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A nonlinear viscoelastic plate equation with $\overrightarrow{p}(x,t)-$ Laplacian operator: Blow up of solutions with negative initial energy

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Abstract

This paper concerns about with a class nonlinear viscoelastic plate equation with lower order perturbation of $\overrightarrow{p}(x,t)-$Laplacian operator

$$u_{tt} + \Delta^2 u - \Delta_{\overrightarrow{p}(x,t)} u + \int_0^t g(t-s)\Delta u(s)ds - \epsilon \Delta u_t + f(u) = 0,$$

$$(x,t) \in Q_T = \Omega \times (0,T), \Omega \subset \mathbb{R}^n, n \geq 2$$

associated with initial and Dirichlet-Neumann boundary conditions. Under suitable conditions on g, $f(u)$ and the variable exponent of

$\overrightarrow{p}(x,t)-$Laplacian operator, it is proved that any weak solution with negative initial energy blows up in finite time, assuming a strong damping $\epsilon\Delta u_t$ ($\epsilon > 0$) acting in the domain. This equation corresponds to a viscoelastic version arising in dynamics of elastoplastic flows and plate vibrations.

Keywords: non-linear viscoelastic equation, anisotropy, strong damping, blow up in finite time, nonstandard growth conditions.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with Lipschitz-continuous boundary $\Gamma = \partial\Omega, Q_T = \Omega \times (0,T),\ \Gamma_T = \partial\Omega \times (0,T)$. In this paper we consider the following initial-boundary value problem

$$u_{tt} + \Delta^2 u - \Delta_{\overrightarrow{p}(x,t)} u + \int_0^t g(t-s)\Delta u(s)ds - \epsilon\Delta u_t + f(u) = 0 \text{ in } Q_T, \quad (1)$$

$$u = \partial u/\partial\nu = 0,\ \Gamma_T = \partial\Omega \times (0,T), \quad (2)$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \quad (3)$$

where

$$\Delta_{\overrightarrow{p}(x,t)} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2} \frac{\partial u}{\partial x_i}\right), \overrightarrow{p}(x,t) = (p_1, p_2, ..., p_n)$$

called the $\overrightarrow{p}(x,t)-$Laplacian operator, $g \geq 0$ is a memory kernel that decays exponentially and $f(u)$ is a nonlinear function. Here $\epsilon \geq 0$ is a constant and $\partial u/\partial\nu$ denotes the normal derivative directed outside of Ω.
The problem (1)-(3) without the viscoelastic term, and with the usual p -Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, ($p = const \geq 2$) has been extensively studied concerning existence, nonexistence and long-time dynamics. For results of the nature, we refer the reader to [33]-[36] related to the equation

$$u_{tt} + \Delta^2 u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \Delta u_t = h(x,t,u,u_t), \quad (4)$$

in bounded domains of $\mathbb{R}^n$. In the absence of dissipation terms $-\Delta u_t$ or u_t, the authors [16], [25] and [33] considered the question of blow up in finite time and global existence with small data to (4).

In one dimension, (4) without damping or forcing terms is related to the model

$$\rho u_{tt} + \varsigma u_{xxxx} + a(u_x^2)_x = 0,\ a = const > 0 \text{ and } \varsigma = const > 0,$$

which describes elastoplastic-microstructures flows as discussed in [2, 3].

In two dimensions, with $p = 4$ and weak damping, (4) corresponds to the so called model for nonlinear plates

$$u_{tt} + \Delta^2 u - \operatorname{div}(|\nabla u|^2 \nabla u) + k u_t = \sigma \Delta(u^2) - f(u).$$

This is indeed a limit of the Mindlin-Timoshenko plates as the shear modulus tends to infinity, as shows in [13]. Remarkable results were obtained by [12, 13] who proved the existence of finite-dimensional global attractors under a weak damping ku_t instead of $-\Delta u_t$.

In the presence of the viscoelastic term ($g \neq 0$), the first study of the problem (5) below with memory effects has been done in [4]. The authors considered the following of the problem

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta_p u - (g * \Delta u)(t) - \epsilon \Delta u_t + f(u) = 0 \text{ in } Q_T, \\ u = \Delta u = 0 \text{ on } \Gamma_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \end{cases} \tag{5}$$

where a fading memory term is given by

$$(g * \Delta u)(t) = \int_0^t g(t-s)\Delta u(s) ds,$$

and proved the well-posedness and asymptotic stability, but they did not study the blow up of the system (5).

Equation (1) can be viewed as a generalization of the evolutional p-Laplacian type

$$u_{tt} = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right). \tag{6}$$

with the constant exponent of nonlinearity $p \in (1, \infty)$. During the last decades equation (6) was intensively studied and was casted for the role of a touchstone in the nonlinear PDEs. There is extensive literature devoted to equation (6). In the dissipative case, by adding viscosity term $-\Delta u_t$, pioneering papers were given by [18, 19] and [32]. The existence of global solution without an additional dissipation term is an open problem.

We also mention the very important contribution by [5]. The author proved the existence and blow up for the weak solution of wave equation with $p(x,t)$-Laplacian and damping term given by

$$u_{tt} = \operatorname{div}(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u + \varepsilon \nabla u_t) + b(x,t)|u|^{\sigma(x,t)-2}u + f(x,t) \text{ in } Q_T, \tag{7}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{8}$$

$$u|_{\Gamma_T} = 0 \text{ on } \Gamma_T = \partial\Omega \times (0,T). \tag{9}$$

where the coefficients $a,\ b,\ f$ and exponents p, σ are given measurable function of their arguments, and $\varepsilon = const > 0$. Such equations (with variable exponents of nonlinearities) are usually referred as equations with nonstandard growth conditions.

Equations with nonstandard growth conditions occur in the mathematical modelling of various physical phenomena, e.g., the flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, processes of filtration through a porous media and the image processing - see, [10] and the further references therein.

It is to be noted here that in all papers (referring to the case $p \neq 2$) the viscous term $\varepsilon \Delta u_t$ plays a key role in the proof of global or local existence (even if $p = \text{const} \neq 2$). The principal difficulty remains in proving an existence theorem by considering the term $-\Delta_{\overrightarrow{p}(x,t)} u$. The viscous term $\varepsilon \Delta u_t$ $(\varepsilon > 0)$ facilities the proof of existence theorem but complicates to prove blow up effect.

