B. Lovat [11] obtained results concerning the long time behavior of solutions for a reaction-diffusion equation on a bounded domain with a nonlocal diffusion term, but say nothing about the maximal compact invariant global attractor.

A. A. Ovono [13] considered a nonlinear reaction-diffusion equation on a bounded domain with a nonlocal diffusion term working with a linear form restricted to bounded balls

$$l(u) = l_g(u) = \int_{\Omega \cap B(x,\delta)} g(x)u(x)dx, \text{ for all } u \in L^2(\Omega).$$

He comments in Remark 4.5 on the difficulty to show that the semigroup, associated with this kind of nonlocal problems, is unifomly compact for large t (see [14]). Furthermore, it is remarked that the result can be proved, but don't make any computations.

We consider in our work a nonlinear reaction-diffusion equation on a bounded domain with a nonlocal diffusion term working with a more general linear form l of the form (5). To the best of our knowledge, this is the first work proving the existence of a global attractor for a nonlinear parabolic equation of reaction-diffusion with a nonlocal diffusion term working with the whole physical domain Ω.

This paper is organized as follows. Before the main result, in Section 2 we briefly outine the notation and terminology to be used subsequently. In Section 3 we present existence and uniqueness of the solution, in the Section 4 we prove the exponential stability of the solution of the equation and in Section 5 we prove continuity of the solution with respect to the initial values. Finally, Section 6 is devoted to proving the existence of the maximal compact invariant global attractor in $L^2(\Omega)$ for problem (2).

2. Preliminaries

In what follows, in sections 3, 4 and 5, we consider the evolution problem (2) in which $a : R \to R$ and $f : R \to R$ satisfies the conditions, $f : R \to R$ is a Lipschitz continuous function, that is, there exists $\gamma > 0$ such that

$$|f(s) - f(t)| \leq \gamma |s - t|, \forall s, t \in R, \text{ and } f(0) = 0. \tag{3}$$

$$0 < m \leq a(t) \leq M, \text{ for all } t \in R \text{ with } \lambda_1 > \frac{\gamma}{m}, \tag{4}$$

wher λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and $l : L^2(\Omega) \to R$ is a continuous linear form, i.e., there exists $g \in L^2(\Omega)$ such that

$$l(u) = l_g(u) = \int_\Omega g(x)u(x)dx, \text{ for all } u \in L^2(\Omega). \tag{5}$$

By $< \cdot, \cdot >$ we will represent the duality pairing between X and X', being X' the topological dual of the space X. We represent by $H^m(\Omega)$ the usual Sobolev space of ordem m, by $H_0^m(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$, and by

$L^2(\Omega)$ the class of square Lebesgue integrable real functions. In particular, $H_0^1(\Omega)$ has inner product $((\cdot,\cdot))$ and norm $\|\cdot\|$ given by

$$((u,v)) = \int_\Omega \nabla u.\nabla v dx \quad \text{and} \quad \|u\|^2 = \int_\Omega |\nabla u|^2 \, dx.$$

For the Hilbert space $L^2(\Omega)$ we represent its inner product and norm, respectively, by $(\cdot,\cdot)$ and $|\cdot|_2$ defined by

$$(u,v) = \int_\Omega u(x)v(x)dx \text{ and } |u|_2^2 = \int_\Omega u^2 dx.$$

Throughout this paper C is a generic positive constant, not necessarily the same at each ocasion. We take the initial conditions as following

$$u_0 \in L^2(\Omega).$$

We denote the potencial energy associated to (2) by

$$E(t) = \frac{1}{2}\int_\Omega |u|^2 \, dx. \tag{6}$$

3. Existence and uniqueness of a global solution

We should point out that the main tools in proving the existence and uniqueness of solution to the evolution equation will be the well-known Galerkin's Method and the Aubin-Lions's Theorem. For the sake of completeness we will enunciate the last one (see [15]).

