On the blowup of solutions of a Schrödinger equation with an inhomogeneous damping coefficient

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Abstract

We consider the Cauchy problem for the nonlinear self-focusing Schrödinger equation in $\mathbb{R}^N$ with an inhomogeneous smooth damping coefficient and we prove, for suitable initial data, and in the spirit of the seminal work [4], a blowup result for the corresponding local solutions. We also give some lower bound estimates for the blowing-up solutions, following the ideas in [2] to the undamped case.

Keywords: nonlinear Schrödinger equations, damping, blowup

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1. Introduction and main results

Let us consider the following Cauchy problem:

$$\begin{aligned} i\,u_t \;&=\; \Delta u + |u|^{p-1}\,u - \frac{i}{2}\,a\,u\,, \qquad i=\sqrt{-1}\,, \quad x\in\mathbb{R}^N\,, \quad t\ge 0\,, \\ u(x,0) \;&=\; u_0(x)\,, \qquad x\in\mathbb{R}^N\,, \quad u_0\in H^1(\mathbb{R}^N)\,, \end{aligned} \tag{1.1}$$

with $a\in C^1(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$, $a\ge 0$, $\frac{4}{N}<p-1<\frac{4}{N-2}$ if $N>2$, $\frac{4}{N}<p-1<+\infty$ if $N=1,2$.

In this paper we study the effect of an inhomogeneous damping coefficient on the singularity formation for the nonlinear self-focusing Schrödinger equation, corresponding to an electromagnetic wave absorbed by an inhomogeneous medium.

The study of the possible blowup of the solutions of the Cauchy problem (1.1), in the particular case where a is a positive constant, was made in [6] and [7] (cf. also [3]) for a class of initial data.

By a a well known result of T. Kato (cf. [1], Theo. 4.4.6), the Cauchy problem (1.1) is well posed in $H^1(\mathbb{R}^N)$, that is there exists a $T>0$ and a unique strong solution $u\in C([0,T];H^1(\mathbb{R}^N))$. Moreover, this solution is defined in a maximal time interval $[0,T_{\max}[$ and, if $T_{\max}<+\infty$, then $|u(t)|_{H^1}\xrightarrow[t\to T_{\max}]{}+\infty$. In particular, if $a(x)=a(|x|)$ and $u_0(x)=u_0(|x|)$, the solution $u(x,t)$ has also radial symmetry, that is $u(x,t)=u(|x|,t)$, $t\in[0,T_{\max}[$.

If we set, with $\int=\int_{\mathbb{R}^N}$, for u solution of (1.1),

$$E(t) \;=\; \int |\nabla u(x,t)|^2\,dx \;-\; \frac{2}{p+1}\int |u(x,t)|^{p+1}\,dx\,, \tag{1.2}$$

it is easy to derive from (1.1)

$$\frac{d}{dt}\int |u(t)|^2\,dx \;+\; \int a\,|u(t)|^2\,dx \;=\; 0 \tag{1.3}$$

and

$$\begin{aligned} \frac{d}{dt}E(t) \;+\; \int a\,|\nabla u(t)|^2\,dx \;-\; \int a\,|u(t)|^{p+1}\,dx \\ +\;\mathrm{Re}\int \big(\nabla u(t)\cdot\nabla a\big)\,\bar u(t)\,dx \;=\; 0 \end{aligned} \tag{1.4}$$

for $t\in[0,T_{\max}[$.

From (1.3) we obtain, since $a \geq 0$, and with $|u(t)|_q = |u(t)|_{L^q(\mathbb{R}^N)}$,

$$|u(t)|_2 \;\leq\; |u_0|_2\,, \qquad t \in [0, T_{\max}[\ . \tag{1.5}$$

Now, if we assume

$$|x|\, u_0 \;\in\; L^2(\mathbb{R}^N)\ , \tag{1.6}$$

we derive (cf. [1], §6.5, for the case $a = 0$ and [7] for the case $a > 0$ constant) $|x|\, u(t) \in L^2(\mathbb{R}^N)$, $t \in [0, T_{\max}[$, and with

$$V(t) \;=\; -4\ \mathrm{Im}\!\int \big(x\cdot\nabla u(t)\big)\, \bar u(t)\, dx \tag{1.7}$$

we obtain

$$\frac{d}{dt}\int |x^2|\, |u(t)|^2\, dx \;+\; \int a\, |x|^2\, |u(x,t)|^2\, dx \;=\; V(t)\ . \tag{1.8}$$

