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A reaction-diffusion model for a class of nonlinear parabolic equations with moving boundaries: existence, uniqueness, exponential decay and simulation

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Abstract

The aim of this paper is to establish the existence, uniqueness and asymptotic behaviour of a strong regular solution for a class of nonlinear equations of reaction-diffusion nonlocal type with moving boundaries:

$$\begin{cases} u_t - a\left(\int_{\Omega_t} u(x,t)dx\right) u_{xx} = f\left(x,t\right) \ , \quad (x,t) \in Q_t \\ u\left(\alpha(t),t\right) = u\left(\beta(t),t\right) = 0 \, , \quad t > 0 \\ u(x,0) = u_0(x) \, , \quad x \in \Omega_0 =]\alpha(0), \beta(0)[\end{cases} ,$$

where Q_t is a bounded non-cylindrical domain defined by

$$Q_t = \left\{(x,t) \in \mathbb{R}^2 : \ \alpha(t) < x < \beta(t), \ \text{ for all } 0 < t < T\right\} \ .$$

Moreover, we study the properties of the solution and implement a numerical algorithm based on the Moving Finite Element Method (MFEM) with polynomial approximations of any degree, to solve this class of problems. Some numerical tests are investigated to evaluate the performance of our Matlab code based on the MFEM and illustrate the exponential decay of the solution.

Keywords: Moving boundary; Nonlocal diffusion term; Strong solution; Moving finite elements; Matlab solver

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1. Introduction

In this paper, we are concerned with the proof of the existence and uniqueness of a strong global solution for a class of non local parabolic problems (P_u) on a bounded domain with lateral moving boundaries. We are also interested in obtaining the exponential energy decay of the system. Problem (P_u) has applicability in a large class of real models, namely, in biology where the solution u could describe the population density subject to evolution [5] and is nonlocal in the sense that the diffusion coefficient a is determined by a global quantity and u depends on the population in the area.

This type of diffusion coefficient was initially proposed by Chipot and Lovat in [4] in an open bounded cylindrical domain. In [8], the authors considered $a = a(l(u))$ and assumed nonlinearity on the right-hand side where one has the nonlinear function $f = f(u)$. The asymptotic behaviour of the solutions as time tends to infinity was studied by Zheng and Chipot [19]

for a nonlinear parabolic equation with two classes of nonlocal terms, in a cylindrical domain. Cavalcanti et al ([3]) worked with a time-dependent form $a = a\left(t, \int_{\Omega_t} |\nabla u(x,t)|^2 \, dx\right)$ to establish the solvability and exponential energy decay of the solution for a model given by a hyperbolic-parabolic equation in a open bounded subset of $\mathbb{R}^n$, with moving boundary.

Moving boundary problems occur in many physical applications involving diffusion, such as in heat transfer where a phase transition occurs, in moisture transport such as swelling grains or polymers, and in deformable porous media problems where the solid displacement is governed by diffusion. These problems have been extensively studied by several authors such as Briozzo [2], Santos [15], Benabidallah [10] and Ferreira [11]. However, they have not addressed the numerical analysis and simulation of the problem and it is our aim to present an algorithm which allows us to perform the numerical analysis directly on the original problem. We use the Moving Finite Element Method with piecewise polynomials of any degree based on an adaptive mesh to study the behaviour and regularity of the solution. The MFEM was originally introduced by Miller [13] and expands the classical Galerkin finite element method on a fixed grid by allowing movement of the space nodes in time. In this work, we apply a Matlab implementation (see [14]) which generalizes the numerical algorithm developed by Sereno [16, 17] and Coimbra [6, 7]. In their work, the spatial domain had fixed boundaries.

To the best of our knowledge, these results are the first in this direction, for a nonlocal reaction-diffusion equation with moving boundaries.

This paper is organized as follows: in section two, we present the formulation of the problem and the hypotheses on the data. In the two following sections, we prove the global existence and uniqueness of a strong solution and investigate the asymptotic behaviour of the solution for large t. In section 5, we give a briefly description of the MFEM and apply it to obtain an approximate numerical solution. To finalize this study, in section 6, we draw some conclusions.

2. Statement of the problem

In this work, we study the solutions of the one-dimensional Dirichlet problem with two moving boundaries defined by

$$(\mathrm{P}_u)\begin{cases} u_t - a\left(\displaystyle\int_{\Omega_t} u(x,t)dx\right) u_{xx} = f\,(x,t)\ , \quad \text{for all } (x,t)\in Q_t \\ u\,(\alpha(t),t) = u\,(\beta(t),t) = 0\,, \quad \text{for all } t\in]0,T[\\ u(x,0) = u_0(x)\,, \quad x\in]\alpha(0),\beta(0)[\end{cases}, \tag{1}$$

where Q_t is a bounded non-cylindrical domain defined by

$$Q_t = \left\{(x,t)\in\mathbb{R}^2 :\ \alpha(t) < x < \beta(t),\ \text{for all } 0<t<T\right\}\ ,$$

T is an arbitrary positive real number, $u_t = \partial u/\partial t$, $u_{xx} = \partial^2 u/\partial x^2$ and a denotes a positive real function. The lateral boundary of Q_t is given by $\Sigma_t = \bigcup_{0\le t<T}\left(\{\alpha(t),\beta(t)\}\times\{t\}\right)$. Moreover, we assume that $\alpha'(t) < 0$ and $\beta'(t) > 0$, for all $t\in[0,T]$.

Note that the hypotheses $\alpha'(t) < 0$ and $\beta'(t) > 0$ imply that Q_t is increasing, in the sense that if $t_2 > t_1$, then the projection of $[\alpha(t_1),\beta(t_1)]$ onto the subspace $t = 0$ is contained in the projection of $[\alpha(t_2),\beta(t_2)]$ onto the same subspace. This also means that the real function $\gamma(t) = \beta(t) - \alpha(t)$ is increasing on $0\le t<T$.

Observe that when (x,t) varies in Q_t, the point (y,t) of $\mathbb{R}^2$, with $y = (x-\alpha(t))/\gamma(t)$, varies in the cylinder $Q =]0,1[\times]0,T[$. Thus, we have the function $\tau : Q_t \longrightarrow Q$ given by $\tau(x,t) = (y,t)$, which is of class $\mathcal{C}^2$. The inverse τ^{-1} is also of class $\mathcal{C}^2$. The change of variable $v(y,t) = u(x,t)$ and $g(y,t) = f(x,t)$ with $x = \alpha(t)+\gamma(t)\,y$ transforms problem (P_u) into problem (P_v), given by

$$(\mathrm{P}_v)\begin{cases} v_t - b_1(y,t)v_y - a\left(\gamma(t)\displaystyle\int_0^1 v(y,t)dy\right) b_2(t)v_{yy} = g(y,t) \quad \text{in } Q \\ v(0,t) = v(1,t) = 0\,, \quad \text{for } 0<t<T \\ v(y,0) = v_0(y), \quad y\in]0,1[\end{cases}.$$

where $g(y,t) = f(\alpha+\gamma\,y,t)$ and $v_0(y) = u_0(\alpha(0)+\gamma(0)\,y)$. The coefficients $b_1(y,t)$ and $b_2(t)$ are defined by

$$b_1(y,t) = \frac{\alpha'(t)+\gamma'(t)y}{\gamma(t)} \quad \text{and} \quad b_2(t) = \frac{1}{(\gamma(t))^2}\,.$$

