Preprint Nº 2013/007
CMAF, Universidade de Lisboa

Existence of optimal controls for a stochastic nonlinear nonlocal parabolic equation ☆

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Abstract

The aim of this paper is to investigate the existence of optimal controls for a stochastic nonlinear parabolic equation of nonlocal type driven by a real Wiener process, controlled by different external forces which are feedback controls.

Keywords: stochastic partial differential equation, stochastic optimal

☆Supported by CNPq Grant 480356/2010-6

control
2008 MSC: 60H15, 60H30, 34K35, 93E20

1. Introduction

In this paper we will study the existence of an optimal control which minimizes the cost function $\mathcal{J}(\Phi)$ with Φ belonging to $\mathcal{U}$, the set of controls associated with the controlled initial-boundary value problem involving a SPDE:

$$\begin{cases} du(t) = (a(\int_D udx)\Delta u + \Phi(t,u))dt + g(t,u)dW(t) \text{ on } t \in]0,T[\,, \\ \quad u(x,\,0) = u_0(x) \text{ on } D \text{ and } u(x,t) = 0 \text{ on } \partial D \times]0,T[\end{cases} \tag{1}$$

where D is a bounded open subset of $\mathbb{R}^n$ with boundary ∂D, $n \geq 1$, $a = a(s)$ is a continuous function with Lipschitz constant L such that $0 < p \leq a(s) \leq P$ where p and P are constants and $W(t)$ is a real Wiener process.

The problem of the existence of an optimal control for SPDE has been studied by several authors, for example, by Nagase [9], Buckdahn and Răşcanu [3], Gatarek and Sobczyk [8]. The results of these papers cannot be applied in the study of the equation in (1) because they assume semilinearity or boundedness for the nonlinearities. In [2], the existence of optimal controls for the stochastic Navier - Stokes equation was studied.

The existence and uniqueness of the solutions for the deterministic non local problem:

$$\begin{cases} u_t = \; a\left(\int_D u\;dx\right)\Delta u + \Phi(x,t) \text{ on } D \times]0,T[\,, \\ u(x,\;0) = u_0(x) \text{ on } D \text{ and } u(x,t) = 0 \text{ on } \partial D \times]0,T[\,, \end{cases} \tag{2}$$

was studied by M. Chipot and B. Lovat [4]. This problem arises in various situations, for instance u could describe the density of a population (of bacteria, for example) subject to spreading. The diffusion coefficient a is then supposed to depend on the entire population in the domain rather than on the local density, that is, movements are guided by considering the global state of the medium.

A possible inerpretation of the random forcing term "$g(t,u)dW(t)$" in the model (1) is that there is rapidly fluctuations affecting the state.

The article is organized in the following way: in Section 2, we present the basic spaces, the norms, properties and notations which we are going to work with in the subsequent sections; in Section 3, we formulate the control problem, which is the goal of this work; in Section 4, we prove the existence of an optimal control. The idea is to prove that a minimizing sequence has a subsequence which converges weakly (see Lemma 4.1). Then we prove that weak convergence of the feedback controls implies strong convergence of a subsequence of the corresponding solutions (see Theorems 4.1 and 4.2).

2. Preliminares and notations

To simplify notation we use the letter $\mathbb{T}$ for the interval $[0, T]$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(\mathcal{F}_t)_{t\in\mathbb{T}}$ a right-continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}-$null sets and let $E(X)$ denotes the mathematical expectation of the random variable X. We abbreviate "*almost surely* $\omega \in \Omega$." to a.s.

Let B be a Banach space with norm $\|\cdot\|_B$ and let $\mathcal{B}(B)$ denote the Borel $\sigma-$algebra of B. The space $L^2(\Omega\times\mathbb{T}; B)$ is the set of all $\mathcal{F}\otimes\mathcal{B}(\mathbb{T})-$measurable processes $u : \Omega \times \mathbb{T} \to B$ which are $\mathcal{F}_t-$ adapted and $E(\int_{\mathbb{T}} \|u\|_B^2 dt) < \infty$. In this work, $(W(t))_{t\in\mathbb{T}}$ is a real Wiener process $\mathcal{F}_t-$adapted. Various constants will be denoted by c and $1_D := \int_D dx$.

