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Existence, uniqueness and blow up for Hyperbolic equations with nonstandard growth conditions

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Abstract

In this paper we consider the following problem

$$u_{tt} = \Delta u + b(x,t)\,|u|^{\sigma(x,t)-2}\,u + f(x,t),\ x,t \in Q_T = \Omega \times (0,T),$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \geq 0,\ x \in \Omega,$$

$$u_{\Gamma_T} = 0,\ \Gamma_T = \partial\Omega \times (0,T).$$

under conditions

$$0 \leq b^-(t) \leq b(x,t) \leq b^+(t) < \infty,$$

$$1 \leq \sigma^-(t) \leq \sigma(x,t) \leq \sigma^+(t) < \infty.$$

We study the existence and uniqueness of weak and strong solutions. We prove the blow up of nonnegative solutions. Also we consider the blow up effect for more generalized problem

$$\begin{cases} u_{tt} = Lu + f(x,t,u) \quad \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega,\ u_t(x,0) = u_1(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases}$$

where L is the linear elliptic operator

$$Lu = D_i\left(a_{ij}(x)D_j u + a_i(x)u\right) + a_0(x)u,$$

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and the right-hand side has the form

$$f(x,t,u) = \sum_{k=1}^{N} b_k(x,t) u^{\sigma_k(x,t)-1} + \sum_{i=N}^{K} c_i(x,t) \int_\Omega d_i(s,t) u^{\sigma_i(s,t)-1} ds,$$

with $b_k \geq 0$, $c_i \geq 0$, $d_i \geq 0$.

Keywords: Faedo-Galerkin method, nonlinear hyperbolic equation with anisotropy and nonstandard grows conditions, existence and uniqueness of weak solutions and blow up.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary Γ and $Q_T = \Omega \times (0,T]$. We consider the following initial boundary value problem

$$u_{tt} = Lu + f(x,t), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

$$Lu = \Delta u + b(x,t) |u|^{\sigma(x,t)-2} u,$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \geq 0, \ x \in \Omega, \tag{1.2}$$

$$u|_{\Gamma_T} = 0, \ \Gamma_T = \partial\Omega \times (0,T). \tag{1.3}$$

The coefficient $b(x,t)$, and the exponent $\sigma(x,t)$ and the source term $f(x,t)$ are given measurable functions of their arguments. We discuss existence, uniqueness of weak and strong solutions to problem (1.1)-(1.3). Also we prove the blow up of nonnegative solutions to problem (1.1)-(1.3), concentrating our attention on difficulties caused by variable exponent $\sigma(x,t)$. First we make a remark about the problem

$$u_{tt} = \operatorname{div}(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u + \varepsilon \nabla u_t) + b(x,t)|u|^{\sigma(x,t)-2}u + f(x,t), \tag{1.4}$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \tag{1.5}$$

$$u|_{\Gamma_T} = 0, \ \Gamma_T = \partial\Omega \times (0,T). \tag{1.6}$$

where the coefficients a, b, f and exponents p, σ are given measurable function of their arguments and $\varepsilon = const > 0$. Such equations (with variable exponents of nonlinearities) are usually referred to as equations with nonstandard growth conditions.

Properties of energy weak solutions for elliptic and parabolic equations of the type (1.4) have been investigated sufficiently completely (see, e.g., [1-3],[7],[9], [11],[13-14]). Such elliptic, parabolic and hyperbolic equations occur in the mathematical modeling of various physical phenomena, e.g., the flows of electrorheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, processes of filtration through a porous media and the image processing, theory of nonlinear viscoelasticity (see, e.g., [8-10],[33-34]) and the further references therein.

Returning to equation (1.4) it is to be noted that local and global existence and blow up for equations in the form (1.4) with constant exponents p, σ have been investigated in many papers (see, e.g., [12],[16-19],[22-25],[28-31],[36],[38-40]) and the further references therein.

It is to be noted here that in all these papers (referring to the case $p \neq 2$) the viscous term $\varepsilon \Delta u_t$ plays a key role in the proof of global or local existence (even if $p = \text{const} \neq 2, \sigma = \text{const}$). The principal difficulty to prove an existence theorem define the term $\operatorname{div}(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u)$. The viscous term $\varepsilon \Delta u_t$ $(\varepsilon > 0)$ facilities the proof of existence theorem but complicates to prove blow up effect.

To the best of our knowledge, the equations with $\varepsilon = 0$ were studied only in papers (see, e.g., [4-5],[27],[35]) where proved existence of measure solutions.

The theory of equations (1.4) with nonstandard growth conditions (p, σ are variable) accounts only for the following papers.

An existence result was proved for the autonomous equation

$$u_{tt} = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + \alpha \Delta u_t + f(x,u),\ \alpha > 0$$

in [21]. In the paper [5],[6] were proved existence and blow up results for the problem (1.4)-(1.6) with $a(x,t), p(x,t), b(x,t), \sigma(x,t)$. The dependence of the coefficients and exponents of the equation on t essentially complicates this problem. In all refereing here papers the method of energy estimates was used and assumed that the the initial energy is negative.

If $p \equiv 2,\ \varepsilon = 0$ the problem (1.4)-(1.6) is simplified essentially. In paper [32] the authors considered existence and blow up of solutions for the parabolic

$$u_t = \Delta u + f(x,u), \quad (x,t) \in Q_T = \Omega \times (0,T) \tag{1.7}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.8}$$

$$u|_{\Gamma_T} = 0,\ \Gamma_T = \partial\Omega \times (0,T) \tag{1.9}$$

and hyperbolic problems

$$u_{tt} = \Delta u + f(x,u), \quad (x,t) \in Q_T = \Omega \times (0,T) \tag{1.10}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{1.11}$$

$$u|_{\Gamma_T} = 0,\ \Gamma_T = \partial\Omega \times (0,T) \tag{1.12}$$

where

$$f(x,u) = b(x)u^{p(x)} \text{ or } f(u) = b(x)\int_\Omega u^{q(y)}(y,t)dy. \tag{1.13}$$

The authors proved blow up results for parabolic and hyperbolic problem. The existence result was proved only for the parabolic problem, but the result does not prove the existence of solution to the hyperbolic equation.