A special case $a(x,t) = 1, b(x,t) = 0$ and $p(x,t) = p(x)$ was studied in [20]. In the case $a(x,t) = 1$, $b(x,t)|u|^{\sigma(x,t)-2}u = b(x)u^{p(x)}$ and $p(x,t) = 2$, the question of existence and blow-up of nonnegative solutions of problem (7)-(9) were discussed in [29]. Recently, in [8], improve the result obtained in [29].

In [9], the authors improve the results from [4] related with the study of local and global existence, as well as the uniqueness of the weak solution $u(x,t)$ to problem (1).

In [17], the authors have proved results for phenomena of stability, unstability, blow-up and asymptotic behavior for the Cauchy problem for abstract hyperbolic equation in a Banach space X of the form

$$u_{tt} = f'(u),\ t > 0;\ u(0) = u_0,\ u_t(0) = u_1,$$

where $f : X \to \mathbb{R}$ is a C^1-function was considered.

To the best of the knowledge, this is the first result dealing with equation (1) subject to the interaction of the memory term with lower order perturbation of $\overrightarrow{p}(x,t)-$ Laplacian type and presenting blowup.

In this paper, our main objective is to study the blow up in finite time of the solution by considering (1) in a context associated with the $\overrightarrow{p}(x,t)-$Laplacian operator and boundary conditions of the type $u = \partial u / \partial \nu = 0$ on Γ_T. We pay special attention of the specific properties caused by the variable exponents $p_i(x,t)$, $i = 1, 2, ..., n$, of the equation (1).

The paper is organized as follows. First of all we introduce the function spaces of the Orlicz-Sobolev in Sections 2, 3 present a brief description of their main properties. In Section 4 we present the statement of the problem (1)-(3). We give a definition of weak solution and local result existence theorem established in [9]. Finally, Section 5 is devoted to proof that any weak solution with negative initial energy blows up in finite.

2 The functions space

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with Lipschitz-continuous boundary $\Gamma = \partial\Omega$.

We denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with a compact support contained in Ω. The inner products and norms in $L^2(\Omega)$ and $H_0^1(\Omega)$ are represented by $(.,.), \|.\|$ respectively and they are given by

$$(u,v)_\Omega = \int_\Omega u(x)v(x)dx \text{ and } \|u\|^2_{L^2(\Omega)} = \|u\|^2_{2,\Omega} = \int_\Omega u^2 dx,$$

$$\|u\|^2_{H_0^1(\Omega)} = \|\nabla u\|^2_{2,\Omega} = \int_\Omega |\nabla u|^2 \, dx.$$

We recall some known facts from the theory of the Sobolev spaces with variable exponent (see [10, 14]).

Let $L^{p(\cdot)}(\Omega)$ be the set of measurable functions f on Ω such that

$$A_{p(.)}(f) = \int_\Omega |f(x)|^{p(x)} \, dx < \infty.$$

The set $L^{p(.)}$ equipped with the Luxemburg norm

$$\|f\|_{p(.),\Omega} = \|f\|_{p(.)} = \inf \left\{ \lambda > 0; A_{p(.)}(\frac{f}{\lambda}) \leq 1 \right\}$$

is a Banach space.

Let us list some basic properties of the spaces $L^{p(\cdot)}(\Omega)$ used in the rest of this paper. It follows directly from the definition of the norm that

$$\min \left(\|f\|^{p-}_{p(.)}, \|f\|^{p+}_{p(.)} \right) \leq A_{p(.)}(f) \leq \max \left(\|f\|^{p-}_{p(.)}, \|f\|^{p+}_{p(.)} \right),$$

where

$$p^- = \inf_\Omega p(x), \; (p')^- = \inf_\Omega p'(x), \; p' = p(x)/(p(x)-1).$$

We have following Hölder-type inequality

$$\int_\Omega |fg| \, dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|f\|_{p(.)} \|g\|_{p'(.)} \leq 2 \|f\|_{p(.)} \|g\|_{p'(.)},$$

which holds for all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with $p(x) \in (1,\infty)$. The Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ with $p(x) \in [p^-,p^+] \subset (1,\infty)$ is defined by

$$\begin{cases} W_0^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : \, |\nabla u|^{p(x)} \in L^1(\Omega), \; u = 0 \text{ on } \partial\Omega\right\}, \\ \|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \sum_i \|D_i u\|_{p(\cdot),\Omega} + \|u\|_{p(\cdot),\Omega}. \end{cases} \tag{10}$$

Let $p(x)$ be log-continuous in Ω: $\forall\, x, y \in \Omega$ such that $|x-y| < \frac{1}{2}$

$$|p(x) - p(y)| \leq \omega(|x-y|) \quad \text{with} \quad \overline{\lim}_{\tau\to 0+}\left(\omega(\tau)\ln\frac{1}{\tau}\right) = C < \infty. \tag{11}$$

Throughout the paper we use the following properties of the functions from the spaces $W_0^{1,p(\cdot)}(\Omega)$:

• if condition (11) is fulfilled, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$, and the space $W_0^{1,p(\cdot)}(\Omega)$ can be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (10) – see [14, 30, 37, 38, 39];

• if $p(x) \in C^0(\overline{\Omega})$, the the space $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive;

• if $1 < q(x) \leq \sup_\Omega q(x) < \inf_\Omega p_*(x)$ with

$$p_*(x) = \begin{cases} \dfrac{p(x)n}{n-p(x)} & \text{if } p(x) < n, \\ \infty & \text{if } p(x) > n, \end{cases}$$

then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact if $q < p_*(x)$.

3 Anisotropic spaces of functions depending on x and t

Consider the cylinder

$$Q_T = \{z = (x,t) : \, x \in \Omega, \, t \in [0,T]\}$$

of a finite height T. Wherever it doesn't cause a confusion we will use the notation $z = (x,t)$ for the points of the cylinder Q_T and drop the sub-index T. The lateral boundary of the cylinder Q is $\Gamma = \partial\Omega \times (0,T)$.

