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A reaction-diffusion model for the nonlinear coupled system: existence, uniqueness, long time behavior and localization properties of solutions

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Abstract

In this work, we study the Dirichlet problem for a class of nonlinear coupled systems of reaction-diffusion nonlocal type:

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1 |u|^{p-2}u = f_1(x,t) & \text{in} \quad \Omega\times]0,T] \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2 |v|^{p-2}v = f_2(x,t) & \text{in} \quad \Omega\times]0,T] \\ u(x,t) = v(x,t) = 0 & \text{on} \quad \partial\Omega\times]0,T] \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \text{in} \quad \Omega \end{cases}.$$

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We prove the existence and uniqueness of weak and strong solutions of these systems and localization properties of the solutions, including the waiting time effect. Moreover important results on polynomial and exponential decay and vanishing of the solutions in finite time are also presented. We improve the results obtained by Chipot and Lovart [1], Corrêa, Menezes and Ferreira [2], Raposo et al. [3] and Simsen and Ferreira [4] for coupled systems.

Keywords: nonlinear parabolic system, nonlocal diffusion term, reaction-diffusion, strong solutions, localization properties, waiting time, polynomial and exponential decay

1. Introduction

For the last several decades, various types of equations have been employed as some mathematical model describing physical, chemical, biological and ecological systems. Among them, the most successful and crucial one is the following model of a semilinear parabolic partial differential equation, called the reaction-diffusion equation,

$$\frac{\partial u}{\partial t} - a\Delta u - f(u) = 0, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function and a is a real function called the diffusion coefficient.
In 1997, M. Chipot and B. Lovat [1] studied the existence and uniqueness of the solutions for nonlocal problems

$$\begin{cases} u_t - a(l(u))\Delta u = f(x,t) & \text{in} \quad \Omega \times (0,T) \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega \end{cases} \tag{2}$$

where Ω is a bounded open subset in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, T is some arbitrary time and a is some function from $\mathbb{R}$ into $(0, +\infty)$. In problem (2), a and f are both continuous functions and $l : L_2(\Omega) \to \mathbb{R}$ is a continuous linear form. This problem arises in various situations, for instance, u could describe the density of a population (for example, of bacteria) subject to spreading. The diffusion coefficient a is then supposed to depend on the entire population in the domain rather than on the local density, that is, movements are guided by considering the global state of the medium. The problem studied is nonlocal in the sense that the diffusion coefficient is determined by a global quantity. In 2004, Corrêa, Menezes and Ferreira [2] gave an extension of the result obtained by M. Chipot and B. Lovat [1], considering $a = a(l(u))$ and $f = f(x, u)$ continuous functions. Indeed, in [2], the authors improved the results in [1, 5, 6] by considering both stationary and evolutionary situations where the nonlinearity appears not only in the operator $u \to a(l(u))\Delta u$, but also in the right-hand side where one has the nonlinear function f. Recently, Simsen and Ferreira [4] studied the reaction-diffusion problem

$$\left\{\begin{array}{lll} u_t - a(l(u))\Delta u + |u|^{p-2}u = f(u) & \text{in} & \Omega \times (0, T) \\ u(x, t) = 0 & \text{on} & \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) & \text{in} & \Omega \end{array}\right. .$$

They investigate the existence, uniqueness, continuity with respect to the initial values, exponential stability of the weak solutions, continuity of the solution and an important result on the existence of the global attractor.
The differential equations are very useful in many areas of science, but the most interesting real life problems involve more than one unknown function. In this case, we have the reaction-diffusion system

$$\frac{\partial \mathbf{u}}{\partial t} - A\Delta\mathbf{u} - f(\mathbf{u}) = 0, \tag{3}$$

where $\mathbf{u} = (u_1, \ldots, u_m)$ is the vector of unknowns, $f : \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinear function, and A is an $m \times m$ real matrix of diffusion. In 1998, L.A.F. Oliveira [7] considered the reaction-diffusion system where A is an $m \times m$ real matrix and $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^2 function. In particular, he studied the exponential decay for some cases. Except for some publications on the subject, such as the search for travelling-wave solutions and some problems in ecology and epidemic theory, most authors assume that the diffusion matrix A is diagonal, so that the coupling between the equations is present only in the nonlinearity of the reaction term f. However, cross-diffusion phenomena are not uncommon (see [2] and references therein) and (3) can be treated as a system of equations in which A is not even diagonalizable.
Raposo et al. [3], in 2008, studied the reaction-diffusion coupled system in a parallel way, via a parameter $\alpha = const > 0$, of the form

$$\left\{\begin{array}{lll} u_t - a(l(u))\Delta u + f(u - v) = \alpha(u - v) & \text{in} & \Omega\times]0, T] \\ v_t - a(l(v))\Delta v - f(u - v) = \alpha(v - u) & \text{in} & \Omega\times]0, T] \end{array}\right. ,$$

with $a(\xi) > 0$, f a Lipschitz-continuous function and l a continuous linear form. They proved the existence, uniqueness and exponential decay of solutions.
We consider, in our work, a nonlinear coupled system of reaction-diffusion on a bounded domain with a more general nonlocal diffusion term working on two linear forms l_1 and l_2.

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1 |u|^{p-2}u = f_1(x,t) & \text{in} \quad \Omega\times]0,T] \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2 |v|^{p-2}v = f_2(x,t) & \text{in} \quad \Omega\times]0,T] \end{cases}. \tag{4}$$

In our case, u and v could describe the densities of two populations that interact through the functions a_1 and a_2. We assume that death in species u is proportional to $|u|^{p-2}u$ by the factor $\lambda_1 \geq 0$ and that death in species v is proportional to $|v|^{p-2}v$ by the factor $\lambda_2 \geq 0$ with $p > 1$. The supply of being by external sources is denoted by f_1 and f_2. In this paper, we are concerned with the proof of the existence and uniqueness of smooth global solutions for system (4). The other aim of our work is to study some localization properties and the asymptotic behavior when $t \to \infty$. We improve the results obtained by Chipot and Lovart [1], Corrêa, Menezes and Ferreira [2] and Raposo et al. [3]. To the best of our knowledge, these results are the first in this direction, for a nonlocal coupled reaction-diffusion system.
This paper is organized as follows. Following the formulation of the problem and the hypotheses on the data in Section 2, in Section 3, we prove the global existence and uniqueness of strong solutions. Moreover, in Section 4, we present the localization properties of local solutions. To finalize this study, we investigate, in Section 5, the asymptotic behavior of the solutions for large t and the vanishing in finite time property.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^N$ be a domain with smooth boundary $\partial\Omega$. We consider the problem of finding the couple (u, v) which satisfies the following conditions:

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1 |u|^{p-2}u = f_1(x,t) & \text{in} \quad \Omega\times]0,T] \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2 |v|^{p-2}v = f_2(x,t) & \text{in} \quad \Omega\times]0,T] \\ u(x,t) = v(x,t) = 0 & \text{on} \quad \partial\Omega\times]0,T] \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \text{in} \quad \Omega \end{cases}, \tag{5}$$

where λ_1, λ_2 are nonnegative constants and $p > 1$. In what follows, we make the following hypotheses:

Hyp1: $v_0, u_0 \in L_2(\Omega)$;

Hyp2: $a_i : \mathbb{R}^2 \to \mathbb{R}$ is bounded with $0 < m \leq a_i(s,r) \leq M$, $s, r \in \mathbb{R}$, $i = 1, 2$;

Hyp3: $a_i : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz-continuous with $|a_i(s_1, r_1) - a_i(s_2, r_2)| \leq A_i|s_1 - s_2| + B_i|r_1 - r_2|$, $s_1, s_2, r_1, r_2 \in \mathbb{R}$, $i = 1, 2$;

Hyp4: $l_i : L_2(\Omega) \to \mathbb{R}$ is a continuous linear form, that is, there exists a positive function $g_i \in L_2(\Omega)$ such that

$$l_i(u) = l_{g_i}(u) = \int_\Omega g_i(x)u(x)dx, \text{ for all } u \in L_2(\Omega), \ i = 1, 2; \tag{6}$$

Hyp5: $f_1, f_2 \in L_2(0, T; L_2(\Omega))$.