We will denote by $H^1(D)$ the usual Sobolev space with norm $\|\cdot\|_1$, and inner product $(\cdot,\cdot)_1$, $H_0^1(D)$ the Sobolev space with zero boundary condition with dual space $H^{-1}(D)$, $L^2(D)$ with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$.

Now, we will state the assumptions on the SPDE in (1):

i. $\Phi : \mathbb{T} \times L^2(D) \to L^2(D)$ is a mapping such that

- **a.** $\|\Phi(t, u) - \Phi(s, v)\|^2 \leq J|t - s|^2 + K\|u - v\|^2$ for all $t, s \in \mathbb{T}$, $u, v \in L^2(D)$, where J and K are positive constants,
- **b.** $\|\Phi(0, 0)\|^2 \leq \mu$ with $\mu > 0$ a constant,
- **c.** $\Phi(\cdot, u) \in L^2(\mathbb{T}, L^2(D))$ for all $u \in L^2(D)$;

ii. $g : \mathbb{T} \times L^2(D) \to L^2(D)$ is a mapping such that

- **a.** $\|g(t, u) - g(t, v)\|^2 \leq \eta\|u - v\|^2$ for all $t \in \mathbb{T}$, $u, v \in L^2(D)$, where η is a positive constant,
- **b.** $g(t, 0) = 0$ for all $t \in \mathbb{T}$,
- **c.** $g(\cdot, u) \in L^2(\mathbb{T}, L^2(D))$ for all $u \in L^2(D)$;

iii. u_0 is a $L^2(D)-$valued $\mathcal{F}_0-$measurable random variable such that $E\|u_0\|^4 < \infty$.

Remark 2.1. *The condition $g(t,0) = 0$ (see (**ii.b**)) is not a restriction. This can be omitted, in which case we can use the estimate $\|g(t,u)\|^2 \leq 2K\|u\|^2 + 2\|g(t,0)\|^2$.*

We define the map $\mathcal{A} : H_0^1(D) \times H_0^1(D) \to H^{-1}(D)$ by

$$\langle \mathcal{A}(v,u), \eta\rangle = a\left(\int_D v\, dx\right)(\nabla u, \nabla \eta)$$

for $v, u, \eta \in H_0^1(D)$ and we assume that

$$(\nabla u, \nabla u) = \|u\|_1^2 \quad \text{for all } u \in H_0^1(D).$$

In this work, we understand that the stochastic process u_Φ is a solution to the problem in (1), in the following sense.

Definition 2.1. *Let u_0 be a random variable which does not depend on $W(t)$. The stochastic process $(u_\Phi(t))_{t\in\mathbb{T}} \in L^2(\Omega \times \mathbb{T}; H_0^1(D))$, $\mathcal{F}_t-$ adapted, with a.s. sample paths continuous in $L^2(D)$, is a solution to (1) if it satisfies the equation:*

$$\begin{aligned}(u_\Phi(t), v) + \int_0^t \langle \mathcal{A}(u_\Phi(s), u_\Phi(s)), v\rangle\, ds = (u_0, v) + \int_0^t (\Phi(s, u_\Phi(s)), v) ds + \\ + \int_0^t (g(s, u_\Phi(s)), v) dW(s)\end{aligned} \tag{3}$$

a.s. for all $v \in H_0^1(D)$ and $t \in \mathbb{T}$, where the stochastic integral is in the Itô sense.

Uniqueness means indistinguishability.

Theorem 2.1. *The equation in (3) has a unique solution u_Φ which satisfies:*

$$E(\sup_{t\in\mathbb{T}} \|u_\Phi(t)\|^2) + E(\int_0^T \|u_\Phi(s)\|_1^2 ds) \leq cE(\|u_0\|^2) \tag{4}$$

and

$$E(\sup_{t\in\mathbb{T}} \|u_\Phi(t)\|^4) + E\left(\int_0^T \|u_\Phi(s)\|^2 ds\right)^2 \leq cE(\|u_0\|^4) \tag{5}$$

where $c = c(K, L, \eta, \mu, p, T)$ is a positive constant.