Lemma 1. *(Aubin-Lions) Let B_0, B and B_1 be Banach spaces, where B_0 and B_1 are reflexive so that $B_0 \subset B \subset B_1$ and the immersion $B_0 \subset B$ is compact. If $1 < p_0, p_1 < \infty$, let*

$$W = \{w; w \in L^{p_0}(0,T;B_0) \text{ and } w_t \in L^{p_1}(0,T;B_1))\}$$

the Banach space with the norm

$$\|w\| = \|w\|_{L^{p_0}(0,T;B_0)} + \|w_t\|_{L^{p_1}(0,T;B_1)},$$

then the immersion $W \subset L^{p_0}(0,T;B)$ *is compact.*

Proof: (see [15])

Theorem 2. *(Existence) Let* $u_0 \in L^2(\Omega)$ *and* $0 < T < +\infty$. *If the conditions (3), (4) and (5) hold, then problem (2) possesses a global solution, i.e., there exists a function* u *such that*

$$u \in L^2(0,T;H_0^1(\Omega) \cap L^p(\Omega)) \cap C([0,T];L^2(\Omega)),$$

$$u_t \in L^2(0,T;H^{-1}(\Omega)),$$

$$u(0) = u_0,$$

$$\frac{d}{dt}(u,v) + a(l(u)) \int_\Omega \nabla u.\nabla v dx + (|u|^{p-2} u, v) = \int_\Omega f(u)v, \ \forall \ v \in H_0^1(\Omega), \quad (7)$$

where (7) must be understood as an equality in $D'(0,T)$.

Proof: . Let $\{w_n\}_{n\in N}$ be a Hilbertian basis in $H_0^1(\Omega)$ and V_n be the space generated by $w_{1,}w_2, ..., w_n$, $n = 1, 2......$

Let us consider

$$u_n(t) = \sum_{i=1}^{n} r_{in}(t) w_i$$

be the weak solution of the following approximate problem corresponding with (2), where $r_{in}(t)$ are the solution of the nonlinear ODE system in the variant t

$$\int_\Omega u_n'(t) w dx + a(l(u_n)) \int_\Omega \nabla u_n.\nabla w dx + \int_\Omega |u_n|^{p-2} u_n w dx = \int_\Omega f(u_n) w dx, \quad (8)$$

for all $w \in V_n$, with initial condition

$$u_n(0) = u_{0n} = \sum_{i=1}^{n} p_{in} w_i \to u_0 \text{ in } L^2(\Omega), \tag{9}$$

where $p_{in} = \int_\Omega u_0 w_i dx$ and $u' = \frac{\partial u}{\partial t}$.

As it is well known, the system (8)-(9) has a local solution $u_n(t)$ on some interval $[0, t_n)$, $0 < t_n < T$. We Claim that for any $T > 0$, such a solution can be extended to the whole interval $[0, T]$ by using the first a prior estimate below.

Setting $w = u_n(t)$ in (8), we obtain

$$\frac{1}{2}\frac{d}{dt} |u_n(t)|_2^2 + a(l(u_n)) \|u_n(t)\|^2 + \int_\Omega |u_n(t)|^p \, dx = \int_\Omega f(u_n(t)) u_n(t) dx. \tag{10}$$

From (3) and (4), we can write (10) as

$$\frac{1}{2}\frac{d}{dt} |u_n(t)|_2^2 + m \|u_n(t)\|^2 + |u_n(t)|_{L^p(\Omega)}^p \leq \gamma \int_\Omega |u_n(t)|^2 \, dx.$$

Let λ_1 be the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Then, thanks to Poincare's inequality

$$\frac{d}{dt} |u_n(t)|_2^2 + (2\lambda_1 m - 2\gamma) \|u_n(t)\|^2 + |u_n(t)|_{L^p(\Omega)}^p \leq 0. \tag{11}$$

Integrating (11) over $(0, t)$, and using that $u_n(0) \to u_0$ strongly in $L^2(\Omega)$, we obtain

$$|u_n(t)|_2^2 + (2\lambda_1 m - 2\gamma) \int_0^t \int_\Omega \|u_n(t)\|^2 \, dx + \int_0^t \int_\Omega |u_n(t)|_{L^p(\Omega)}^p \, dx \leq |u_n(0)|_2^2 \leq C, \tag{12}$$

where C is a positive constant that does not depend on t and n.