Definition 1 (Weak solution)**.** *We say that the couple (u, v) is a weak solution of System (5) if*

$$u, v \in L_2(0, T; H_0^1(\Omega) \cap L_p(\Omega)) \cap C([0, T]; L_2(\Omega)), \tag{7}$$

$$u_t, v_t \in L_2(0, T; H^{-1}(\Omega)), \tag{8}$$

$$\frac{d}{dt}\int_\Omega u\chi \ dx + a_1(l_1(u), l_2(v))\int_\Omega \nabla u.\nabla\chi \ dx + \lambda_1\int_\Omega |u|^{p-2}u\chi \ dx = \\ = \int_\Omega f_1\chi \ dx, \tag{9}$$

$$\frac{d}{dt}\int_\Omega v\psi \ dx + a_2(l_1(u), l_2(v))\int_\Omega \nabla v.\nabla\psi \ dx + \lambda_2\int_\Omega |v|^{p-2}v\psi \ dx = \\ = \int_\Omega f_2\psi \ dx, \tag{10}$$

for all $\chi, \psi \in H_0^1(\Omega)$, where (9) and (10) must be understood as an equality in $D'(0, T)$,

$$u(x, t) = v(x, t) = 0, \quad (x, t) \in \partial\Omega\times]0, T] \tag{11}$$

and

$$u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), \quad x \in \Omega. \tag{12}$$

3. Existence and uniqueness of global solutions

This section is devoted to the proof of the existence and uniqueness of a strong solution of System (5). We should point out that the main tools used in proving the existence and uniqueness of the solution to this system are the well-known Faedo-Galerkin method and Aubin-Lions lemma. For the sake of completeness, we state the last one and for simplicity, when there is no danger of confusion, we omit the domain in the norm's index.

Lemma 2 (Aubin-Lions). *Let B_0, B and B_1 be Banach spaces, where B_0 and B_1 are reflexive, $B_0 \subset B \subset B_1$ and the immersion $B_0 \subset B$ is compact. Let*

$$W = \{w : w \in L_{p_0}(0, T; B_0) \text{ and } w_t \in L_{p_1}(0, T; B_1)\}$$

be the Banach space with norm

$$\|w\| = \|w\|_{L_{p_0}(0,T;B_0)} + \|w_t\|_{L_{p_1}(0,T;B_1)} .$$

If $p_0, p_1 < \infty$, then the immersion $W \subset L_{p_0}(0, T; B)$ is compact.

Theorem 3 (Existence). *Let $p > 1$ and $0 < T < +\infty$. If Hyp1-Hyp5 hold, then there exists a weak solution (u, v) of System (5) in the sense of Definition 1.*

Proof. Let $\{w_n(x)\}_{n\in N}$ be a Hilbertian basis in $H_0^1(\Omega)$ and S_n the space generated by $w_1, w_2, \ldots, w_n$, $n = 1, 2, \ldots$.
Let

$$u_n(x,t) = \sum_{i=1}^{n} U_{in}(t)w_i(x) \text{ and } v_n(x,t) = \sum_{i=1}^{n} V_{in}(t)w_i(x)$$

be the weak solutions of the following approximate problem corresponding to (5), where $U_{in}(t)$ and $V_{in}(t)$ are the solutions of the nonlinear ODE system in the variable t:

$$\int_\Omega (u_n)_t w \, dx + a_1(l_1(u_n), l_2(v_n)) \int_\Omega \nabla u_n.\nabla w \, dx + \lambda_1 \int_\Omega |u|^{p-2} uw \, dx =$$

$$= \int_\Omega f_1 w \, dx \text{ for all } w \in S_n, \tag{13}$$

$$\int_\Omega (v_n)_t w \, dx + a_2(l_1(u_n), l_2(v_n)) \int_\Omega \nabla v_n.\nabla w \, dx + \lambda_2 \int_\Omega |v|^{p-2} vw \, dx =$$

$$= \int_\Omega f_2 w \, dx \text{ for all } w \in S_n, \tag{14}$$

with initial conditions

$$u_n(0) = u_{0n} = \sum_{i=1}^{n} U_{0in} w_i \to u_0 \text{ in } L_2(\Omega), \tag{15}$$

$$v_n(0) = v_{0n} = \sum_{i=1}^{n} V_{0in} w_i \to v_0 \text{ in } L_2(\Omega). \tag{16}$$

As is well known, System (13)-(16) has a local solution $(u_n(t), v_n(t))$ in some interval $[0, t_n[$, $0 < t_n < T$. We claim that, for any $T > 0$, such a solution can be extended to the whole interval $[0, T]$ by using the first a priori estimate below.

Setting $w = u_n(x,t)$ in (13), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L_2}^2 + a_1(l_1(u_n), l_2(v_n))\|u_n\|_{H_0^1}^2 + \lambda_1\|u_n\|_{L_p}^p = \int_\Omega f_1 u_n \, dx. \tag{17}$$

Using the Poincaré and Holder inequalities, we have that

$$\int_\Omega |f_1 u_n| \, dx \leq C\|f_1\|_{L_2}^2 + \frac{m}{2}\|u_n\|_{H_0^1}^2$$

and using Hyp2, we can write Equation (17) as

$$\frac{d}{dt}\|u_n\|_{L_2}^2 + m\|u_n\|_{H_0^1}^2 + 2\lambda_1\|u_n\|_{L_p}^p \leq C\|f_1\|_{L_2}^2. \tag{18}$$

Integrating (18) over $[0, t]$ and using the fact that $u_n(0) \to u_0$ strongly in $L_2(\Omega)$, we obtain

$$\|u_n\|_{L_2}^2 + m\int_0^t \|u_n\|_{H_0^1}^2 \, dt + 2\lambda_1 \int_0^t \|u_n\|_{L_p}^p \, dt \leq$$

$$\leq \|u_n(x,0)\|_{L_2}^2 + C\int_0^t \|f_1\|_{L_2}^2 \, dt \leq C, \tag{19}$$

where C is a constant that does not depend on t and n.
Setting $w = v_n(x,t)$ in (14) and applying the same arguments, we obtain

$$\|v_n\|_{L_2}^2 + m\int_0^t \|v_n\|_{H_0^1}^2 \, dt + 2\lambda_2 \int_0^t \|v_n\|_{L_p}^p \, dt \leq$$

$$\leq \|v_n(x,0)\|_{L_2}^2 + C\int_0^t \|f_2\|_{L_2}^2 \, dt \leq C, \tag{20}$$

where C is a constant that does not depend on t and n.
By (19) and (20), we have:

$$(u_n) \text{ and } (v_n) \text{ are bounded in } L_\infty(0,T;L_2(\Omega)), \tag{21}$$

$$(u_n) \text{ and } (v_n) \text{ are bounded in } L_2(0,T;H_0^1(\Omega)), \tag{22}$$

$$(u_n) \text{ and } (v_n) \text{ are bounded in } L_p(0,T;L_p(\Omega)). \tag{23}$$