We deduce from (1.1) and (1.8) (assuming u smooth enough for the computations)

$$\begin{aligned}
\frac{dV}{dt} &= -\ 4\ \mathrm{Im}\!\int x\cdot\nabla u_t\, \bar u\, dx \;-\; 4\ \mathrm{Im}\!\int x\cdot\nabla \bar u\, u_t\, dx \\
&= 4\, N\ \mathrm{Im}\!\int u_t\, \bar u\, dx \;+\; 8\ \mathrm{Im}\!\int x\cdot\nabla \bar u\, u_t\, dx \\
&= 4\, N\left[\int |\nabla u|^2\, dx \;-\; \int |u|^{p+1}\, dx\right] + 8\int |\nabla u|^2\, dx \\
&\quad -\ 4\, N\int |\nabla u|^2\, dx \;+\; \frac{8\, N}{p+1}\int |u|^{p+1}\, dx \;-\; 4\ \mathrm{Im}\!\int \big(x\cdot\nabla \bar u\big)\, a\, u\, dx\ .
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{dV}{dt} \;=\; 8&\left[\int |\nabla u|^2\, dx \;-\; \frac{N(p-1)}{2(p+1)}\int |u|^{p+1}\, dx\right] \\
&-\ 4\ \mathrm{Im}\!\int \big(x\cdot\nabla \bar u\big)\, a\, u\, dx\ .
\end{aligned} \tag{1.9}$$

We derive, from (1.8),

$$\frac{d^2}{dt^2}\int |x|^2\, |u|^2\, dx \;+\; \frac{d}{dt}\int a\, |x|^2\, |u|^2\, dx \;=\; \frac{dV}{dt} \tag{1.10}$$

and, from (1.1),

$$\frac{1}{2}\,\frac{d}{dt}\int a\,|x|^2\,|u|^2\,dx \;=\; \mathrm{Im}\!\int \Delta u\,a\,|x|^2\,\bar u\,dx \;-\; \frac{1}{2}\int a^2\,|x|^2\,|u|^2\,dx$$

and so, integrating by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\int a\,|x|^2\,|u|^2\,dx \;+\; \int a^2\,|x|^2\,|u|^2\,dx \;=&\\ =\; -2\,\mathrm{Im}\!\int \big(\nabla u\cdot\nabla a\big)\,|x|^2\,\bar u\,dx \;-\; 4\,\mathrm{Im}\!\int \big(x\cdot\nabla u\big)\,a\,\bar u\,dx\;.\end{aligned}$$

Hence, from (1.10) we deduce

$$\begin{aligned}\frac{d^2}{dt^2}\int |x|^2\,|u|^2\,dx \;=\; \frac{dV}{dt} \;+&\int a^2\,|x|^2\,|u|^2\,dx\\ &+\;2\,\mathrm{Im}\!\int \big(\nabla u\cdot\nabla a\big)\,|x|^2\,\bar u\,dx \;+\; 4\,\mathrm{Im}\!\int \big(x\cdot\nabla u\big)\,a\,\bar u\,dx\;.\end{aligned} \tag{1.11}$$

Finally, by (1.9) and (1.11) we obtain

$$\begin{aligned}\frac{d^2}{dt^2}\int |x|^2\,|u|^2\,dx \;=\;& \int a^2\,|x|^2\,|u|^2\,dx \;+\; 2\,\mathrm{Im}\!\int \big(\nabla u\cdot\nabla a\big)\,|x|^2\,\bar u\,dx\\ &+\;8\,\mathrm{Im}\!\int \big(x\cdot\nabla u\big)\,a\,\bar u\,dx \;+\; 2\,N\,(p-1)\,E(t)\\ &-\;\theta(p,N)\int |\nabla u|^2\,dx\;,\end{aligned} \tag{1.12}$$

with $\theta(p,N) = 2\,N\,(p-1) - 8 > 0$.

We will prove the following results (where $\operatorname{supp} a$ means support of a):

Theorem 1.1. *Assume, in addition to the initial hypothesis in* (1.1)*, that* a *and* u_0 *are radial symmetric,* $|x|a$*,* $|x|^2\,|\nabla a| \in L^\infty(\mathbb{R}^N)$*,* $|x|\,u_0 \in L^2(\mathbb{R}^N)$*,* $2 \le p \le \frac{2}{N} + 3$*,* $a, |\nabla a| \in L^{\frac{2}{3-p}}(\mathbb{R}^N)$ *if* $p < 3$ *and that there exists* $\varepsilon > 0$ *such that* $\operatorname{supp} a \subset \big\{x \in \mathbb{R}^N \mid |x| \ge \varepsilon\big\}$*. Then, if* $E_0 = E(u_0) < 0$*, there exists* $T_0 = T_0(u_0) \in \,]0, +\infty[$ *and* $\delta > 0$ *such that, if* $\max\big(|a|_{W^{1,q}}, \big||x|\,a\big|_\infty, \big||x|^2\,|\nabla a|\big|_\infty\big) < \delta$*, with* $q = \frac{2}{3-\min(3,p)} \in [1, +\infty]$*, then the local solution* $u(t)$ *of the Cauchy problem* (1.1) *blows up in finite time with* $T_{\max} \le T_0$*.*

Remark 1.1. It is easy to see that, for fixed u_0 and T_0, δ decreases with ε (cf. the proof of Theorem 1.1).