By (12) we have

$$(u_n) \text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \tag{13}$$

$$(u_n) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \tag{14}$$

$$(u_n) \text{ is bounded in } L^p(0,T;L^p(\Omega)). \tag{15}$$

From (8) we have that

$$u_n' = a(l(u_n))\Delta u_n - |u_n|^{p-2}\, u_n + f(u_n) \in H^{-1}(\Omega).$$

Notice that $-a(l(u_n))\Delta u_n$ defines an element of $H^{-1}(\Omega)$, given by the duality

$$< -a(l(u_n))\Delta u_n, w >= a(l(u_n)) \int_\Omega \nabla u_n.\nabla w dx, \text{ for all } w \in H_0^1(\Omega).$$

Since $u_n \in L^2(0,T;L^2(\Omega))$ and

$$\int_0^T |f(u_n(t))|_2^2\, dt \leq \gamma^2 \int_0^T |u_n(t)|_2^2\, dt$$

we have

$$f(u_n) \in L^2(0,T;L^2(\Omega)) \subset L^1(0,T;L^2(\Omega)),$$

and we concludes that

$$(u_n') \text{ is bounded in } L^2(0,T;H^{-1}(\Omega)). \tag{16}$$

From (13), (14) and (16) we can extract a subsequence (which we denote by the same symbol) such that

$$u_n \overset{*}{\rightharpoonup} u \;\text{ weak star in } L^\infty(0,T;L^2(\Omega)), \tag{17}$$

$$u_n \rightharpoonup u \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \tag{18}$$

$$u_n' \rightharpoonup u' \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \tag{19}$$

Consequently

$$\int_0^T (u_n, w_1)dt \to \int_0^T (u, w_1)dt,\ \forall w_1 \in L^1(0,T;L^2(\Omega)),$$

$$\int_0^T ((u_n, w_2))dt \to \int_0^T ((u, w_2))dt,\ \forall w_2 \in L^2(0,T;H_0^1(\Omega)),$$

$$\int_0^T < u_n', w_3 > dt \to \int_0^T < u', w_3 > dt,\ \forall w_3 \in L^2(0,T;H_0^1(\Omega)).$$

On the orther hand, $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. From the convergences (17) and (19) and from Aubin-Lions's Compactness Lemma, we get

$$u_n \to u \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{20}$$

hence, passing if necessary, to a subequence, still denote by (u_n), one has

$$u_n \to u \ \text{ a.e. in } \Omega \times (0,T).$$

Since that $s \mapsto |s|^{p-2}\, s$ is a continonus function, we obtain

$$|u_n|^{p-2}\, u_n \to |u|^{p-2}\, u \ \text{ a.e. in } \Omega \times (0,T). \tag{21}$$

From (15), observing that p and p' are conjugates, we conclude

$$(|u_n|^{p-2}\, u_n) \text{ is bounded in } L^{p'}(0,T;L^{p'}(\Omega)). \tag{22}$$

Therefore, from (21)-(22) and [[15], Chapter 1, Lemma 3], we infer that

$$|u_n|^{p-2}\, u_n \rightharpoonup |u|^{p-2}\, u \ \text{ weakly in } L^{p'}(0,T;L^{p'}(\Omega)),$$

which implies that

$$\int_0^T\!\!\int_\Omega |u_n(t)|^{p-2} u_n(t) w dx dt \to \int_0^T\!\!\int_\Omega |u(t)|^{p-2} u(t) w dx dt,\ \forall\, w \in L^p(0,T;L^p(\Omega)).$$

We choose $w = \theta v$ with $\theta \in D(0,T)$, $v \in L^2(\Omega)$ and we will show that for $\theta \in D(0,T)$ and for all $v \in L^2(\Omega)$,

$$\int_0^T [(f(u_n), v) - (f(u), v)]\theta(t)dt \to 0.$$

Indeed, using (20) and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\int_0^T [(f(u_n), v) - (f(u), v)]\theta(t)dt &= \int_0^T (f(u_n) - f(u), v)\theta(t)dt \\
&\leq \int_0^T \int_\Omega |f(u_n) - f(u)|\,|v|\,|\theta(t)|\,dxdt \\
&\leq C_\gamma \left(\int_0^T \int_\Omega |u_n - u|^2\,dxdt \right)^{\frac{1}{2}} \\
&\times \left(\int_0^T \int_\Omega |v|^2\,dxdt \right)^{\frac{1}{2}} \to 0.
\end{aligned}$$