Then we can extend the solution to the interval $[0,T]$. Now we need to pass to the limit when $n \to \infty$. We have

$$(u_n)_t = a_1(l_1(u_n), l_2(v_n))\Delta u_n - \lambda_1 |u_n|^{p-2} u_n + f_1 \in H^{-1}(\Omega).$$

Notice that $-a_1(l_1(u_n), l_2(v_n))\Delta u_n$ defines an element of $H^{-1}(\Omega)$, given by the duality

$$\langle -a_1(l_1(u_n), l_2(v_n))\Delta u_n, w\rangle = a(l(u_n))\int_\Omega \nabla u_n.\nabla w dx, \text{ for all } w \in H_0^1(\Omega).$$

Then

$$(u_n)_t \text{ is bounded in } L_2(0,T;H^{-1}(\Omega)). \tag{24}$$

Due to the Banach-Alouglu corollary, from (21), (22) and (24), we can extract subsequences $u_{mk} = u_m$ and $v_{mk} = v_m$(which we denote by the same symbol) such that

$$u_n \overset{*}{\rightharpoonup} u,\ v_n \overset{*}{\rightharpoonup} v \text{ in } L_\infty(0,T;L_2(\Omega)), \tag{25}$$

$$u_n \rightharpoonup u,\ v_n \rightharpoonup v \text{ in } L_2(0,T;H_0^1(\Omega)), \tag{26}$$

$$(u_n)_t \rightharpoonup u_t,\ (v_n)_t \rightharpoonup v_t \text{ in } L_2(0,T;H^{-1}(\Omega)). \tag{27}$$

On the other hand, $H_0^1(\Omega) \overset{c}{\hookrightarrow} L_2(\Omega) \hookrightarrow H^{-1}(\Omega)$. From (25), (27) and Aubin-Lions compactness lemma, we obtain

$$u_n \to u \text{ strongly in } L_2(0,T;L_2(\Omega)). \tag{28}$$

Hence, passing if necessary to a subsequence, still denoted by (u_n), one has

$$u_n \to u \text{ a.e in } \Omega \times (0,T).$$

Since $s \mapsto |s|^{p-2}\, s$ is a continuous function, we obtain

$$|u_n|^{p-2}\, u_n \longrightarrow |u|^{p-2}\, u \text{ a.e in } \Omega \times (0,T). \tag{29}$$

From (23) and observing that $p' = \frac{p}{p-1} > 1$ is the conjugate of $p > 1$, we conclude that

$$(|u_n|^{p-2}\, u_n) \text{ is bounded in } L_{p'}(0,T; L_{p'}(\Omega)). \tag{30}$$

Therefore, from (29) and (30), we infer that

$$|u_n|^{p-2}\, u_n \rightharpoonup |u|^{p-2}\, u \text{ in } L_{p'}(0,T; L_{p'}(\Omega))$$

and this implies that

$$\int_0^T \int_\Omega |u_n|^{p-2}\, u_n w \; dxdt \rightarrow \int_0^T \int_\Omega |u|^{p-2}\, uw \; dxdt,\; \forall w \in L_p(0,T; L_p(\Omega)).$$

To conclude the proof, we need to show that

$$\int_0^T a_1(l_1(u_n), l_2(v_n)) \int_\Omega \nabla u_n . \nabla w \; dxdt \rightarrow$$

$$\rightarrow \int_0^T a_1(l_1(u), l_2(v)) \int_\Omega \nabla u . \nabla w \; dxdt.$$

For this purpose, it is sufficient to prove that

$$a_1(l_1(u_n), l_2(v_n)) \rightarrow a_1(l_1(u), l_2(v)) \text{ in } L_2(0,T).$$

Since a is continuous, we will show that

$$l_1(u_n) \rightarrow l_1(u) \text{ in } L_2(0,T).$$

In fact,

$$\int_0^T |l_1(u_n) - l_1(u)|^2\, dt = \int_0^T |l_1(u_n - u)|^2\, dt \leq C \int_0^T \|u_n - u\|_{L_2}^2\, dt.$$

This last result is a consequence of the convergence in (28). For the second equation the process is identical. Now we will verify the initial data. Indeed, using the regularity result,

$$u \in C^0(0,T; L_2(\Omega)).$$

In this form, it makes sense to calculate $u(0)$. Let us consider $\theta \in C^1(0,T;\mathbb{R})$, with $\theta(0) = 1$ and $\theta(T) = 0$. As a result of the convergence in (27),

$$\int_0^T (u_n', \eta)\theta dt \to \int_0^T (u', \eta)\theta dt,\ \eta \in L_2(\Omega). \tag{31}$$

Performing integration by parts in (31),

$$-(u_n(0), \eta) - \int_0^T (u_n, \eta)\theta' dt \to -(u(0), \eta) - \int_0^T (u, \eta)\theta' dt. \tag{32}$$

Using the convergence in (26) and (32), $(u_n(0), \eta) \to (u(0), \eta)$ for all $\eta \in H_0^1(\Omega)$. But $u_n(0)$ converges strongly to u_0 in $L_2(\Omega)$ and consequently weakly in $L_2(\Omega)$. Therefore $(u_n(0), \eta) \to (u_0, \eta)$ for all $\eta \in H_0^1(\Omega)$. From the uniqueness of the limit, $(u(0), \eta) \to (u_0, \eta)$ for all $\eta \in H_0^1(\Omega)$. Thus $u(0) = u_0$. In a similar manner we can conclude that $v(0) = v_0$. Hence Problem (5) has a solution. □

Clearly, the regularity of (u, v) can be improved if we assume more regular initial data.

Theorem 4. *If $u_0, v_0 \in L_\infty(\Omega)$ and $\int_0^t \|f\|_{L_\infty}\, dt \le C$, then the weak solution (u, v) of System (5) satisfies*

$$u, v \in L_\infty(0, T; L_\infty(\Omega)).$$

Proof. If $k > 0$ and if we multiply the first equation by u^{2k-1} and integrate in Ω, we obtain

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega u^{2k}\, dx + (2k-1)a_1(l_1(u), l_2(v))\int_\Omega |\nabla u|^2 u^{2(k-1)}\, dx +$$

$$+\lambda_1 \int_\Omega |u|^{p+2k-2}\, dx = \int_\Omega f_1 u^{2k-1}\, dx.$$

Then

$$\frac{1}{2k}\frac{d}{dt}\|u\|_{L_{2k}}^{2k} \le \|f\|_{L_{2k}}\|u\|_{L_{2k}}^{2k-1} \ \Leftrightarrow\ \|u\|_{L_{2k}}^{2k-1}\frac{d}{dt}\|u\|_{L_{2k}} \le \|f\|_{L_{2k}}\|u\|_{L_{2k}}^{2k-1}.$$

Simplifying the factor $\|u\|_{L_{2k}}^{2k-1}$ and integrating in t results in

$$\|u\|_{L_{2k}} \le \|u_0\|_{L_{2k}} + \int_0^t \|f\|_{L_{2k}}\, dt.$$

Making $k \to \infty$, we then obtain the estimate for u. With the same process we can obtain the estimate for v. □

In fact, if the conditions of the last theorem are valid, then the solution is a strong solution in the sense that $u_t, \Delta u \in L_2(0, T; L_2(\Omega))$.

Theorem 5. *Suppose that* $u_0, v_0 \in L_\infty(\Omega) \cap H_0^1(\Omega)$ *and* $\int_0^t \|f\|_{L_\infty}\ dt \leq C$. *Then System (5) admits a strong solution* (u, v).