Since we are interested in proving the existence of a strong solution in Q_t, let us consider the following hypotheses:

$$
\begin{array}{ll}
(H1) & \alpha,\, \beta \in \mathcal{C}^2\left([0,T]\right) \text{ and } 0<\gamma_0<\gamma(t)<\gamma_1<\infty\,, \text{ for all } t\in[0,T] \\
(H2) & \alpha',\, \beta' \in L^1\left(]0,T[\right)\cap L^2\left(]0,T[\right) \\
(H3) & u_0 \in H_0^1\left(\Omega_0\right)\,, \quad \Omega_0 =]\alpha(0),\beta(0)[\\
(H4) & f \in L^2\left(0,T;L^2\left(\Omega_t\right)\right)\cap L^1\left(0,T;L^2\left(\Omega_t\right)\right)\,, \quad \Omega_t =]\alpha(t),\beta(t)[\\
(H5) & a:\mathbb{R}\longrightarrow\mathbb{R}^+ \text{ is lipschitz-continuous} \\
 & \qquad\qquad\qquad\qquad \text{with } 0<m_a\le a(s)\le M_a\,, \text{ for all } s\in\mathbb{R}
\end{array}
$$

The main result of this paper is as follows:

Theorem 1. *Under the assumptions $(H1)-(H5)$, there exists a unique strong solution $u:Q_t\longrightarrow\mathbb{R}$ for problem (P_u), that is,*

$$u_t - a\left(l(u)\right)u_{xx} = f\left(x,t\right) \quad \text{in } L^2\left(0,T;L^2\left(\Omega_t\right)\right)\,,$$

satisfying the regularity conditions:

$$
\begin{array}{l}
u\in L^\infty\left(0,T;H_0^1\left(\Omega_t\right)\cap H^2\left(\Omega_t\right)\right) \\
u_t\in L^2\left(0,T;L^2\left(\Omega_t\right)\right)
\end{array}\,,
$$

where $l:L^2\left(\Omega_t\right)\longrightarrow\mathbb{R}$ is a continuous linear form defined by $l(u)=\int_{\alpha(t)}^{\beta(t)}u(x,t)dx$.

In what follows, let $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $\|\cdot\|$ be, respectively, the scalar product and the norms in $L^2(\Omega)$ and $H_0^1(\Omega)$. Moreover, let

$$l_1(v)=\int_0^1\gamma(t)\,v(y,t)dy\,.$$

3. Existence of a solution of the transformed problem

In order to demonstrate the existence of a solution in Theorem 1, we first prove the existence of a solution of problem (P_v) applying the Faedo-Galerkin method and use the diffeomorphism to establish the existence of a solution of the original problem. So consider the following hypotheses:

$$
\begin{array}{ll}
(H3') & v_0\in H_0^1\left(\Omega\right) \\
(H4') & g\in L^2\left(0,T;L^2\left(\Omega\right)\right)\cap L^1\left(0,T;L^2\left(\Omega\right)\right)\,.
\end{array}
$$

Theorem 2. *Under the hypotheses* $(H1)-(H2)$, $(H3')-(H4')$ *and* $(H5)$ *there exists a solution* $v : Q \longrightarrow \mathbb{R}$ *of problem* (P_v), *that is,*

$$v_t - b_1(y,t)v_y - a\left(l_1(v)\right) b_2(t) v_{yy} = g(y,t) \quad \text{in } L^2\left(0,T;L^2\left(\Omega\right)\right)$$

which satisfies the following conditions:

$$\begin{array}{l} v \in L^\infty\left(0,T;H_0^1\left(\Omega\right)\cap H^2\left(\Omega\right)\right) \\ v_t \in L^2\left(0,T;L^2\left(\Omega\right)\right) \end{array} .$$

Proof. We construct Galerkin's approximations. Let $B = \{w_n(y)\}_{n\in\mathbb{N}}$ be a Hilbertian basis in $H_0^1(\Omega)$ and V_m the subspace spanned by the first m vectors of B. If $v_m(t) \in V_m$, then it can be represented by

$$v_m(t) = \sum_{i=1}^{m} c_{im}(t) w_i(y)\,, \quad 0 \le t < t_m\,, \quad t_m < T\,,$$

where v_m is the solution of the system of ordinary differential equations (approximate problem related with v_m)

$$\left\{ \begin{array}{l} \left(\dfrac{\partial v_m}{\partial t}, w\right) - \left(b_1 \dfrac{\partial v_m}{\partial y}, w\right) - a\left(l_1(v_m)\right) b_2 \left(\dfrac{\partial^2 v_m}{\partial y^2}, w\right) = (g,w)\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } w \in V_m \\ v_m(0) = v_{0m} \to v_0\,, \quad \text{strongly in } H_0^1(\Omega) \end{array} \right. , \tag{2}$$

which we will denote by (P_v^m). Rewriting this problem as $Y' = F(t,Y)$ with $Y(0) = Y_0$, it can be proved that $F(\cdot,Y)$ is measurable (in t) for Y fixed; $F(t,\cdot)$ is continuous (in Y) for t fixed and for every compact K contained in the domain of F, there is a real integrable function h_K such that $|F(t,Y)| \le h_K(t)$, for all $(t,Y) \in K$. Thus, by Caratheodory's theorem, it follows that problem (P_v^m) admits a local solution $c_{im}(t) \in \mathcal{C}^2([0,t_m[)$. To extend such a solution to the interval $[0,T]$, independent of m and t, the following a priori estimate is needed.

A priori estimates.

Estimate I: Setting $w = v_m$ in (P_v^m) and taking into account that

$$\left(-\frac{\partial^2 v_m}{\partial y^2}, w\right) = \left(\frac{\partial v_m}{\partial y}, \frac{\partial w}{\partial y}\right)\,,$$

we obtain

$$\left(\frac{\partial v_m}{\partial t}, v_m\right) - \left(b_1 \frac{\partial v_m}{\partial y}, v_m\right) + a\left(l_1(v_m)\right) b_2 \left|\frac{\partial v_m}{\partial y}\right|^2 = (g, v_m)\,. \tag{3}$$

Integrating by parts the second term of (3) and using the boundary conditions, we obtain

$$\begin{aligned}\int_0^1 b_1 \frac{\partial v_m}{\partial y} v_m dy = \frac{1}{2}\int_0^1 b_1 \frac{\partial}{\partial y} \left|v_m(y,t)\right|^2 dy &= -\frac{1}{2}\int_0^1 \frac{\gamma'}{\gamma} \left|v_m(y,t)\right|^2 dy \\ &= -\frac{1}{2}\frac{\gamma'}{\gamma} \left|v_m\right|^2\end{aligned} . \quad (4)$$

Using the equivalence of norms in $H_0^1(\Omega)$ and the Schwarz inequality, the approximate differential equation (3) yields the inequality

$$\frac{1}{2}\frac{d}{dt}\left|v_m\right|^2 + \frac{1}{2}\frac{\gamma'}{\gamma}\left|v_m\right|^2 + m_a b_2 \left\|v_m\right\|^2 \leq \left|(g, v_m)\right| \leq \frac{1}{2}\left(\left|g\right|^2 + \left|v_m\right|^2\right) . \quad (5)$$

Integrating from 0 to t, we get

$$\left|v_m(t)\right|^2 + 2m_a \int_0^t b_2 \left\|v_m\right\|^2 ds \leq \int_0^t \left|g\right|^2 ds + \int_0^t \left(1 + \frac{\gamma'}{\gamma}\right)\left|v_m\right|^2 ds + \left|v_{0m}\right|^2 .$$