In the present we generalize the results of the paper [32] on the cases when the coefficient $b(x,t)$ and the exponent $\sigma(x,t)$ may depend on temporal variable

t. Also we consider some limit cases when $b(x,t)$ may vanishes in Q_T and $\sigma(x,t) \searrow 2$ as $t \to \infty$.

The present paper is organized as follows. Before the main results, in Section 2 we introduce preliminaries on function spaces of Sobolev type. In Section 3 we present the statement of the problem and we give a definition of weak solution. Section 4 is devoted to proof of the weak and strong solutions and uniqueness to problem (1.1)-(1.3). The weak and strong solution is obtained as the limit of the sequence of Galerkin's approximations.

First we derive estimates for an energy functional. Section 5 is devoted to the investigation of the blow-up of energy strong solutions with nonpositive energy functional in both cases $\sigma \geq \sigma^- > 2$ and the limit cases when $b(x,t)$ may vanishes and $\sigma(x,t) \searrow 2$ as $t \to \infty$. Finally, in Section 6 we present one important generalization of Section 5, considering the case when L is the linear elliptic operator

$$Lu = D_i\left(a_{ij}(x)D_j u + a_i(x)u\right) + a_0(x)u,$$

and the right-hand side has the form

$$f(x,t,u) = \sum_{k=1}^{N} b_k(x,t)u^{\sigma_k(x,t)-1} + \sum_{i=N}^{K} c_i(x,t)\int_\Omega d_i(s,t)u^{\sigma_i(s,t)-1}ds,$$

with $b_k \geq 0$, $c_i \geq 0$, $d_i \geq 0$.

2. Preliminaries. Function spaces

We denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with a compact support contained in Ω The inner products and norms in $L^2(\Omega)$ and $H_0^1(\Omega)$ we be represented by $(.,.), |.|, ((.,.)), \|.\|$ respectively, given by

$$(u,v) = \int_\Omega u(x)v(x)dx \text{ and } |u|^2 = \int_\Omega u^2 dx,$$

$$((u,v)) = \int_\Omega \nabla u.\nabla v dx \text{ and } \|u\|^2 = \int_\Omega |\nabla u|^2 dx.$$

If X is a Banach Space, then we denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ the Banach Space of the vector valued functions $u : (0,T) \to X$, which are measurable and $\|u(t)\|_X \in L^p(0,T)$, with the norms:

$$\|u(t)\|_{L^p(0,T;X)} = \left[\int_0^T \|u(t)\|_X^p \, dt\right]^{\frac{1}{p}}, \; 1 \leq p < \infty,$$

$$\|u(t)\|_{L^p(0,T;X)} = ess \sup_{0 \leq t < T} \|u(t)\|_X \, , \; p = \infty.$$

We recall some known facts from the theory of the Sobolev spaces with variable exponent. The reader is referred to the monograph [14] for the detailed exposition of this theory.

Let $L^{p(.)}(\Omega)$ be the set of measurable functions $f(x)$ on Ω such that

$$A_{p(.)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx < \infty.$$

The set $L^{p(.)}$ equipped with the Luxemburg norm

$$\|f\|_{p(.),\Omega} = \|f\|_{p(.)} = \inf\left\{\lambda > 0; A_{p(.)}(\frac{f}{\lambda}) \le 1\right\}$$

becomes a Banach space.

Let us indicate the basic properties of the spaces $L^{p(.)}(\Omega)$ used in the rest of this paper.

· It follows directly from the definition of the norm that

$$\min\left(\|f\|_{p(.)}^{p-}, \|f\|_{p(.)}^{p+}\right) \le A_{p(.)}(f) \le \max\left(\|f\|_{p(.)}^{p-}, \|f\|_{p(.)}^{p+}\right). \tag{2.14}$$

· Hölder inequality. For all $f \in L^{p(.)}(\Omega)$, $g \in L^{p'(.)}(\Omega)$ with

$$p(x) \in (1,\infty),\ p' = \frac{p(x)}{p(x)-1},\ p^- = \inf_\Omega p(x),\ (p')^- = \inf_\Omega p'(x),$$

the following inequality holds:

$$\int_\Omega |fg|\,dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|f\|_{p(.)}\|g\|_{p'(.)} \le 2\|f\|_{p(.)}\|g\|_{p'(.)}. \tag{2.15}$$

We should point out that the main tools in proving the existence and uniqueness of solution to the evolution equation will be the well-known Galerkin's Method and compactness arguments.

3. Statement of the problem

Now we consider the following problem

$$u_{tt} = \Delta u + b(x,t)\,|u|^{\sigma(x,t)-2}\,u + f(x,t)\ (x,t) \in Q_T = \Omega \times (0,T) \tag{3.1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \ge 0,\ x \in \Omega, \tag{3.2}$$

$$u_{\Gamma_T} = 0,\ \Gamma_T = \partial\Omega \times (0,T) \tag{3.3}$$

under conditions

$$0 \le b^-(t) \le b(x,t) \le b^+(t) < \infty \tag{3.4}$$

$$2 \le \sigma^-(t) \le \sigma(x,t) \le \sigma^+(t) < \infty. \tag{3.5}$$

Let us introduce the notations

$$\begin{aligned} &\sigma^{+}(t)=\sup_{\Omega}\sigma(x,t), && \sigma^{-}(t)=\inf_{\Omega}\sigma(x,t),\\ &\sigma^{+}=\sup_{Q_T}\sigma(x,t), && \sigma^{-}=\inf_{Q_T}\sigma(x,t),\\ &b^{+}(t)=\sup_{\Omega}b(x,t), && b^{-}(t)=\inf_{\Omega}\sigma(x,t),\\ &b^{+}=\sup_{Q_T}b(x,t), && b^{-}=\inf_{Q_T}b(x,t). \end{aligned} \tag{3.6}$$

We assume that

$$u_0\in L^2(\Omega),\ u_1\in L^2(\Omega),\ f\in L^2(Q_T). \tag{3.7}$$

The solution of problem of (3.1)-(3.3) is undestood in the following sense.