Proof. Multiplying the first equation by Δu and integrating in Ω, we have

$$\int_\Omega u_t \Delta u\ dx - \int_\Omega a_1(l_1(u), l_2(v))(\Delta u)^2\ dx + \lambda_1 \int_\Omega |u|^{p-2} u \Delta u\ dx =$$

$$= \int_\Omega f_1 \Delta u\ dx.$$

Applying Green's theorem to the first and last terms of the left-hand side, we obtain

$$\int_\Omega \nabla u_t \cdot \nabla u\ dx + \int_\Omega a_1(l_1(u), l_2(v))(\Delta u)^2\ dx +$$

$$+\lambda_1(p-1) \int_\Omega |u|^{p-2} |\nabla u|^2\ dx = -\int_\Omega f_1 \Delta u\ dx,$$

and hence

$$\frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla u|^2\ dx + \int_\Omega a_1(l_1(u), l_2(v))(\Delta u)^2\ dx +$$

$$+\lambda_1(p-1) \int_\Omega |u|^{p-2} |\nabla u|^2\ dx = -\int_\Omega f_1 \Delta u\ dx.$$

Integration in $[0, t]$ leads to

$$\frac{1}{2} \int_\Omega |\nabla u|^2\ dx + \int_0^t \int_\Omega a_1(l_1(u), l_2(v))(\Delta u)^2\ dxdt +$$

$$+\lambda_1(p-1) \int_0^t \int_\Omega |u|^{p-2} |\nabla u|^2\ dxdt = \frac{1}{2} \int_\Omega |\nabla u_0|^2\ dx - \int_0^t \int_\Omega f_1 \Delta u\ dxdt.$$

By Cauchy's inequality and the lower bound of a_1, we conclude that

$$\frac{1}{2} \int_\Omega |\nabla u|^2\ dx + m \int_0^t \int_\Omega (\Delta u)^2\ dxdt + \lambda_1(p-1) \int_0^t \int_\Omega |u|^{p-2} |\nabla u|^2\ dxdt \leq$$

$$\leq \frac{1}{2} \int_\Omega |\nabla u_0|^2\ dx + C \int_0^t \int_\Omega f_1^2\ dxdt + \frac{m}{2} \int_0^t \int_\Omega (\Delta u)^2\ dxdt.$$

Simplifying,

$$\int_\Omega |\nabla u|^2\ dx + m\int_0^t\int_\Omega (\Delta u)^2\ dxdt \le C\left(\int_\Omega |\nabla u_0|^2\ dx + \int_0^t\int_\Omega f_1^2\ dxdt\right).$$

Using the same arguments for v and the hypothesis of the theorem, we can prove that

$$\Delta u, \Delta v \in L_2(0,T;L_2(\Omega)).$$

Recalling the first two equations we can now show that

$$u_t = a_1(l_1(u), l_2(v))\Delta u - \lambda_1|u|^{p-2}u + f_1 \in L_2(0,T;L_2(\Omega))$$

and

$$v_t = a_2(l_1(u), l_2(v))\Delta v - \lambda_2|v|^{p-2}v + f_2 \in L_2(0,T;L_2(\Omega)).$$

□

Next we prove the uniqueness of strong solutions.

Theorem 6 (Uniqueness)**.** *Let $p > 1$ and $0 < T < +\infty$. If Hyp1-Hyp5 hold, then there is at most one strong solution to Problem (5).*

Proof. Let (u_1, v_1) and (u_2, v_2) be two solutions of System (5). Then

$$\begin{cases} (u_i)_t - a_1(l_1(u_i)l_2(v_i))\Delta u_i + \lambda_1|u_i|^{p-2}u_i = f_1 \\ (v_i)_t - a_2(l_1(u_i)l_2(v_i))\Delta v_i + \lambda_2|v_i|^{p-2}v_i = f_2 \end{cases},\ i = 1,2.$$

Subtracting, integrating in Ω and multiplying the first equation by $r = u_1 - u_2$ and the second equation by $s = v_1 - v_2$, we obtain

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|r\|_{L_2}^2 + a_1(l_1(u_1), l_2(v_1))\|r\|_{H_0^1}^2 + \lambda_1(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, u_1 - u_2) \le \\ \qquad \le |a_1(l_1(u_2), l_2(v_2)) - a_1(l_1(u_1), l_2(v_1))| \int \nabla u_2 \cdot \nabla r\ dx, \\ \frac{1}{2}\frac{d}{dt}\|s\|_{L_2}^2 + a_2(l_1(u_1), l_2(v_1))\|s\|_{H_0^1}^2 + \lambda_2(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2, v_1 - v_2) \le \\ \qquad \le |a_2(l_1(u_2), l_2(v_2)) - a_2(l_1(u_1), l_2(v_1))| \int \nabla v_2 \cdot \nabla s\ dx. \end{cases}$$

Adding and using the fact that $(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, u_1 - u_2) \ge 0$ and $(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2, v_1 - v_2) \ge 0$, we have

$$\frac{1}{2}\frac{d}{dt}(\|r\|_{L_2}^2 + \|s\|_{L_2}^2) + m(\|r\|_{H_0^1}^2 + \|s\|_{H_0^1}^2) \leq$$
$$\leq |a_1(l_1(u_2), l_2(v_2)) - a_1(l_1(u_1), l_2(v_1))| \int_\Omega \nabla u_2 \cdot \nabla r \, dx +$$
$$+ |a_2(l_1(u_2), l_2(v_2)) - a_2(l_1(u_1), l_2(v_1))| \int_\Omega \nabla v_2 \cdot \nabla s \, dx.$$

From the properties of l_i, it follows that

$$\frac{1}{2}\frac{d}{dt}(\|r\|_{L_2}^2 + \|s\|_{L_2}^2) + m(\|r\|_{H_0^1}^2 + \|s\|_{H_0^1}^2) \leq$$
$$\leq (A_1|l_1(u_2) - l_1(u_1)| + B_1|l_2(v_2) - l_2(v_1)|)\|u_2\|_{H_0^1}\|r\|_{H_0^1} +$$
$$+ (A_2|l_1(u_2) - l_1(u_1)| + B_2|l_2(v_2) - l_2(v_1)|)\|v_2\|_{H_0^1}\|s\|_{H_0^1}$$
$$\leq (A_1|l_1(u_2 - u_1)| + B_1|l_2(v_2 - v_1)|)\|u_2\|_{H_0^1}\|r\|_{H_0^1} +$$
$$+ (A_2|l_1(u_2 - u_1)| + B_2|l_2(v_2 - v_1)|)\|v_2\|_{H_0^1}\|s\|_{H_0^1}$$
$$\leq C_1(\|u_2 - u_1\|_{L_2} + \|v_2 - v_1\|_{L_2})\|u_2\|_{H_0^1}\|r\|_{H_0^1} +$$
$$+ C_2(\|u_2 - u_1\|_{L_2} + \|v_2 - v_1\|_{L_2})\|v_2\|_{H_0^1}\|s\|_{H_0^1}.$$

Thus

$$\frac{1}{2}\frac{d}{dt}(\|r\|_{L_2}^2 + \|s\|_{L_2}^2) + \frac{m}{2}(\|r\|_{H_0^1}^2 + \|s\|_{H_0^1}^2) \leq \frac{M(t)}{2}(\|r\|_{L_2}^2 + \|s\|_{L_2}^2),$$

with

$$M(t) = \frac{C_1^2}{4m}\|u_2\|_{H_0^1}^2 + \frac{C_2^2}{4m}\|v_2\|_{H_0^1}^2 \in L_1(0, T).$$