From the hypotheses $(H1)$ and $(H2)$ it follows that

$$\int_0^t \left(1 + \frac{\gamma'}{\gamma}\right)\left|v_m\right|^2 ds \leq \left(1 + \frac{c_0}{\gamma_0}\right)\int_0^t \left|v_m\right|^2 ds ,$$

where c_0 is independent of m. Let $c^* = \min\{1, \frac{2m_a}{\gamma_1^2}\}$. Then, using $(H4')$ and the convergence in (2), we get

$$\left|v_m(t)\right|^2 + \int_0^t \left\|v_m\right\|^2 ds \leq C + C\int_0^t \left|v_m\right|^2 ds , \quad (6)$$

where $C = \max\{\frac{c_1}{c^*}, \frac{1+\frac{c_0}{\gamma_0}}{c^*}\}$. Applying Gronwall's Lemma to the last inequality, we obtain

$$\begin{array}{lll} (v_m)_{m\in\mathbb{N}} & \text{is bounded in} & L^\infty\left(0,T;L^2(\Omega)\right) \\ (v_m)_{m\in\mathbb{N}} & \text{is bounded in} & L^2\left(0,T;H_0^1(\Omega)\right) \end{array} . \quad (7)$$

Then we can extend the solution to the interval $[0, T]$.

Estimate II: Now we want to estimate the derivative $\partial v_m/\partial t$. Multiplying by $w = \partial v_m/\partial t$ in (P_v^m), we obtain

$$\begin{aligned}\left(\frac{\partial v_m}{\partial t}, \frac{\partial v_m}{\partial t}\right) - \left(b_1 \frac{\partial v_m}{\partial y}, \frac{\partial v_m}{\partial t}\right) + a\left(l_1(v_m)\right) b_2 \left(\frac{\partial v_m}{\partial y}, \frac{\partial^2 v_m}{\partial y \partial t}\right) \\ = \left(g, \frac{\partial v_m}{\partial t}\right)\end{aligned} . \quad (8)$$

The second term in (8) can be estimated as

$$\left|\left(b_1\frac{\partial v_m}{\partial y},\frac{\partial v_m}{\partial t}\right)\right| \leq |b_1|\left|\frac{\partial v_m}{\partial y}\right|\left|\frac{\partial v_m}{\partial t}\right| \leq \frac{|\alpha'|+|\gamma'|}{\gamma_0}\|v_m\|\left|\frac{\partial v_m}{\partial t}\right|, \tag{9}$$

using the Schwarz inequality and hypothesis $(H1)$. The third term yields the inequality

$$a\left(l_1(v_m)\right)b_2\left(\frac{\partial v_m}{\partial y},\frac{\partial^2 v_m}{\partial y\partial t}\right) = a\left(l_1(v_m)\right)b_2\frac{1}{2}\frac{d}{dt}\left|\frac{\partial v_m}{\partial y}\right|^2 \geq \frac{m_a}{\gamma_1^2}\frac{1}{2}\frac{d}{dt}\left|\frac{\partial v_m}{\partial y}\right|^2. \tag{10}$$

Substituting (9) and (10) in (8), we obtain

$$\left|\frac{\partial v_m}{\partial t}\right|^2 + \frac{m_a}{\gamma_1^2}\frac{1}{2}\frac{d}{dt}\left|\frac{\partial v_m}{\partial y}\right|^2 \leq \frac{|\alpha'|+|\gamma'|}{\gamma_0}\|v_m\|\left|\frac{\partial v_m}{\partial t}\right| + \frac{1}{2}|g|^2 + \frac{1}{2}\left|\frac{\partial v_m}{\partial t}\right|^2. \tag{11}$$

We now apply the inequality $2cd \leq c^2+d^2$ to the first term of the right hand side of (11). We obtain

$$\frac{1}{4}\left|\frac{\partial v_m}{\partial t}\right|^2 + \frac{m_a}{\gamma_1^2}\frac{1}{2}\frac{d}{dt}\|v_m\|^2 \leq \left(\frac{|\alpha'|+|\gamma'|}{\gamma_0}\right)^2\|v_m\|^2 + \frac{1}{2}|g|^2,$$

Integrating from 0 to t, we have

$$\begin{aligned}\int_0^t\left|\frac{\partial v_m}{\partial s}\right|^2 ds + \frac{2m_a}{\gamma_1^2}\|v_m(t)\|^2 \leq \frac{4}{\gamma_0^2}\int_0^t\left(|\alpha'|+|\gamma'|\right)^2\|v_m\|^2 ds \\ +2\int_0^t|g|^2 ds + \frac{2m_a}{\gamma_1^2}\|v_{0m}\|^2\end{aligned}. \tag{12}$$

From hypothesis $(H2)$ we can ensure that

$$\int_0^t\left(|\alpha'|+|\gamma'|\right)^2\|v_m\|^2 ds \leq c_1\int_0^t\|v_m\|^2 ds.$$

Therefore, in analogy to what we did for the first estimate, there is a positive constant C which does not depend on t and m, such that

$$\int_0^t\left|\frac{\partial v_m}{\partial s}\right|^2 ds + \|v_m(t)\|^2 \leq C + C\int_0^t\|v_m\|^2 ds, \tag{13}$$

and using the Gronwall inequality yields

$$\begin{array}{lll} (v_m)_{m\in\mathbb{N}} & \text{is bounded in} & L^\infty\left(0,T;H_0^1(\Omega)\right) \\ (\frac{\partial v_m}{\partial t})_{m\in\mathbb{N}} & \text{is bounded in} & L^2\left(0,T;L^2(\Omega)\right) \end{array} . \tag{14}$$

Estimate III: Finally, we get an upper bound for the second order spatial derivative. Taking $w = -\partial^2 v_m/\partial y^2$ in (P_v^m) and after integrating the first term by parts, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|v_m\right\|^2 - \left(b_1\frac{\partial v_m}{\partial y}, -\frac{\partial^2 v_m}{\partial y^2}\right) - a\left(l_1(v_m)\right) b_2\left(\frac{\partial^2 v_m}{\partial y^2}, -\frac{\partial^2 v_m}{\partial y^2}\right) \\ = \left(g, -\frac{\partial^2 v_m}{\partial y^2}\right) \end{aligned} . \tag{15}$$

Applying the arguments used in *estimate II*, we obtain

$$\begin{aligned} \left(b_1\frac{\partial v_m}{\partial y}, -\frac{\partial^2 v_m}{\partial y^2}\right) &\leq \frac{|\alpha'|+|\gamma'|}{\gamma_0}\left\|v_m\right\|\left|\frac{\partial^2 v_m}{\partial y^2}\right| \\ &\leq \frac{\varepsilon}{2}\left(\frac{|\alpha'|+|\gamma'|}{\gamma_0}\right)^2\left\|v_m\right\|^2 + \frac{1}{2\varepsilon}\left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 \end{aligned} , \tag{16}$$

for all $\varepsilon > 0$. The third term of equation (15) implies that

$$-a\left(l_1(v_m)\right) b_2\left(\frac{\partial^2 v_m}{\partial y^2}, -\frac{\partial^2 v_m}{\partial y^2}\right) = a\left(l_1(v_m)\right) b_2\left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 \geq \frac{m_a}{\gamma_1^2}\left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 \tag{17}$$

and from the last term of (15) it follows that

$$\left|-\left(g, \frac{\partial^2 v_m}{\partial y^2}\right)\right| \leq |g|\left|\frac{\partial^2 v_m}{\partial y^2}\right| \leq \frac{\varepsilon}{2}|g|^2 + \frac{1}{2\varepsilon}\left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 . \tag{18}$$