The next result is a partial extension of Theorem 6.5.13 in [1] (first proved in [2]) for the case $a = 0$:

Theorem 1.2. *Assume, in addition to the initial hypothesis in* (1.1), *that* $a \in L^{\frac{p+1}{p-1}}(\mathbb{R}^N)$ *and* $|\nabla a| \in L^{\frac{2(p+1)}{p-1}}(\mathbb{R}^N)$. *Then, if* $u(t)$ *is a solution of the Cauchy problem* (1.1) *such that* $T_{\max} < +\infty$, *there exists a positive constant* c_0 *such that*

$$|\nabla u|_{L^\infty(0,t;L^2)} \geq \frac{c_0}{(T_{\max} - t)^\gamma}\,, \qquad \textit{for} \quad t \in \,]0, T_{\max}[\;, \tag{1.13}$$

where $\gamma = \frac{1}{p-1} - \frac{N-2}{4} > 0$.

2. Proof of Theorem 1.1

From (1.4) we derive

$$E(t) = E_0 + \int_0^t \left[\int a\,|u|^{p+1}\,dx - \int a\,|\nabla u|^2\,dx - \mathrm{Re} \int \big(\nabla u \cdot \nabla a\big)\,\bar{u}\,dx \right] d\tau \tag{2.1}$$

and, in order to estimate the right-hand side of (1.12), we obtain, by (1.5),

$$\begin{cases} \displaystyle\int a^2\,|x|^2\,|u|^2\,dx \,\leq\, \big|a\,|x|\big|_\infty^2\,|u_0|_2^2\,, \\[2ex] \displaystyle\left| 2\ \mathrm{Im} \int \big(\nabla u \cdot \nabla a\big)\,|x|^2\,\bar{u}\,dx \right| \,\leq\, 2\,|\nabla u|_2\,\big||x|^2\,|\nabla a|\big|_\infty\,|u_0|_2 \,\leq \\[1ex] \qquad \leq\, \delta_1\,|\nabla u|_2^2 \,+\, c(\delta_1)\,\big||x|^2\,|\nabla a|\big|_\infty^2\,|u_0|_2^2\,, \quad \delta_1 > 0 \text{ to be choosed}\,, \\[2ex] \displaystyle\left| 8\ \mathrm{Im} \int \big(x \cdot \nabla u\big)\,a\,\bar{u}\,dx \right| \,\leq\, 8\,|\nabla u|_2\,\big||x|\,a\big|_\infty\,|u_0|_2 \,\leq \\[1ex] \qquad \leq\, \delta_1\,|\nabla u|_2^2 \,+\, c(\delta_1)\,\big||x|\,a\big|_\infty^2\,|u_0|_2^2\,. \end{cases} \tag{2.2}$$

To estimate the right-hand side of (2.1) we assume a and u radial symmetrics and, for a certain $\varepsilon > 0$, $\operatorname{supp} a \subset \big\{x \in \mathbb{R}^N \mid |x| \geq \varepsilon\big\}$. For the sake of simplicity, we take $\varepsilon = 1$ and we apply an idea introduced in [5], Remark 4.7, to obtain for $\sigma \in \mathbb{R}^N$, $|\sigma| = 1$, $r = |x|$ and assuming first $2 \le p-1 \le \frac{2}{N} + 2$,

$$
\begin{aligned}
a(r\sigma)\,|u(r\sigma)|^{p-1} \;&=\; -\,(p-1)\ \mathrm{Re}\int_r^\infty a(z\sigma)\,\sigma\cdot\nabla u(z\sigma)\,\bar u(z\sigma)\,|u(z\sigma)|^{p-3}\,dz \\
&\quad -\int_r^\infty \sigma\cdot\nabla a(z\sigma)\,|u(z\sigma)|^{p-1}\,dz \\
&\le\; (p-1)\,\left(\int_1^\infty z^{-(N-1)}\,a(z\sigma)\,|u(z\sigma)|^{2(p-2)}\,dz\right)^{\frac12}\cdot \\
&\quad \cdot\left(\int_1^\infty z^{N-1}\,a(z\sigma)\,|\nabla u(z\sigma)|^2\,dz\right)^{\frac12} \\
&\quad +\int_1^\infty |\sigma\cdot\nabla a(z\sigma)|\,|u(z\sigma)|^{p-1}\,dz \\
&\le\; (p-1)\,|a|_\infty\,|u|_{2(p-2)}^{p-2}\,|\nabla u|_2 + |\nabla a|_\infty\,|u|_{p-1}^{p-1}\ .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int a\,|u|^{p+1}\,dx \;&\le\; \left(\int |u|^2\,dx\right)\,\left|a\,|u|^{p-1}\right|_\infty \\
&\le\; |u_0|_2^2\,(p-1)\,|a|_\infty\,|u|_{2(p-2)}^{p-2}\,|\nabla u|_2 + |u_0|_2^2\,|\nabla a|_\infty\,|u|_{p-1}^{p-1}\ .
\end{aligned}
\tag{2.3}
$$