Proof: The proof of the existence and uniqueness of a solution u_Φ uses the Galerkin approximation and is similar to the proof of Theorem 3.1 of [6]. To obtain the estimates in (4) and (5), we use the Burkholder and Schwarz inequalities. □

Let $\{\alpha_j; w_j\}_1^\infty$ be the eigensystem of $-\Delta$ with domain $H_0^1(D) \cap H^2(D)$. We remark that $\{w_j\}_{j=1}^\infty$ is an orthonormal set in $L^2(D)$ and orthogonal set in $H_0^1(D)$. For each $n \in \mathbb{Z}^+$, we consider $(L^2(D))_n := span\,\{w_1, w_2, \ldots, w_n\}$ equipped with the norm induced from $L^2(D)$. We define $\Pi_n : L^2(D) \to (L^2(D))_n$ as the orthogonal projection such that, $\Pi_n h := \sum_{i=1}^n (h, w_i)w_i$ for $h \in L^2(D)$ and we can extend Π_n to a projection operator $\Pi_n : H^{-1}(D) \to (H^{-1}(D))_n$ by $\Pi_n v := \sum_{i=1}^n \langle v, w_i \rangle\, w_i$. We define the following truncations:

$$\mathcal{A}^n v := \Pi_n \mathcal{A} v,\ \Phi^n := \Pi_n \Phi, \quad g^n := \Pi_n g,\ u_{0n} := \Pi_n u_0.$$

Now, for each $n \in \mathbb{Z}^+$, we consider the sequence of finite dimensional evolution equations

$$\begin{aligned}(u_{n,\Phi^n}(t), v) \ &= (u_{0n}, v) - \int_0^t \langle \mathcal{A}^n(u_{n,\Phi^n}(s), u_{n,\Phi^n}(s)), v\rangle\, ds + \\ &+ \int_0^t (\Phi^n(s, u_{n,\Phi^n}(s), v)\, ds + \int_0^t (g^n(s, u_{n,\Phi^n}(s)), v) dW(s)\end{aligned} \tag{6}$$

a.s., $v \in (L^2(D))_n$, $t \in \mathbb{T}$.

Theorem 2.2. *For each $n \in \mathbb{Z}^+$, the equation in (6) has a unique solution $u_{n,\Phi^n} \in L^2(\Omega \times \mathbb{T}, H_0^1(D))$ with a.s. sample paths continuous in $L^2(D)$, which satifies:*

$$E(\sup_{t\in\mathbb{T}} \|u_{n,\Phi^n}(t)\|^2) + E(\int_0^T \|u_{n,\Phi^n}(s)\|_1^2 ds) \leq cE(\|u_0\|^2) \tag{7}$$

and

$$E(\sup_{t\in\mathbb{T}} \|u_{n,\Phi^n}(t)\|^4) + E\left(\int_0^T \|u_{n,\Phi^n}(s)\|^2 ds\right)^2 \leq cE(\|u_0\|^4) \tag{8}$$

where $c = c(K, L, \eta, \mu, p, T)$ is a positive constant.

Proof: The proof of the existence and uniqueness of the solution u_{n,Φ^n} follows from Theorem 3 (p. 45) of [8] or Corollary 6.3.1 (p. 112) of [1]). To obtain the estimates in (7) and (8) we use the Burkholder and Schwarz inequalities. □

3. Formulation of the control problem

We consider the SPDE (1) controlled by *continuous feedback controls* and denote by $\mathcal{U} := \{\Phi : \mathbb{T} \times L^2(D) \to L^2(D)\}$ the set of the admissible controls satisfying (**i**). We define the *cost functional*

$$\mathcal{J}(\Phi) := E(\int_0^T \Big(\mathcal{L}(s, u_\Phi(s)) + \mathcal{K}(\Phi(s, u_\Phi(s)))\Big)ds) + E(\mathcal{H}(u_\Phi(T))), \;\; \Phi \in \mathcal{U} \tag{9}$$

whenever the integral in (9) exists and is finite, with $\mathcal{L} : \mathbb{T} \times H_0^1(D) \to \mathbb{R}_+$, $\mathcal{K} : L^2(D) \to \mathbb{R}_+$, and $\mathcal{H} : L^2(D) \to \mathbb{R}_+$. It is required that the mappings $\mathcal{K}$, $\mathcal{H}$, and $u \in L^2(\mathbb{T}; H_0^1(D)) \longmapsto \displaystyle\int_0^T \mathcal{L}(s, u(s))ds$ are weak sequentially lower semicontinuos.