Substituting (16) to (18) in equation (15), we obtain

$$\frac{d}{dt}\left\|v_m\right\|^2 + 2\left(\frac{m_a}{\gamma_1^2} - \frac{1}{\varepsilon}\right)\left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 \leq \varepsilon\left(\frac{|\alpha'|+|\gamma'|}{\gamma_0}\right)^2\left\|v_m\right\|^2 + \varepsilon|g|^2 . \tag{19}$$

Observe that for $\varepsilon > \gamma_1^2/m_a$ one has $\left(\frac{m_a}{\gamma_1^2} - \frac{1}{\varepsilon}\right) > 0$. So set $\varepsilon = 2\gamma_1^2/m_a$, for example. Then integrating from 0 to t, we obtain

$$\begin{aligned} \int_0^t \frac{d}{dt}\left\|v_m\right\|^2 ds + \frac{m_a}{\gamma_1^2}\int_0^t\left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 ds \leq \\ \leq \int_0^t \frac{2\gamma_1^2}{m_a}\left(\frac{|\alpha'|+|\gamma'|}{\gamma_0}\right)^2\left\|v_m\right\|^2 ds + \int_0^t \frac{2\gamma_1^2}{m_a}|g|^2 ds \end{aligned} . \tag{20}$$

Then, using $(H4')$ and the strong convergence in (2), we have that

$$\|v_m(t)\|^2 + \int_0^t \left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 ds \leq C + C\int_0^t \|v_m\|^2 \, ds \,, \tag{21}$$

where C is a positive constant that does not depend on t and m. By Gronwall's inequality we conclude that

$$(\tfrac{\partial^2 v_m}{\partial y^2})_{m\in\mathbb{N}} \quad \text{is bounded in} \quad L^2\left(0,T;L^2(\Omega)\right) \ . \tag{22}$$

By the estimates obtained in (7), (14) and (22) there exists a subsequence of $(v_m)_{m\in\mathbb{N}}$, which we still denote by (v_m), such that

$$\begin{array}{ll} v_m \rightharpoonup v & \text{in } \ L^2\left(0,T;H_0^1(\Omega)\right) \\ v_m \overset{\star}{\rightharpoonup} v & \text{in } \ L^\infty\left(0,T;H_0^1(\Omega)\right) \\ \dfrac{\partial v_m}{\partial t} \rightharpoonup \dfrac{\partial v}{\partial t} & \text{in } \ L^2\left(0,T;L^2(\Omega)\right) \\ \dfrac{\partial^2 v_m}{\partial y^2} \rightharpoonup \dfrac{\partial^2 v}{\partial y^2} & \text{in } \ L^2\left(0,T;L^2(\Omega)\right) \end{array} \ . \tag{23}$$

From the Aubin-Lions Compactness Lemma (see [12]), as $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) = (L^2(\Omega))' \hookrightarrow L^2(\Omega)$, we have

$$v_m \to v \quad \text{in} \quad L^2\left(0,T;L^2(\Omega)\right) \ , \tag{24}$$

hence, passing if necessary to a subsequence (still denote by (v_m)), one has

$$v_m \to v \quad \text{a.e. in} \quad \Omega\times]0,T[\,. \tag{25}$$

Passage to the limit as $m \to \infty$.

To pass to the limit in the nonlinear part, it is required to prove that $a\left(l_1\left(v_m\right)\right) \to a\left(l_1(v)\right)$ in $L^2(0,T)$. Since the function a is continuous by hypothesis $(H5)$, it is sufficient to check that $l_1\left(v_m\right) - l_1(v) \to 0$ in $L^2(0,T)$, for each fixed t. In fact we have that

$$\int_0^T \left|l_1\left(v_m\right) - l_1(v)\right|^2 dt = \int_0^T \left|l_1\left(v_m - v\right)\right|^2 dt \leq C\int_0^T \left|v_m - v\right|^2 dt < \epsilon \,.$$

The last inequality follows from the convergency in (24). Now we can pass to the limit as $m \to \infty$ in the differential equation of the approximate problem (P_v^m) and obtain the following :

$$\frac{\partial v}{\partial t} - b_1(y,t)\frac{\partial v}{\partial y} - a\left(l_1(v)\right) b_2(t)\frac{\partial^2 v}{\partial y^2} = g(y,t) \quad \text{em } L^2\left(0,T;L^2\left(\Omega\right)\right) \ .$$

To verify the initial condition, we consider the third convergency in (23), integrate by parts and use the second convergency in (23). Since $\lim_{m\to\infty} v_{0m} = v_0$ in $L^2(\Omega)$, we obtain $v(0) = v_0$. Hence problem (P_v) has a solution. $\square$

In the next section, we address the uniqueness of the solution of (P_v).

4. Uniqueness of the solution

The uniqueness of the global strong solution of the transformed problem with fixed boundaries is guaranteed by the following theorem.

Theorem 3. *Let $v : Q \longrightarrow \mathbb{R}$ be a global strong solution of (P_v) given by Theorem (2), $v_0 \in H_0^1(\Omega)$ and $0 < T < \infty$. Suppose that the function a is Lipschitzian with constant $A > 0$, that is,*

$$|a(s_1) - a(s_2)| \leq A\,|s_1 - s_2|\,, \quad \textit{for all} \;\; s_1, s_2 \in \mathbb{R}\,.$$

If $(H1)$ and $(H2)$ hold, then problem (P_v) has a unique solution.

Proof. Let v_1 and v_2 be two solutions of problem (P_v), that is,

$$\begin{aligned} \frac{\partial v_1}{\partial t} - b_1(y,t)\frac{\partial v_1}{\partial y} - a\left(l_1(v_1)\right) b_2(t)\frac{\partial^2 v_1}{\partial y^2} &= g(y,t) \\ \frac{\partial v_2}{\partial t} - b_1(y,t)\frac{\partial v_2}{\partial y} - a\left(l_1(v_2)\right) b_2(t)\frac{\partial^2 v_2}{\partial y^2} &= g(y,t) \end{aligned}\,,$$

with $v_1(0,t) = v_1(1,t) = 0$ and $v_2(0,t) = v_2(1,t) = 0$. Subtracting the two differential equations, we get

$$\frac{\partial v_1}{\partial t} - \frac{\partial v_2}{\partial t} - b_1\frac{\partial v_1}{\partial y} + b_1\frac{\partial v_2}{\partial y} - a\left(l_1(v_1)\right) b_2\frac{\partial^2 v_1}{\partial y^2} + a\left(l_1(v_2)\right) b_2\frac{\partial^2 v_2}{\partial y^2} = 0\,,$$

and it follows that $r = v_1 - v_2$ is solution of

$$\frac{\partial r}{\partial t} - b_1\frac{\partial r}{\partial y} - a\left(l_1(v_1)\right) b_2\frac{\partial^2 v_1}{\partial y^2} + a\left(l_1(v_2)\right) b_2\frac{\partial^2 v_2}{\partial y^2} = 0\,,$$

in $L^2\left(0,T;L^2\left(\Omega\right)\right)$, with $r(0) = 0$. Taking the inner product in $L^2\left(\Omega\right)$ with r, integrating by parts and adding and subtracting $a\left(l_1\left(v_1\right)\right) b_2\left(\frac{\partial v_2}{\partial y}, \frac{\partial r}{\partial y}\right)$, we

obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|r|^2+\frac{1}{2}\frac{\gamma'}{\gamma}|r|^2+a\left(l_1\left(v_1\right)\right)b_2\left[\left(\frac{\partial v_1}{\partial y},\frac{\partial r}{\partial y}\right)-\left(\frac{\partial v_2}{\partial y},\frac{\partial r}{\partial y}\right)\right]\\ =\left[a\left(l_1\left(v_2\right)\right)-a\left(l_1\left(v_1\right)\right)\right]b_2\left(\frac{\partial v_2}{\partial y},\frac{\partial r}{\partial y}\right)\end{aligned}. \tag{26}$$

