Complex Ginzburg-Landau equation with absorption: existence, uniqueness and localization properties

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Dedicated to Professor Hugo Beirão de Veiga on his 70th birthday

Abstract. In this paper we study the time-dependent complex Ginzburg-Landau equation in $\Omega \times (0, T)$, Ω open bounded set in $\mathbb{R}^n$, with a nonlinear absorbing term and we prove global existence of solution for the initial boundary value problem and some properties as localization and extinction of solutions in special cases.

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1. Introduction.

The time dependent complex Ginzburg-Landau (CGL) equation is very important as a model for the study of the pattern formation and the onset of instabilities in non equilibrium fluid dynamic systems (cf.[5]). The mathematical study of the time dependent CGL equation has been investigated in [6] ,[7] and [4] in the whole space $\mathbb{R}^n$ for non absorbing nonlinearities. Localization and finite time extinction properties of the solutions in the absorbing case, for the heat equation, has been established in [1] and, for the Schrödinger equation, in [2] and in [3]. In this paper we study the time dependent CGL equation in a bounded domain of $\mathbb{R}^n$ in the case of an absorbing nonlinearity. Existence of a global solution for the initial boundary value problem is proved by application of the Galerkin's approximation method. Unfortunately, we cannot prove a uniqueness result for the weak solutions. In sections 5 and 6 we prove some localization properties for the solutions and also finite time extinction in special cases.

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary Γ and $Q_T = \Omega \times (0, T]$. We consider the following initial boundary value problem for complex function $u = \mathrm{Re}\ u + i \,\mathrm{Im}\, u$

$$e^{-i\gamma} u_t = \triangle u - |u|^{\sigma-2} u + f(x,t),\ 1 < \sigma < 2, -\pi/2 < \gamma < \pi/2, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u|_{\Gamma_T} = 0,\ \Gamma_T = \partial\Omega \times (0,T). \tag{1.3}$$

Here f and u_0 are given complex value functions such that

$$f \in L^2(Q_T), u_0 \in L^2(\Omega) \tag{1.4}$$

The contents of the paper is as follows:
2. Galerkin's approximation. A priori estimates.
3. Passage to the limit as $m \to \infty$. Existence of weak and strong solutions.
4.About the uniqueness.
5. Stable localization.
6. Waiting time property.
7. Finite time extinction for equations with a given source $f(x,t)$.
8. Further extensions.

2. Galerkin's approximation. A priori estimates.

A weak solution of problem (1.1)-(1.3) will be obtained as the limit of the sequence of Galerkin's approximations $u^{(m)}$ as $m \to \infty$.The Galerkin's approximations of solutions to problem (1.1)-(1.3) are sought in the form

$$u^{(m)} \equiv \sum_{k=1}^{m} u_k(t)\psi_k(x),\ u_k(t) = (u(x,t),\, \psi_k(x))_\Omega \ = \mathrm{Re}\, u_k(t) + i\, \mathrm{Im}\, u_k(t),$$

where

$$-\triangle\psi_k \equiv \lambda_k\psi_k \ \text{ in } \Omega,\ \psi_k \ = 0 \text{ on } \partial\Omega, (\psi_k \ = \overline{\psi_k}\ ,\ \psi_k \ \geq 0,\ (\,\psi_j,\, \psi_k)_\Omega \ = |\Omega|\, \delta_{jk}).$$

We assume also

$$u_0^{(m)} \to u_0 \text{ strongly in } L^2(\Omega) \text{ (or in } W_0^{1,2}(\Omega)). \tag{2.1}$$

The coefficients $u_k(t)$ are defined from the relations

$$\int_\Omega \left(u_t^{(m)} + e^{i\gamma} \left(-\triangle u^{(m)} + \left| u^{(m)} \right|^{\sigma-2} u^{(m)} - f \right) \right) \psi_k dx = 0,\ k = 1, ..., m. \tag{2.2}$$

Last equalities and the initial conditions lead us to the Cauchy problem for the system of m ordinary differential equations of the second order for the coefficients $u_k(t)$

$$u_k' = F_k(t, u_1(t), ..., u_m(t)), \tag{2.3}$$

$$u_k(0) = \int_\Omega u_0\psi_k dx,\ k = 1, ..., m, \tag{2.4}$$

where

$$F_k = \int_\Omega \left(e^{i\gamma} \left(\triangle u^{(m)} \psi_k \left(- \left| u^{(m)} \right|^{\sigma-2} u^{(m)} + f \right) \psi_k \right) \right) dx$$

By Peano's Theorem, for every finite m the problem (2.3),(2.4) has a solution $u_k(t),\ k = 1, ..., m$ on an interval $(0, T_m)$ for each m. The estimates below allow one to take $T_m = T$ for all m.

Multiplying (2.3) by $\overline{u_k}$ and summing , we arrive at the first energy relation

$$\sum_{k=1}^{m} u_k' \overline{u_k} = \int_\Omega \left(e^{i\gamma} \left(\triangle u^{(m)} \overline{u^{(m)}} + \left(- \left| u^{(m)} \right|^{\sigma-2} u^{(m)} + f \right) \overline{u^{(m)}} \right) \right) dx = \tag{2.5}$$

$$= -e^{i\gamma} \int_\Omega \left(\left| \nabla u^{(m)} \right|^2 + \left| u^{(m)} \right|^\sigma - f \overline{u^{(m)}} \right) dx.$$

Taking in the last relation the real part, we obtain

$$\frac{1}{2} \frac{d}{dt} \left\| u^{(m)} \right\|_{2,\Omega}^2 + \cos\gamma \int_\Omega \left(\left| \nabla u^{(m)} \right|^2 + \left| u^{(m)} \right|^\sigma \right) dx = \mathrm{Re} \left(e^{i\gamma} \int_\Omega f \overline{u^{(m)}} dx \right).$$