If X is a Banach space, then we denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ the Banach space of measurable vector valued functions $u : (0,T) \to X$, such that $\|u(t)\|_X \in L^p(0,T)$, together with the norms:

$$\|u(t)\|_{L^p(0,T;X)} = \left[\int_0^T \|u(t)\|_X^p \, dt\right]^{\frac{1}{p}}, \ 1 \le p < \infty,$$

$$\|u(t)\|_{L^p(0,T;X)} = ess \sup_{0\le t<T} \|u(t)\|_X, \ p = \infty.$$

We will use the following function spaces (see [5],[10])

$$W = W(Q_T) = \left\{u : u \in L^2(0,T; H_0^2(\Omega)); u_t \in L^2(0,T; H_0^1(\Omega))\right\},$$

$$W^\infty = W^\infty(Q_T) = \left\{u : u \in W(Q_T); u \in L^\infty(0,T; H_0^2(\Omega)); u_t \in L^\infty(0,T; L^2(\Omega))\right\},$$

endowed with the norms

$$\|u\|_{W(Q)} = \|u\|_{L^2(0,T;H_0^2(\Omega))} + \|u_t\|_{L^2(0,T;H_0^1(\Omega))},$$

$$\|u\|_{W^\infty(Q)} = \|u\|_{W(Q_T)} + \|u\|_{L^\infty(0,T;H_0^2(\Omega))} + \|u_t\|_{L^\infty(0,T;L^2(\Omega))}.$$

Note that $\|u\|_{W(Q)}$ may be used in the equivalent form

$$\|u\|_{W(Q)} = \|u\|_{L^2(Q)} + \|\triangle u\|_{L^2(Q)} + \|\nabla u_t\|_{L^2(Q)}.$$

Let $\mathbf{p}(z) = \overrightarrow{p} = (p_1(z), \ldots, p_n(z))$ be a vector-valued function defined on $Q = Q_T$. We assume that the components of $\mathbf{p}(z)$ satisfy the conditions

$$\left\{\begin{array}{l} p_i(z) \text{ are measurable functions defined on } Q, \\ p_i(z): \ Q \mapsto (1,\infty), \\ \text{there exist constants } p_i^\pm, \ p^\pm \text{ such that} \\ \qquad p_i(z) \in [p_i^-, p_i^+] \subseteq [p^-, p^+] \subset (1,\infty). \end{array}\right. \tag{12}$$

For every fixed $t \in (0,T)$, we introduce the anisotropic Banach space

$$\mathbf{V}_t(\Omega) = \left\{u(x): \ u(x) \in L^2(\Omega) \cap W_0^{1,1}(\Omega), \ |D_i u(x)|^{p_i(x,t)} \in L^1(\Omega)\right\},$$

$$\|u\|_{\mathbf{V}_t(\Omega)} = \|u\|_{2,\Omega} + \sum_{i=1}^n \|D_i u\|_{p_i(\cdot,t),\Omega} + \|\triangle u\|_{2,\Omega}.$$

The elements of the space $\mathbf{V}_t(\Omega)$ depend on $t \in (0,T)$ as a parameter and the norms $\|u\|_{\mathbf{V}_t(\Omega)}$ are functions of t. By $\mathbf{V}_t'(\Omega)$ we denote the dual space to $\mathbf{V}_t(\Omega)$ with respect to the scalar product in $L^2(\Omega)$. For every $t \in (0,T)$ the inclusion

$$\mathbf{V}_t(\Omega) \subset \mathbf{X} = W_0^{1,p^-}(\Omega) \cap L^2(\Omega)$$

holds. This is why $\mathbf{V}_t(\Omega)$ is reflexive and separable as a closed subspace of $\mathbf{X}$.

By $\mathbf{W}_{\overrightarrow{p}}(Q)$ we denote the Banach space

$$\mathbf{W}_{\overrightarrow{p}}(Q) = \left\{ u: \ (0,T) \mapsto \mathbf{V}_t(\Omega) |\ u \in L^2(Q),\ |D_i u|^{p_i(x,t)} \in L^1(Q),\ u = 0 \text{ on } \Gamma \right\},$$

$$\|u\|_{\mathbf{W}_{\overrightarrow{p}}(Q)} = \|u\|_{2,Q} + \sum_{i=1}^{n} \|D_i u\|_{p_i(\cdot),Q}.$$

$\left(\mathbf{W}_{\overrightarrow{p}}(Q)\right)'$ is the dual of $\mathbf{W}_{\overrightarrow{p}}(Q)$ (the space of linear functionals over $\mathbf{W}(Q)$).

We have the following characterization

$$w \in \left(\mathbf{W}_{\overrightarrow{p}}(Q)\right)' \ \Leftrightarrow \ \begin{cases} \exists\, (w_0, w_1, \dots, w_n), \quad w_0 \in L^2(Q), \quad w_i \in L^{p_i'(\cdot)}(Q), \\ \forall \phi \in \mathbf{W}(Q) \quad \langle\langle w, \phi \rangle\rangle = \displaystyle\int_Q \left(w_0 \phi + \sum_{i=1}^{n} w_i D_i \phi \right) dz. \end{cases}$$

The norm in $\mathbf{W}'(Q)$ is defined by

$$\|v\|_{\mathbf{W}'(Q)} = \sup \left\{ \langle v, \phi \rangle |\ \phi \in \mathbf{W}(Q),\ \|\phi\|_{\mathbf{W}(Q)} \le 1 \right\}.$$

Let $\mathbf{v} = (v_1, \dots, v_n)$ be a vector-valued function defined in Q. Assume that $p_i(z)$ satisfy conditions (12). Introduce the modular

$$A_{\mathbf{p}(\cdot)}(\mathbf{v}) = \sum_{i=1}^{n} \int_Q |v_i|^{p_i(z)}\ dz.$$