The third term of this equation implies that

$$a\left(l_1\left(v_1\right)\right)b_2\left(\frac{\partial r}{\partial y},\frac{\partial r}{\partial y}\right)=a\left(l_1\left(v_1\right)\right)b_2\left\|r\right\|^2\geq\frac{m_a}{\gamma_1^2}\left\|r\right\|^2. \tag{27}$$

As a is A-Lipschitzian and using the hypothesis $(H1)$ and the Schwarz inequality, one obtains the following upper bound for the term on the right hand side of (26) :

$$\frac{A}{\gamma_0^2}\left|l_1\left(v_2\right)-l_1\left(v_1\right)\right|\left|\frac{\partial v_2}{\partial y}\right|\left|\frac{\partial r}{\partial y}\right|\leq\frac{Ac_0}{\gamma_0^2}\left|r\right|\left|\frac{\partial v_2}{\partial y}\right|\left\|r\right\|. \tag{28}$$

Substituting (27) and (28) in equation (26) and multiplying by 2, similarly to (19), we obtain

$$\frac{d}{dt}|r|^2+\left(\frac{2m_a}{\gamma_1^2}-\frac{1}{\varepsilon}\right)\left\|r\right\|^2\leq\varepsilon\left(\frac{Ac_0}{\gamma_0^2}\right)^2|r|^2\left|\frac{\partial v_2}{\partial y}\right|^2+\frac{|\gamma'|}{\gamma_0}|r|^2, \tag{29}$$

for some $\varepsilon>\gamma_1^2/m_a/2$. Setting $\varepsilon=\gamma_1^2/m_a$, for example, and integrating from 0 to t in both members of (29), it follows that

$$|r(t)|^2\leq\int_0^t\psi(s)\left|r\right|^2ds, \tag{30}$$

where the function $\psi\in L^1(0,T)$ is defined by

$$\psi(s)=\frac{\gamma_1^2}{m_a}\frac{A^2c_0^2}{\gamma_0^4}\left|\frac{\partial v_2}{\partial y}(s)\right|^2+\frac{|\gamma'(s)|}{\gamma_0}.$$

By the Gronwall inequality we obtain $r=0$. Then $v_1=v_2$, which completes the proof. □

Now we are in position to prove Theorem 1.

Proof. Let v be the solution of problem (P_v) with initial data $v_0(y) = u_0(\alpha(0) + \gamma(0)y)$. As $u(x,t) = v(y,t)$, where $x = \alpha(t) + \gamma(t)y$, in order to verify that $u(x,t)$ given by Theorem 1 is the solution of problem (P_u), it is sufficient to observe that the transformation $\tau : Q_t \longrightarrow Q$ is (a diffeomorphism) of class $\mathcal{C}^2$. In fact, by the equalities $u_t = v_t - b_1(y,t)v_y$ and $u_{xx} = b_2(t)v_{yy}$, the existence of a solution of problem (P_v) and the regularity of $v(y,t)$ given by Theorem 2, we can conclude that $u(x,t)$ is a solution of (P_u). Finally, the uniqueness of the solution of (P_u) is a consequence of the uniqueness of the solution of (P_v), because $u = v$. □

5. Exponential decay of the solution

The goal of this section is to establish a rate decay for the energy associated to problem (P_u). Therefore, we obtain the asymptotic behaviour, for a large t, of the natural energy

$$E(t) = \frac{1}{2}\,|u(t)|^2_{L^2(\Omega_t)}\,, \tag{31}$$

inside the time dependent domain Q_t. Thus, we can state:

Theorem 4. *Assuming the hypotheses of Theorem 1, if $f(x,t) = 0$ in (P_u), then the function E satisfies*

$$E(t) \le E(0)e^{-\delta t}\,, \quad \textit{for all } t \ge 0\,, \ \textit{with } \delta > 0\,.$$

In order to prove this Theorem one needs to establish the Poincaré's inequality in Ω_t. Thus one has:

Lemma 5. *If $u \in H^1_0(\Omega_t)$ then*

$$|u(t)|^2_{L^2(\Omega_t)} \le \gamma^2(t)\,|u_x(t)|^2_{L^2(\Omega_t)}\,.$$

Proof. In fact, from the fundamental theorem of calculus we have that

$$u(x,t) = \int_{\alpha(t)}^{x} \frac{\partial}{\partial \xi} u(\xi,t)d\xi\,.$$

From this and Schwarz's inequality, we obtain

$$|u(x,t)|^2_{\mathbb{R}} \le \gamma(t)\,|u_x(t)|_{L^2(\Omega_t)}\,.$$

Integrating in Ω_t, we get

$$|u(t)|^2_{L^2(\Omega_t)} \le \gamma^2(t)\,|u_x(t)|^2_{L^2(\Omega_t)}\,.$$

□

Hence, we are in a position to prove Theorem 4.

Proof. Consider the first equation in (1). Taking the inner product in $L^2(\Omega_t)$, when $f = 0$, with $u(x,t)$, we have

$$(u_t, u) - a(l(u))\,(u_{xx}, u) = 0\,. \tag{32}$$

Applying the Leibnitz rule and using the null Dirichlet boundary conditions in the first term of (32), yields

$$(u_t, u) = \frac{1}{2}\frac{d}{dt}\int_{\Omega_t} |u(x,t)|^2\,dx\,. \tag{33}$$

Integrating by parts in the second term of (32) and from the boundary conditions, we obtain

$$(u_{xx}, u) = -\,|u_x(t)|^2_{L^2(\Omega_t)}\,. \tag{34}$$

Substituting (33) and (34) in (32) and using (31), one gets

$$\frac{d}{dt}E(t) + a(l(u))\,|u_x(t)|^2_{L^2(\Omega_t)} = 0\,. \tag{35}$$

From the last equation, we have that $\frac{d}{dt}E(t) \leq 0$, for all $t \geq 0$, since a is a positive form. So, the energy E is a nonnegative decreasing function.

By the Lemma 5 and hypothesis $(H1)$, we get the Poincaré's inequality

$$|u(t)|^2_{L^2(\Omega_t)} \leq \gamma_1^2\,|u_x(t)|^2_{L^2(\Omega_t)}\,.$$

Thus, using the lower bound of a, from equation (35), we obtain

$$\frac{d}{dt}E(t) + \frac{m_a}{\gamma_1^2}\,|u(t)|^2_{L^2(\Omega_t)} \leq 0\,, \quad \text{for all } t \geq 0\,, \tag{36}$$

and

$$\frac{d}{dt}\left(E(t)e^{2m_a t/\gamma_1^2}\right) \leq 0\,. \tag{37}$$

Integrating from 0 to t, we conclude that

$$E(t) \leq E(0)e^{-\delta t}\,, \tag{38}$$

with $\delta = 2m_a/\gamma_1^2$, which proves the exponential decay of the solution when the reaction force f is null. □

Remark 1: When f decays in an appropriate way (see [9]), we can obtain the same result as in Theorem 4 with $f \neq 0$.