As a consequence of the convergence (20), we have

$$\int_0^T |l(u_n) - l(u)|^2\,dt = \int_0^T |l(u_n - u)|^2\,dt \leq C \int_0^T |u_n - u|^2\,dt \to 0.$$

Therefore,

$$l(u_n) \to l(u) \text{ strongly in } L^2(0,T).$$

Since a is continuous, we have

$$a(l(u_n)) \to a(l(u)) \text{ in } L^2(0,T).$$

So, we get

$$\int_0^T a(l(u_n)) \int_\Omega \nabla u_n.\nabla v\theta(t)dxdt \to \int_0^T a(l(u)) \int_\Omega \nabla u.\nabla v\theta(t)dxdt,$$

for every $\theta \in D(0,T)$ and for every $v \in L^2(\Omega)$.

These convergence implies that we may take limits in the approximate problem (8) and (7) is verified. Now, we will make verify of the initial data.

In fact, using the result of regularity in [16] we have that $u \in C([0,T]; L^2(\Omega))$. Therefore, makes sense calculate $u(0)$. Let us consider $\theta \in C^1((0,T); R)$, with $\theta(0) = 1$ and $\theta(T) = 0$. Using the convergence (19) we have

$$\int_0^T (u_n', \eta)\theta dt \to \int_0^T (u', \eta)\theta dt, \ \eta \in L^2(\Omega). \tag{23}$$

Performing integration by parts in (23) we obtain

$$-(u_n(0), \eta) - \int_0^T (u_n, \eta)\theta' dt \to -(u(0), \eta) - \int_0^T (u, \eta)\theta' dt. \tag{24}$$

Using the convergence (18) in (24) we have $(u_n(0), \eta) \to (u(0), \eta)$, for all $\eta \in H_0^1(\Omega)$. But $u_n(0)$ converges strong for u_0 in $L^2(\Omega)$, consequently weak in $L^2(\Omega)$. Therefore $(u_n(0), \eta) \to (u_0, \eta)$, for all $\eta \in H_0^1(\Omega)$. From uniqueness of the limit we have $(u(0), \eta) = (u_0, \eta)$, for all $\eta \in H_0^1(\Omega)$. Thus, $u(0) = u_0$. So our problem (2) possesses a solution. □

To finish this section we will show the uniqueness of the weak solution. For this end, we need the

Lemma 3. *[17] (Tartar's inequality) If $p \geq 2$, hence for all $a, b \in R$, there exists a constant $\gamma_0 = \gamma_0(p) > 0$ such that*

$$< |a|^{p-2} a - |b|^{p-2} b, a - b > \ \geq \ \gamma_0 |a - b|^p .$$

Theorem 4. *(Uniqueness) Let $u_0 \in L^2(\Omega)$ and $0 < T < +\infty$. Let us assume that a is Lipschitz continuous in the sense that there exists a constant A such that*

$$|a(t) - a(s)| \leq A |t - s|, \ \forall t, s \in R. \tag{25}$$

If (3), (4) and (5) hold, then there is a unique solution of problem (2).

Proof: . Let us denote by u_1 and u_2 two solutions of (2). Then

$$(\frac{d}{dt}u_1, v) + a(l(u_1)) \int_\Omega \nabla u_1.\nabla v dx + (|u_1|^{p-2} u_1, v) = (f(u_1), v),$$

and

$$(\frac{d}{dt}u_2, v) + a(l(u_2)) \int_\Omega \nabla u_2.\nabla v dx + (|u_2|^{p-2} u_2, v) = (f(u_2), v).$$

Thus,

$$\begin{aligned}
&\left(\frac{d}{dt}(u_1 - u_2, v)\right) + a(l(u_1)) \int_\Omega \nabla u_1.\nabla v dx - a(l(u_2)) \int_\Omega \nabla u_2.\nabla v dx \\
= &\int_\Omega (|u_2|^{p-2} u_2 - |u_1|^{p-2} u_1, v) dx + (f(u_1) - f(u_2), v),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\left(\frac{d}{dt}(u_1 - u_2), v\right) + a(l(u_1))\int_\Omega \nabla(u_1 - u_2).\nabla v + \int_\Omega (|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2, v) dx \\
= &\ (a(l(u_2)) - a(l(u_1))) \int_\Omega \nabla u_2.\nabla v dx\gamma + (f(u_1) - f(u_2), v).
\end{aligned}$$