By Gagliardo–Nirenberg inequality we deduce, with $\alpha = \frac{N(p-3)}{2(p-2)}$, $\alpha_1 = \frac{N(p-3)}{2(p-1)}$, $\beta = \frac{N(p-3)}{2} \le 1$,

$$
\begin{aligned}
|u|_{2(p-2)}^{p-2} \;&\le\; c_0\,|\nabla u|_2^{\alpha(p-2)}\,|u|_2^{(1-\alpha)(p-2)} \;\le\; c_0\,|\nabla u|_2^{\beta}\,|u_0|_2^{(1-\alpha)(p-2)}\ , \\
|u|_{p-1}^{p-1} \;&\le\; c_1\,|\nabla u|_2^{\alpha_1(p-1)}\,|u|_2^{(1-\alpha_1)(p-1)} \;\le\; c_1\,|\nabla u|_2^{\beta}\,|u_0|_2^{(1-\alpha_1)(p-1)}\ .
\end{aligned}
\tag{2.4}
$$

Hence, by (2.3) we deduce

$$
\begin{aligned}
\int a\,|u|^{p+1}\,dx \;&\le\; c_0\,|u_0|_2^{2+(1-\alpha)(p-2)}\,(p-1)\,|a|_\infty\,|\nabla u|_2^{\beta+1} \\
&\quad +\;|u_0|_2^{2+(1-\alpha_1)(p-1)}\,|\nabla a|_\infty\,|\nabla u|_2^{\beta}\ .
\end{aligned}
\tag{2.5}
$$

We derive, by (2.1) and (2.5),

$$
\begin{aligned}
&2\,N\,(p-1)\,E(t) \;\le \\
&\quad\le\; 2\,N\,(p-1)\,E_0 \;+\; 2\,N\,(p-1)\,\int_0^t \Big[c_2(a,u_0)\,|\nabla u|_2^{\beta+1} \\
&\qquad +\; c_3(a,u_0)\,|\nabla u|_2^{\beta} \;+\; \frac12\,|\nabla a|_\infty\,|u_0|_2^2 \;+\; \frac12\,|\nabla a|_\infty\,|\nabla u|_2^2 \Big]\,d\tau\ ,
\end{aligned}
\tag{2.6}
$$

with

$$\left.\begin{array}{rcl} c_2(a,u_0) &=& c_0\,|u_0|_2^{2+(1-\alpha)(p-2)}\,(p-1)\,|a|_\infty \;\to\; 0 \\ c_3(a,u_0) &=& c_1\,|u_0|_2^{2+(1-\alpha_1)(p-1)}\,|\nabla a|_\infty \;\to\; 0 \end{array}\right. ,\quad \text{when}\ \ |a|_{W^{1,\infty}} \to 0\ .$$

Since $\beta \le 1$, we deduce, from (2.6),

$$2\,N\,(p-1)\,E(t) \;\le\; 2\,N\,(p-1)\,E_0 + c_4(a,u_0)\,t + c_5(a,u_0)\int_0^t |\nabla u|_2^2\,d\tau\ , \tag{2.7}$$

with $c_4(a,u_0), c_5(a,u_0) \to 0$, when $|a|_{W^{1,\infty}} \to 0$.

In the case $1 \le p-1 < 2$, and since $2(p-2) < 2$, $a, |\nabla a| \in L^{\frac{2}{3-p}}(\mathbb{R}^N)$, a similar estimate, by application of Hölder's inequality and (1.5), can be derived (avoiding the use of Gagliardo–Nirenberg inequality).