We will study the problem of minimizing $\mathcal{J}$ among the admissible controls which will be denoted by $(\mathcal{P})$.

4. Existence of optimal controls

Lemma 4.1. *Let Φ_n be a minimizing sequence for problem $\mathcal{P}$. There exists a subsequence n_k of n and a mapping $\Phi \in \mathcal{U}$ such that for all $t \in \mathbb{T}$, x, $y \in L^2(D)$, we have*

$$\lim_{k\to\infty} (\Phi_{n_k}(t, x), y) = (\Phi(t, x), y). \tag{10}$$

Proof: Let $\{t_i\}_{i=1}^\infty$ and $\{f_i\}_{i=1}^\infty$ be dense subsets of $\mathbf{T}$ and $L^2(D)$, respectively. Because the sequence $\{\Phi_n(t_j, d_i)\}_{n=1}^\infty$ is bounded for $i, j \in \mathbb{Z}^+$ we can find a subsequence (n_k) of (n) such that, for all $y \in L^2(D)$ and $i, j \in \mathbb{Z}^+$ we have

$$\lim_{k\to\infty} (\Phi_{n_k}(t_j, d_i), y) = (\Phi(t_j, d_i), y). \tag{11}$$

For $t \in \mathbb{T} \setminus \{t_i\}_{i=1}^\infty$ and $x \in L^2(D) \setminus \{d_i\}_{i=1}^\infty$, there exist sequences $\{t_{i_k}\}_{k=1}^\infty$ and $\{f_{j_k}\}_{k=1}^\infty$ such that $\lim_{k\to\infty} t_{i_k} = t$ and $\lim_{k\to\infty} f_{j_k} = x$. Then, thanks to (11) and (**i.a**), it follows that the limit

$$\Phi(t, x) := \lim_{k\to\infty} (\Phi(t_{i_k}, d_{j_k}) \tag{12}$$

exists. From this, (11) and (**i.a**) we have (10). The weak convergence and (**i.a**) imply that Φ satisfies (**i.a**) and (**i.b**). □

For simplicity the subsequence of $\{\Phi_{n_k}\}_{k=1}^\infty$ obtained in the previous lemma will be relabeled as the same, for this sequence and Φ as in the last lemma let us consider the equation

$$\begin{aligned}(\hat{u}_{\Phi_n}(t), v) &= (u_0, v) - \int_0^t \langle \mathcal{A}(u_\Phi(s), \hat{u}_{\Phi_n}(s)), v\rangle \, ds + \\ &+ \int_0^t (\Phi_n(s, u_\Phi(s)), v) \, ds + \int_0^t (g(s, u_\Phi(s)), v) dW(s)\end{aligned} \tag{13}$$

a.s., $v \in (L^2(D))_n$, $t \in \mathbb{T}$ and for $n \in \mathbb{Z}^+$. There is a unique process $\hat{u}_{\Phi_n} \in L^2(\Omega \times \mathbb{T}; H_0^1(D))$ which is a solution of (13) with a.s. continuous trajectories in $L^2(D)$ (see Theorem 7.2 [5]) satisfying:

$$E(\sup_{t\in\mathbb{T}} \|\hat{u}_{\Phi_n}(t)\|^4) + E\left(\int_0^T \|\hat{u}_{\Phi_n}(s)\|_1^2 ds\right)^2 \leq c(E(\|u_0\|^4) + E(\int_0^T \|u_\Phi(s)\|^4 ds)) \tag{14}$$

where c is a positive constant independent of n.
To obtain the estimates in (14) we use the Burkholder and Schwarz inequalities.