Definition 1. *A function u: $\Omega_T\to\mathbf{R}$ is called a weak solution to (3.1)-(3.3) if:*

1.
$$(\nabla u,\ u_t)\in L^{\infty}\left(0,T;L^2(\Omega)\right),\ u\in L^{\infty}\left(0,T;L^p(\Omega)\right), \tag{3.8}$$
$$1\le p\le\frac{2n}{\sigma^{+}(n-2)}\ \textit{if}\ n>2;1\le p<\infty\ \textit{if}\ n\le 2, \tag{3.9}$$

2.
$$u(x,t)\to u_0(x)\ \textit{in}\ L^2(\Omega),\ \int_{\Omega}(u_t(\cdot,t)-u_1(\cdot))\varphi(\cdot)dx\to 0 \tag{3.10}$$
as $t\to 0$, $\forall\varphi\in C^{\infty}(\Omega)$,

3. $\forall\varphi\in C^{\infty}\left(0,T;C_0^{\infty}(\Omega)\right),\ \varphi(x,T)=0,$ *the following integral identity holds*
$$\int_{Q_T}\left(-u_t\varphi_t+\nabla u\cdot\nabla\varphi-b\left|u\right|^{\sigma-2}u\varphi\right)= \tag{3.11}$$
$$=\int_{\Omega}u_1\varphi(\cdot,0)+\int_{Q_T}f\varphi.$$

We will assume that

$$u_0\in W_0^{1,2}(\Omega),\ u_1\in L^2(\Omega),\ f\in L^2(Q_T),|b|\le b^{+}<\infty. \tag{3.12}$$

Now, we are in a position to state our existence results.

4. Local existence of weak and strong solutions

Theorem 1. *(**Local existence of weak solutions**) Under conditions (3.4), (3.5), (3.9). Then there exist a function u weak solution of the problem (3.1)-(3.3) in the sense of Definition 1 for a small time $t\in[0,T_{\max})$, ($T_{\max}>0$ is small).*

4.1 Step 1. Galerkin's approximations.

Proof. A weak solution of problem (1.1)-(1.3) will be obtained as the limit of the sequence of Galerkin's approximations $u^{(m)}$ as $m \to \infty$. The Galerkin's approximations of solutions to problem (1.1)-(1.3) are sought in the form

$$u^{(m)} \equiv \sum_{k=1}^{m} u_k(t)\psi_k(x),\ u_k(t) = (u(x,t), \psi_k(x))_\Omega, \tag{4.13}$$

where

$$-\Delta\psi_k(x) = \lambda_k \,\text{in}\, \Omega,\ \psi_k(x) = 0 \,\text{on}\, \Gamma,\ (\lambda_k, \psi_k) \geq 0,\ (\psi_k, \psi_i) = \delta_{k,i}$$

We assume also

$$u_1^{(m)} \to u_1 \text{ strongly in } L^2(\Omega),\ u_0^{(m)} \to u_0 \text{ strongly in } W_0^{1,2}(\Omega). \tag{4.14}$$

The coefficients $u_k(t)$ are defined from the relations

$$\int_\Omega (u_{tt}^{(m)} - \triangle u^{(m)} - b\left|u^{(m)}\right|^{\sigma-2} u^{(m)} + f)\psi_k dx = 0,\ k = 1, ..., m. \tag{4.15}$$

Last equalities and the initial conditions lead us to the Cauchy problem for the system of m ordinary differential equations of the second order for the coefficients $u_k(t)$

$$u_k'' = F_k(t, u_1(t), ..., u_m(t)), \tag{4.16}$$

$$u_k(0) = \int_\Omega u_0\psi_k dx,\ u_k'(0) = \int_\Omega u_1\psi_k dx,\ k = 1, ..., m, \tag{4.17}$$

where

$$F_k = \int_\Omega \left[-\left(\nabla u^{(m)}\nabla\psi_k + b\left|u^{(m)}\right|^{\sigma-2} u^{(m)}\psi_k + f\psi_k\right)\right] dx.$$

By Peano's Theorem, for every finite m the problem (4.16),(4.17) has a solution $u_k(t),\ k = 1, ..., m$ on an interval $(0, T_m)$ for each m. The estimates below allow one to take $T_m = T$ for all m.

4.2 Step 2. A priori estimates.

Now we derive estimates for an energy functional and for approximated solutions which do not depend on m.

Multiplying (4.16) by u_{kt} summing and omitting for simplicity the index m, we arrive at the energy relation

$$\frac{d}{2dt}\left(\int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx\right) - \int_\Omega b(x,t)\,|u|^{\sigma(x,t)-2} uu_t dx = \int_\Omega fu_t dx. \tag{4.18}$$

Next, we will analyze terms of (4.18).

Estimate for $I_1 = \int_\Omega b(x,t) |u|^{\sigma(x,t)-2} uu_t dx$. We have, by the Cauchy-Schwarz inequality and using (3.4) and (3.5) one gets

$$|I_1| \le \left| \int_\Omega b(x,t) |u|^{\sigma(x,t)-2} uu_t dx \right| \le C \left(\int_\Omega |u_t|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |u|^{2\sigma-2} dx \right)^{\frac{1}{2}} \le \tag{4.19}$$

$$\le C \left(\int_\Omega |u_t|^2 dx \right)^{\frac{1}{2}} \left[\left(\int_\Omega |u|^{2\sigma^+-2} dx \right)^{\frac{1}{2}} + 1 \right] \le$$

$$C \left(\int_\Omega |u_t|^2 dx \right)^{\frac{1}{2}} \left[\left(\int_\Omega |\nabla u|^2 \right)^{\frac{\sigma^+-1}{2}} + 1 \right],$$

where

$$\sigma^+ \le 1 + \frac{n}{n-2} \text{if } n > 2;\ \sigma^+ < \infty \text{ if } n = 2.$$