We obtain, from (1.12), (2.2) and (2.7),

$$\begin{aligned} \frac{d^2}{dt^2}\int |x|^2\,|u|^2\,dx \;\le& \\ \le\; & \big(c(\delta_1)+1\big)\,\big||x|\,a\big|_\infty^2\,|u_0|_2^2 \;+\; c(\delta_1)\,\big||x|^2\,|\nabla a|\big|_\infty\,|u_0|_2^2 \\ & +\;\big(2\,\delta_1 - \theta(p,N)\big)\,|\nabla u|_2^2 \;+\; 2\,N\,(p-1)\,E_0 \\ & +\; c_4(a,u_0)\,t \;+\; c_5(a,u_0)\int_0^t |\nabla u|_2^2\,d\tau\ . \end{aligned} \tag{2.8}$$

Now, we choose $\delta_1 > 0$ such that $\gamma = 2\,\delta_1 - \theta(p,N) < 0$ and we put

$$c_6(a,u_0) \;=\; \big(c(\delta_1)+1\big)\,\big||x|\,a\big|_\infty^2\,|u_0|_2^2 \;+\; c(\delta_1)\,\big||x|^2\,|\nabla a|\big|_\infty\,|u_0|_2^2 \;\to\; 0\ ,$$

when $\max\big(\big||x|\,a\big|_\infty,\ \big||x|^2\,|\nabla a|\big|_\infty\big) \to 0$.

We deduce from (2.8) and (1.8), by integrating in time (2.8),

$$\begin{aligned} |\gamma|\int_0^s |\nabla u(\tau)|_2^2\,d\tau \;+\; \frac{d}{ds}\int |x|^2\,|u(s)|^2\,dx \;\le& \\ \le\; & V(0) \;+\; 2\,N\,(p-1)\,E_0\,s \;+\; c_6(a,u_0)\,s \;+\; c_4(a,u_0)\,\frac{s^2}{2} \\ & +\; c_5(a,u_0)\int_0^s\left(\int_0^\tau |\nabla u(\xi)|_2^2\,d\xi\right)d\tau\ . \end{aligned} \tag{2.9}$$

Then, since $E_0 < 0$,

$$
\begin{aligned}
|\gamma| \int_0^t \left(\int_0^s |\nabla u(\tau)|_2^2 \, d\tau \right) ds \ &\leq \\
&\leq \int |x|^2 \, |u_0|^2 \, dx \, + \, |V(0)| \, t \, + \, c_6(a, u_0) \, \frac{t^2}{2} \, + \, c_4(a, u_0) \, \frac{t^3}{3} \\
&\quad + \, c_5 \int_0^t \left(\int_0^s \left(\int_0^\tau |\nabla u(\xi)|_2^2 \, d\xi \right) d\tau \right) ds \ .
\end{aligned}
$$

Applying Gronwall's inequality we derive

$$
\int_0^t \left(\int_0^s |\nabla u(\tau)|_2^2 \, d\tau \right) ds \ \leq \ \widetilde{c}(a, u_0, t) \ , \tag{2.10}
$$

with $\widetilde{c}(a, u_0, t)$ bounded in each bounded interval.

Now, let us set

$$
f(t) \ = \ \frac{N(p-1)}{2} \, E_0 \, t^2 \, + \, V(0) \, t \, + \int |x|^2 \, |u_0|^2 \, dx \ . \tag{2.11}
$$

Let us fix $T_0 > 0$ such that $f(T_0) < 0$, and choose $a = a(x)$ such that

$$
\begin{cases}
c_6(a, u_0) \, + \, \dfrac{1}{2} \, c_4(a, u_0) \, T_0 \ < \ \dfrac{N(p-1)}{2} \, |E_0| \ , \\[2ex]
c_5(a, u_0) \, \widetilde{c}(a, u_0, T_0) \ < \ \dfrac{N(p-1)}{4} \, |E_0| \, T_0 \ .
\end{cases} \tag{2.12}
$$

We deduce from (2.9), (2.10) and (2.12), with $t \leq T_0$,

$$
\begin{aligned}
\frac{d}{dt} \int |x|^2 \, |u(t)|^2 \, dx \ &\leq \ V(0) \, + \, 2 \, N \, (p-1) \, E_0 \, t \\
&\qquad + \, \frac{N \, (p-1)}{2} \, |E_0| \, t \, + \, \frac{N \, (p-1)}{4} \, |E_0| \, T_0 \ .
\end{aligned}
$$

Then, for $t \leq T_0$,

$$
\begin{aligned}
\int |x|^2 \, |u(t)|^2 \, dx \ &\leq \ \int |x|^2 \, |u_0|^2 \, dx \, + \, V(0) \, t \, + \, N \, (p-1) \, E_0 \, t^2 \\
&\qquad + \, \frac{N \, (p-1)}{4} \, |E_0| \, t^2 \, + \, \frac{N \, (p-1)}{4} \, |E_0| \, T_0 \, t \ .
\end{aligned}
$$

In particular, for $t = T_0$ we derive

$$\begin{aligned}\int |x|^2\,|u(t)|^2\,dx \;&\le\; \int |x|^2\,|u_0|^2\,dx \;+\; V(0)\,T_0 \;+\; \frac{N\,(p-1)}{2}\,E_0\,T_0^2 \\ &=\; f(T_0) \;<\; 0\;,\end{aligned}$$

which is absurd. Hence $T_{\max} \le T_0$ and the theorem is proved.