Assuming

$$0 < \kappa \leq \cos\gamma \tag{2.6}$$

we derive the estimate

$$\left\| u^{(m)}(\cdot, t) \right\|_{2,\Omega}^2 + \kappa \int_0^t \int_\Omega \left(\left| \nabla u^{(m)} \right|^2 + \left| u^{(m)} \right|^\sigma \right) dx d\tau \leq \tag{2.7}$$

$$\leq \left(\int_0^t \|f\|_{2,\Omega} \, d\tau \right)^2 + \left\| u^{(m)}(\cdot, 0) \right\|_{2,\Omega}^2.$$

Analogously, multiplying (2.3) by $\lambda_k \overline{u_k}$ and summing, we arrive at the second energy relation

$$\sum_{k=1}^{m} \lambda_k u_k' \overline{u_k} = \int_\Omega \left(e^{i\gamma} \left(\triangle u^{(m)} + \left(- \left| u^{(m)} \right|^{\sigma-2} u^{(m)} + f \right) \right) \right) \overline{\triangle u^{(m)}} dx.$$

Taking in the last relation the real part, we obtain

$$\frac{1}{2} \frac{d}{dt} \left\| \nabla u^{(m)} \right\|_{2,\Omega}^2 + \cos\gamma \int_\Omega \left| \triangle u^{(m)} \right|^2 dx = \tag{2.8}$$

$$= \mathrm{Re} \left(e^{i\theta} \int_\Omega \left(\left| u^{(m)} \right|^{\sigma-2} u^{(m)} \triangle \overline{u^{(m)}} - f \triangle \overline{u^{(m)}} \right) dx \right) \leq$$

$$\leq \frac{\kappa}{2} \int_\Omega \left| \triangle u^{(m)} \right|^2 dx + \frac{1}{\kappa} \int_\Omega \left| u^{(m)} \right|^{2\sigma-2} dx + \frac{1}{\kappa} \int_\Omega |f|^2 \, dx \, .$$

Using inequality

$$\int_\Omega \left| u^{(m)} \right|^{2\sigma-2} dx \leq \left(\int_\Omega \left| u^{(m)} \right|^2 dx \right)^{\sigma-1} |\Omega|^{2-\sigma}$$

and (2.7), we derive

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla u^{(m)}\right\|_{2,\Omega}^{2}+\frac{\kappa}{2}\int_{\Omega}\left|\triangle u^{(m)}\right|^{2}dx\leq \tag{2.9}$$

$$\leq\frac{1}{\kappa}\left(\int_{\Omega}\left|u^{(m)}\right|^{2}dx\right)^{\sigma-1}|\Omega|^{2-\sigma}+\frac{1}{\kappa}\int_{\Omega}|f|^{2}dx\ \leq$$

$$\leq\frac{1}{\kappa}\left(\left(\int_{0}^{t}\|f\|_{2,\Omega}d\tau\right)^{2}+\left\|u^{(m)}(\cdot,0)\right\|_{2,\Omega}^{2}\right)^{\sigma-1}|\Omega|^{2-\sigma}+\frac{1}{\kappa}\int_{\Omega}|f|^{2}dx=M(t).$$

Integrating last inequality with respect to t,we arrive at estimate

$$\left\|\nabla u^{(m)}(\cdot,t)\right\|_{2,\Omega}^{2}+\kappa\int_{0}^{t}\int_{\Omega}\left|\triangle u^{(m)}\right|^{2}dxd\tau\leq \tag{2.10}$$

$$\leq\left\|\nabla u^{(m)}(\cdot,0)\right\|_{2,\Omega}^{2}+2\int_{0}^{t}M(\tau)d\tau.$$

Multiplying (2.3) by $\overline{u_k'}$ and summing , we arrive at the relation

$$\sum_{k=1}^{m}u_k'\overline{u_k'}=\int_{\Omega}\left(e^{i\gamma}\left(\triangle u^{(m)}\overline{u_t^{(m)}}+\left(-\left|u^{(m)}\right|^{\sigma-2}u^{(m)}+f\right)\overline{u_t^{(m)}}\right)\right)dx \tag{2.11}$$

which implies

$$\left\|u_t^{(m)}\right\|_{2,\Omega}^{2}\leq 3\left(\left\|\triangle u^{(m)}\right\|_{2,\Omega}^{2}+\int_{\Omega}\left|u^{(m)}\right|^{2\sigma-2}dx+\|f\|_{2,\Omega}^{2}\right)\ . \tag{2.12}$$

Applying

$$\left\|u^{(m)}(\cdot,t)\right\|_{2,\Omega}^{2}\leq C(n)\left\|\nabla u^{(m)}(\cdot,t)\right\|_{2,\Omega}^{2},$$

estimates (2.7), (2.10) and assuming, without loss of generality that

$$\left\|u^{(m)}(\cdot,0)\right\|_{2,\Omega}^{2}\leq\|u_0\|_{2,\Omega}^{2},\ \left\|\nabla u^{(m)}(\cdot,0)\right\|_{2,\Omega}^{2}\leq\|\nabla u_0\|_{2,\Omega}^{2}$$

we come to

$$\sup_{t\in[0,T]}\left\|u^{(m)}(\cdot,t)\right\|_{2,\Omega}^{2}+\int_{0}^{T}\int_{\Omega}\left(\left|\nabla u^{(m)}\right|^{2}+\left|u^{(m)}\right|^{\sigma}\right)dxd\tau\leq \tag{2.13}$$

$$\leq\frac{1}{\kappa}\left[\left(\int_{0}^{T}\|f\|_{2,\Omega}d\tau\right)^{2}+\|u_0\|_{2,\Omega}^{2}\right]=M_1.$$

$$\sup_{t\in[0,T]}\left\|\nabla u^{(m)}(\cdot,t)\right\|_{2,\Omega}^{2}+\int_{0}^{T}\int_{\Omega}\left(\left|u_t^{(m)}\right|^{2}+\left|\triangle u^{(m)}\right|^{2}\right)dxd\tau\leq \tag{2.14}$$

$$\leq C(\kappa,|\Omega|,\sigma)\left(\|\nabla u_0\|_{2,\Omega}^{2}+\|f\|_{2,Q_T}^{2}+\left(\|\nabla u_0\|_{2,\Omega}^{2}+\|f\|_{2,Q_T}^{2}\right)^{\sigma-1}\right)=M_2.$$