Estimate for $I_2 = \int_\Omega f(x,t) u_t dx$. We obtain

$$|I_2| \le \left| \int_\Omega f(x,t) u_t dx \right| \le C \left(\int_\Omega |u_t|^2 dx \right)^{\frac{1}{2}}. \tag{4.20}$$

Introducing

$$X(t) = \left(\int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx \right),$$

and using (4.19)-(4.20), we obtain the ordinary differential inequality (ODI)

$$\frac{dX}{dt} \le C \left[X^{\frac{\sigma^+}{2}} + 1 \right].$$

It is easy to see that, $X(t)$ satisfies the following estimate

$$X(t) = \int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx \le$$

$$\le (1+X_0)[1 - t(\varkappa - 1)C(1+X_0)^{\varkappa-1}]^{\frac{1}{\varkappa-1}} := M < \infty, \varkappa = \frac{\sigma^+}{2},$$

with

$$0 \le t < T_{\max} = \frac{1}{(\varkappa-1)C(1+X_0)^{\varkappa-1}}.$$

Combining (4.18)-(4.20), finally we come at estimate

$$X^{(m)}(t) = \int_\Omega \left[\left| u_t^{(m)} \right|^2 dx + \left| \nabla u^{(m)} \right|^2 + \left| u^{(m)} \right|^{2\sigma^+-2} \right] dx \le M, \tag{4.21}$$

with constant M independent of m. Last estimate is valid for $t \in [0, T_{\max})$.

Let $d_k(t) \in C^2(0,T)$ be arbitrary functions. Multiplying (4.15) by $d_k(t)$, integrating over $[0,T]$ and summing with respect to k, we arrive at the identity

$$\int_{Q_T} (u_{tt}^{(m)}\varphi + \nabla u^{(m)}\nabla\varphi)dxdt -$$

$$- \int_{Q_T} \left(b\left|u^{(m)}\right|^{\sigma-2} u^{(m)} + f \right) \varphi dxdt = 0,$$

which is valid for any function

$$\varphi = \sum_{k=1}^{N} d_k(t)\psi_k(x),\ N \le m, \varphi \in P_m.$$

Integrating (4.15) over $(0,t)$ and applying the integration by parts for the first term, we can rewrite last one in the form

$$\int_{Q_t} \left(-u_t^{(m)}\varphi_t + \nabla u^{(m)}\nabla\varphi \right) dxd\tau - \tag{4.22}$$

$$- \int_{Q_t} \left(b\left|u^{(m)}\right|^{\sigma-2} u^{(m)} + f \right) \varphi dxd\tau - (u_t^{(m)}, \varphi)_\Omega \Big|_0^t = 0.$$

The estimate (4.21) allows us conclude that there exist u and a subsequence of $\{u^{(m)}\}$, still denoted by $\{u^{(m)}\}$, such that

$$\begin{array}{c} |u^{(m)}|^{\sigma(\cdot,\cdot)-1} \to |u|^{\sigma(\cdot,\cdot)-1} \text{ strongly in } L^p(Q_T),\ p \le \frac{2n}{\sigma^+(n-2)}, \\ u_t^{(m)} \to u_t \text{ weakly in } L^2(Q_T) \cap L^\infty(0,T;L^2(\Omega)), \\ \nabla u^{(m)} \to \nabla u \text{ weakly in } L^2(Q_T) \cap L^\infty(0,T;L^2(\Omega)). \end{array} \tag{4.23}$$

Remark 1. *It should be noted that*

$$\left\|u^{(m)}(\cdot,t) - u_0(\cdot)\right\|_{2,\Omega}^2 \le 2\left(\left\|u^{(m)}(\cdot,t) - u_0^{(m)}(\cdot)\right\|_{2,\Omega}^2 + t\left\|u_t^{(m)}\right\|_{2,Q_T}^2 \right) \le \tag{4.24}$$

$$\le C\left(\left\|u^{(m)}(\cdot,t) - u_0^{(m)}(\cdot)\right\|_{2,\Omega}^2 + t \right),$$

and $\forall\ \varphi \in C^\infty(\Omega)$

$$|(u_{tt},\varphi)_\Omega| \le \|\nabla u\|_{2,\Omega} \|\nabla\varphi\|_{2,\Omega} + (b^+ \|u\|_{\sigma^+-1,\Omega}^{\sigma^+-1} + 1) \|\varphi\|_{\infty,\Omega}. \tag{4.25}$$

4.3. Step 3. Passing to the limit as $m \to \infty$.

Passing to the limit in (4.22) as $m \to \infty$, we obtain for any $\varphi \in P_m$

$$\int_{Q_t} (-u_t\varphi_t + \nabla u\nabla\varphi)\, dxd\tau - \tag{4.26}$$

$$-\int_{Q_t}\left(b\,|u|^{\sigma-2}u+f\right)\varphi dxd\tau-(u_t,\varphi)_\Omega|_0^t=0,\ \text{a.e. } t\in[0,T_{\max}).$$

But the space of the functions $\varphi\in P_m$ is dense in $C^\infty(Q_T)$. Then, the proof of Theorem 1 is complete.

Remark 2. *It should be noted that according to* (4.24), (4.25)

$$\|u(\cdot,t)-u_0(\cdot)\|_{2,\Omega}^2\le Ct,$$

and

$$(u_t(\cdot,t)-u_1(\cdot),\varphi(\cdot))_\Omega\to 0 \text{ as } t\to\infty,$$

for $\forall\ \varphi\in C^\infty(\Omega)$.