3. Proof of Theorem 1.2

We follow the ideas in the proof of Theorem 6.5.13 in [1] (cf. also [2]).

Let u be a solution of the Cauchy problem (1.1) in the interval $[0, T_{\max}[$ with $T_{\max} < +\infty$.

We have for $t \in [0, T_{\max}[$, by (1.4),

$$\begin{aligned}&\int |\nabla u(t)|^2\,dx \;-\; \frac{2}{p+1}\int |u(t)|^{p+1}\,dx \;-\; E_0 \;= \\ &\quad= \int_0^t \left[\int a\,|u|^{p+1}\,dx \;-\; \int a\,|\nabla u|^2\,dx \;-\; \mathrm{Re}\int \big(\nabla u\cdot\nabla a\big)\,\bar u\,dx\right] d\tau\;.\end{aligned} \tag{3.1}$$

Since

$$\begin{aligned}\left|\int \big(\nabla u\cdot\nabla a\big)\,\bar u\,dx\right| \;&\le\; |\nabla u|_2\,|\nabla a|_\infty\,|u|_2 \\ &\le\; |\nabla u|_2\,|\nabla a|_\infty\,|u_0|_2 \\ &\le\; \frac12\,|\nabla u|_2^2 \;+\; \frac12\,|\nabla a|_\infty^2\,|u_0|_2^2\;,\end{aligned}$$

we derive from (3.1) with $c = c(a, u_0, T_{\max}) > 0$ a (general) constant,

$$\begin{aligned}\int |u|^{p+1}\,dx \;&\le\; c\left(1 + \int |\nabla u|^2\,dx\right) \;+\; c\int_0^t |\nabla u|_2^2\,d\tau\;, \\ |u|_{p+1}^{p-1} \;&\le\; c\left(1 + |\nabla u|_2^2\right)^{\frac{p-1}{p+1}} \;+\; c\left(\int_0^t |\nabla u|_2^2\,d\tau\right)^{\frac{p-1}{p+1}}\;.\end{aligned} \tag{3.2}$$

Set $r = p+1$ and let q such that (q, r) is an admissible pair in the Strichartz's sense (cf. [1], §2.3), that is such that

$$\frac{2}{q} \;=\; N\left(\frac12 - \frac1r\right).$$

We have $r' = \frac{p+1}{p}$ (conjugate exponent of r), and by (6.5.44) in [1] and (3.2) we derive

$$\begin{aligned}\left|\nabla\left(|u|^{p-1}u\right)\right|_{r'} &\le c\,|u|_r^{p-1}\,|\nabla u|_r\\ &\le \left[c\left(1+|\nabla u|_2\right)^{\frac{2(p-1)}{r}} + c\left(\int_0^t |\nabla u|_2^2\,d\tau\right)^{\frac{p-1}{r}}\right]|\nabla u|_r\ .\end{aligned}$$

Hence we obtain, for $0 < t < \tau < T_{\max}$,

$$\begin{aligned}&\left|\nabla\left(|u|^{p-1}u\right)\right|_{L^{q'}(t,\tau;L^{r'})} \le\\ &\quad\le c\left(1+|\nabla u|_{L^\infty(t,\tau;L^2)}\right)^{\frac{2(p-1)}{r}}|\nabla u|_{L^{q'}(t,\tau;L^r)}\\ &\qquad + c\,|\nabla u|^{\frac{2(p-1)}{r}}_{L^\infty(0,\tau;L^2)}|\nabla u|_{L^{q'}(t,\tau;L^r)}\\ &\quad\le c\,(\tau-t)^{\frac{q-q'}{q\,q'}}\left(1+|\nabla u|_{L^\infty(t,\tau;L^2)}+|\nabla u|_{L^\infty(0,t;L^2)}\right)^{\frac{2(p-1)}{r}}|\nabla u|_{L^q(t,\tau;L^r)}\ .\end{aligned} \tag{3.3}$$

Now we set

$$f_t(\tau) = 1 + |\nabla u|_{L^\infty(t,\tau;L^2)} + |\nabla u|_{L^q(t,\tau;L^r)}\ . \tag{3.4}$$

We derive from (3.3) and (3.4)

$$\begin{aligned}&\left|\nabla\left(|u|^{p-1}u\right)\right|_{L^{q'}(t,\tau;L^{r'})} \le\\ &\qquad\le c\,(\tau-t)^{\frac{q-q'}{q\,q'}}\left[f_t(\tau)^{1+\frac{2(p-1)}{r}} + |\nabla u|^{\frac{2(p-1)}{r}}_{L^\infty(0,t;L^2)}f_t(\tau)\right].\end{aligned} \tag{3.5}$$