Taking $v = (u_1 - u_2)(t)$, for $a.e.\ t$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} |u_1 - u_2|^2 + a(l(u_1)) \int_\Omega |\nabla(u_1 - u_2)|^2 dx \\
+ &\int_\Omega (|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2)(u_1 - u_2) dx \\
\leq &\ |a(l(u_2)) - a(l(u_1))| \int_\Omega |\nabla u_2| |\nabla(u_1 - u_2)| dx + \gamma \int_\Omega |u_1 - u_2|^2 dx.
\end{aligned}$$

Using Lemma 3 we have

$$\int_\Omega (|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2)(u_1 - u_2) dx \geq \gamma_0 \|u_1 - u_2\|^p_{L^p(\Omega)} \geq 0.$$

So,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} |u_1 - u_2|^2 + a(l(u_1)) \int_\Omega |\nabla(u_1 - u_2)|^2 dx \\
\leq &\ |a(l(u_2)) - a(l(u_1))| \int_\Omega |\nabla u_2| |\nabla(u_1 - u_2)| dx + \gamma \int_\Omega |u_1 - u_2|^2 dx.
\end{aligned}$$

Now, the proof follows completely analogous as in the proof of Theorem 4 in [8]. □

4. Exponential Stability

In this section we show that the energy (6) associated to problem (2) decay exponential to zero as t tends to infinity.

Theorem 5. *Let u be the global solution of problema (2) given by the Theorems 2 and 4. For each $u_0 \in L^2(\Omega)$ the solution of the Cauchy problem (2) satisfies*

$$\lim_{t \to +\infty} |u(t)|_2 = 0, \tag{26}$$

and

$$|u(t)|_2 \leq |u_0|_2 \exp^{-(m\lambda_1 - \gamma)t}, \ 0 < t < \infty. \tag{27}$$

Proof: . Multiplying (2) by u we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 \, dx + a(l(u)) \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u|^p \, dx = \int_\Omega f(u) u dx.$$

Using (3)-(4) and Poincaré inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 \, dx + m\lambda_1 |u|_2^2 + |u|^p_{L^p(\Omega)} \leq \gamma |u|_2^2 .$$

Hence

$$\frac{d}{dt}|u|_2^2 + (2m\lambda_1 - 2\gamma) |u|_2^2 \leq 0.$$

Then (26) and (27) follows easily. □

5. Continuity of the solution with respect to the initial values

In this section we prove the joint continuity of the solution of problem (2). In particular, we get continuity of the solution with respect to the initial values. With this purpose we need the following

Theorem 6. *Let be u the solution of problem (2). For all $T > 0$, we have:*

$$\int_0^T ||u(s)||^2 ds \leq \mathcal{C}_1(|u_0|_2), \tag{28}$$

where $\mathcal{C}_1$ is a locally bounded function.

Proof: Multiplying the equation in (2) by u we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + a(l(u))\int_\Omega |\nabla u|^2\,dx + \int_\Omega |u|^p\,dx = \int_\Omega f(u)u dx.$$

Using (3)-(4) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + m\,\|u\|^2 + |u|^p_{L^p(\Omega)} \leq \gamma\,|u|_2^2\,.$$

Integrating over $(0,T)$, we have

$$|u|_2^2 + 2m\int_0^T \|u\|^2\,ds + 2\int_0^T |u|^p_{L^p(\Omega)}\,ds \leq |u_0|_2^2 + 2\gamma\int_0^T |u|_2^2\,ds. \tag{29}$$

Using the Poincaré inequality, we get

$$|u|_2^2 + (2m\lambda_1 - 2\gamma)\int_0^T |u|_2^2\,ds + 2\int_0^T |u|^p_{L^p(\Omega)}\,ds \leq |u_0|_2^2\,.$$