Theorem 2. ***(Local existence and uniqueness of strong solutions)*** *If in addition to Theorem 1*

$$\left(\|\nabla b\|_{\infty,\ \Omega},\|\nabla\sigma\|_{\infty,\ \Omega},\|b\|_{\infty,\ \Omega},\|\nabla f\|_{2,\ Q_T}\right)\le C<\infty \tag{4.27}$$

and

$$\nabla^2 u_0,\nabla u_1\in L^2(\Omega)), \tag{4.28}$$

then any weak solution possess the following properties

$$\nabla^2 u,\nabla u_t\in L^\infty(0,T;L^2(\Omega)), \tag{4.29}$$

$$u\in L^\infty(0,T;L^\infty(\Omega))\ if\ n<4,\ u\in L^\infty(0,T;L^{\frac{2n}{n-4}}(\Omega))\ if\ n>4\ . \tag{4.30}$$

This solution is unique if

$$\sigma^+\le 2+\frac{2}{(n-2)}\text{ if } n>2;\sigma^+<\infty \text{ if } n\le 2. \tag{4.31}$$

The idea of the proof of Theorem 2 consists in obtaining more high order a priori estimate for the approximations solutions in Galerkin Method.

Proof. Multiplying (4.16) by Δu_{kt} summing and omitting for simplicity the index m, and integrating over Ω we obtain

$$\frac{d}{2dt}\left(\int_\Omega|\nabla u_t|^2dx+\int_\Omega|\triangle u|^2dx\right)-\int_\Omega b(x,t)\,|u|^{\sigma-2}u\triangle u_tdx=\int_\Omega f\triangle u_tdx \tag{4.32}$$

Considering the expressions for

$$I_3=\int_\Omega f\triangle u_tdx=-\int_\Omega\nabla f\nabla u_tdx,$$

and

$$I_4=\int_\Omega b(x,t)\,|u|^{\sigma-2}u\triangle u_tdx.$$

Using the Cauchy-Schwarz inequality and (4.27) in I_3 we have

$$|I_3| = \left| \int_\Omega \nabla f \nabla u_t dx \right| \leq C \left(\int_\Omega |\nabla u_t|^2 \, dx \right)^{\frac{1}{2}}. \tag{4.33}$$

Applying Green's theorem in I_4, we obtain

$$I_4 := \int_\Omega b(x,t) \, |u|^{\sigma-2} \, u \bigtriangleup u_t dx = -\int_\Omega \nabla \left(b(x,t) \, |u|^{\sigma-2} \, u \right) \nabla u_t dx =$$
$$= -\int_\Omega \left(|u|^{\sigma-2} \, u \nabla b + b(x,t) \left(\nabla u \, |u|^{\sigma-2} + |u|^{\sigma-2} \, u \frac{1}{2} \ln u \nabla \sigma + |u|^{\sigma-2} \, (\sigma - 2) \nabla u \right) \right) \nabla u_t dx,$$

with $C = C(\|\nabla b\|_{\infty,\ \Omega}, \|\nabla \sigma\|_{\infty,\ \Omega}, \|b\|_{\infty,\ \Omega}, \|\sigma\|_{\infty,\ \Omega})$, and using Holder's inequality we get

$$|I_4| \leq C \left(\int_\Omega |\nabla u_t|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \left(|u|^{2\sigma-2} + |u|^{2\sigma-2} \ln^2 u + 2 \, |u|^{2\sigma-4} \, |\nabla u|^2 \right) dx \right)^{\frac{1}{2}},$$

furthermore, observing that

$$I_5 = \left(\int_\Omega |u|^{2\sigma-4} \, |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \leq \left(\int_\Omega |\nabla u|^{2q} \, dx \right)^{\frac{1}{2q}} \left(\int_\Omega |u|^{\frac{2\sigma-4}{q-1} q} \, dx \right)^{\frac{q-1}{2q}} \leq$$
$$\leq C \left(\int_\Omega |\nabla u|^{2q} \, dx \right)^{\frac{1}{2q}} \left(\int_\Omega |u|^{\frac{2\sigma^+-4}{q-1} q} \, dx + 1 \right)^{\frac{q-1}{2q}}.$$

Next we use the embedding inequalities

$$\left(\int_\Omega |\nabla u|^{2q} \, dx \right)^{\frac{1}{2q}} \leq C \left(\int_\Omega |\triangle u|^2 \, dx \right)^{\frac{1}{2}}, \text{ with } 2q \leq \frac{2n}{n-2}$$

$$\left(\int_\Omega |u|^{\frac{2\sigma^+-4}{q-1} q} \, dx \right)^{\frac{q-1}{2q}} \leq C \left(\int_\Omega |\nabla u|^{2q} \, dx \right)^{\frac{\sigma^+-2}{2q}} \leq C \left(\int_\Omega |\Delta u|^2 \, dx \right)^{\frac{\sigma^+-2}{2}}$$

if

$$\frac{2\sigma^+ - 4}{q-1} q \leq \frac{2qn}{n-2q} \Leftrightarrow \sigma^+ \leq 2 + \frac{n(q-1)}{n-2q} \text{ if } 2q < n \Leftrightarrow \sigma^+ \leq 2 + \frac{2}{n-4} \text{ if } n > 4,$$

$$\sigma^+ < \infty \text{ if } 2q \geq n \Leftrightarrow n \leq 4.$$

Finally we obtain

$$I_5 = \left(\int_\Omega |u|^{\frac{2\sigma^+-4}{q-1} q} \, |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \leq C \left(\int_\Omega |\triangle u|^2 \, dx \right)^{\frac{1}{2} + \frac{2\sigma^+-4}{4}}. \tag{4.34}$$

In analogous way applying the Young inequality and inequality $|u|^{2\sigma^+-2}\ln^2 u \leq C(\delta)u^{2\sigma^+-2+\delta}$, $\delta > 0$, we have

$$I_6 := \left(\int_\Omega \left(|u|^{2\sigma-2} + |u|^{2\sigma-2}\ln^2 u\right) dx\right)^{\frac{1}{2}} \leq$$

$$\leq C\left(\int_\Omega \left(|u|^{2\sigma^+-2} + |u|^{2\sigma^+-2}\ln^2 u\right) dx + 1\right)^{\frac{1}{2}} \tag{4.35}$$

$$\leq C\left[\left(\int_\Omega |\triangle u|^2\, dx\right)^{\frac{\sigma^+-1}{2}} + \left(\int_\Omega |\triangle u|^2\, dx\right)^{\frac{\sigma^+-1}{2}+\delta} + 1\right] \tag{4.36}$$

$$\leq C\left[\left(\int_\Omega |\triangle u|^2\, dx\right)^{\frac{\sigma^+-1}{2}+\delta} + 1\right],$$

with

$$\forall\delta > 0, \sigma^+ < 2 + \frac{8}{n-4} \text{ if } n > 4; \sigma^+ < \infty \text{ if } n \leq 4.$$