By applying Strichartz's integral estimates for the Schrödinger equation (cf. [1]), we derive

$$\begin{aligned}&|\nabla u|_{L^\infty(t,\tau;L^2)} + |\nabla u|_{L^q(t,\tau;L^r)} \le\\ &\qquad\le c\,|\nabla u(t)|_2 + c\left[|\nabla(|u|^{p-1}u)|_{L^{q'}(t,\tau;L^{r'})} + |\nabla(au)|_{L^{q'}(t,\tau;L^{r'})}\right].\end{aligned} \tag{3.6}$$

Moreover, we have

$$|\nabla(au)|_{L^{q'}(t,\tau;L^{r'})} \le |(\nabla a)\,u|_{L^{q'}(t,\tau;L^{r'})} + |a\,\nabla u|_{L^{q'}(t,\tau;L^{r'})}\ ,$$

and, since $\frac{2\,r'}{2-r'} = \frac{2(p+1)}{p-1}$, $\frac{r\,r'}{r-r'} = \frac{p+1}{p-1}$, we derive, by (1.5),

$$|(\nabla a)\,u|_{L^{q'}(t,\tau;L^{r'})} \le |\nabla a|_{\frac{2(p+1)}{p-1}}\,|u_0|_2\,|\tau-t|^{\frac{1}{q'}} \le c\ ,$$

$$\begin{aligned}|a\,\nabla u|_{L^{q'}(t,\tau;L^{r'})} &\le |a|_{\frac{p+1}{p-1}}\,|\nabla u|_{L^{q'}(t,\tau;L^r)} \le |a|_{\frac{p+1}{p-1}}\,(\tau-t)^{\frac{q-q'}{q\,q'}}\,|\nabla u|_{L^q(t,\tau;L^r)}\\ &\le c\,(\tau-t)^{\frac{q-q'}{q\,q'}}\,f_t(\tau) \le c\,(\tau-t)^{\frac{q-q'}{q\,q'}}\,f_t(\tau)^{1+\frac{2(p-1)}{r}}\ .\end{aligned}$$

Hence, by (3.4), (3.5) and (3.6), we deduce

$$\begin{aligned} f_t(\tau) \;\le\; & c_1\left(1+|\nabla u(t)|_2\right) \;+\; c\,(\tau-t)^{\frac{q-q'}{q\,q'}}\, f_t(\tau)^{1+\frac{2(p-1)}{r}} \\ & + c\,(\tau-t)^{\frac{q-q'}{q\,q'}}\, f_t(\tau)\,|\nabla u|_{L^\infty(0,t;L^2)}^{\frac{2(p-1)}{r}} \;. \end{aligned} \tag{3.7}$$

Since we have $f_t(\tau) \xrightarrow[\tau\to T_{\max}]{} +\infty$ and

$$f_t(t) \;\le\; 1+|\nabla u|_{L^\infty(0,t;L^2)} \;,$$

there exists $\tau_0 \in \,]t, T_{\max}[$ such that

$$f_t(\tau_0) \;=\; (c_1+1)\left(1+|\nabla u|_{L^\infty(0,t;L^2)}\right) \;.$$

By (3.7) we derive

$$\begin{aligned} 1 \;+\; & |\nabla u|_{L^\infty(0,t;L^2)} \;\le \\ & \le\; c\,(\tau_0-t)^{\frac{q-q'}{q\,q'}}\, f_t(\tau_0)^{1+\frac{2(p-1)}{r}} \;+\; c\,(\tau_0-t)^{\frac{q-q'}{q\,q'}}\, f_t(\tau_0)\,|\nabla u|_{L^\infty(0,t;L^2)}^{\frac{2(p-1)}{r}} \\ & \le\; c\,(\tau_0-t)^{\frac{q-q'}{q\,q'}}\left(1+|\nabla u|_{L^\infty(0,t;L^2)}\right)^{1+\frac{2(p-1)}{r}} \;. \end{aligned}$$

Hence, we obtain

$$1+|\nabla u|_{L^\infty(0,t;L^2)} \ge \frac{c}{(T_{\max}-t)^{\gamma}}\,, \qquad \gamma = \frac{r}{2(p-1)}\,\frac{q-q'}{q\,q'} = \frac{1}{p-1} - \frac{N-2}{4}\;,$$

for $t \in \,]0, T_{\max}[$, and this achieves the proof of Theorem 1.2.