For the elements of $\mathbf{W}_{\overrightarrow{p}}(Q)$ the following inequality

$$\begin{aligned} &\min \left\{ \sum_{i=1}^{n} \|D_i u\|_{p_i(\cdot,\cdot),Q}^{p^+}, \sum_{i=1}^{n} \|D_i u\|_{p_i(\cdot,\cdot),Q}^{p^-} \right\} \\ &\qquad \le A_{\mathbf{p}(\cdot,\cdot),Q}(\nabla u) \le \max \left\{ \sum_{i=1}^{n} \|D_i u\|_{p_i(\cdot,\cdot),Q}^{p^-}, \sum_{i=1}^{n} \|D_i u\|_{p_i(\cdot,\cdot),Q}^{p^+} \right\}. \end{aligned} \tag{13}$$

holds. We also use the space $\mathbf{W}_{\overrightarrow{p}}^{\infty}(Q)$

$$\mathbf{W}_{\overrightarrow{p}}^{\infty}(Q) = \left\{ u : u \in \mathbf{W}_{\overrightarrow{p}}(Q),\ |u_{x_i}|^{p_i(x,t)} \in L^{\infty}\left(0,T; L^1(\Omega)\right) \right\}.$$

Note that

$$W^{\infty}(Q) \subseteq \mathbf{W}^{\infty}_{\overrightarrow{p}}(Q) \text{ if } p^+ \leq \frac{2n}{n-2}.$$

We introduce also the functional space

$$\mathbf{U}(Q) = \mathbf{W}(Q) \cap \mathbf{W}_{\overrightarrow{p}}(Q)$$

endowed with the norm

$$\|u\|_{\mathbf{U}(Q)} = \|u\|_{\mathbf{W}(Q)} + \|u\|_{\mathbf{W}_{\overrightarrow{p}}(Q)}$$

and

$$\mathbf{U}^{\infty}(Q) = \mathbf{W}^{\infty}(Q) \cap \mathbf{W}^{\infty}_{\overrightarrow{p}}(Q).$$

For the exponents $p_i(x,t)$ depending on $(x,t) \in Q$ we will use the notation $p_i \in C_{log}(Q)$ if p_i satisfies condition (12) in the cylinder Q and:

$$C_{log}(Q) := \left\{ p_i \in C^0(\overline{Q}) \;\middle|\; \begin{array}{l} \forall\, z = (x,t),\, \zeta = (y,\tau) \in Q \\ \text{such that } |x-y| + |t-\tau| < \frac{1}{2}, \\ |p_i(z) - p_i(z)| \leq \omega(|z-\zeta|) \end{array} \right\} \tag{14}$$

with a continuous function ω satisfying the condition

$$\overline{\lim_{\tau \to 0^+}}\, \omega(\tau) \ln \frac{1}{\tau} = C < +\infty.$$

4 Statement of the problem

We consider a class of plate equations with memory term and with $\overrightarrow{p}(x,t)$ Laplacian type

$$u_{tt} + \Delta^2 u - \Delta_{\overrightarrow{p}(x,t)} u + \int_0^t g(t-s)\Delta u(s)ds - \epsilon \Delta u_t + f(u) = 0 \text{ in } Q_T, \tag{15}$$

$$u = \partial u/\partial \nu = 0 \text{ on } \Gamma_T = \partial\Omega \times (0,T), \tag{16}$$

$$u(x,0) = u_0(x),\; u_t(x,0) = u_1(x),\; x \in \Omega, \tag{17}$$

here $\epsilon \geq 0$ is a constant, $\partial u/\partial \nu$ represents the normal derivative and

$$\Delta_{\overrightarrow{p}(x,t)} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial u}{\partial x_i} \right),\; \overrightarrow{p} = (p_1, p_2, ..., p_n)$$

under conditions

$$1 \leq p_i^- \leq p_i(x,t) \leq p_i^+ < \infty,\;\; |p_{it}| \leq C_{p_i},\; i = 1, ..., n. \tag{18}$$

We assume that

$(H.1)$ $g:\mathbb{R}_+ \to \mathbb{R}_+$ is a bounded differential function such that

$$g(0) > 0,\ g(s) \geq 0,\ \gamma = 1 - \lambda_1 \int_0^\infty g(s)ds > 0, \tag{19}$$

where $\lambda_1 > 0$ is the embedding constant for

$$\|\nabla u\|_{2,\Omega}^2 \leq \lambda_1 \|\Delta u\|_{2,\Omega}^2 . \tag{20}$$

So, there exists a constant $\gamma_1 > 0$ such that

$$g'(t) \leq -\gamma_1 g(t),\ \text{for all } t \geq 0. \tag{21}$$

$(H.2)$ Concerning the source term $f(u)$, we assume that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that

$$|f(u)| \leq \gamma_2 |u|^{\sigma-1},\ \forall\ u \in \mathbb{R},\ \gamma_2 > 0.$$

and

$$1 < \sigma \leq \frac{2n-4}{n-4} \text{ if } n \geq 5 \text{ and } \sigma < \infty \text{ if } 1 \leq n \leq 4.$$

Suppose also that f satisfies the following condition

$$-\gamma_3 |u|^\sigma \geq G(u) = \int_0^u f(s)ds, \forall u \in \mathbb{R} \text{ and } \sigma > 2, \tag{22}$$

where γ_3 is a positive constant.

Lemma 1 *Suppose that $g \in C^1(\mathbb{R})$ and $u \in C^1((0,T);H^1(\Omega))$. Then the following identity holds*

$$\int_\Omega \int_0^t g(t-s)\nabla u(s)ds \cdot \nabla u_t(s)dx = \tag{23}$$

$$= -\frac{1}{2}\frac{d}{dt}\left\{(g\circ\nabla u)(t) - \left(\int_0^t g(s)ds\right)\|\nabla u(t)\|_{2,\Omega}^2\right\} +$$

$$+\frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_{2,\Omega}^2$$

where

$$(g\circ\nabla u)(t) = \int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|_{2,\Omega}^2 ds.$$

The proof of Lemma 1 readily from differentiating the term $(g \circ \nabla u)(t)$ with respect to t.