Using estimate (2.14) and the formula

$$u^{(m)}(x,t)-u_0(x)=\int_{0}^{t}u_t^{(m)}(x,\tau)d\tau+u^{(m)}(x,0)-u_0(x)$$

we derive that

$$\left\|u^{(m)}(x,t)-u_0(x)\right\|_{2,\Omega}^2 \leq C\left(t+\left\|u^{(m)}(x,0)-u_0(x)\right\|_{2,\Omega}^2\right). \qquad (2.15)$$

3. Passage to the limit as $m \to \infty$. Existence of weak and strong solutions

First we prove the existence a strong solution, assuming that the data satisfy (2.14). The estimate (2.14) allows us conclude that there exist u and a subsequence of $\{u^{(m)}\}$, still denoted by $\{u^{(m)}\}$, such that

$$\begin{gathered} u^{(m)} \to u \ \text{ a.e. in } Q_T \\ u^{(m)} \to u \text{ strongly in } L^\sigma(Q_T)\cap L^2(Q_T), \\ \triangle u^{(m)} \rightharpoonup \triangle u \text{ weakly in } L^2(Q_T), \\ u_t^{(m)} \rightharpoonup u_t \text{ weakly in } L^2(Q_T), \end{gathered} \qquad (3.1)$$

Let $c_k(t) \in C^1[0,T]$ be arbitrary complex value function. Multiplying (2.2) by $c_k(t)$, summing and integrating over $[0,T]$,we come to

$$\int_{Q_T}\left(u_t^{(m)}+e^{i\gamma}\left(-\triangle u^{(m)}+\left|u^{(m)}\right|^{\sigma-2}u^{(m)}-f\right)\right)\Phi dx=0, \qquad (3.2)$$

for any function $\Phi^N=\sum_{k=1}^{N} c_k(t)\psi_k(x),\ N\leq m$. Note that the functions $\Phi^N$are dense in $L^2(Q_T)$. Fixing N and passing to the limit as $m\to\infty$ we obtain

$$\int_{Q_T}\left(u_t+e^{i\gamma}\left(-\triangle u+|u|^{\sigma-2}u-f\right)\right)\Phi^N dx=0, N\leq\infty \qquad (3.3)$$

Moreover according to (2.15)

$$\|u(x,t)-u_0(x)\|_{2,\Omega}^2 \leq Ct \to 0 \text{ as } t\to 0. \qquad (3.4)$$

Hence u is a strong solution of the problem (1.1)-(1.3): $(u_t, \triangle u)\in L^2(Q_T)$ and satisfy (1.1) almost everywhere in Q_T.

Now we prove the existence a weak solution assuming that

$$u_0 \in L^2(\Omega). \qquad (3.5)$$

Let $u^{(m)}\in W^{1,2}(\Omega)$ be such that

$$\left\|u^{(m)}-u_0\right\|_{2,\Omega} \to 0 \text{ as } m\to\infty, \ \left\|u^{(m)}\right\|_{2,\Omega} \leq \|u_0\|_{2,\Omega}.$$

Then there exists a subsequence of strong solutions $u^{(m)}$ satisfying to estimate (2.13) and moreover

$$\left|\left(u_t^{(m)},\Phi\right)_{Q_T}\right| \leq C\|\nabla\Phi\|_{2,Q_T}, \ \left|\frac{d}{dt}\int_\Omega\left|u^{(m)}\right|^2dx\right| \in L^1[0,T]. \qquad (3.6)$$

Strong solution $u^{(m)}$ satisfies

$$\int_{Q_T}\left(-u^{(m)}\Phi_t+e^{i\gamma}\left(\nabla u^{(m)}\nabla\Phi+\left|u^{(m)}\right|^{\sigma-2}u^{(m)}\Phi-f\Phi\right)\right)dx= \tag{3.7}$$
$$=-\int_\Omega u^{(m)}(\cdot,t)\Phi(\cdot,t)+\int_\Omega u^{(m)}(\cdot,0)\Phi(\cdot,0).$$

Estimates (2.13) , (3.7) make possible us to conclude that there exist u and a subsequence of $\{u^{(m)}\}$, still denoted by $\{u^{(m)}\}$, such that

$$\begin{gathered} u^{(m)}\to u \ \text{ a.e. in } Q_T\\ u^{(m)}\to u \text{ strongly in } L^\sigma(Q_T)\cap L^2(Q_T),\\ u^{(m)}\rightharpoonup u \text{ weakly in } L^2(\Omega),\ \text{a.e in } [0,T],\\ \nabla u^{(m)}\rightharpoonup\nabla u \text{ weakly in } L^2(Q_T), \end{gathered} \tag{3.8}$$

Passing to the limit in (3.7) as $m\to\infty$, we obtain that

$$\int_{Q_T}\left(-u\Phi_t+e^{i\gamma}\left(\nabla u\nabla\Phi+|u|^{\sigma-2}u\Phi-f\Phi\right)\right)dx= \tag{3.9}$$
$$=-\int_\Omega u(\cdot,T)\Phi(\cdot,T)dx+\int_\Omega u(\cdot,0)\Phi(\cdot,0)dx.$$

This concludes the proof of existence of weak solution.

Theorem 3.1. *Under conditions (*1.4*), (*2.6*) the problem (*1.1*)- (*1.3*) has at least one weak solution* $u\in L^\infty\left(0,T;L^2(\Omega)\right)$, $\nabla u\in L^2(Q_T)$, *satisfying (*3.9*). If, in addition* $\nabla u_0\in L^2(\Omega)$, *then the problem (*1.1*)- (*1.3*) has at least one strong solution* $(u_t,\triangle u)\in L^2(Q_T)$ *satisfying (*1.1*) almost everywhere in* Q_T.

Remark 3.2. *In fact any weak solution be a strong solution for* $t>0$.