Now defining $y(t) = (\|r\|_{L_2}^2 + \|s\|_{L_2}^2)$,

$$\frac{dy}{dt} \leq M(t)y, \quad y(0) = 0.$$

Hence $y(t) = 0$ and consequently $u_1 = u_2$ and $v_1 = v_2$. □

4. Localization properties

In this section, we study some localization properties of solutions for these systems. We assume that all the hypotheses of the existence and uniqueness

theorems are fulfilled.
Take $x_0 \in \Omega$, $\rho_0 \in]0, dist(x_0, \partial\Omega)]$ and define

$$B_\rho \equiv B_\rho(x_0) = \{x \in \mathbf{R}^n : |x - x_0| < \rho\} \subset \Omega,$$

$$S_\rho \equiv S_\rho(x_0) = \partial B_\rho(x_0).$$

The first property we study is stable localization. In what follows, we assume that

$$u(x,0) = 0 \text{ and } v(x,0) = 0 \text{ a.e. in } B_{\rho_0}. \tag{33}$$

Definition 7 ([8])**.** *A function $\rho(t) : [0, t^+[\to [0, +\infty[$, $\rho(0) \le \rho_0$, is called a rate at the point x_0 if for each $t \in [0, t^*[$,*

$$u(x,t) = 0 \text{ a.e. in } B_{\rho(t)}(x_0) = \{x : |x - x_0| < \rho(t)\} \subset \Omega.$$

Definition 8 ([8])**.** *A function $u(x,t)$ is said to have the property of **stable localization** if for some $x_0 \in \Omega$ there exists a strictly positive rate $\rho(t)$, at the point x_0, defined on the whole of $[0, \infty[$, such that*

$$\liminf_{t\to\infty} \rho(t) > 0.$$

We introduce the local energy functions for u:

$$b_u(\rho,t) = ||u(\cdot,t)||^2_{L_2(B_\rho)}, \qquad \overline{b}_u(\rho) = \sup_{0\le\tau\le T} b_u(\rho,\tau),$$

$$E_u(\rho,t) = \int_0^t \int_{B_\rho} |\nabla u|^2 \; dxd\tau, \qquad \overline{E}_u(\rho) = \sup_{0\le\tau\le T} E_u(\rho,\tau),$$

$$D_u(\rho,t) = \int_0^t \int_{B_\rho} |u|^p \; dxd\tau \text{ and } \quad \overline{D}_u(\rho) = \sup_{0\le\tau\le T} D_u(\rho,\tau).$$

The local energy functions for v are defined analogously. By *sup* we mean *ess sup*. We set

$$\mathbf{b} = b_u + b_v, \quad \mathbf{E} = E_u + E_v \quad \text{and} \quad \mathbf{D} = D_u + D_v.$$

Without loss of generality, we always assume that

$$\overline{b}_u(\rho) + \overline{E}_u(\rho) + \overline{D}_u(\rho) + \overline{b}_v(\rho) + \overline{E}_v(\rho) + \overline{D}_v(\rho) \le M_3, \; \rho \le \rho_0, \; t \le T \tag{34}$$

Some steps are very similar for both functions, so, in those cases, we only present the theory for u and, if there is no danger of confusion, we omit the index u. Since E and D are monotone and non-decreasing in ρ and t, the following weak derivatives exist:

$$\begin{cases} E_\rho(\rho,t) = \int_0^t \int_{S_\rho} |\nabla u|^2 \ dsd\tau, & D_\rho(\rho,t) = \int_0^t \int_{S_\rho} |u|^p \ dsd\tau \\ E_t(\rho,t) = \int_{B_\rho} |\nabla u|^2 \ dx, & D_t(\rho,t) = \int_{B_\rho} |u|^p \ dx \\ E_{\rho t}(\rho,t) = \int_{S_\rho} |\nabla u|^2 \ ds, & D_{\rho t}(\rho,t) = \int_{S_\rho} |u|^p \ ds. \end{cases}$$

Theorem 9 (Stable localization)**.** *Let (u,v) be a solution of Problem (5) in $B_{\rho_0} \times (0,T), (B_{\rho_0} \subset \Omega)$ with $1 < p < 2$ and assume that*

$$u(x,0)=0,\ v(x,0)=0,\ f_1(x,0)=0,\ f_2(x,t)=0, \quad (x,t) \in B_{\rho_0} \times [0,T]. \tag{35}$$

Then

$$u(x,t) = 0 \text{ and } v(x,t) = 0 \text{ for a.a. } (x,t) \in B_\rho \times \mathbb{R}^+, \tag{36}$$

where ρ is defined by the formula in (45).

Proof. If we multiply the first equation of (5) by u and integrate in B_ρ for $\rho \leq \rho_0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{B_\rho} u^2 \ dx + \int_{B_\rho} a_1|\nabla u|^2 + \lambda_1|u|^p \ dx = \int_{S_\rho} a_1 u \nabla u \cdot n \ ds.$$

Integrating in $]0,t[$ and since we assume (33), we conclude that

$$\frac{1}{2}\int_{B_\rho} u^2 \ dx + \int_0^t \int_{B_\rho} a_1|\nabla u|^2 \ dxd\tau + \int_0^t \int_{B_\rho} \lambda_1|u|^p \ dxd\tau =$$

$$= \int_0^t \int_{S_\rho} a_1 u \nabla u \cdot n \ dsd\tau.$$

Changing the notation, we have

$$\frac{1}{2}b(\rho) + mE(\rho,t) + \lambda_1 D(\rho,t) \leq I. \tag{37}$$

Now we evaluate I in the following way:

$$
\begin{aligned}
|I| &\leq M\int_0^t\int_{S_\rho}|\nabla u|\,|u|\;\;dsd\tau \\
&\leq M\left(\int_0^t\int_{S_\rho}|\nabla u|^2\;\;dsd\tau\right)^{\frac{1}{2}}\left(\int_0^t\int_{S_\rho}|u|^2\;\;dsd\tau\right)^{\frac{1}{2}} \\
&\leq M\left(E_\rho\right)^{\frac{1}{2}}\left(\int_0^t\int_{S_\rho}|u|^2\;\;dsd\tau\right)^{\frac{1}{2}}. \qquad (38)
\end{aligned}
$$

Next, we apply the multiplicative inequality

$$
\|u\|_{L_2(S_\rho)} \leq C\left(\|\nabla u\|_{L_2(B_\rho)} + \rho^{-\delta}\|u\|_{L_q(B_\rho)}\right)^\theta \times \left(\|u\|_{L_q(B_\rho)}\right)^{1-\theta}, \qquad (39)
$$

with

$$
\theta = \frac{n(2-q)+q}{n(2-q)+2q} < 1,\;\; 1<q<2,\;\; \delta = -\left(1+\frac{2-q}{2q}n\right)
$$

and we use

$$
\|u\|_{L_q(B_\rho)} \leq \|u\|_{L_2(B_\rho)}^{\frac{2(q-p)}{q(2-p)}}\|u\|_{L_p(B_\rho)}^{\frac{p(2-q)}{q(2-p)}} = \|u\|_{L_2(B_\rho)}^{\frac{(2-p)}{2}}\|u\|_{L_p(B_\rho)}^{\frac{p}{2}},\;\; p<q=\frac{4}{4-p}<2.
$$