Remark 2: Results of Theorems 1 and 4 can be easily generalized for

$$u_t - a\left(\int_{\Omega_t} u(x,t)dx\right)\Delta u = f(x,t)\,, \quad \text{in } Q_t \subset \mathbb{R}^{n+1} \ (n \geq 1)$$

where Q_t is a bounded non-cylindrical domain defined by

$$Q_t = \bigcup_{0 \leq t < \infty} \Omega_t \times \{t\}\,,$$

with lateral boundary $\Sigma_t = \cup_{0 \leq t < \infty}(\Gamma_t \times \{t\})$. For a more details see [11, 15].

6. Numerical study

The purpose of this section is to determine the approximate solution of problem (P_u) using the MFEM with high degree approximations. This method could be applied to problem (P_v) just as some authors did; instead, our numerical algorithm allows us to solve the problem immediately in the non cylindrical domain. The manifestation of the properties of the solution as the exponential decay, the finite speed of propagation or the occurrence of blow-up are dependent on the initial data. So, we consider examples of the functions a, f, α and β to illustrate the regularity of the solution, particularly in the neighbourhood of the two moving boundaries. In the appendix we give examples of domains Q_t.

Now we give a brief description of some relevant aspects of our formulation of the MFEM.

6.1. Approximation scheme

For the space discretization, consider the following partition of the spatial domain Ω_t in N finite elements by $N-1$ interior space nodes,

$$\mathcal{P} \ : \ \alpha(t) = X_1(t) < X_2(t) < \ldots < X_N(t) < X_{N+1}(t) = \beta(t) \quad t \geq 0\,.$$

The most significant feature of the MFEM is the time dependence of the nodal positions allowing the adaptivity of the spatial mesh. These are treated as unknown time-dependent variables which must be evaluated as part of the solution procedure. The method automatically updates the length of each

finite element so that the solution becomes suitably represented. To solve efficiently time-dependent problems with moving boundaries, a special boundary technique is developed by the introduction of two moving nodes $X_1(t)$ and $X_{N+1}(t)$ describing the position of the moving ends of the spatial domain at each instant t. In each finite element of $\mathcal{P}$, we define the numerical solution by a polynomial of arbitrary degree. Let S_r denote the set of continuous piecewise polynomial functions of degree r on $\bar{\Omega}_t$ (closure of Ω_t) which vanish on the boundary $\partial\Omega_t$,

$$S_r = \left\{ U \in \mathcal{C}_0^0(\bar{\Omega}_t) : \; U|_{\Omega_e} \text{ is a polynomial of degree at most } r \,, \; \forall \Omega_e \in \mathcal{P} \right\} \,,$$

where $\Omega_e = \left[X_e^{(1)}, X_e^{(2)} \right] = [X_e(t), X_{e+1}(t)]$. For a fixed time $t > 0$, any function in S_r is uniquely determined by its values at the global interpolation points $\{\xi^j(t) : j = 1, 2, ..., \tilde{N}\}$ in Ω_t. Given a function u on Ω_t such that $u(\alpha(t), t) = u(\beta(t), t) = 0$, we define its interpolant, denoted by $\mathcal{I}_r(u)$, as

$$\mathcal{I}_r(u) = U = \sum_{j=1}^{\tilde{N}} u(\xi^j(t), t)\Phi^j(x) \,,$$

where $\Phi^j(\xi^i) = \delta_{ij}$, δ_{ij} is the Kronecker symbol. On the canonical element Ω_e, denote

$$\boldsymbol{X}_e = (X_e^{(1)}, X_e^{(2)})^T \,, \quad \boldsymbol{U}_e = (u(\xi_e^1), ..., u(\xi_e^{r+1})^T \,.$$

Then, locally, the element interpolation polynomial is defined by

$$V_e(x, t) = \Psi_e \, \boldsymbol{U}_e \,,$$

where $\Psi_e = \left(\Phi_e^1|_{\Omega_e} \,, ..., \, \Phi_e^{r+1}|_{\Omega_e} \right)$ represents the local polynomial basis functions. The positions of the interior interpolating points to Ω_e are obtained by exploiting the element boundary information [1]. This strategy allows us to minimize the maximum absolute error of the local approximation. The semi-discrete problem consists in finding $U \in S_r$ such that $(\mathcal{R}, \mathcal{R})$ is minimum, where $\mathcal{R}$ is the PDE residual. So, in order to define the numerical approximation U of u we must integrate, in time, the system of ordinary differential equations generated by the minimization of

$$\min \left(|\mathcal{R}|^2_{L^2(\Omega_t)} + P_{enalty} \right) \,, \tag{39}$$

over all possible choices of first time derivative of each effective parameter of the method. It is necessary to introduce Miller's penalty term into the objective function to avoid the singularities due to parallelism or element folding. To get the explicit form of the general equations, it is necessary to evaluate the integrals that result from (39). It should be noted at this point, however, that some of these integrals are not properly defined. To overcome this difficulty, it is necessary to define approximations of spatial derivatives at the separation nodes X_e, $e = 2, ..., N$. This can be achieved by applying a smoothing strategy based on cubic Hermite polynomials in a small neighbourhood of each node X_e. By the minimization process, we obtain a system of ordinary differential equations (ODE) that may be written in matrix form

$$M(t,\, \boldsymbol{Y})\frac{d\,\boldsymbol{Y}}{dt} = \boldsymbol{F}(t,\, \boldsymbol{Y})\,. \tag{40}$$

Our formulation originates sparse mass matrices strongly dependent on $\boldsymbol{Y}$. Nodal amplitudes and nodal positions are found interlaced in vector $\boldsymbol{Y}$, ordered in such way that M is a quasi-diagonal block matrix. Let A be a square matrix of order $r+1$ defined by

$$A(i,j) = \int_0^1 \Phi_e^i\big|_{\Omega_e}(\upsilon)\;\Phi_e^j\big|_{\Omega_e}(\upsilon)d\upsilon\,.$$

Denote

$$C_e^i = \sum_{k=1}^{r+1} A(k,i)\upsilon^k\;\frac{\partial V_e}{\partial x}\bigg|_{\xi_e^k}\,,\; D_e^i = \sum_{k=1}^{r+1} A(k,i)(1-\upsilon^k)\;\frac{\partial V_e}{\partial x}\bigg|_{\xi_e^k}\,,\; i = 1, ..., r+1\,,$$

where $\xi_e^k = X_e^{(1)} + Z_e\upsilon^k$, with $Z_e = X_e^{(2)} - X_e^{(1)}$.