Theorem 4.1. *The solution to (3) and (13) satisfies:*

$$\lim_{n\to\infty} E(\int_0^T \|(u_\Phi - \hat{u}_{\Phi_n})(s)\|_1^2 ds) = \lim_{n\to\infty} E(\|(u_\phi - \hat{u}_{\Phi_n})(T)\|^2) = 0.$$

Proof: Let us consider the equation

$$(z(t), v) = (u_0, v) - \int_0^t \langle \mathcal{A}(u_\Phi(s), z(s)), v\rangle \, ds + \int_0^t (g(s, u_\Phi(s)), v) dW(s) \tag{15}$$

a.s., $v \in H_0^1(D)$ and $t \in \mathbb{T}$. There exists a unique solution $z \in L^2(\Omega \times \mathbb{T}; H_0^1(D))$ of (15), which has a.s. continuous trajectories in $L^2(D)$. By using the Gronwall lemma, we get the estimate

$$E(\sup_{t\in\mathbb{T}} \|z(t)\|^2) + 2pE(\int_0^T \|z(s)\|_1^2 ds) \leq k\left(E(\|u_0\|^2) + E(\int_0^T \|u_\Phi(s)\|^2 ds)\right).$$

From this and (7), there exists $k_2(\omega) > 0$ independent of n, such that for all $n \in \mathbb{Z}^+$ and a.s.,

$$\begin{aligned}\sup_{t\in\mathbb{T}} \quad \|\Pi_n z(t)\|^2 &\leq \sup_{t\in\mathbb{T}} \|z(t)\|^2 \leq k_2(\omega), \\ \int_0^T \|\Pi_n z(s)\|_1^2 \quad ds &\leq \int_0^T \|z(s)\|_1^2 ds \leq k_2(\omega)\end{aligned} \tag{16}$$

and

$$
\begin{aligned}
\sup_{t\in\mathbb{T}} \quad \|\Pi_n u_\Phi(t)\|^2 &\le \sup_{t\in\mathbb{T}} \|u_\Phi(t)\|^2 \le k_2(\omega), \\
\int_0^T \|\Pi_n u_\Phi(s)\|_1^2 \quad ds &\le \int_0^T \|u_\Phi(s)\|_1^2 ds \le k_2(\omega).
\end{aligned} \tag{17}
$$

Using stochastic integral properties and (5), we obtain that for all $s, t \in \mathbb{T}$, $t > s$,

$$
E(\| \int_s^t g(r, u_\Phi(r)) dW(r)\|_{-1}^4) \le c(t-s)^2 E(\|u_0\|^4)
$$

As a result of the Kolmogorov continuity test, we get a random variable H such that

$$
\| \int_s^t g(r, u_\Phi(r)) dW(r)\|_{-1}^2 \le H(\omega)|t-s|^{2\gamma} \tag{18}
$$

a.s. with $0 < \gamma < \frac{1}{4}$ and for every $t, s \in \mathbb{T}$.

Let $\bar{\Omega} \subset \Omega$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for $\omega \in \bar{\Omega}$ the equations in (3) and (15) are satisfied and, for each $n \in \mathbb{Z}^+$, (13) is also satisfied and the inequalities in (16), (17) and (18) are satisfied.

From (13), (15), (17) and properties $\mathcal{A}$ and Φ_n, it follows that for $\omega \in \bar{\Omega}$,

$$
\sup_{t\in\mathbb{T}} \|(\hat{u}_{\Phi_n} - z)(t)\|^2 + p \int_0^T \|(\hat{u}_{\Phi_n} - z)(s)\|_1^2 ds \le \frac{JT+\eta}{p} + \frac{1}{p} \int_0^T \|u_{\Phi(s)}\|_1^2 ds \le k(\omega),
$$

where $k(\omega)$ is independent of n. Hence, for all $n \in \mathbb{Z}^+$, we obtain

$$
\sup_{t\in\mathbb{T}} \|\hat{u}_{\Phi_n}(t)\|^2 + p \int_0^T \|\hat{u}_{\Phi_n}(s)\|_1^2 ds \le k(\omega) \tag{19}
$$

for $\omega \in \bar{\Omega}$, where $k(\omega)$ is a positive constant independent of n.

For $\omega \in \bar{\Omega}$, we consider the sequence

$$
F(\omega) := \{\hat{u}_{\Phi_n}(\omega, \cdot)\}_{n=1}^{\infty},
$$

which is bounded because of (19).