Introducing

$$Y(t) = \int_\Omega |\nabla u_t|^2\, dx + \int_\Omega |\triangle u|^2\, dx,$$

and substituting the estimates (4.33)-(4.35) in (4.32), we obtain the ordinary differential inequality (ODI)

$$\frac{dY}{dt} = \frac{d}{dt}\left(\int_\Omega |\nabla u_t|^2\, dx + \int_\Omega |\triangle u|^2\, dx\right) \leq C\left[Y^{\varkappa} + 1\right]$$

where

$$\varkappa = \max\left(\frac{\sigma^+-1}{2} + \delta, 1 + \frac{\sigma^+-2}{2}\right) > 1,\ \forall\delta > 0.$$

It implies

$$Y(t) = \int_\Omega |\nabla u_t|^2\, dx + \int_\Omega |\triangle u|^2\, dx \leq$$

$$\leq (1+Y_0)[1 - t(\varkappa-1)C(1+Y_0)^{\varkappa-1}]^{\frac{1}{\varkappa-1}} < \infty,$$

with

$$0 \leq t < T_{\max} = \frac{1}{(\varkappa-1)C(1+Y_0)^{\varkappa-1}}.$$

Combining (4.32)-(4.35), finally we come at estimate

$$Y^{(m)}(t) = \int_\Omega \left[\left|\nabla u_t^{(m)}\right|^2 dx + \left|\Delta u^{(m)}\right|^2\right] dx \leq M_1, \tag{4.37}$$

with constant M_1 independent of m. Last estimate is valid for $t \in [0, T_{\max})$. Now, proceeding in the same way as in Theorem 1, using the convergence (4.23), the estimate (4.37), it follows that we can pass to the limit in the approximate solutions and therefore, the proof of local existence strong solutions is complete.

We prove now the uniqueness of strong solutions. Let u and v be two strong solutions of problem (3.1)-(3.3), thus (u, v) satisfy (4.29)-(4.30), so all integrations below are justified and with the same initial data, in fact $(u - v)(0) = 0$ and $(u_t - v_t)(0) = 0$. Then, u and v satisfies the identities,

$$u_{tt} - \Delta u - b\,|u|^{\sigma-2}\,u = f,\ v_{tt} - \Delta v - b\,|v|^{\sigma-2}\,v = f \text{ in } Q_T.$$

By subtraction, we conclude that the function $w = u - v$ satisfies the equations

$$w_{tt} - \Delta w - b(\,|u|^{\sigma-2}\,u - |v|^{\sigma-2}\,v) = 0 \text{ in } Q_T. \tag{4.38}$$

Multiplying (4.38) by w_t, we arrive at the energy relation

$$\frac{d}{2dt}\left(\int_\Omega |w_t|^2\,dx + \int_\Omega |\nabla w|^2\,dx\right) - \int_\Omega b[(|u|^{\sigma-2}\,u - |v|^{\sigma-2}\,v)]w_t dx = 0.$$

Integrating over Ω, using the embedding theorems and the conditions (3.5)-(3.12) and (4.27), we obtain

$$\frac{d}{2dt}\left(\int_\Omega |w_t|^2\,dx + \int_\Omega |\nabla w|^2\,dx\right) \le b^+(\sigma^+ - 1)\int_\Omega (1 + |u| + |v|)^{\sigma^+-2} w w_t dx \le \tag{4.39}$$

$$\le C\left(\int_\Omega |w_t|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |w|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}}\left(\int_\Omega (1 + |u| + |v|)^{n(\sigma^+-2)} dx\right)^{\frac{1}{n}} \le$$

$$\le C\left(\int_\Omega |w_t|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega (1 + |u| + |v|)^{n(\sigma^+-2)} dx\right)^{\frac{1}{n}} \le$$

$$\le C\left(\int_\Omega |w_t|^2\,dx + \int_\Omega |\nabla w|^2\,dx\right)\left(\int_\Omega (1 + |u| + |v|)^{n(\sigma^+-2)} dx\right)^{\frac{1}{n}}$$

$$\le C\left(\int_\Omega |w_t|^2\,dx + \int_\Omega |\nabla w|^2\,dx\right).$$

Now, using the inequality (4.38) with initial conditions $w_t(x, 0) = \nabla w(x, 0) = 0$ we obtain $w(x, t) = 0$, that is, $u = v$. Thus, we conclude the proof of the uniqueness of strong solution. Therefore, the proof of Theorem 2 is complete.

Remark 3. *Repeating the arguments used in the proof of theorems 1 and 2, we can see that the solution of the problem (3.1)-(3.3) is global for all time $t \ge 0$ considering the function $\sigma(x, t) \le 2$.*

5. Blow up of solutions

5.1. Ordinary differential inequality

We will consider the following second ordinary differential inequality (ODI) for unknown function $\mu \geq 0$

$$\mu''(t) \geq h(t,\mu) := -\lambda\mu + \alpha_0 \min\left[\mu^{\sigma^-(t)-1}, \mu^{\sigma^+(t)-1}\right] \tag{5.40}$$

with positive constants λ and α_0. First we consider ODI, assuming that

$$2 < \sigma^- = \text{const} \leq \sigma(x,t) \leq \sigma^+ = \text{const} < \infty. \tag{5.41}$$