Assume that

$$u_0 \in H_0^2(\Omega) \cap W^{1,p_i(.,0)}(\Omega), u_1 \in L^2(\Omega). \tag{24}$$

Definition 2 *A function $u : \Omega_T \to \mathbb{R}$ is called an energy weak solution to (15)-(17) if:*

$$u \in \mathbf{U}(Q_T) = \mathbf{W}(Q_T) \cap \mathbf{W}_{\overrightarrow{p}}(Q_T) \cap \mathbf{W}^{\infty}(Q_T) \cap \mathbf{W}_{\overrightarrow{p}}^{\infty}(Q_T),$$

$$u(x,t) \to u_0(x) \text{ in } H_0^1(\Omega), \int_\Omega (u_t(\cdot,t) - u_1(\cdot))\psi(\cdot)dx \to 0$$

as $t \to 0$, $\forall \psi \in C_0^\infty(\Omega)$. The following integral identity holds

$$\int_{Q_T} \left(-u_t\varphi_t + \Delta u \Delta \varphi + \sum_{i=1}^n |u_{x_i}|^{p_i(x,t)-2} u_{x_i}\varphi_{x_i} + \epsilon \nabla u_t \nabla \varphi \right) dxdt +$$

$$\begin{aligned} &+ \int_{Q_T} \left(\int_0^t g(t-s)\Delta u(x,s)ds + f(u) \right) \varphi dxdt \\ &= (u_1, \varphi(0))_\Omega - (u_t(T), \varphi(T))_\Omega, \forall \varphi \in \mathbf{U}(Q_T). \end{aligned}$$

In [9] was proved that any weak solution satisfies the energy relation

$$E(t) + \epsilon \int_0^t \|\nabla u_t(s)\|_{2,\Omega}^2 ds \tag{25}$$

$$= \int_0^t \left(\frac{1}{2}(g' \circ \nabla u)(s) - \frac{1}{2} g(s) \|\nabla u(s)\|_{2,\Omega}^2 + Q(s) \right) ds + E(0)$$

where the energy functional is given by

$$E(t) = \frac{1}{2} \|u_t(t)\|_{2,\Omega}^2 + \frac{1}{2} \|\Delta u\|_{2,\Omega}^2 + \sum_{i=1}^n \int_\Omega \frac{1}{p_i} |u_{x_i}|^{p_i} dx + \int_\Omega G(u(t))dx$$

$$-\frac{1}{2} \left(\int_0^t g(s)ds \right) \|\nabla u(t)\|_{2,\Omega}^2 + \frac{1}{2}(g \circ \nabla u)(t),$$

and

$$Q = \int_\Omega \left(\sum_{i=1}^n \frac{1}{p_i^2} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x,t)} \left(1 - p_i \ln \left| \frac{\partial u}{\partial x_i} \right| \right) |p_{it}| \right) dx,$$

$$(g\circ\nabla u)(t)=\int_0^t g(t-s)\,\|\nabla u(s)-\nabla u(t)\|_{2,\Omega}^2\,ds.$$

Computation of (25) involves the easily verified formula

$$|u_{x_i}|^{p_i-2}u_{x_i}\ u_{x_it}=\frac{\partial}{\partial t}\left(\frac{|u_{x_i}\ |^{p_i}}{p_i}\right)+|u_{x_i}\ |^{p_i}\left(\frac{1}{p_i^2}-\frac{\ln|u_{x_i}\ |}{p_i}\right)p_{it},$$

together with the formula (23).

First we prove two Lemmas for the energy functional. These Lemmas will to used for investigate the blow up effect for solutions with nonpositive initial energy functional.

Lemma 3 *Assume that*

$$p_{it}=0\to Q=0,\ \ i=1,...,n., \tag{26}$$

and $(H.1)$ *hold. Then*

$$E(t)+\epsilon\int_0^t\|\nabla u_t(s)\|_{2,\Omega}^2\,ds \tag{27}$$

$$+\int_0^t\left(\frac{\gamma_1}{2}(g\circ\nabla u)(s)+\frac{1}{2}g(s)\,\|\nabla u(s)\|_{2,\Omega}^2\right)ds\le E(0),\ \text{for all } t\ge 0.$$

Proof. According to (25) and (21), (26) we have

$$E(t)+\epsilon\int_0^t\|\nabla u_t(s)\|_{2,\Omega}^2\,ds=\int_0^t\left(\frac{1}{2}(g'\circ\nabla u)(s)-\frac{1}{2}g(s)\,\|\nabla u(s)\|_{2,\Omega}^2\right)ds+E(0)$$

$$\le-\int_0^t\left(\frac{\gamma_1}{2}(g\circ\nabla u)(s)+\frac{1}{2}g(s)\,\|\nabla u(s)\|_{2,\Omega}^2\right)ds+E(0).$$

Lemma 4 *Assume that*

$$p_{it}\le 0,\ |p_{it}|\le C_p,\quad i=1,...,n., \tag{28}$$

and $(H.1)$ *hold. Then*

$$E(t)+\epsilon\int_0^t\|\nabla u_t(s)\|_{2,\Omega}^2\,ds+\int_0^t\left(\frac{\gamma_1}{2}(g\circ\nabla u)(s)+\frac{1}{2}g(s)\,\|\nabla u(s)\|_{2,\Omega}^2\right)ds$$
$$\le E(0)+t\,|\Omega|\,nC_p\left(p_i^-\right)^{-2}. \tag{29}$$

Proof. We use the inequality

$$E(t)+\epsilon\int_0^t \|\nabla u_t(s)\|_{2,\Omega}^2\,ds \tag{30}$$
$$+\int_0^t\left(\frac{\gamma_1}{2}(g\circ\nabla u)(s)+\frac{1}{2}g(s)\|\nabla u(s)\|_{2,\Omega}^2\right)ds\le\int_0^t Q(s)ds+E(0)$$

and evaluate Q in the following way

$$Q=\sum_{i=1}^n\int_{\Omega\cap(1\le p_i\ln|u_{x_i}|)}\left(\frac{1}{p_i^2}|u_{x_i}|^{p_i(x,t)}(1-p_i\ln|u_{x_i}|)|p_{it}|\right)dx+ \tag{31}$$
$$+\sum_{i=1}^n\int_{\Omega\cap(1>p_i\ln|u_{x_i}|)}\left(\frac{1}{p_i^2}|u_{x_i}|^{p_i(x,t)}(1-p_i\ln|u_{x_i}|)|p_{it}|\right)dx\le$$
$$\le\sum_{i=1}^n\int_{\Omega\cap(1>p_i\ln|u_{x_i}|)}\left(\frac{1}{p_i^2}|u_{x_i}|^{p_i(x,t)}(1-p_i\ln|u_{x_i}|)|p_{it}|\right)dx\le|\Omega|\frac{nC_p}{(p_i^-)^2}.$$

Above we have used the following properties of the function

$$F(\eta)=\frac{\eta^p}{p^2}(1-p\ln\eta),\ 0\le\eta\le\exp(\frac{1}{p}),$$

$$F(0)=F(\exp(\frac{1}{p}))=0,\ F'(\eta)=-\eta^{p-1}\ln\eta,\ \max_{0\le\eta\le\exp(\frac{1}{p})}F(\eta)=F(1)=\frac{1}{p^2}.$$

By substituting (31) into (30) we obtain the desired estimate.