From (13), we obtain

$$
\begin{aligned}
\|\hat{u}_{\Phi_n}(t) - \hat{u}_{\Phi_n}(s)\|_{-1}^2 \quad &\le (t-s) \int_s^t \left(\|\mathcal{A}(u_\Phi(r), \hat{u}_{\Phi_n}(r))\|_{-1}^2 + \|\Phi_n(r, u_\Phi(r))\|\|_{-1}^2 \right) dr + \\
&+ \| \int_s^t g(r, u_\Phi(r)) dW(r)\|_{-1}^2,
\end{aligned}
$$

for each $t, s \in \mathbb{T}$, $t > s$. From this, (18), (19) and the properties of $\mathcal{A}$, Φ_n, we get

$$\|\hat{u}_{\Phi_n}(t) - \hat{u}_{\Phi_n}(s)\|_{-1}^2 \leq k(\omega)(t-s) + H(\omega)(t-s)^{2\gamma}$$

for $\gamma \in (0, \frac{1}{4})$ and where $k(\omega) > 0$ is independent of n.

Consequently, $F(\omega)$ is equi-continuous in $C([0,T], H^{-1}(D))$. Now, using Dubinsky's Theorem, (see [10], Theorem 4.1, p. 132), it follows that $F(\omega)$ is relatively compact in $L^2(\mathbb{T}; L^2(D))$.Thus there exists a subsequence n_k of n and $\hat{u} \in L^2(\mathbb{T}; L^2(D))$ such that

$$\lim_{k\to\infty} \int_0^T \|(\hat{u}_{\Phi_{n_k}} - \hat{u})(s)\|^2 ds = 0. \tag{20}$$

From (13), (3) and the properties of $\mathcal{A}$, we obtain

$$\begin{aligned}\|\hat{u}_{\Phi_{n_k}}(T) - u_\Phi(T)\|^2 \quad &+2p\int_0^T \|(\hat{u}_{\Phi_{n_k}} - u_\Phi)(t)\|_1^2 dt \leq \\ &\leq \int_0^T \left(\Phi_{n_k}(t, u_\Phi(t)) - \Phi(t, u_\Phi(t)), (\hat{u}_{\Phi_{n_k}} - \hat{u})(t)\right) dt + \\ &+ \int_0^T (\Phi_{n_k}(t, u_\Phi(t)) - \Phi(t, u_\Phi(t)), (\hat{u} - u)(t))\, dt.\end{aligned}$$

We use Lemma 4.1, (20) and the properties of Φ_n and Φ to obtain

$$\lim_{k\to\infty} \|(\hat{u}_{\Phi_{n_k}} - u_\Phi)(T)\|^2 = \lim_{k\to\infty} \int_0^T \|(\hat{u}_{\Phi_{n_k}} - u_\Phi)(t)\|_1^2 dt = 0.$$

Since every subsequence of $(\hat{u}_{\Phi_n}(\omega, \cdot)$ has a subsequence which converges to the same limit $u_{\Phi(\omega,\cdot)}$ in the space $L^2(0, T; H_0^1(D))$, it follows that the sequence $(\hat{u}_{\Phi_n}(\omega, \cdot)$ converges to $u_\Phi(\omega, \cdot)$. Similarly, we can conclude that $(\hat{u}_{\Phi_n}(\omega, T)$ converges to $u_\Phi(\omega, T)$ in $L^2(D)$.

From Theorem 2.1 and (14), the processes $(\hat{u}_{\Phi_n})_{t\in\mathbb{T}}$ and $(u_\Phi)_{t\in\mathbb{T}}$ are uniformly integrable and thus the theorem follows. □

Let $(Q(t))$ be a $H_0^1(D)-$valued process with

$$\int_0^T \|Q(s)\|_1^2 ds < \infty \text{ and } \sup_{t\in\mathbb{T}} \|Q(t)\|^2 < \infty \ a.s.$$

For each M, a nonnegative integer, we define the following stopping times:

$$\mathcal{T}_M^Q := \begin{cases} \inf\left\{t \in \mathbb{T} : \int_0^t \|Q(s)\|_1^2 ds \geq M\right\}, \\ T, \text{ if } \int_0^T \|Q(s)\|_1^2 ds < M, \end{cases}$$

$$\widehat{\mathcal{T}}_M^Q := \begin{cases} \inf\{t \in \mathbb{T} : \sup_{t\in\mathbb{T}} \|Q(t)\|^2 \geq M\} \\ T, \text{ if } \sup_{t\in\mathbb{T}} \|Q(t))\|^2 < M \end{cases}$$

and $\mathcal{T}_M^Q := min\left\{\bar{\mathcal{T}}_M^Q, \widehat{\mathcal{T}}_M^Q\right\}$.