Lemma 1. *Let $\mu \geq 0$ satisfies to ODI* (5.39), *conditions* (5.40) *are valid and*

$$\mu(0) = a_0 > 0,\ \ \mu'(0) = a_1 > 0,\ \ h(\mu(0)) \geq h_0 > 0.$$

Then, there exists a finite time $T_{\max} < \infty$ such that

$$\sup_{0 \leq t \leq T\max} \mu(t) = +\infty.$$

Proof. It's very easy verify that $h(\mu)$ is an increasing function and $h(\mu) \geq 0$ for all $\mu \geq \mu_0$. It implies that $\mu(t) \geq a_0 t + a_1$ and

$$h(\mu) \geq \frac{\alpha_0}{2}\mu^{\sigma^+-1} \text{ for } t \geq t_0 = \max[\frac{1}{a_0}, \frac{1}{a_0}\left(\frac{2\lambda}{\alpha_0}\right)^{\frac{1}{\sigma^+-2}}].$$

Hence we arrive to ordinary differential inequality

$$\mu''(t) \geq \frac{\alpha_0}{2}\mu^{\sigma-1} \text{ for } t \geq t_0.$$

Multiplying last inequality by μ' and integrating we obtain

$$\left(\frac{\mu'^2(t)}{2}\right) \geq \frac{\alpha_0}{2\sigma^-}\mu^{\sigma^-} + C,\ \ \mu'(t) \geq \sqrt{2\left(\frac{\alpha_0}{2\sigma^-}\mu^{\sigma^-} + C\right)}.$$

It means that there exists a finite

$$\mu(t) \to \infty \text{ as } t \to T_{\max}.$$

This completes the proof of the Lemma 1.

Now we assume (in contrast with (5.41)) that

$$2 \leq \sigma^-(t) \leq \sigma(x,t) \leq \sigma^+(t) = \text{const} < \infty \tag{5.42}$$

and more exactly $\sigma^-(t) \searrow 2$ as $t \to \infty$. Let us assume that $0 \leq (\sigma^-(t) - 2) = \epsilon(t) \searrow 0$ and $\epsilon(t)$ is decreasing function such that

$$\epsilon(\tau) \leq \epsilon(t),\ \ \tau \geq t \geq 0. \tag{5.43}$$

Lemma 2. *Let* $\mu \geq 0$ *be a solution to ODI* (5.39). *Let conditions* (5.41), (5.42) *hold and*

$$\mu_0 = \mu(0) = a_0 \geq 1,\ \mu_0' = \mu'(0) = a_1 > 0,$$

$$h_0(0, \mu_0) \geq \lambda - \alpha_0 = C_1 > 0,$$

$$\int_{\mu_0}^{\infty} \left(\int_{\mu_0}^{\mu} s^{\epsilon(\ln\ \frac{s}{\gamma})+1} \right)^{\frac{1}{2}} ds d\mu < \infty.$$

Then, there exists a finite time $T_{\max} < \infty$ *such that*

$$\sup_{0 \leq t \leq T_{\max}} \mu(t) = \infty.$$

Proof. It's very easy verify that

$$\frac{\partial^2 h(t,\mu)}{\partial^2 \mu} \geq 0, \frac{\partial h(t,\mu)}{\partial \mu} \geq 0,\ 0 \leq \mu < \infty, \forall t \geq 0,$$

$$h(t,\mu) \geq (\lambda - \alpha_0)\mu_0 = C_0 > 0 \text{ if } \mu_0 \geq 1,\ \forall t \geq 0.$$

Last relations permit us conclude that $h(t,\mu) \geq C_1 \mu$ for sufficient large μ. Let us consider ordinary differential inequality

$$\mu''(t) \geq C_1 \mu^{\sigma^-(t)-1},\ \mu \geq 1, C_1 > 0,\ \sigma^-(t) \geq 2. \tag{5.44}$$

letting later without loss of generality that $C_1 = 1$. Then integrating inequality

$$\mu''(t) \geq \mu^{\sigma^-(t)-1} \geq \mu,$$

with initial conditions

$$\mu(0) = \mu_0,\ \mu^{'}(0) = \mu_0^{'},$$

$$\mu(t) \geq \frac{\mu_0 + \mu_0^{'}}{2} e^t + \frac{\mu_0 - \mu_0^{'}}{2} e^{-t} \geq \frac{\mu_0}{2} e^t = \gamma e^t,$$

we obtain or

$$\ln\ \frac{\mu(t)}{\gamma} \geq t. \tag{5.45}$$

Then according to (5.43), (5.45) we can to write

$$\epsilon(\ln\ \frac{\mu(t)}{\gamma}) \leq \epsilon(t),$$

and (5.44) transforms to the autonomous ODI in the form

$$\mu'' \geq \mu^{\epsilon(\ln\ \frac{\mu}{\gamma})+1}.$$

Multiplying by μ' and integrating we have

$$\frac{\mu'^2}{2} \geq \int_{\mu_0}^{\mu} s^{\epsilon(\ln\ \frac{s}{\gamma})+1} ds + C.$$

Solutions of last ODI blow up if

$$\int_{\mu_0}^{\infty} \frac{d\mu}{\sqrt{\int_{\mu_0}^{\mu} s^{\epsilon(\ln \ \frac{s}{\gamma})+1} ds}} < \infty. \tag{5.46}$$

This completes the proof of the Lemma 2.