4. About the uniqueness

Let u and v be two weak solutions of the problem (1.1)-(1.3) with the same initial data and $w=u-v$. Then according to (3.9)

$$\int_{Q_T}\left(-w\Phi_t+e^{i\gamma}\left(\nabla w\nabla\Phi+(|u|^{\sigma-2}u-|v|^{\sigma-2}v)\Phi\right)\right)dx \tag{4.1}$$
$$=-\int_\Omega w(\cdot,T)\Phi(\cdot,t)dx,$$

and

$$\int_{Q_T}\left(-w\overline{w_t}+e^{i\gamma}\left(|\nabla w|^2+(|u|^{\sigma-2}u-|v|^{\sigma-2}v)(\overline{u-v})\right)\right)dx= \tag{4.2}$$
$$=-\int_\Omega w(\cdot,T)\overline{w}(\cdot,t)dx.$$

Taking the real part we have

$$\frac{1}{2}\int_{\Omega}|w(\cdot,T)|^2\,dx+\cos\gamma\int_{Q_T}|\nabla w|^2\,dx+$$

$$+\operatorname{Re}\Big(\int_{Q_T}\left(e^{i\gamma}(|u|^{\sigma-2}\,u-|v|^{\sigma-2}\,v)(\overline{u-v})\right)dx\Big)=0$$

Unfortunately the inequality

$$\operatorname{Re}\Big(\int_{Q_T}\left(e^{i\gamma}(|u|^{\sigma-2}\,u-|v|^{\sigma-2}\,v)(\overline{u-v})\right)dx\Big)\geq 0$$

is valid only for $\gamma=0$. (see [3]).Thus uniqueness of the solution remains an open problem.

5. Localization properties

5.1. Notation and assumptions

In this section we analyze local qualitative properties of weak solutions to (1.1)-(1.3). The analysis relies on the methods developed in [1]. We will use the following notations: given a point $(x_0,t_0)\in Q_T\equiv\Omega\times(0,T)$

$$\begin{aligned}&B_\rho\equiv B_\rho(x_0)=\{x\in\mathbb{R}^n: |x-x_0|<\rho\}\subset\Omega,\\ &S_\rho\equiv S_\rho(x_0)=\partial B_\rho(x_0).\end{aligned}$$

We introduce the local energy functions

$$b(\rho,t)=||u(\cdot,t)||^2_{2,B_\rho},\qquad \overline{b}(\rho)=\sup_{0\leq\tau\leq t}b(\rho,\tau),$$

$$E(\rho,t)=\int_0^t\int_{B_\rho}|\nabla u|^2dxd\tau,\ D(\rho,t)=\int_0^t\int_{B_\rho}|u|^\sigma\,dxd\tau$$

We will consider a local weak solutions in the cylinder $(0,T)\times B_{\rho_0}(x_0)$.Without loss of generality we always assume that

$$\overline{b}(\rho)+\overline{E}(\rho)+\overline{D}(\rho)\leq M_3,\ \rho\leq\rho_0,\ t\leq T \tag{5.1}$$

Remind that according to estimate (2.13)

$$\overline{b}(\rho)+\overline{E}(\rho)+\overline{D}(\rho)=\sup_{t\in[0,T]}\|u(\cdot,t)\|^2_{2,\Omega}+\int_0^T\int_{\Omega}\left(|\nabla u|^2+|u|^\sigma\right)dxd\tau\leq \tag{5.2}$$

$$\leq\frac{1}{\kappa}\left[\left(\int_0^T\|f\|_{2,\Omega}\,d\tau\right)^2+\|u_0\|^2_{2,\Omega}\right]=M_1.$$

Since $E(\rho,t)$ is monotone non-decreasing in ρ and t, there exist the weak derivatives

$$\begin{cases} E_\rho(\rho,t) = \int_0^t \int_{S_\rho} |\nabla u|^2 dS d\tau, & D_\rho(\rho,t) = \int_0^t \int_{S_\rho} |u|^\sigma \, dS d\tau, \\ E_t(\rho,t) = \int_{B_\rho} |\nabla u|^2 dx, & E_{\rho t}(\rho,t) = \int_{S_\rho} |\nabla u|^2 dS, \\ D_t(\rho,t) = \int_{B_\rho} |u|^\sigma \, dx, & D_{\rho t}(\rho,t) = \int_{S_\rho} |u|^\sigma \, dS. \end{cases}$$

5.2. Stable localization

First we consider the case $f = 0$. Assume that

$$\int_{B_{\rho_0}} |u_0|^2 \, dx = 0, \ f(x,t) = 0, \ (x,t) \in [0,T] \times B_{\rho_0}. \tag{5.3}$$

Multiplying (1.1) by $e^{i\gamma}\overline{u}$ and integrating over $B_\rho = \{x : |x - x_0| < \rho\}$, $\rho \leq \rho_0$, we obtain

$$\int_{B_\rho} u_t \overline{u} dx = e^{i\gamma} \int_{B_\rho} \left(\left| \nabla u^2 \right| + |u|^\sigma \right) dx + \left(e^{i\gamma} \int_{S_\rho} \nabla u \overline{u} \overrightarrow{n} dS \right), \quad \rho \leq \rho_0. \tag{5.4}$$

Taking the real part, we have

$$\frac{d}{dt} \int_{B_\rho} |u|^2 \, dx = -2\cos\gamma \int_{B_\rho} \left(\left| \nabla u^2 \right| + |u|^\sigma \right) dx + 2 \operatorname{Re} \left(e^{i\gamma} \int_{S_\rho} \nabla u \overline{u} \overrightarrow{n} dS \right) \tag{5.5}$$

Integrating last relation with respect to t, we come to

$$\int_{B_\rho} |u(\cdot,\tau)|^2 \, dx \Big|_0^t + 2\cos\gamma \int_0^t \int_{B_\rho} \left(\left| \nabla u^2 \right| + |u|^\sigma \right) dx d\tau = I(\rho,t) \tag{5.6}$$

where

$$I(\rho,t) = 2 \int_0^t \operatorname{Re} \left(e^{i\gamma} \int_{S_\rho} \nabla u \overline{u} \overrightarrow{n} dS \right) d\tau. \tag{5.7}$$