Theorem 4.2. *Let $\{\Phi_n\}_{n\in\mathbb{N}}$ be as in the last theorem. There is a subsequence n_k of n such that*

$$\lim_{k\to\infty} E(\int_0^T \|(u_\Phi - u_{\Phi_{n_k}})(s)\|_1^2 ds) = \lim_{k\to\infty} E(\|(u_\Phi - u_{\Phi_{n_k}})(T)\|^2) = 0.$$

Proof: For the sake of convenience, we use the abbreviations,$u := u_\Phi$ and $\mathcal{T}_M := \mathcal{T}_M^u$ for $M = 1, 2, \ldots$.
Let $e(t) := \displaystyle\int_0^t exp(-\frac{2K}{p} - 2\eta - 2L^2 1_D\|u(s)\|_1^2)ds$. As a result of the Itô formula, we get

$$\begin{aligned} &e(\mathcal{T}_M)\|\hat{u}_{\Phi_n}(\mathcal{T}_M) - u_{\Phi_n}(\mathcal{T}_M)\|^2 + \\ +2\int_0^{\mathcal{T}_M} &e(s)\left\langle \mathcal{A}(u(s), \hat{u}_{\Phi_n}(s)) - \mathcal{A}(u_{\Phi_n}(s), u_{\Phi_n}(s)), (\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\right\rangle ds = \\ &+2\int_0^{\mathcal{T}_M} e(s)(\Phi_n(s, u(s)) - \Phi_n(s, u_{\Phi_n}(s)), (\hat{u}_{\Phi_n} - u_{\Phi_n})(s))ds + \\ &+2\int_0^{\mathcal{T}_M} e(s)(g(s, u(s)) - g(s, u_{\Phi_n}(s)), (\hat{u}_{\Phi_n} - u_{\Phi_n})(s))dW(s) + \\ &+\int_0^{\mathcal{T}_M} e(s)\|g(s, u(s)) - g(s, u_{\Phi_n}(s)\|^2 ds + \\ &+\int_0^{\mathcal{T}_M} e'(s)\|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|^2 ds. \end{aligned}$$

We first observe that

$$\begin{aligned} \langle \mathcal{A}(u, \hat{u}_{\Phi_n}) - \ &\mathcal{A}(u_{\Phi_n}, u_{\Phi_n}), (\hat{u}_{\Phi_n} - u_{\Phi_n})\rangle = \\ &= \langle \mathcal{A}(u, \hat{u}_{\Phi_n}) - \mathcal{A}(u_{\Phi_n}, \hat{u}_{\Phi_n}), (\hat{u}_{\Phi_n} - u_{\Phi_n})\rangle + \\ &+ \langle \mathcal{A}(u_{\Phi_n}, \hat{u}_{\Phi_n} - u_{\Phi_n}), (\hat{u}_{\Phi_n} - u_{\Phi_n})\rangle = \\ &\left((a(\int_D u(x)dx - a(\int_D u_{\Phi_n}(x)dx))\nabla\hat{u}_{\Phi_n} - \nabla u, \nabla\hat{u}_{\Phi_n} - u_{\Phi_n}\right) + \\ &+ \left((a(\int_D u(x)dx - a(\int_D u_{\Phi_n}(x)dx))\nabla u, \nabla\hat{u}_{\Phi_n} - u_{\Phi_n}\right) + \\ &+ \langle \mathcal{A}(u_{\Phi_n}, \hat{u}_{\Phi_n} - u_{\Phi_n}), (\hat{u}_{\Phi_n} - u_{\Phi_n})\rangle. \end{aligned}$$