In our notation, we have

$$
\|u\|_{L_q(B_\rho)} \leq b^{\frac{(2-p)}{4}}\;\; D_t^{\frac{1}{2}}
$$

and (39) takes the form

$$
\|u\|_{L_2(S_\rho)} \leq C\left(E_t^{\frac{1}{2}} + \rho^{-\delta} b^{\frac{(2-p)}{4}}\;\; D_t^{\frac{1}{2}}\right)^\theta\left(b^{\frac{(2-p)}{4}}\;\; D_t^{\frac{1}{2}}\right)^{1-\theta}. \qquad (40)
$$

Recalling (38), we have successively

$$
\begin{aligned}
|I| &\leq M\left(E_\rho\right)^{\frac{1}{2}}\left(\int_0^t\int_{S_\rho}|u|^2\;\;dsd\tau\right)^{\frac{1}{2}} \\
&\leq C\left(E_\rho\right)^{\frac{1}{2}}\left(\int_0^t\left(E_t^{\frac{1}{2}} + \rho^{-\delta} b^{\frac{(2-p)}{4}}\;\; D_t^{\frac{1}{2}}\right)^{2\theta}\left(b^{\frac{(2-p)}{4}}\;\; D_t^{\frac{1}{2}}\right)^{2(1-\theta)}\;\; d\tau\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq C\rho^{-\delta\theta}K_1\left(E_\rho\right)^{\frac{1}{2}}\ \overline{b}^{\frac{(2-p)(1-\theta)}{4}}\left(\int_0^t \left(E_t+ D_t\right)^{\theta}\left(\ D_t\right)^{(1-\theta)}\ d\tau\right)^{\frac{1}{2}}$$

$$\leq C\rho^{-\delta\theta}K_1\left(E_\rho\right)^{\frac{1}{2}}\ \overline{b}^{\frac{(2-p)(1-\theta)}{4}}\left(\int_0^t \left(E_t+ D_t\right)\ d\tau\right)^{\frac{1}{2}}$$

$$\leq C\rho^{-\delta\theta}K_1\left(E_\rho\right)^{\frac{1}{2}}\ \overline{b}^{\frac{(2-p)(1-\theta)}{4}}\left(E+ D\right)^{\frac{1}{2}}$$

$$\leq C\rho^{-\delta\theta}K_1\left(E_\rho\right)^{\frac{1}{2}}\ \left(\overline{b}+E+ D\right)^{\frac{1}{2}+\frac{(2-p)(1-\theta)}{4}},$$

with $K_1=\max(1,\overline{b}^{\frac{(2-p)\theta}{4}}(\rho_0))\max(1,\rho_0^{\delta\theta})$.
Substituting the last inequality in (37), we obtain

$$\overline{b}+\overline{E}+\overline{D}\leq C\rho^{-\delta\theta}K_1\left(\overline{E}_\rho\right)^{\frac{1}{2}}\ \left(\overline{b}+\overline{E}+\overline{D}\right)^{\frac{1}{2}+\frac{(2-p)(1-\theta)}{4}}$$

or

$$\left(\overline{b}+\overline{E}+\overline{D}\right)^{\frac{1}{2}-\frac{(2-p)(1-\theta)}{4}}\leq C\rho^{-\delta\theta}K_1\left(\overline{E}_\rho\right)^{\frac{1}{2}}.$$

If we set

$$\nu=\frac{(2-p)(1-\theta)}{2}<1,$$

then

$$\overline{E}^{1-\nu}\leq\rho^{-2\delta\theta}(CK_1)^2\overline{E}_\rho \tag{41}$$

or, in a more complete notation,

$$\overline{E}_u^{1-\nu}\leq\rho^{-2\delta\theta}(CK_1)^2(\overline{E}_u)_\rho. \tag{42}$$

Similarly,

$$\overline{E}_v^{1-\nu}\leq\rho^{-2\delta\theta}(CK_2)^2(\overline{E}_v)_\rho. \tag{43}$$

Adding these two inequalities, we have

$$(\overline{E}_u+\overline{E}_v)^{1-\nu}\leq\rho^{-2\delta\theta}K_3(\overline{E}_u+\overline{E}_v)_\rho\Leftrightarrow\overline{\mathbf{E}}^{1-\nu}\leq\rho^{-2\delta\theta}K\overline{\mathbf{E}}_\rho \tag{44}$$

and integrating this inequality in $]\rho,\rho_0[$, we obtain

$$\rho_0^\omega-\rho^\omega\leq\frac{K\omega}{\nu}\left(\overline{\mathbf{E}}^{\nu}(\rho_0)-\overline{\mathbf{E}}^{\nu}(\rho)\right),\ \omega=1+2\delta\theta.$$

Thus $\overline{\mathbf{E}}^{\nu}(\rho)=0$ if the function $\rho(t)$ satisfies

$$\rho^\omega(t)=\rho_0^\omega-\frac{K\omega}{\nu}\overline{\mathbf{E}}^{\nu}(\rho_0). \tag{45}$$

This result implies that if the global energy

$$\left(\overline{b}_u(\rho_0) + \overline{E}_u(\rho_0) + \overline{D}_u(\rho_0) + \overline{b}_u(\rho_0) + \overline{E}_u(\rho_0) + \overline{D}_u(\rho_0)\right)$$

is not large, then $\rho(t) > 0$ and there exists a cylinder $B_\rho \times \mathbb{R}^+$ where u and v are zero a.e.. □

Now we investigate the waiting time property in the sense of the following definition.

Definition 10 ([8])**.** *Given $x_0 \in \Omega$, let*

$$\rho_0 = \sup\{\rho > 0 : u(x,0) = 0 \text{ a.e. in } B_\rho(x_0) \subset \Omega\}.$$

We say that $u(x,t)$ has the ***generalized waiting time property*** *if, for some $t^* > 0$, the function $\rho(t) \equiv \rho_0$ is a rate at the point x_0 on the interval $[0, t^*]$.*

In this case, we assume that for $0 < \rho_0 < R$,

$$\int_{B_{\rho_0}} |u_0|^2 \, dx = 0, \ \int_{B_{\rho_0}} |v_0|^2 \, dx = 0, \ x \in B_{\rho_0}, \tag{46}$$

$$f_1(x,t) = 0, \ f_2(x,t) = 0, \ (x,t) \in B_{\rho_0} \times [0,T] \tag{47}$$

and, in addition,

$$\int_{B_\rho} |u_0|^2 \, dx + \int_{B_\rho} |v_0|^2 \, dx + \int_0^T \int_{B_\rho} |f_1|^{\frac{p}{p-1}} \, dx d\tau +$$

$$+ \int_0^T \int_{B_\rho} |f_2|^{\frac{p}{p-1}} \, dx d\tau \le \varepsilon \left(\rho - \rho_0\right)_+^{\frac{1}{\nu}}, \tag{48}$$

where ν is defined in the previous theorem.
We assume that

$$\sup_{t \in [0,T]} \int_{B_R} |u|^2 + |v|^2 \, dx +$$

$$+ \int_0^T \int_{B_R} |\nabla u|^2 + |\nabla v|^2 + |u|^p + |v|^p \ dx dt \le M_R < \infty \tag{49}$$

and that the parameters $M_R, \ \nu, \ R \ , \rho_0$ satisfy

$$G = M_R - C_5 \left(\frac{M_R}{\nu}\right)^{\frac{1}{1-\nu}} (R - \rho_0)^{-\frac{\nu}{1-\nu}} - C_6 \varepsilon \left(R - \rho_0\right)^{\frac{1}{\nu}} = 0, \tag{50}$$

where $C_5 = C_5(\overline{\mathbf{b}}(R))$ and $C_6(p,n)$ is a constant in embedding inequality.