In our notations we can rewrite (5.6) in the form

$$b(\rho,t) + 2\cos\gamma \left(E(\rho,t) + D(\rho,t) \right)) = I(\rho,t). \tag{5.8}$$

Last inequality implies

$$\overline{b}(\rho) + E(\rho,t) + D(\rho,t) \leq C(\kappa) \left| I \right|. \tag{5.9}$$

Next we evaluate I in the following way

$$|I| = \left| 2 \int_0^t \operatorname{Re} \left(e^{i\gamma} \int_{S_\rho} \nabla u \overline{u} \overrightarrow{n} dS \right) d\tau \right| \leq 2 \int_0^t \int_{S_\rho} |\nabla u| \, |u| \, dS d\tau \leq \tag{5.10}$$

$$\leq 2 \left(\int_0^t \int_{S_\rho} |\nabla u|^2 \, dS d\tau \right)^{\frac{1}{2}} \left(\int_0^t \int_{S_\rho} |u|^2 \, dS d\tau \right)^{\frac{1}{2}} \leq 2 \left(E_\rho \right)^{\frac{1}{2}} \left(\int_0^t \int_{S_\rho} |u|^2 \, dS d\tau \right)^{\frac{1}{2}}$$

To estimate the integral on the right-hand side of last inequality we proceed in the usual way and apply the following inequality:

$$\|u\|_{L^2(S_\rho)} \le C\left(\|\nabla u\|_{L^2(B_\rho)} + \rho^{-\delta}\|u\|_{L^q(B_\rho)}\right)^{\theta}\left(\|u\|_{L^q(B_\rho)}\right)^{1-\theta} \tag{5.11}$$

(the constant C is independent of ρ) with

$$\theta = \frac{n(2-\sigma)+2}{n(2-\sigma)+4} < 1,$$

Also we use

$$\|u\|_{L^q(B_\rho)} \le \|u\|_{L^2(B_\rho)}^{\frac{2-\sigma}{2}}\|u\|_{L^\sigma(B_\rho)}^{\frac{\sigma(2-q)}{q(2-\sigma)}} = b^{\frac{2-\sigma}{4}}\ C_t^{\frac{1}{2}},\ \sigma < q = \frac{4}{4-\sigma} < 2. \tag{5.12}$$

In our notations (5.11) takes the form

$$\|u\|_{L^2(S_\rho)} \le C\left(E_t^{\frac{1}{2}} + \rho^{-\delta} b^{\frac{2-\sigma}{4}}\ C_t^{\frac{1}{2}}\right)^{\theta}\left(b^{\frac{2-\sigma}{4}}\ C_t^{\frac{1}{2}}\right)^{1-\theta} \tag{5.13}$$

Raising both sides (5.13) to the power 2 and integrating in $\tau \in (0,t)$ and raising to the power 1/2, we come to the inequalities

$$\left(\int_0^t \int_{S_\rho} |u|^2\, dS d\tau\right)^{\frac{1}{2}} \le C\left(\int_0^t \left(E_t + \rho^{-2\delta} b^{\frac{2-\sigma}{2}}\ D_t\right)^{\theta}\left(b^{\frac{2-\sigma}{2}}\ D_t\right)^{(1-\theta)}\right)^{\frac{1}{2}} \le \tag{5.14}$$

$$\le C\rho^{-\theta\delta}\max(1,\rho^{\theta\delta})\max(1,\overline{b}^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}^{\frac{2-\sigma}{4}(1-\theta)}\left(\int_0^t (E_t + D_t) d\tau\right)^{\frac{1}{2}} \le$$

$$\le C\rho^{-\theta\delta}\max(1,\rho^{\theta\delta})\max(1,\overline{b}^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}^{\frac{2-\sigma}{4}(1-\theta)}(E + D)^{\frac{1}{2}}.$$

Joining (5.9), (5.10) and (5.14), we have

$$|I| \le C\,(E_\rho)^{\frac{1}{2}}\,\rho^{-\theta\delta}\max(1,\rho^{\theta\delta})\max(1,\overline{b}^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}^{\frac{2-\sigma}{4}(1-\theta)}(E + D)^{\frac{1}{2}} \tag{5.15}$$

and correspondingly

$$\overline{b} + E + D \le C\,(E_\rho)^{\frac{1}{2}}\,\rho^{-\theta\delta}\max(1,\rho^{\theta\delta})\max(1,\overline{b}^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}^{\frac{2-\sigma}{4}(1-\theta)}(E + D)^{\frac{1}{2}} \le \tag{5.16}$$

$$\le C\,(E_\rho)^{\frac{1}{2}}\,\rho^{-\theta\delta}\max(1,\rho^{\theta\delta})\max(1,\overline{b}^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}^{\frac{2-\sigma}{4}(1-\theta)-\frac{\nu}{2}}(\overline{b} + E + D)^{\frac{1}{2}+\frac{\nu}{2}}.$$

with some ν defined by

$$\mathbf{0} < \nu < \frac{2-\sigma}{2}(\mathbf{1}-\theta) = \frac{2-\sigma}{n(2-\sigma)+4} < \mathbf{1}. \tag{5.17}$$

Dividing both sides of (5.16) by $(\overline{b} + E + D)^{\frac{1}{2}+\frac{\nu}{2}}$ and raising to the power 2, we come to inequality

$$E^{1-\nu} \le \left(\overline{b} + E + D\right)^{1-\nu} \le K\rho^{-2\theta\delta}E_\rho \tag{5.18}$$

with

$$K = C\left(\max(1,\rho_0^{\theta\delta})\max(1,\overline{b}(\rho_0)^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}(\rho_0)^{\frac{2-\sigma}{4}(1-\theta)-\frac{\nu}{2}}\right)^2. \tag{5.19}$$

Integrating (5.18) we get the inequality

$$\rho_0^{\omega} - \rho^{\omega} \leq \frac{K\omega}{\nu}\left(E^{\nu}(\rho_0) - E^{\nu}(\rho)\right),\ \omega = 1 + 2\theta\sigma. \tag{5.20}$$

Thus, $E^{\nu}(\rho) = 0$ if the function $\rho(t)$ satisfies

$$\rho^{\omega}(t) = \rho_0^{\omega} - \frac{K\omega}{\nu}E^{\nu}(\rho_0). \tag{5.21}$$

Hence, we have proved the following

Theorem 5.1. *Let u be a local weak solution of the problem (1.1)-(1.3) in $Q_T = (0,T) \times B_{\rho_0}, (B_{\rho_0} \subset \Omega)$ and*

$$u(x,0) = 0,\ f = 0,\quad x \in B_{\rho_0} \tag{5.22}$$

Then

$$u(x,t) = 0 \textit{ for a.a. } (x,t) \in \mathbf{R}^+ \times B_{\rho} \tag{5.23}$$

where ρ is defined by the formula (5.21) and ω, K, ν are defined by (5.19), (5.20),(5.17).