Then,

$$\int_0^T |u|_2^2\,ds \leq \frac{|u_0|_2^2}{(2m\lambda_1 - 2\gamma)}.$$

Using this last estimate in (29), we get

$$\int_0^T \|u\|^2\,ds \leq \frac{1}{2m}\left[|u_0|_2^2 + \frac{2\gamma\,|u_0|_2^2}{(2m\lambda_1 - 2\gamma)}\right] =: \mathcal{C}_1(|u_0|_2).$$

□

Theorem 7. *Let be $T > 0$ and $v_0 \in H$ fixed. If u and v are solutions of problem (2) with $u(0) = u_0$ and $v(0) = v_0$, then there exists a positive constant $C(T, |v_0|)$ such that*

$$\sup_{t\in[0,T]} |u(t) - v(t)|_2 \le C(T, |v_0|_2)|u_0 - v_0|_2.$$

Proof: Let u and v the solutions of problem (2) with $u(0) = u_0$ and $v(0) = v_0$. We have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|u-v|_2^2 + (-a(l(u))\Delta u + a(l(u))\Delta v, u-v) \\ + & \ (-a(l(u))\Delta v + a(l(v))\Delta v, u-v) + (|u|^{p-2}u - |v|^{p-2}v, u-v) \\ = & \ (f(u) - f(v), u-v). \end{aligned}$$

Using Lemma 3, (3), (4) and (25) we get

$$\frac{1}{2}\frac{d}{dt}|u-v|_2^2 + \frac{m}{2}\|u-v\|^2 \le \left(\gamma - \frac{C^2}{2m}\|v\|^2\right)|u-v|_2^2.$$

So,

$$\frac{d}{dt}|u-v|_2^2 \le \eta(t)|u-v|_2^2,$$

where $\eta(t) := 2\left(\gamma - \frac{C^2}{2m}\|v(t)\|^2\right)$. Multiplying by $\exp\{-\int_0^t \eta(s)ds\}$ we obtain

$$\frac{d}{dt}\left(\exp\{-\int_0^t \eta(s)ds\}|u-v|_2^2\right) \le 0.$$

Then, with some more computation we get

$$\sup_{t\in[0,T]} |u(t) - v(t)|_2 \le C(T, |v_0|_2)|u_0 - v_0|_2,$$

with $C(T, |v_0|_2) := \exp\{\gamma T + \frac{C^2}{2m}\mathcal{C}_1(|v_0|_2)\}$. □

Theorem 8. *(Joint Continuity) Let be $T > 0$. The map*

$$u : [0, T] \times H \to H$$

$$(t, u_0) \mapsto u(t, u_0)$$

is a continuous function, where $u(t, u_0)$ is the solution of problem (2) with initial value u_0 evaluated at time t.

Proof: Let $u(t, u_0)$ the solution of problem (2) with initial value u_0 evaluated at time t. Let be $(t_0, v_0) \in [0, T] \times H$ fixed. Given an arbitrarily $\epsilon > 0$ there is $\delta_1 > 0$ such that $|u(t, v_0) - u(t_0, v_0)|_2 < \frac{\epsilon}{2}$ whenever $|t - t_0| < \delta_1$. Taking $\delta := \min\left\{\frac{\epsilon}{2C(T, |v_0|_2)}, \delta_1\right\} > 0$, we have

$$\begin{aligned} |u(t, u_0) - u(t_0, v_0)|_2 &\leq |u(t, u_0) - u(t, v_0)|_2 + |u(t, v_0) - u(t_0, v_0)|_2 \\ &< C(T, |v_0|_2)\,|u_0 - v_0|_2 + \frac{\epsilon}{2} \\ &< C(T, |v_0|_2)\delta + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

whenever

$$\|(t, u_0) - (t_0, v_0)\|_{[0,T]\times H} < \delta.$$

□

6. Global attractor

For convenience to the reader we recall some definitions from Ladyzhenskaya [18] on (nonlinear) semigroups theory.

Definition 9. *Let* (X, d) *a complete metric space. A semigroup is a family of single-valued continuous operators* $T(t) : X \to X$ *depending on a parameter* $t \in \mathbb{R}^+$ *and enjoying the semigroup property:*

$$T(t_1)T(t_2)(x) = T(t_1 + t_2)(x), \text{ for all } t_1, t_2 \in \mathbb{R}^+ \text{ and } x \in X;$$

and $T(0) = I_d$.