Theorem 11 (Waiting time effect)**.** *Let (u,v) be a local weak solution of Problem (5) in $B_R\times]0,T]$ with $1<p<2$ and suppose that (46)-(50) hold. Then*

$$u(x,t)=0 \text{ and } v(x,t)=0,\quad (x,t)\in B_{\rho_0}\times[0,T]. \tag{51}$$

Proof. Repeating the arguments of the proof of Theorem 9 for $\rho_0\le\rho\le R$, we arrive at the inequalities

$$\overline{b}_u+E_u(\rho,t)+D_u(\rho,t)\le C_1(E_u)_\rho^{\frac{1}{1-\nu}}+C_2\varepsilon\,(\rho-\rho_0)_+^{\frac{1}{\nu}}, \tag{52}$$

$$\overline{b}_v+E_v(\rho,t)+D_v(\rho,t)\le C_3(E_v)_\rho^{\frac{1}{1-\nu}}+C_4\varepsilon\,(\rho-\rho_0)_+^{\frac{1}{\nu}}. \tag{53}$$

Adding the last two inequalities, we obtain

$$\overline{\mathbf{b}}+\mathbf{E}(\rho,t)+\mathbf{D}(\rho,t)\le C_5\mathbf{E}_\rho^{\frac{1}{1-\nu}}+C_6\varepsilon\,(\rho-\rho_0)_+^{\frac{1}{\nu}} \tag{54}$$

and, using the properties

$$\mathbf{E}_\rho\le\Psi_\rho,\quad \Psi=\mathbf{E}+\mathbf{D}=\int_0^t\int_{B_\rho}\left(\left|\nabla u^2\right|+\left|\nabla v^2\right|+|u|^p+|v|^p\right)dxd\tau,$$

$$\sup_{\tau\in[0,t]}\frac{\partial}{\partial\rho}\Psi_\rho(\rho,\tau)=\sup_{\tau\in[0,t]}\left(\int_0^t\int_{S_\rho}\left(\left|\nabla u^2\right|+\left|\nabla v^2\right|+|u|^p+|v|^p\right)dsd\tau\right)=$$

$$=\left(\int_0^T\int_{S_\rho}\left(\left|\nabla u^2\right|+\left|\nabla v^2\right|+|u|^p+|v|^p\right)dsd\tau\right)=\frac{\partial}{\partial\rho}\left(\sup_{\tau\in[0,t]}\Psi(\rho,t)\right),$$

we can rewrite (54) in the form

$$W(\rho)\le C_5W_\rho^{\frac{1}{1-\nu}}+C_6\varepsilon\,(\rho-\rho_0)_+^{\frac{1}{\nu}}, \tag{55}$$

where

$$W(\rho)=\sup_{t\in[0,T]}\Psi(\rho,t)=\int_0^T\int_{B_\rho}\left(\left|\nabla u^2\right|+\left|\nabla v^2\right|+|u|^p+|v|^p\right).$$

Let us now consider the problem

$$\phi(\rho)=C_5\phi_\rho^{\frac{1}{1-\nu}}+C_6\varepsilon\,(\rho-\rho_0)_+^{\frac{1}{\nu}},\ \ \rho_0\le\rho\le R. \tag{56}$$

This problem has a solution of the form

$$\phi(\rho) = M_R \,(R-\rho_0)^{-\frac{1}{\nu}}\,(\rho-\rho_0)^{\frac{1}{\nu}} \tag{57}$$

if the parameters $M_R,\ \nu,\ R\ ,\rho_0$ satisfy

$$G = M_R - C_5 \left(\frac{M_R}{\nu}\right)^{\frac{1}{1-\nu}} (R-\rho_0)^{-\frac{\nu}{1-\nu}} - C_6 \varepsilon\,(R-\rho_0)^{\frac{1}{\nu}} = 0.$$

For fixed parameters $M_R,\ \nu,\ \rho_0$ it is sufficient to choose R sufficiently large and ε sufficiently small. It is easy to verify that $\phi(\rho)$ is an upper bound for $W(\rho)$ and, since $\phi(\rho_0) = 0$ and due to the monotony in ρ of W, the proof is completed. □

5. Asymptotic behavior for large t

We terminate with the study of the asymptotic behavior of the solutions when $t \to \infty$. For this purpose, we introduce the global energy function

$$\mathfrak{b}(t) = \frac{1}{2}\int_\Omega u^2 + v^2 \; dx.$$

Theorem 12 (Exponencial decay)**.** *If $f_1 \equiv f_2 \equiv 0$ and $\lambda_1 = \lambda_2 = 0$, then function $\mathfrak{b}$, with (u,v) a solution of Problem (5), satisfies*

$$\mathfrak{b}(t) \leq \mathfrak{b}(0) e^{-\alpha t}$$

for α a positive constant which depends on m and Ω.

Proof. Multiplying the first equation by u, the second by v, integrating in Ω and adding these two equations, we obtain

$$\frac{d}{dt}\mathfrak{b}(t) + \int_\Omega a_1(l_1(u), l_2(v))|\nabla u|^2 \; dx + \int_\Omega a_2(l_1(u), l_2(v))|\nabla v|^2 \; dx = 0.$$

By the lower bound of a_i and applying Poincaré's inequality, we conclude that

$$\frac{d}{dt}\mathfrak{b}(t) + 2Cm\mathfrak{b}(t) \leq 0.$$

So

$$\mathfrak{b}(t) \leq \mathfrak{b}(0) e^{-2Cmt}.$$

Thus the claim is proved with $\alpha = 2Cm > 0$. □

Theorem 13 (Polynomial decay). *Suppose that $p > 2$ and that (u,v) is solution of Problem (5).*

i) If $f_1 \equiv f_2 \equiv 0$, then $\mathfrak{b}$ satisfies the estimate

$$\mathfrak{b}^{\mu-1}(t) \leq \frac{\mathfrak{b}(0)^{\mu-1}}{1 + tC(\mu-1)\mathfrak{b}(0)^{\mu-1}},$$

where $\mu = \frac{p}{2}$ and C depends only on N, p and Ω.

ii) If $f_1 \not\equiv 0$ or $f_2 \not\equiv 0$, there exists $K > 0$ and $B > 0$, depending only on N, p and Ω, such that

$$\|f_i\|_{L_2} \leq \frac{K}{(Bt+1)^{\frac{2\mu-1}{2(\mu-1)}}}, \quad i = 1, 2 \tag{58}$$

implies that

$$\mathfrak{b}(t) \leq \frac{\mathfrak{b}(0)}{(Bt+1)^{\frac{1}{\mu-1}}}.$$

Proof. Multiplying the first equation by u and integrating in Ω,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\, dx + \int_\Omega a_1(l_1(u), l_2(v))|\nabla u|^2\, dx + \lambda_1 \int_\Omega |u|^p\, dx = \int_\Omega f_1 u\, dx. \tag{59}$$

By Holder's theorem,

$$\int_\Omega |u|^p\, dx \geq C\left(\int_\Omega u^2\, dx\right)^\mu, \quad \mu = \frac{p}{2} \geq 1.$$

Substituting this last inequality in (59) and ignoring the second term, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\, dx + C\left(\int_\Omega u^2\, dx\right)^\mu \leq \left|\int_\Omega f_1 u\, dx\right|. \tag{60}$$