Remark 5.2. *The result of the Theorem implies that if the global energy $b + E + D$ is not large , then $\rho(t) > 0$ and there exists a cylinder $\mathbf{R}^+ \times B_{\rho}$ where u is zero a.e.*

5.3. Waiting time property

Now we consider local solution u to (1.1), (1.2) in $(0,T) \times B_R, .(B_R \subset \Omega)$. We assume that (as well as (5.3))

$$\int_{B_{\rho_0}} |u_0|^2\, dx = 0,\ f(x,t) = 0\ (x,t) \in [0,T] \times B_{\rho_0}, 0 < \rho_0 < R \tag{5.24}$$

and in addition

$$\left(\int_{B_{\rho}} |u_0|^2\, dx + \int_0^T \int_{B_{\rho}} |f(\cdot,t)|^{\frac{\sigma}{\sigma-1}}\, dx d\tau\right) \leq \varepsilon\,(\rho - \rho_0)_+^{\frac{1}{\nu}}, \tag{5.25}$$

where ν is defined by (5.17).

We assume that

$$\left(\sup_{t\in[0,T]} \int_{B_R} |u|^2\, dx + \int_0^T \int_{B_R} (|\nabla u|^2 + |u|^{\sigma}) dx d\tau\right) \leq W_R < \infty \tag{5.26}$$

and the parameters $W_R,\ \nu,\ R\ , \rho_0$ of the problem (5.38), (5.39) satisfy

$$G = W_R - K\left(\frac{W_R}{\nu}\right)^{\frac{1}{1-\nu}} (R - \rho_0)^{-\frac{\nu}{1-\nu}} - C\varepsilon\,(R - \rho_0)^{\frac{1}{\nu}} = 0 \tag{5.27}$$

where $C(\sigma,n)$ is a constant in embeding inequality (5.11).

Theorem 5.3. *Let u be a local weak solution of the problem (1.1)-(1.3) in $Q_T = (0,T) \times B_R, (B_R \subset \Omega)$ and (5.24), (5.25), (5.26), (5.27) hold. Then*

$$u(x,t) = 0,\quad (x,t) \in [0,T] \times B_{\rho_0}. \tag{5.28}$$

Proof. Repeating the calculations (5.4)-(5.8) for $\rho \geq \rho_0$, we come to energy relation

$$b(\rho,t)+2\cos\gamma\ \left(E(\rho,t)+D(\rho,t)\right)) = I(\rho,t)+b(\rho,0)+I_f. \tag{5.29}$$

where

$$I(\rho,t)=2\int_0^t \mathrm{Re}\left(e^{i\gamma}\int_{S_\rho}\nabla u\overline{u}\overrightarrow{n}dS\right)d\tau,\ I_f=\mathrm{Re}(e^{i\gamma}\int_0^t\int_{B_\rho} fudxd\tau)$$

The energy relation implies

$$\overline{b}(\rho)+\ E(\rho,t)+D(\rho,t)\leq C(\kappa)\left(|I|+|I_f|+b(\rho,0)\right). \tag{5.30}$$

Applying the estimates (5.16), (5.15), we evaluate I, in the following way

$$|I|\leq C\left(E_\rho\right)^{\frac{1}{2}}\rho^{-\theta\delta}\max(1,\rho^{\theta\delta})\max(1,\overline{b}^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}^{\frac{2-\sigma}{4}(1-\theta)-\frac{\nu}{2}}(E+\ D)^{\frac{1}{2}+\frac{\nu}{2}}. \tag{5.31}$$

On other hand

$$|\ I_f|\leq\delta D+C(\delta)\int_0^t\int_{B_\rho}|f|^{\frac{\sigma}{\sigma-1}}dxd\tau,\ \delta\in(0,1). \tag{5.32}$$

Substituting last inequalities in (5.29), choosing a suitable δ, we come to inequality

$$\overline{b}+E(\rho,t)+D(\rho,t)\leq\Lambda+C\left(b(\rho,0)+\int_0^t\int_{B_\rho}|f|^{\frac{\sigma}{\sigma-1}}dxd\tau\right) \tag{5.33}$$

where

$$\Lambda=C\left(E_\rho\right)^{\frac{1}{2}}\max(1,\rho_0^{-\theta\delta})\max(1,\overline{b}^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}^{\frac{2-\sigma}{4}(1-\theta)-\frac{\nu}{2}}(\overline{b}+E+\ D)^{\frac{1}{2}+\frac{\nu}{2}}$$

with ν defined by (5.17). First term in the right side of (5.33) we evaluate, applying the Young inequality, in the following way

$$\Lambda\leq\delta(\overline{b}+E+\ D)+CE_\rho^{\frac{1}{1-\nu}}\left(\max(1,\rho_0^{-\theta\delta})\max(1,\overline{b}^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}^{\frac{2-\sigma}{4}(1-\theta)-\frac{\nu}{2}}\right)^{\frac{2}{1-\nu}}. \tag{5.34}$$