Lemma 5 ***Local estimates***
Assume that (22), i.e.,

$$-\gamma_3|u|^\sigma\ge G(u)=\int_0^u f(s)ds,\ \forall\ u\in\mathbb{R},$$

(24) and $(H.1)-(H.2)$ hold. Suppose the number p^+ satisfies the inequality

$$p^+<2\left(1+\frac{2(\sigma-2)}{2(\sigma+2)+(\sigma-2)n}\right),\ \sigma>2, \tag{32}$$

and under conditions (18), (26) (or (18),(28); or (18);(32)) and in addition

$$\sigma<p^-\left(1+\frac{2}{n}\right)-1,$$

or

$$2 < \sigma < 2 + \frac{8}{n}.$$

Then, there exists $T_{\max} > 0$ *which depend on* $\Pi = \Pi(\|u_1\|_{2,\Omega}^2, \ \|\Delta u_0\|_{2,\Omega}^2, \|u_0\|_\sigma, C_p)$, *such that*

$$\Psi(t) := \|u_t(t)\|_{2,\Omega}^2 + \|\Delta u(t)\|_{2,\Omega}^2 + \sum_{i=1}^n \int_\Omega |u_{x_i}(t)|^{p_i} \, dx + \int_\Omega |u(t)|^\sigma \, dx$$

$$+\epsilon \int_0^t \|\nabla u_t(s)\|_{2,\Omega}^2 \, d\tau \leq C(\Pi), \ \ 0 \leq t < T_{\max}$$

holds with a constant $C(\Pi)$ *independent of* m.

Proof. See [9].

We first state a local existence theorem

Theorem 6 *(Local in time existence [9]) Let* $u_0 \in H_0^2(\Omega) \cap W^{1,p_i(.,0)}(\Omega)$, $u_1 \in L^2(\Omega)$. *Under conditions of Lemma 5, (14) and* $(H.1) - (H.2)$ *the problem the problem (15)-(17) has at least one weak solution in the sense of Definition 2 for a small time* $t \in [0, T_{\max}), (T_{\max} > 0$ *is small).*

Proof. See [9].

5 Blow up property

Let u weak solutions of problem (15)-(17). We consider the energy functional defined by

$$E(t) = \frac{1}{2} \|u_t(t)\|_{2,\Omega}^2 + \frac{1}{2} \|\Delta u\|_{2,\Omega}^2 + \sum_{i=1}^n \int_\Omega \frac{1}{p_i} |u_{x_i}|^{p_i} \, dx$$
$$+ \int_\Omega G(u(t))dx + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{2} \int_0^t g(s)ds \, \|\nabla u(t)\|_{2,\Omega}^2,$$

Recall that according to the inequality (27), (29) is valid for weak solutions.

Suppose that there exists $\lambda > 2$ such that

$$2 \left(\lambda \int_0^u f(s)ds - uf(u) \right) \geq \gamma_4 |u|^\sigma, \ \forall u \in \mathbb{R} \text{ where } \gamma_4 > 0, \qquad (33)$$

and

$$E(0) \leq 0, \ \ 0 < (u_0, u_1)_\Omega, \qquad (34)$$

$$2 \le p_i^- = const \le p_i(x,t) = p(x) \le p_i^+ = const < \lambda, \ i = 1, 2..., n. \tag{35}$$

First we prove the following Theorem,

Theorem 7 *Let u be an energy weak solution to problem (15)-(17). Let conditions of first assertion in Lemma 3 and (33)-(35) hold. Then there exists a finite time $T_{\max} < \infty$ such that*

$$\phi(t) = \|u(t)\|_{2,\Omega}^2 + \epsilon \int_0^t \int_\Omega |\nabla u|^2 \, dxds \to \infty \ \textit{if} \ t \to T_{\max}.$$

Proof. According to the Lemma 3 (inequality (27)), we have

$$\begin{aligned} & E(t) + \epsilon \int_0^t \|\nabla u_t(s)\|_{2,\Omega}^2 \, ds \\ & + \int_0^t \left(\frac{\gamma_1}{2} (g \circ \nabla u)(s) + \frac{1}{2} g(s) \|\nabla u(s)\|_{2,\Omega}^2 \right) ds \\ & \le E(0). \end{aligned} \tag{36}$$

It is very easy to verify that

$$\phi'(t) = 2 \left(u, u_t\right)_\Omega + \epsilon \int_\Omega |\nabla u|^2 \, dx,$$

$$\begin{aligned} \phi''(t) &= 2 \|u_t(t)\|_{2,\Omega}^2 + 2 \left(u, u_{tt}\right)_\Omega + 2\epsilon \left(\nabla u, \nabla u_t\right)_\Omega = \\ &= 2 \|u_t(t)\|_{2,\Omega}^2 - 2 \int_\Omega \left(\sum_{i=1}^n |u_{x_i}|^{p_i} + u f(u) + (\Delta u)^2 \right) dx \\ &+ 2 \int_\Omega \int_0^t g(t-s) \nabla u(s) . \nabla (t) ds dx. \end{aligned} \tag{37}$$