If $f_1 \equiv f_2 \equiv 0$, adding the last inequality to the similar inequality for v, we arrive at the following differential inequality for $\mathfrak{b}$:

$$\mathfrak{b}' + C\mathfrak{b}^\mu \leq 0.$$

Integrating, we obtain

$$\mathfrak{b}^{\mu-1} \leq \frac{\mathfrak{b}(0)^{\mu-1}}{1 + Ct(\mu-1)\mathfrak{b}(0)^{\mu-1}}.$$

If $f_1 \not\equiv 0$, then we can write (60) as

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\ dx + C\left(\int_\Omega u^2\ dx\right)^\mu \le \left(\int_\Omega f_1^2\ dx\right)^{\frac{1}{2}}\left(\int_\Omega u^2\ dx\right)^{\frac{1}{2}}. \qquad (61)$$

Adding (61) to the equivalent for v, we arrive at a new differential inequality for $\mathfrak{b}$:

$$\mathfrak{b}' + C_1\mathfrak{b}^\mu \le C_2(t)\mathfrak{b}^{\frac{1}{2}}.$$

Here $C_2(t) = C\max\{\|f_1\|_{L_2}, \|f_2\|_{L_2}\}$ and introducing the function $\mathfrak{g}(t) = \mathfrak{b}^{\frac{1}{2}}$ and using (58), we can write the last inequality in the form

$$\mathfrak{g}' + \frac{C_1}{2}\mathfrak{g}^{2\mu-1} \le \frac{K}{(Bt+1)^{\frac{2\mu-1}{2(\mu-1)}}}.$$

Now we consider the ODE

$$\mathfrak{h}' + \frac{C_1}{2}\mathfrak{h}^{2\mu-1} = \frac{K}{(Bt+1)^{\frac{2\mu-1}{2(\mu-1)}}}. \qquad (62)$$

If the constants satisfy

$$K < C\mathfrak{b}(0)^{\frac{2\mu-1}{2}}, \quad B = (\mathfrak{b}(0)^{\frac{2\mu-1}{2}} - K)(\mu-1)\mathfrak{b}(0)^{-\frac{1}{2}},$$

then

$$\mathfrak{h} = \frac{\mathfrak{b}_0^{\frac{1}{2}}}{(Bt+1)^{\frac{1}{2(\mu-1)}}}$$

are solutions of this equation and are upper bounds for the functions $\mathfrak{g}$. Reverting to function $\mathfrak{b}$, we get the desired estimate. □

Theorem 14 (Vanishing in a finite time). *Suppose that $1 < p < 2$ and that (u, v) is solution of Problem (5).*

i) If $f_i \equiv 0$, i=1,2, then (u, v) vanishes in a finite time, that is,

$$u(x,t) \equiv 0 \text{ and } v(x,t) \equiv 0 \text{ in } \Omega \text{ for } t > t^*,$$

where t^* depends only on $\|u_0\|_{L_2}$, $\|v_0\|_{L_2}$, p and Ω.

ii) If $f_1 \not\equiv 0$ or $f_2 \not\equiv 0$, then there exist $\epsilon' > 0$ and $t' > t^*$ such that

$$\|f_i\|_{L_2} \leq \epsilon \left[1 - \frac{t}{t_f}\right]_+^{\frac{2\mu-1}{2-2\mu}}, \ i = 1,2, \tag{63}$$

with $t_f \geq t'$, μ defined by (65) and where $0 < \epsilon < \epsilon'$ implies that (u,v) vanishes in a finite time.

Proof. Multiplying the first equation by u and integrating in Ω,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega a_1(l_1(u),l_2(v))|\nabla u|^2\,dx + \lambda_1\int_\Omega |u|^p\,dx = \int_\Omega f_1 u\,dx$$

or

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + C\int_\Omega |\nabla u|^2 + |u|^p\,dx = \int_\Omega f_1 u\,dx. \tag{64}$$

By the multiplicative inequality,

$$\int_\Omega u^2\,dx \leq \left(\int_\Omega |\nabla u|^2\,dx\right)^\theta \left(\int_\Omega |u|^p\,dx\right)^{\frac{2(1-\theta)}{p}} \leq$$

$$\leq \left(\int_\Omega |\nabla u|^2\,dx + |u|^p\,dx\right)^{\theta+\frac{2(1-\theta)}{p}}, \ \theta \in]0,1[.$$

Therefore

$$\int_\Omega |\nabla u|^2\,dx + |u|^p\,dx \geq \left(\int_\Omega u^2\,dx\right)^\mu,$$

where

$$\mu = \frac{p}{p\theta + 2(1-\theta)} < 1. \tag{65}$$

Substituting this last inequality in (64), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + C\left(\int_\Omega u^2\,dx\right)^\mu \leq \int_\Omega f_1 u\,dx. \tag{66}$$

If $f_i \equiv 0$, $i = 1,2$, adding the last inequality to the equivalent inequality for v, we arrive at the following differential inequality for $\mathfrak{b}$:

$$\mathfrak{b}' + C\mathfrak{b}^\mu \leq 0.$$

Integrating, we obtain

$$\mathfrak{b}^{1-\mu} \leq \mathfrak{b}(0)^{1-\mu} - Ct(1-\mu),$$

whence

$$\mathfrak{b} = 0 \text{ for } t \geq t^* = \frac{\mathfrak{b}(0)^{1-\mu}}{C(1-\mu)} < \infty$$

and the same occurs for u and v.
Now suppose that $f_1 \not\equiv 0$ and write (66) as

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \, dx + C\left(\int_\Omega u^2 \, dx\right)^\mu \leq \left(\int_\Omega f_1^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega u^2 \, dx\right)^{\frac{1}{2}}. \tag{67}$$

Adding (67) to the homologous inequality for v, we arrive at a new differential inequality for $\mathfrak{b}$:

$$\mathfrak{b}' + C_1\mathfrak{b}^\mu \leq C_2(t)\mathfrak{b}^{\frac{1}{2}}.$$

Here, $C_2(t) = C\max\{\|f_1\|_{L_2}, \|f_2\|_{L_2}\}$. Using (63) and introducing the function $\mathfrak{g}(t) = \mathfrak{b}^{\frac{1}{2}}$, we can write the last inequality in the form

$$\mathfrak{g}' + \frac{C_1}{2}\mathfrak{g}^{2\mu-1} \leq \frac{\epsilon}{2}\left[1 - \frac{t}{t_f}\right]_+^{\frac{2\mu-1}{2-2\mu}}.$$

Now we consider the ODE

$$\mathfrak{h}' + \frac{C_1}{2}\mathfrak{h}^{2\mu-1} = \frac{\epsilon}{2}\left[1 - \frac{t}{t_f}\right]_+^{\frac{2\mu-1}{2-2\mu}}. \tag{68}$$

The solutions of this equation are upper bounds for the functions $\mathfrak{g}$. It is easy to verify that if $\mathfrak{h}(0)$, ϵ and t_f satisfy

$$\frac{C}{2}\mathfrak{h}(0)^{2\mu-1} - \frac{\mathfrak{h}(0)}{2(1-\mu)t_f} - \frac{\epsilon}{2} \geq 0, \tag{69}$$

then the functions

$$\mathfrak{h} = \mathfrak{h}(0)\left[1 - \frac{t}{t_f}\right]_+^{\frac{1}{2-2\mu}}$$

are solutions of (68).
For a given $\mathfrak{h}(0)$, we can always choose ϵ and t_f such that (69) is true and hence the claim is proved. □

6. Conclusions

We studied a nonlinear system of parabolic equations. The existence and uniqueness of strong global solutions was proved and we derived some localization and asymptotic properties of the solution. The application of the Euler-Galerkim finite element method to this problem is in progress.

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