Joining (5.25), (5.29), (5.33), we have

$$E(\rho,t)+D(\rho,t)\leq\overline{b}+E(\rho,t)+D(\rho,t)\leq \tag{5.35}$$
$$\leq KE_\rho^{\frac{1}{1-\nu}}+C\varepsilon\left(\rho-\rho_0\right)_+^{\frac{1}{\nu}},\ \rho_0\leq\rho\leq R$$

with

$$K=C\left(\max(1,\rho_0^{-\theta\delta})\max(1,\overline{b}^{\frac{2-\sigma}{2}})^{\frac{\theta}{2}}\overline{b}^{\frac{2-\sigma}{4}(1-\theta)-\frac{\nu}{2}}\right)^{\frac{2}{1-\nu}}.$$

Using the properties

$$E_\rho\leq\Psi_\rho,\ \Psi=E+D=\int_0^t\int_{B_\rho}\left(\left|\nabla u^2\right|+|u|^\sigma\right)dxd\tau,$$

$$\sup_{\tau\in[0,t]} \Psi_\rho(\rho,\tau) = \sup_{\tau\in[0,t]} \left(\int_0^t \int_{S_\rho} \left(\left|\nabla u^2\right| + |u|^\sigma \right) dx d\tau \right) =$$

$$= \int_0^t \int_{S_\rho} \left(\left|\nabla u^2\right| + |u|^\sigma \right) dx d\tau = \frac{\partial}{\partial\rho} \left(\sup_{\tau\in[0,t]} \Psi(\rho,t) \right),$$

we can rewrite (5.35) in the form

$$W(\rho) \leq K W_\rho^{\frac{1}{1-\nu}} + C\varepsilon \left(\rho-\rho_0\right)_+^{\frac{1}{1-\gamma}}, \; \rho_0 \leq \rho \leq R \tag{5.36}$$

where

$$W(\rho) = \sup_{\sup_{t\in[0,T]}} \Psi(\rho,t) = \int_0^T \int_{B_\rho} \left(\left|\nabla u^2\right| + |u|^\sigma \right) dx d\tau.$$

We complete (5.36) with

$$W(R) = \int_0^T \int_{B_R} \left(\left|\nabla u^2\right| + |u|^\sigma \right) dx d\tau = W_R. \tag{5.37}$$

Let us consider the problem

$$\varphi(\rho) = K \varphi_\rho^{\frac{1}{1-\nu}} + C\varepsilon \left(\rho-\rho_0\right)_+^{\frac{1}{\nu}}, \; \rho_0 \leq \rho \leq R; \varphi(R) = W_R. \tag{5.38}$$

Last problem has unique solution in the form

$$\varphi(\rho) = W_R \left(R-\rho_0\right)^{-\frac{1}{\nu}} \left(\rho-\rho_0\right)^{\frac{1}{\nu}} \tag{5.39}$$

if the parameters $W_R,\ \nu,\ R\ ,\rho_0$ of the problem (5.38), (5.39) satisfy

$$G = W_R - K \left(\frac{W_R}{\nu}\right)^{\frac{1}{1-\nu}} \left(R-\rho_0\right)^{-\frac{\nu}{1-\nu}} - C\varepsilon \left(R-\rho_0\right)^{\frac{1}{\nu}} = 0. \tag{5.40}$$

For fixed parameters $W_R,\ \nu,\ \rho_0$ it is sufficient to choice R sufficiently large and ε sufficiently small. It is easy verify that $\varphi(\rho)$ is a majorant for $W(\rho)$.The Theorem is proved. □

6. Finite time extinction for equations with a given source $f(x,t)$

Let us consider

$$e^{-i\gamma} u_t = \triangle u - |u|^{\sigma-2} u + f(x,t),\, 1<\sigma<2, |\gamma| < \gamma/2, \tag{6.1}$$

$$u(x,0) = u_0(x), \quad x\in\Omega, \tag{6.2}$$

$$u|_{\Gamma_T} = 0,\ \Gamma_T = \partial\Omega \times (0,T). \tag{6.3}$$

Theorem 6.1. *Let u be any weak solution to (6.1)-(6.3). Then:*

1. (Finite time extinction.) If $f=0$ then there exists $t^*= t^*(\sigma, \|u_0\|_{2,\Omega}, n, \Omega) <$

∞ such that

$$u(x,t) = 0, \quad x\in\Omega,\ t\geq t^*. \tag{6.4}$$

2. (Extinction in a prescribed moment.) If $f \neq 0$ *satisfies*

$$\int_\Omega |f(\cdot,t)|^{\frac{\sigma}{\sigma-1}}\,dx \leq \varepsilon\left(1-\frac{t}{t_f}\right)_+^{\frac{\mu}{1-\mu}},\ t_f > t^*, \tag{6.5}$$

$$\mu = \frac{\sigma}{\theta\sigma + 2(1-\theta)} < 1,\ \theta = \frac{(2-\sigma)n}{2n - \sigma(n-2)} < 1$$

then there exists $\varepsilon^* = \varepsilon^*(\sigma, \|u_0\|_{2,\Omega}, n, \Omega) < \infty$ *such that*

$$u(x,t) = 0, \quad x \in \Omega,\ t \geq t_f \tag{6.6}$$

if $0 < \varepsilon < \varepsilon^*$.

Proof. Let us introduce the global energy functions

$$b(t) = ||u(\cdot,t)||^2_{2,\ \Omega}, \qquad E(t) = \int_0^t \int_\Omega |\nabla u|^2 dx d\tau,$$

$$D(t) = \int_0^t \int_\Omega |u|^\sigma\,dx d\tau$$

Using the boundary conditions (1.2), analogously with (5.5) we derive for any weak solution the following energy relation

$$\frac{d}{dt}\int_\Omega |u|^2\,dx = -2\cos\gamma\int_\Omega \left(\left|\nabla u^2\right| + |u|^\sigma\right)dx + 2\,\mathrm{Re}\left(\int_\Omega e^{i\gamma} f\overline{u}dx\right) \tag{6.7}$$