Taking into account (27) and first inequality in (34), we evaluate $\phi''(t)$ from below in the following way

$$\begin{aligned} \phi''(t) &\ge 2 \|u_t(t)\|_{2,\Omega}^2 - 2 \int_\Omega \left(\sum_{i=1}^n |u_{x_i}|^{p_i} + u f(u) + (\Delta u)^2 \right) dx \\ &+ 2 \int_\Omega \int_0^t g(t-s) \nabla u(s) . \nabla u(t) ds dx \\ &+ 2\lambda E(t) + 2\lambda\epsilon \int_0^t \|\nabla u_t(s)\|_{2,\Omega}^2 \, ds. \end{aligned}$$

Thus,

$$\phi''(t) \geq (2+\lambda)\|u_t(t)\|_{2,\Omega}^2 + (\lambda-2)\|\Delta u\|_{2,\Omega}^2 + 2\sum_{i=1}^{n}\int_\Omega\left(\frac{\lambda}{p_i}-1\right)|u_{x_i}|^{p_i}dx +$$
$$2\left(\int_\Omega\left(\lambda\int_0^u f(s)ds - uf(u)\right)dx)\right) + \lambda(g\circ\nabla u)(t) + I,$$

where

$$I = 2\int_\Omega\int_0^t g(t-s)\nabla u(s).\nabla u(t)dsdx - \lambda\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|_{2,\Omega}^2.$$

Next we present I in the form

$$I = -(\lambda-2)\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|_{2,\Omega}^2 + 2\int_\Omega\int_0^t g(t-s)\nabla u(t).[\nabla u(s)-\nabla u(t)]dsdx.$$

and evaluate from below in the following way

$$I \geq -(\lambda-2)\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|_{2,\Omega}^2 - 2\int_0^t g(t-s)\|\nabla u(t)\|_{2,\Omega}\|\nabla u(s)-\nabla u(t)\|_{2,\Omega}ds.$$

Using the Young inequality in the last expression we can conclude that

$$I \geq -(g\circ\nabla u)(t) - (\lambda-1)\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|_{2,\Omega}^2. \tag{38}$$

From (33) and (38) we have.

$$\begin{aligned}\phi''(t) &\geq (2+\lambda)\|u_t(t)\|_{2,\Omega}^2 + 2(\lambda-1)\|\Delta u\|_{2,\Omega}^2 + 2\int_\Omega\sum_{i=1}^{n}\left(\frac{\lambda}{p_i}-1\right)|u_{x_i}|^{p_i}dx + \\ &+ (\lambda-1)\left((g\circ\nabla u)(t) - \left(\int_0^t g(s)ds\right)\|\nabla u(t)\|_{2,\Omega}^2\right) + \gamma_4\int_\Omega|u|^\sigma dx \\ &+ 2\lambda\epsilon\int_0^t\|\nabla u_t(s)\|_{2,\Omega}^2 ds.\end{aligned} \tag{39}$$

Now, using (19) and (20), it is very easy to verify that

$$(\lambda-1)\left[2\|\Delta u\|_{2,\Omega}^2 - \left(\int_0^t g(s)ds\right)\|\nabla u(t)\|_{2,\Omega}^2\right] > 0.$$

Thus, $\phi''(t) > 0$.
It follows

$$\phi'(t) > 0,\ t > 0,\ \text{if}\ \phi'(0) \geq 2(u_0, u_1)_\Omega + \epsilon \int_\Omega |\nabla u_0|^2\, dx > 0.$$

Then we have

$$\phi''(t) > 0, \phi'(t) > 0, \phi(t) > 0.$$

Hence

$$\phi(t) \to \infty \text{ as } t \to T_{\max}.$$

Assume in contrast that

$$T_{\max} = \infty.$$

Using (13), we derive the following inequalities for any fixed t,

$$\|\nabla u(\cdot,t)\|_{2,\Omega} \leq C\, \|\nabla u(\cdot,t)\|_{p_i,\Omega} \leq \tag{40}$$
$$C \max\left\{ \left(\int_\Omega \sum_{i=1}^n |u_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i^-}}, \left(\int_\Omega \sum_{i=1}^n |u_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i^+}} \right\},$$

with $p_i \geq 2$. From (39)

$$\int_\Omega \left(|u_t|^2 + |\Delta u|^2\, dx + \sum_{i=1}^n |u_{x_i}|^{p_i} + |u|^\sigma \right) dx + (g \circ \nabla u)(t) \leq C\phi''.$$

Then we can rewrite (40) in the form

$$\|\nabla u(\cdot,t)\|_{2,\Omega} \leq C \max\left\{ (\phi'')^{\frac{1}{p_i^-}}, (\phi'')^{\frac{1}{p_i^+}} \right\}.$$

Hence we arrive at the inequality

$$0 \leq \phi' = 2\int_\Omega u u_t dx + \int_\Omega |\nabla u|^2\, dx \leq 2\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega u_t^2 dx\right)^{\frac{1}{2}} + \int_\Omega |\nabla u|^2\, dx$$
$$\leq 2\left(\int_\Omega |u|^\sigma\, dx\right)^{\frac{1}{\sigma}} \left(\int_\Omega u_t^2 dx\right)^{\frac{1}{2}} + \int_\Omega |\nabla u|^2\, dx \leq C\left((\phi'')^{\frac{1}{\sigma}+\frac{1}{2}} + (\phi'')^{\frac{2}{p_i^-}} \right) \leq C(\phi'')^{\frac{1}{\mu}},$$

where

$$\mu = \frac{1}{\max[\frac{1}{\sigma}+\frac{1}{2}, \frac{2}{p_i^-}]} > 1 \text{ if } \sigma > 2,\ p_i^- > 2.$$

Taking into account that $\phi'' > 0$, we assume, without loss of generality that $\phi' \geq 1$, and come to the ordinary differential inequality

$$(\phi'(t))^{\mu} \leq C\phi''.$$

Integrating the last ordinary differential equation with respect to t, we arrive to the estimate

$$\phi'(t) \geq \phi'(0)(1 - \frac{t(\mu-1)}{C}(\phi'(0)^{\mu-1})^{-\frac{1}{\mu-1}} \to \infty$$

as

$$t \to T_{\max} = \frac{C}{\mu-1}(\phi'(0))^{-\mu+1} < \infty. \tag{41}$$

Taking into account that

$$\frac{1}{\mu-1} = \frac{\max[\frac{1}{\sigma}+\frac{1}{2}, \frac{2}{p_i^-}]}{1-\max[\frac{1}{\sigma}+\frac{1}{2}, \frac{2}{p_i^-}]} > 1,$$

we obtain that

$$\phi(t) \to \infty, \text{ as } t \to T_{\max}.$$

The theorem is proved.