Using the properties of $\mathcal{A}$, Φ_n and g, we get

$$\begin{aligned}
&E(e(\mathcal{T}_M)\|\hat{u}_{\Phi_n}(\mathcal{T}_M) - u_{\Phi_n}(\mathcal{T}_M)\|^2)+\\
&+2pE(\int_0^{\mathcal{T}_M} e(s)\|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|_1^2 ds) \leq\\
&+2E(\int_0^{\mathcal{T}_M} e(s)(\Phi_n(s,u(s)) - \Phi_n(s,u_{\Phi_n}(s)), (\hat{u}_{\Phi_n} - u_{\Phi_n})(s))ds)+\\
&+E(\int_0^{\mathcal{T}_M} e(s)\|g(s,u(s)) - g(s,u_{\Phi_n}(s)\|^2 ds)+\\
&+E(\int_0^{\mathcal{T}_M} e'(s)\|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|^2 ds)+\\
&+L^2 1_D E(\int_0^{T} e(s)\|(u - u_{\Phi_n})(s)\|^2\|u(s)\|_1^2 ds)+\\
&+PE(\int_0^{T} e(s)\|(\hat{u}_{\Phi_n} - u)(s)\|^2 ds).
\end{aligned} \tag{21}$$

From Theorem 4.1, we can get a subsequence $\{\hat{u}_{\Phi_{n_k}}\}_{k=1}^{\infty}$ that converges to u almost everywhere $(\omega,t) \in \mathcal{F} \times \mathcal{B}(\mathbb{T})$. Thus, from (21), we obtain

$$\begin{aligned}
E((e(\mathcal{T}_M)\|\hat{u}_{\Phi_{n_k}}(\mathcal{T}_M) - u_{\Phi_{n_k}}(\mathcal{T}_M)\|^2) + pE(\int_0^{\mathcal{T}_M} e(s)\|(\hat{u}_{\Phi_{n_k}} - u_{\Phi_{n_k}})(s)\|_1^2 ds) \leq\\
2L^2 1_D\, E(\int_0^{T} e(s)\|(u - \hat{u}_{\Phi_{n_k}})(s)\|^2\|u(s)\|_1^2 ds)+\\
t + (P + \tfrac{2K}{P} + 2\eta)E(\int_0^{T} e(s)\|(\hat{u}_{\Phi_{n_k}} - u)(s)\|^2 ds).
\end{aligned}$$

From this, Theorems 4.1, 2.1 and the triangle inequality, we obtain

$$\lim_{k\to\infty} E(\int_0^{\mathcal{T}_M} \|(u_\Phi - u_{\Phi_{n_k}})(s)\|_1^2 ds) = \lim_{n'\to\infty} E(\|(u_\Phi - u_{\Phi_{n_k}})(\mathcal{T}_M)\|^2) = 0,$$

which implies the desired conclusion. □

Finally, we are in a position to formulate.

Theorem 4.3. *There exists an optimal control for problem $(\mathcal{P})$.*

Proof: Let $\{\Phi_n\}$ be a minimizing sequence for problem $(\mathcal{P})$. We apply Lemma (4.1) and Theorem (4.2) to this sequence. Thus, there exists a subsequence $\{\Phi_{n_k}\}$ of $\{\Phi_n\}$ and $\Phi \in \mathcal{U}$ such that, for all $t \in \mathbb{T}$, $x, y \in L^2(D)$ and a.s. $\omega \in \Omega$, the following hold:

$$\lim_{k\to\infty} \left(\Phi_{n_k}(t, u_{\Phi_{n_k}}), y\right) = (\Phi(t, u_\Phi), y)$$

and

$$\lim_{k\to\infty}\int_0^T \|(u_{\Phi_{n_k}} - u_\phi)(s)\|_1^2 ds = \lim_{k\to\infty}\|(u_{\Phi_{n_k}} - u_\phi)(T)\|^2 = 0.$$

From Theorem (4.2) and the weak sequentially semicontinuity properties of $\mathcal{L}$, $\mathcal{K}$ and $\mathcal{H}$, we get

$$\mathcal{J}(\Phi) \leq \lim\inf_{k\to\infty} \mathcal{J}(\Phi_{n_k}).$$

Since $\{\Phi_n\}$ is a minimizing sequence for problem $(\mathcal{P})$, $\mathcal{J}(\Phi) = \min_{\lambda\in\mathcal{U}} \mathcal{J}(\lambda)$ and thus $\Phi \in \mathcal{U}$ is an optimal feedback control for problem $(\mathcal{P})$. □

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