Finally, we can prove a variant of Corollary 6.5.14 in [1]:

Corollary 3.1. *Under the assumptions of Theorem 1.2 we have* $|u|_{L^\infty(0,t;L^q)} \xrightarrow[t\to T_{\max}]{} +\infty$*, for all* $q > \frac{N(p-1)}{2}$*. Moreover we have, for* $0 < t < T_{\max}$ *and for a certain* $c_0 > 0$*,*

$$|u|_{L^\infty(0,t;L^q)} \ge \frac{c_0}{(T_{\max}-t)^{\gamma_1}}\,, \quad \gamma_1 = \frac{1}{p-1} - \frac{N}{2\,q}\,, \quad \textit{if} \;\; \frac{N(p-1)}{2} < q < p+1\,,$$

and

$$|u|_{L^\infty(0,t;L^q)} \ge \frac{c_0}{(T_{\max}-t)^{\gamma_2}}\,, \quad \gamma_2 = \frac{4-(N-2)\,(p-1)}{(p-1)^2}\left(\frac{1}{2}-\frac{1}{q}\right), \quad \textit{if} \;\; q \ge p+1\,.$$

Proof of Corollary 3.1. From (1.4) and (1.5) we derive, with $c = c(a, u_0, q, T_{\max}) > 0$, for $t < T_{\max}$,

$$\begin{aligned}\int |\nabla u(t)|^2\,dx \;&\le\; E_0 + \frac{2}{p+1}\int |u(t)|^{p+1}\,dx + \int_0^t \left(\int a\,|u|^{p+1}\,dx\right) d\tau \\ &\quad + \int_0^t \left(\int |\nabla a|\,|\nabla u|\,|u|\,dx\right) d\tau \\ &\le\; c + c\,|u(t)|_{p+1}^{p+1} + |a|_\infty\,T_{\max}\,|u|^{p+1}_{L^\infty(0,t;L^{p+1})} \\ &\quad + \int_0^t |\nabla a|_\infty\,|\nabla u|_2\,|u_0|_2\,d\tau \\ &\le\; c + c\,|u|^{p+1}_{L^\infty(0,t;L^{p+1})} + \frac{1}{2}\,|\nabla u|^2_{L^\infty(0,t;L^2)}\ ,\end{aligned}$$

$$|\nabla u|^2_{L^\infty(0,t;L^2)} \;\le\; c + c\,|u|^{p+1}_{L^\infty(0,t;L^{p+1})}\ . \tag{3.8}$$

Assume $\frac{N(p-1)}{2} < q < p+1$ and consider the Gagliardo–Nirenberg inequality (cf. the proof of Corollary 6.5.14 in [1])

$$|u|_{p+1}^{p+1} \;\le\; c_1\,|\nabla u|_2^{2-\mu}\,|u|_q^{p-1+\mu}\,, \qquad \mu \;=\; \frac{4\,q - 2\,N\,(p-1)}{2\,N - (N-2)\,q}\ .$$

By (3.8) we derive

$$|\nabla u|^2_{L^\infty(0,t;L^2)} \;\le\; c + c\,|\nabla u|^{2-\mu}_{L^\infty(0,t;L^2)}\,|u|^{p-1+\mu}_{L^\infty(0,t;L^q)}\ . \tag{3.9}$$

Since $|\nabla u|_{L^\infty(0,t;L^2)} \xrightarrow[t\to T_{\max}]{} +\infty$ by Theorem 1.2, we derive from (3.9) $|\nabla u|^2_{L^\infty(0,t;L^2)} \le c\,|u|^{p-1+\mu}_{L^\infty(0,t;L^q)}$, and so $|u|_{L^\infty(0,t;L^q)} \ge \frac{c_0}{(T_{\max}-t)^{\gamma_1}}$, with $\gamma_1 = \frac{1}{p-1} - \frac{N}{2\,q}$, by the inequality (1.13).

If $q \ge p+1$, we apply Hölder's inequality, as in the proof of Corollary 6.5.14 in [1], to obtain, by (1.5), $|u|_{p+1}^{p+1} \le |u|_q^{\frac{(p-1)q}{q-2}}\,|u_0|_2^{\frac{2\,(q-(p+1))}{q-2}}$, and so, by (3.8) and the inequality (1.13), we obtain the second inequality stated in Corollary 3.1, and this achieves its proof.

Remark 3.1. In the case where a is a positive constant, we easily derive for $u(t)$ a blowup solution of the Cauchy problem (1.1) with $E_0 < 0$, $t < T_{\max}$:

$$\int |u|^{p+1}\,dx \;\le\; c\left(e^{-at}\,|E_0| + \int |\nabla u|^2\,dx\right) \;\le\; c\left(|E_0| + \int |\nabla u|^2\,dx\right).$$

Since the Strichartz's estimates for the damped equation with a positive has been proved in [6], the Theorem 6.5.13 in [1] (and the Corollary 6.5.14) are immediately extended to this case.

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