Remark 8 *Let us notice that constants μ and C (and respectively $T_{\max}$) in (41) depend only on $|\Omega|, n, p_i^{\pm}$.*

Now, we assume that the exponent p_i weakly dependent on t, that is, the constant C_p is small. The proof of the blow up is the same as in the previous Theorem if we guarantee that

$$E(t) + \epsilon \int_0^t \int_{\Omega} |\nabla u_t|^2 \, dxds \leq 0, \; 0 \leq t \leq T_{\max} \tag{42}$$

with $T_{\max}$ already defined in (41). According to the Lemma 4 (see inequality (29)), we have

$$E(t) + \epsilon \int_0^t \int_{\Omega} |\nabla u_t|^2 \, dxds \leq E(0) + T_{\max} |\Omega| \, nC_p(p_i^-)^{-2}.$$

Assuming that

$$C_p \leq \frac{|E(0)| \, (p_i^-)^2}{T_{\max} |\Omega| \, n}, \; E(0) < 0. \tag{43}$$

we arrive to (42). Then we come to

Theorem 9 *Let u be an energy weak solution to problem (15)-(17). Assume the conditions of Lemma 4 together with (42) (with* $T_{\max}$ *defined in (41)) be fulfilled. Let the conditions*

$$E(0) < 0, 0 < (u_0, u_1)_\Omega, \ 2 \le p_i^- \le p_i^+ < \lambda, \ i = 1, 2..., n.$$

hold. Then the solution u blows up (in the sense that $\phi(t)$ *becomes unbounded) on the finite interval* $(0, T_{\max})$.

Now we consider equation (15) with $\epsilon = 0$, assuming that the problem (15)-(17) has at least one local energy solution. Here we follow the paper [17], where the authors have proved the blow up for an abstract hyperbolic equation in a Banach space which includes, the equation of the type (15) with $p = const$, without $\Delta^2 u$ and memory term as an example.

We assume that

$$E(0) \le 0, 0 < (u_0, u_1)_\Omega. \tag{44}$$

By repeating the arguments from paper [17], we prove the following theorem.

Theorem 10 *Let u be an energy weak solution to problem (15)-(17) with* $\epsilon = 0$. *Let conditions of the Lemma 3 be satisfied and assume (44). Then u blows up (in the sense that* $\|u(t)\|_{2,\Omega}^2$ *becomes unbounded) on the finite interval* $(0, T_{\max})$ *with* $T_{\max} = 2\|u_0\|_{2,\Omega}^2 / (\lambda - 2)(u_0, u_1)_\Omega$.

Proof. Consider the following functional

$$G(t) = \|u(t)\|^2 = \|u(t)\|_{2,\Omega}^2 .$$

On the other hand,

$$G'(t) = 2(u, u_t)_\Omega, \ G''(t) = 2\|u(t)\|_{2,\Omega}^2 + 2(u, u_{tt})_\Omega, \tag{45}$$

holds. Taking into account (44), we evaluate $G''(t)$ in the following way

$$\begin{aligned} G''(t) \ge\ & 2\|u_t(t)\|_{2,\Omega}^2 + 2\int_\Omega -\left(\sum_{i=1}^n |u_{x_i}|^{p_i} + uf(u) + (\Delta u)^2\right) dx + \lambda E(t) \\ & + 2\int_\Omega \int_0^t g(t-s)\nabla u(s).\nabla u(t) ds dx \\ & = (2+\lambda)\|u_t(t)\|_{2,\Omega}^2 + (\lambda - 2)\|\Delta u\|_{2,\Omega}^2 + 2\sum_{i=1}^n \int_\Omega \left(\frac{\lambda}{p_i} - 1\right) |u_{x_i}|^{p_i} dx + \\ & + 2\left(\int_\Omega \left(\lambda \int_0^u f(s)ds - uf(u)\right) dx\right) + \lambda(g \circ \nabla u)(t) + I, \end{aligned}$$

for some $\lambda > 2$, $p_i < \lambda$. Then using (39) with $\varepsilon = 0$, one get

$$G''(t) \geq (2+\lambda) \|u_t(t)\|_{2,\Omega}^2, \ t > 0. \tag{46}$$

From the first identity (45) we conclude that

$$G'^2(t) \leq 4 \|u_t\|_{2,\Omega}^2 \|u\|_{2,\Omega}^2 = 4G \|u_t\|_{2,\Omega}^2 \Rightarrow \|u_t\|_{2,\Omega}^2 \geq \frac{G'^2}{4G}. \tag{47}$$

Combining (46) and (47), we come with the following ordinary differential inequality

$$G''(t) \geq (2+\lambda)\frac{G'^2}{4G} \Leftrightarrow \frac{G''(t)}{G'^2(t)} \geq \frac{(2+\lambda)}{4G} \geq 0. \tag{48}$$

Then (44), (48) imply that $G'(t) > 0$ for all $t > 0$. Integrating (48) twice as above, we come with the inequality

$$\|u_0\|_{2,\Omega}^2 \left[1 - t\frac{\lambda-2}{2}\frac{(u_0,u_1)_\Omega}{\|u_0\|_{2,\Omega}^2}\right]^{-\frac{4}{\lambda-2}} \leq \|u(t)\|_{2,\Omega}^2.$$

The theorem is proved.

Now, we assume that the exponent p_i weakly dependent on t, that is, the constant C_p is small.

Repeating the above mentioned arguments (see (42), (43)), we prove

Theorem 11 *Let u an energy weak solution to problem (15)-(17) with $\varepsilon = 0$. Let conditions of the Lemma 4 be fulfilled and (43) (with $T_{\max}$ defined in previous Theorem) and (44) hold. Then the solution u blows up on the finite interval* $(0, T_{\max})$.

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