Each block of M corresponds to a finite element and, setting

$$B_e = (A(:,1)\;,\; -D_e\;,\; A(:,2:r+1)\;,\; -C_e)\;,$$

with dimension $(r+1)\times(r+3)$, we have that

$$M_e = \begin{pmatrix} Z_e\,B_e(1,:) \\ -\frac{1}{2}\left(\frac{\partial\mathcal{I}_r(u)}{\partial x}\Big|_{X_e^+} - \frac{\partial\mathcal{I}_r(u)}{\partial x}\Big|_{X_e^-}\right)Z_e\,B_e(1,:) - P_e^2\vartheta \\ B_e(2:r,:) \\ Z_e\,B_e(r+1,:) \\ \frac{1}{2}\left(\frac{\partial\mathcal{I}_r(u)}{\partial x}\Big|_{X_{e+1}^+} - \frac{\partial\mathcal{I}_r(u)}{\partial x}\Big|_{X_{e+1}^-}\right)Z_e\,B_e(r+1,:) + P_e^2\vartheta \end{pmatrix}\,,$$

where P_e is the viscosity penalty function associated to the e-th finite element, $\vartheta = (0, -1, 0, ..., 0, 1)$ and $B_e(2 : r, :)$ represents the submatrix of B_e, formed by the 2-nd to r-th lines. The solution of the initial value problem (40) can be obtained by an appropriate ODE integrator. We use the function *ode15s* from Matlab ode suite [18], a variable order variable time-step ODE integrator for stiff problems. We included this function in our Matlab code, exploited the sparsity of the mass matrix M and selected the numerical differential formulae methods to perform the integration. For a more detailed description of the MFEM see [7, 17].

6.2. Numerical results

The numerical results presented here are obtained in the Matlab environment using a computer with an Intel Core i7 − 3960X processor at 3.30 GHz. We compute all the integrals without truncation error using Lobatto's quadrature and use the existing standard values of the optional user-modifier method parameters, such as the minimal nodal distance allowed or the ODE solver tolerances.

Example 1: Let $u(x,t) = \frac{1}{\pi^2}\sin(\pi\eta)\cos(\pi t)$, where $\eta = \frac{(t+1)x+t}{3t+1}$ be the exact solution of problem (P_u) for a suitable choice of $f(x,t)$ with a a constant function. Our goal is to compare the exact solution with the approximate numerical solution. Consider the moving boundaries defined by

$$\alpha(t) = -\frac{t}{t+1} \quad \text{and} \quad \beta(t) = \frac{2t+1}{t+1} \tag{41}$$

and the initial condition $u_0(x) = u(x, 0)$. This data satisfies the hypotheses of Theorem 1.

We use $a = 1.71$, four finite elements and local polynomial approximations of degree 5. Figure 1 (left) represents the history of values of the dependent variable for a fixed $x = 0.5$. We see a high rate of decay of values $U(0.5, t)$ as t increase. Figure 1 (right) shows the numerical solution in the non cylindrical domain Q_t for $t \leq 1$. In the norm $L^\infty(0, T; L^2(\Omega_t))$, the numerical error is defined by

$$\begin{aligned} E_{L^\infty} &= \max_{0\leq t_k\leq T}\left(|u(x,t_k) - U(x,t_k)|_{L^2(\Omega_{t_k})}\right) \\ &= \max_{0\leq t_k\leq T}\left(\int_{\Omega_{t_k}} |u(\xi^j,t_k) - U(\xi^j,t_k)|^2 dx\right)^{1/2}, \end{aligned}$$

where $j = 1, ..., \tilde{N}$ and $t_k \in \{0, 0.2, ..., 1.0\}$. Table 1 shows the errors between the exact solution and the MFEM approximate solution. These results

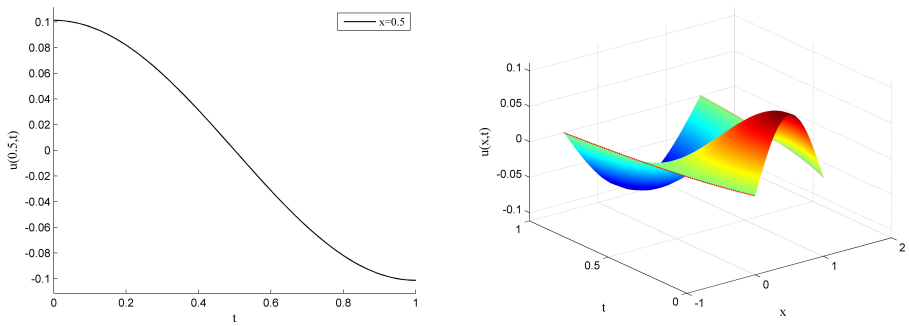

Figure 1: Evolution of the solution at midpoint (left) and approximate solution in $\Omega_t \times [0,1]$ (right)

demonstrate that the MFEM can produce accurate results efficiently using a reduced number of nodes as well as calculation time.

Time (t)	E_{L^∞}
0.0	3.6e-10
0.2	7.0e-09
0.4	2.1e-08
0.6	2.5e-08
0.8	1.7e-08
1.0	1.4e-08

Table 1: Errors obtained for several times.

Example 2: For the second example, consider $\Omega_0 =]0.25, 0.75[$, $a(s) = 1 + \frac{1}{1+s^2}$, $f = \frac{0.001}{1+t}$, $u_0(x) = 16(\alpha(0) - x)(x - \beta(0))$ with $\alpha(t)$ and $\beta(t)$ given by (41). The computed solutions are shown at several time instants in Figure 2 (left). The integration time interval considered is $[0, 1]$. Interior spatial separation nodes are placed to $[3/8, 1/2, 5/8]$ at $t = 0$ and their descendent trajectories are shown in Figure 2 (right). In Figure 3 (left) we plotted $X_e(t)$ versus time. It can be observed that the adaptive mesh has a smooth evolution in spite of the initial situation where the solution has sharp variations. As we use a high degree local approximation, the MFEM does not have to relocate the separation nodes quickly. Figure 3 (right) illustrates the movement of the two boundaries. The approximate numerical solution is shown

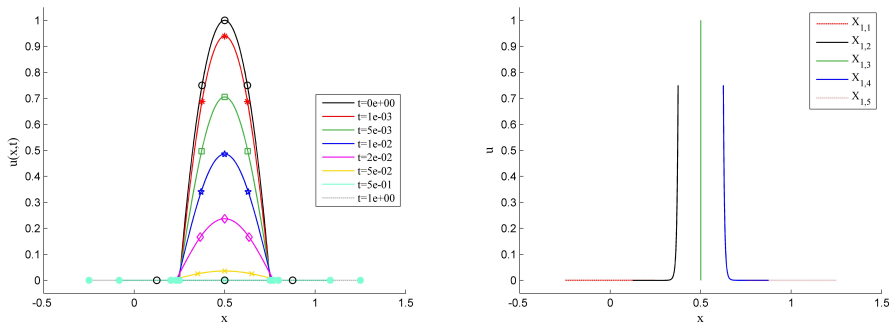

Figure 2: Solution profiles at different times (left) and trajectories of the separation nodes (right)

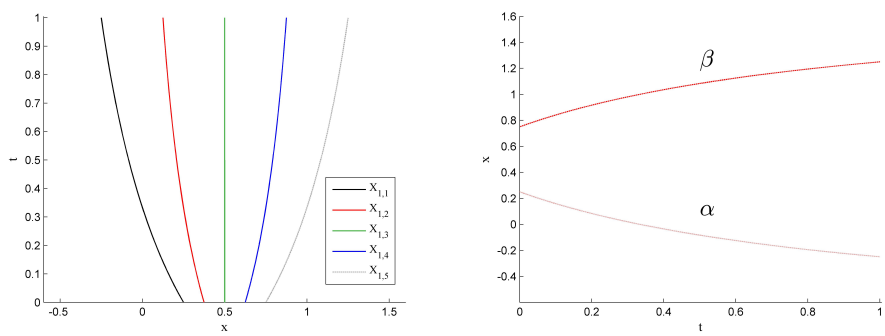

Figure 3: Nodal movement in spatial domain (left) and motion of the boundaries (right)

in the 3D graph of Figure 4 (left). Finally, in Figure 4 (right), we can see the asymptotic behaviour of the solutions.