Definition 10. *Let* A *and* M *be subsets of* X. *We say that* A *attracts* M *or* M *is attracted to* A *by semigroup* $\{T(t)\}_{t\geq 0}$ *if for every* $\epsilon > 0$ *there exists a* $t_1(\epsilon, M) \in \mathbb{R}^+$ *such that* $T(t)M \subset O_\epsilon(A) := \{x \in X; d(x, A) < \epsilon\}$ *for all* $t \geq t_1(\epsilon, M)$. *The set* $A \subset X$ *attracts the point* $x \in X$ *if* A *attracts the one-point set* $\{x\}$.

Definition 11. *If* A *attracts each point* x *of* X *then* A *is called a global attractor (for the semigroup).* A *is called a global B-attractor if* A *attracts each bounded set in* X.

Definition 12. *A semigroup is called bounded dissipative or B-dissipative (respectively pointwise dissipative) if it has a bounded global B-attractor (respectively a bounded global attractor).*

Definition 13. *A set* $A \subset X$ *is called invariant (relative to semigroup* $\{T(t)\}_{t\geq 0}$*) if* $T(t)A = A$, *for all* $t \in \mathbb{R}^+$.

Remark 14. *There exists a variety of descriptions for an attractor (or a global attractor) in the literature. A compact invariant global B-attractor in [18] is exactly a global attractor in [19] and [20] (see [21]).*

By (26) we already see that the null function in $H = L^2(\Omega)$ is the minimal closed global B-attractor, but it isn't sufficient to see that the null function is invariant by the semigroup, i. e., $T(t)(0) = 0$, $\forall t \geq 0$. We can answer this point using results on theory of semigroups.

Definition 15. *A semigroup $\{T(t)\}_{t\geq 0}$ belongs to the class $\mathcal{K}$ if for each $t > 0$ the operator $T(t)$ is compact, i. e., for any bounded set $B \subset X$ its image $T(t)B$ is precompact.*

In this section we will use the following theorem

Theorem 16. *[18] Let $\{T(t) : X \to X, t \geq 0\}$ be a semigroup of class $\mathcal{K}$. If it is B-dissipative, then $\{T(t) : X \to X, t \geq 0\}$ has a minimal closed global B-attractor $\mathcal{M}$, which is compact and invariant.*

Remark 17. *The minimal closed global B-attractor $\mathcal{M}$ in Theorem 16 is in fact the maximal compact invariant global B-attractor.*

Indeed, if $\mathcal{D}$ is an compact invariant global B-attractor, then, by Lemma 3.1 and Lemma 4.10 (ii) in [21] and Proposition 2.2 in [18], we have $\mathcal{D} = \omega(\mathcal{D}) \subset \mathcal{M}$.

For more details on (nonlinear) semigroups theory see [18, 19, 20, 21].

In this section we prove the existence of the minimal closed global B-attractor, which is compact and invariant, for problem (2). To do this we need to prove that the semigroup determined by (2) is of class $\mathcal{K}$ and is bounded dissipative in $\mathcal{H}$.

Theorem 18. *Let $\{T(t)\}$ be the semigroup associated with the problem (2) on $\mathcal{H}$. Then $\{T(t)\}$ is bounded dissipative in $\mathcal{H}$.*

Proof: By (26) we see that the unitary ball with center zero is a bounded global $B-$attractor. □

Theorem 19. *Let $\{T(t)\}$ be the semigroup associated with the problem (2) on $\mathcal{H}$. Then $\{T(t)\}$ is of class $\mathcal{K}$.*

Proof: Using (28), Theorem 8 and repeating the same arguments as in the proof of Theorem 3 in [22] we obtain the result. □

As a consequence of Theorem 16, Remark 17, Theorem 18 and Theorem 19 we obtain the following result

Theorem 20. *The semigroup $\{T(t)\}$ associated with problem (2) has a minimal closed global B-attractor $\mathcal{M}$, which coincides with the maximal compact invariant global B-attractor.*

Remark 21. *All the results in Sections 5 and 6 can also be reproduced for the problem (1).*

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