Next we apply the particular case of inequality (5.11)

$$\begin{aligned} b^{\frac{1}{2}} &= \|u\|_{L^2(\Omega)} \leq C(\sigma,n)\left(\|\nabla u\|_{L^2(\Omega)}\right)^\theta \left(\|u\|_{L^\sigma(\Omega)}\right)^{1-\theta} = \\ &= C\,(E)^{\theta/2}\,D^{(1-\theta)/\sigma} \leq C\,(E+D)^{\theta/2+(1-\theta)/\sigma} \end{aligned} \tag{6.8}$$

with

$$\theta = \frac{(2-\sigma)n}{2n-\sigma(n-2)}.$$

Last inequality implies

$$C(\sigma,n)b^\mu \leq E + D,\ \mu = \frac{\sigma}{\theta\sigma + 2(1-\theta)} < 1. \tag{6.9}$$

Then for $f = 0$ the relation (6.7) gives us the ordinary differential inequality

$$\frac{d}{dt}b + \kappa C b^\mu \leq 0 \tag{6.10}$$

Integrating we obtain

$$b^{1-\mu}(t) \leq b^{1-\mu}(0) - \chi C(1-\mu)t \tag{6.11}$$

then

$$t^* = \frac{b^{1-\mu}(0)}{\chi C(1-\mu)}.$$

First assertion of the Theorem is proved. To prove the second one we use the inequality

$$\frac{d}{dt}\int_\Omega |u|^2\,dx + 2\kappa\int_\Omega \left(\left|\nabla u^2\right| + |u|^\sigma\right)dx \le 2\left|\mathrm{Re}\left(\int_\Omega e^{i\gamma} f\bar{u}dx\right)\right| \le \tag{6.12}$$

$$\le \kappa\int_\Omega |u|^\sigma\,dx + C(\kappa)\int_\Omega |f|^{\frac{\sigma}{\sigma-1}}\,dx.$$

Applying (6.9), (6.5) we come to nonhomogeneous ordinary differential inequality

$$\frac{d}{dt}b + C_1(\kappa,\sigma,n)b^\mu \le C_2(\kappa,\sigma,n)\varepsilon\left(1-\frac{t}{t_f}\right)_+^{\frac{\mu}{1-\mu}}. \tag{6.13}$$

Let us consider the problem

$$w' + C_1 w^\mu = C_2\varepsilon\left(1-\frac{t}{t_f}\right)_+^{\frac{\mu}{1-\mu}},\ w(0) = b(0) = \|u_0\|_{2,\Omega}^2. \tag{6.14}$$

Last problem has unique solution

$$w(t) = w(0)\left(1-\frac{t}{t_f}\right)_+^{\frac{1}{1-\mu}}, \tag{6.15}$$

if

$$-\frac{b(0)}{t_f} + C_1 b(0)^\mu = C_2\varepsilon. \tag{6.16}$$

Notice that if $\varepsilon = 0$ (the source term is absent) then $t_f = t^* = b^{1-\mu}(0)/C_1$. If $t_f > t^* = b^{1-\mu}(0)/C_1$ there exists $\varepsilon > 0$ satisfying (6.16). It is very easy verify that $w(t)$ is a majorant for $b(t)$. Second assertion of the Theorem is proved. □

Remark 6.2. *If $\mu = 1$ in (6.10), then*

$$b(t) \le b(0)e^{-\kappa C t}.$$

If $\mu > 1$ in (6.10), then

$$b^{\mu-1}(t) \le \frac{b^{\mu-1}(0)}{1 + b^{\mu-1}(0)(\mu-1)\kappa C t}.$$

Remark 6.3. *Notice that the relation (6.16) connects the three parameters : the source intensity ε, the instant t_f of vanishing of the source and the initial value $b(0)$. For this reason , given an arbitrary intensity ε, the effect of vanishing of the solution can be provided by an appropriate choice of t_f and $b(0)$.*

7. Further extensions

Above mentioned results can be generalized for equation

$$e^{-i\gamma}u_t = \triangle u - a(x,t)\,|u|^{\sigma-2}\,u + f(x,t) \tag{7.1}$$

in the following cases:

$$1.\ |a| \le a_0 < \infty,\ \mathrm{Re}(e^{i\gamma}a) = \mathrm{Re}\, a\ \cos\gamma - \mathrm{Im}\, a\ sen\ \gamma \ge \kappa > 0; \tag{7.2}$$

$$2.\ 0 \le a = \overline{a} \le a_0 < \infty,\quad \int_0^{\infty} \lambda(t)dt = \infty, \tag{7.3}$$

where

$$\lambda(t) = \left(\int_\Omega a^{-\frac{q}{\sigma-q}}(\cdot,t)dx\right)^{-p},\quad p = \frac{4(\sigma-q)}{(2-q)n\sigma+4q}, 1<q<\sigma.$$

First one is evident. Let us demonstrate the proof of the first assertion of the previous Theorem under conditions (7.3). We use inequality (6.8) in the following form

$$b^{\frac{1}{2}} = \|u\|_{L^2(\Omega)} \le C\left(\|\nabla u\|_{L^2(\Omega)}\right)^{\frac{(2-q)n}{2n-q(n-2)}}\left(\|u\|_{L^q(\Omega)}\right)^{\frac{2q}{2n-q(n-2)}} \le \tag{7.4}$$

$$\le C\left(\int_\Omega\left(|\nabla u|^2 + a\,|u|^\sigma\right)dx\right)^{\frac{\sigma(2-q)n+4q}{2\sigma(2n-q(n-2))}}\left(\int_\Omega a^{-\frac{q}{\sigma-q}}\right)^{\frac{2(\sigma-q)}{\sigma(2n-q(n-2))}},\ q<\sigma.$$

Then we arrive to ordinary differential inequality (compare with (6.10))

$$\frac{d}{dt}b + \kappa C\lambda(t)b^\mu \le 0,\ \mu = \frac{\sigma(2n-q(n-2))}{\sigma n(2-q)+4q} < 1. \tag{7.5}$$

All solutions of the last inequality have a finite time extinction $t^* < \infty$ if

$$\int_0^\infty \lambda(t)dt = \infty. \tag{7.6}$$

The moment t^* is defined by

$$\int_0^{t^*} \lambda(t)dt = \frac{b^{1-\mu}(0)}{\kappa C(1-\mu)}. \tag{7.7}$$

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