Example 3: Consider the data from the previous example with $a(s) = \gamma(t) + \frac{1}{2\gamma(t)}s$. We want to illustrate that the exponential decay of the solution depends on an appropriate rate of decay of function f. So, in Figure 5, we plotted the dependent variable u versus time for two different functions

$$f_1(x,t) = \frac{10x}{t+1} \quad \text{and} \quad f_2(x,t) = \frac{x}{10(t+1)}\,,$$

at two fixed values of the spatial variable x. It can be observed (see Figure 5 (left)) that function f_1 does not have an appropriate decay leading to an asymptotic behaviour of the solution. On the contrary, in Figure 5 (right), we see the asymptotic energy decay for f_2.

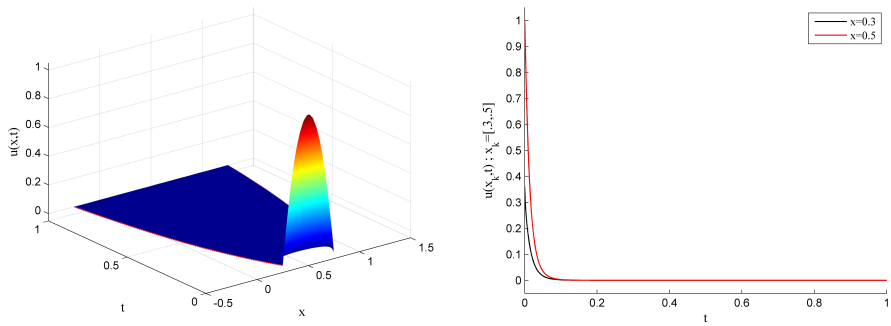

Figure 4: Evolution of the approximate function in space-time domain (left) and asymptotic decay of energy (right)

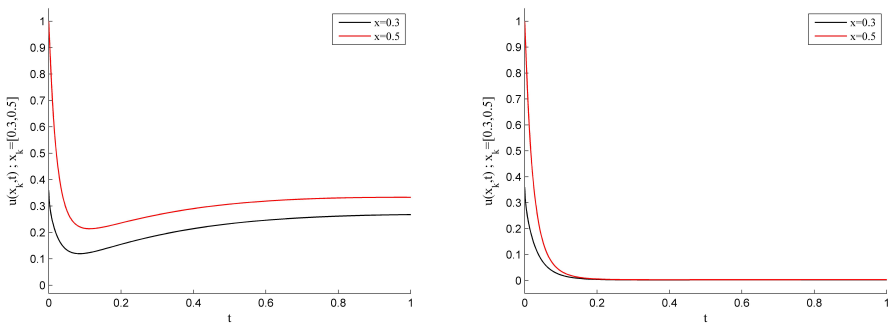

Figure 5: History of $u(x,t)$ at two fixed values of x for the functions f_1 (left) and f_2 (right)

Example 4: In this last example, we use a different limited dilation γ (see appendix) defined by

$$\alpha(t) = -1 + (t+1)^{-1/2} \quad \text{and} \quad \beta(t) = 2 - (t+1)^{-1/2}\,,$$

$a(s) = 1 + \sin(2s)$ and the reaction force $f(x,t) = e^{-x}$. We consider $u_0(x) = 1 - \cos(4\pi x)$ defined in the initial domain $[0,1]$. Initially, the nodes are placed forming a uniform grid. The MFEM automatically relocates moving nodes in order to concentrate them in regions where the solution has sharp profiles. As we use a five degree local approximation, the method is able to move nodes smoothly. This can be seen in Figure 6 (left) where we present the mesh movement. Figure 6 (right) displays the MFEM approximate solution. We can see that the energy associated to this system does not have an exponential decay.

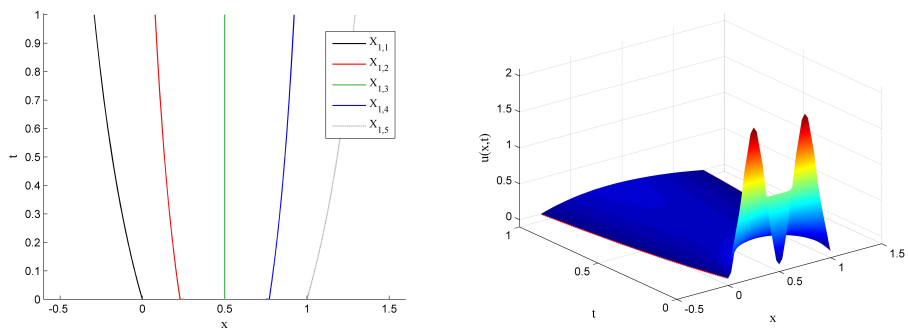

Figure 6: Mesh movement (left) and the approximate numerical solution (right)

7. Conclusions

We prove the existence and uniqueness of strong global solutions for a class of nonlocal problems with moving boundaries. Moreover, we show the exponential decay of the solution. By our numerical algorithm, based on the MFEM with piecewise polynomial of arbitrary degree basis functions in space, we are able to solve the initial problem without using the transformation in the cylindrical domain. Some numerical experiments were presented considering different functions a, f, α and β, to show the moving boundary for the problem. The numerical results demonstrate the accuracy and robustness of our Matlab code based on the MFEM; in particular, they are in agreement with the asymptotic behaviour of the analytic solution. The generalization of the present results to a nonlinear system of two parabolic equations and the application of Euler-Galerkin finite element method to this problem is in progress.

8. Appendix

Examples of domains Q_t

1. We assume that $-\alpha'(t) > 0$ and $\beta'(t) > 0$. So, considering a limited variation (say by K) of the position of both moving boundaries, we must have

$$0 < -\alpha'(t) \leq K\,, \quad 0 < \beta'(t) \leq K\,, \quad \forall t \geq 0\,.$$

Integrating, we obtain Q_t defined by

$$\left| \begin{array}{ll} \alpha(t) = \alpha_0 - \alpha_1 t\,, & 0 < \alpha_1 \leq K \\ \beta(t) = \beta_0 - \beta_1 t\,, & 0 < \beta_1 \leq K \end{array} \right. .$$

Note that, in this case, we have linear boundaries.

2. The lateral boundary

$$\left| \begin{array}{l} \alpha(t) = \alpha_0 + \sqrt[n]{t_0} - \sqrt[n]{t+t_0} \\ \beta(t) = \beta_0 - \sqrt[n]{t_0} + \sqrt[n]{t+t_0} \end{array} \right. ,$$

for $n = 2, 3, ...$ and $t_0 = (nK)^{n/(1-n)}$ is not linear.

3. This example shows that when $t \to \infty$ the domain Q_t is asymptotic to a cylinder

$$\left| \begin{array}{l} \alpha(t) = \alpha_0 - t_0^{-1/2} + (t+t_0)^{-1/2} \\ \beta(t) = \beta_0 + t_0^{-1/2} - (t+t_0)^{-1/2} \end{array} \right. ,$$

with $1/t_0 = K^3$.

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