Remark 4. *Let us construct an explicit example of the function* $\epsilon(t) = (\sigma^-(t) - 2) \geq 0$ *which guarantees the realization (5.45). We apply the next simplest criterion of convergence for last integral (5.45)*

$$\frac{1}{\sqrt{\int_{\mu_0}^{\mu} s^{\epsilon(\ln \ \frac{s}{\gamma})+1} ds}} \leq \frac{C}{\mu \ln^{1+\delta} \mu}, \ \ \delta > 0.$$

Last inequality implies the next chain relations

$$\mu^2 \ln^{2(1+\delta)} \mu \leq C^2 \int_0^{\mu} s^{\epsilon(\ln \ \frac{s}{\gamma})+1} ds \to 2 \ln^{2(1+\delta)} \mu + 2(1+\delta) \ln^{1+2\delta} \mu \leq C^2 \mu^{\epsilon(\ln \ \frac{\mu}{\gamma})},$$

$$C^{-2}(2 \ln^{2(1+\delta)} \mu + 2(1+\delta) \ln^{1+2\delta} \mu) \leq \mu^{\epsilon(\ln \ \frac{\mu}{\gamma})},$$

$$\ln \left\{ C^{-2}(2 \ln^{2(1+\delta)} \mu + 2(1+\delta) \ln^{1+2\delta} \mu) \right\} \leq \epsilon(\ln \ \frac{\mu}{\gamma}) \ln \mu.$$

Assuming for simplicity $\gamma = 1, \ 2C^{-2}(2+\delta) = 1$ *and setting* $\ln \mu = t$, *we arrive to a lower estimate for* $\epsilon(t)$

$$\epsilon(t) = \sigma^-(t) - 2 \geq \frac{\ln t^{2(1+\delta)}}{t}, \ \delta > 0, \ t >> 1$$

which guarantees blow up effect in a finite time.

5.2. Blow up

In this section we consider the non-negative solutions of the problem (3.1)-(3.3), which we take the equation (3.1) in the form

$$u_{tt} = \Delta u + b(x,t) u^{\sigma(x,t)-1}.$$

We will use the Kaplan method of eigenvalue functions. Let $\lambda > 0$ be the first eigenvalue and let nonnegative $\varphi \geq 0$ be the corresponding eigenfunction of the problem

$$-\Delta \varphi = \lambda \varphi \text{ in } \Omega, \ \ u|_{\Gamma} = 0, \ \text{ on } \Gamma = \partial\Omega.$$

We can choose the eigenfunctions φ nonnegative on Ω such that

$$\int_{\Omega} \varphi dx = 1.$$

We introduce functions

$$\mu(t) = \int_\Omega u(\cdot,t)\varphi(\cdot)\ dx,$$

and

$$\alpha(t) = \left(\int_\Omega \left(b(\cdot,t)\varphi^{2-\sigma(\cdot,t)}(\cdot)\right)^{\frac{1}{2-\sigma^-(t)}} dx\right)^{2-\sigma^-(t)} \geq \alpha_0 = const > 0. \quad (5.47)$$

Now, we are in a position to state our results on blow up.

Theorem 3. *Let u be a nonnegative strong solution to problem (3.1)-(3.3). Let conditions* (3.6), (5.40), *(5.47) hold and*

$$\mu(0) = \int_\Omega u_0(x)\phi(x)dx = a_0 > 0,$$

$$\mu'(0) = \int_\Omega u_1(x)\phi(x)dx = a_1 > 0,$$

$$h(\mu(0)) \geq h_0 > 0.$$

Then, there exists a finite time $T_{\max} < \infty$ such that

$$\sup_{0\leq t\leq T\max} \|u(x,t)\|_{\infty,\ \Omega} = +\infty.$$

Proof.
Applying the Lebesgue differentiation theorem, Green's formula and then replaced $\Delta\varphi$ by $-\lambda\varphi$ we find

$$\mu''(t) = \int_\Omega u_{tt}(\cdot,t)\varphi(\cdot)dx = \int_\Omega u\Delta\varphi dx + \int_\Omega bu^{\sigma(x,t)-1}\varphi dx = \quad (5.48)$$

$$= -\lambda\mu + \int_\Omega b(\cdot,t)u^{\sigma(\cdot,t)-1}(\cdot,t)\varphi(\cdot)dx.$$

Applying inverse Hölder's inequality

$$\int_\Omega |u||v|dx \geq \left(\int_\Omega |u|^q dx\right)^{\frac{1}{q}} \left(\int_\Omega |v|^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}},\ q \in (0,1)$$

with $q = \frac{1}{\sigma^-(t)-1}$, and $w = u^{\sigma^-(t)-1}\varphi^{\sigma^-(t)-1}$, $v = b\varphi^{2-\sigma^-(t)}$ and (2.14), we evaluate the last term in (5.48) from below in the following way

$$I = \int_\Omega bu^{\sigma-1}\varphi dx = \int_\Omega (u\varphi)^{\sigma-1} b\varphi^{2-\sigma} dx \geq$$

$$\geq \left(\int_\Omega (u\varphi)^{\frac{\sigma-1}{\sigma^-(t)-1}}\, dx\right)^{\sigma^+-1} \left(\int_\Omega \left(b\varphi^{2-\sigma}\right)^{\frac{1}{2-\sigma^-(t)}}\, dx\right)^{2-\sigma^-(t)} \geq$$

$$\geq \min\left[\|u\varphi\|^{\sigma^-(t)-1}_{\frac{\sigma-1}{\sigma^-(t)-1}}, \|u\varphi\|^{\sigma^-(t)-1}_{\frac{\sigma-1}{\sigma^-(t)-1}}\right] \left(\int_\Omega \left(b\varphi^{2-\sigma}\right)^{\frac{1}{2-\sigma^-(t)}}\, dx\right)^{2-\sigma^-(t)} \geq$$

$$\geq \min\left[\|u\varphi\|_1^{\sigma^-(t)-1}, \|u\varphi\|_1^{\sigma^+(t)-1}\right] \alpha(t) \geq$$

$$\geq \alpha_0(t) \min\left[\mu^{\sigma^+(t)-1}, \mu^{\sigma^+(t)-1}\right].$$

Here were used inequalities (see (2.14))

$$\min\left[\|u\varphi\|^{\sigma^-(t)-1}_{\frac{\sigma-1}{\sigma^-(t)-1}}, \|u\varphi\|^{\sigma^+(t)-1}_{\frac{\sigma-1}{\sigma^-(t)-1}}\right] \leq \left(\int_\Omega (u\varphi)^{\frac{\sigma-1}{\sigma^-(t)-1}}\, dx\right)^{\sigma^-(t)-1},$$

$$\|u\varphi\|_{1,\Omega} \leq \|1\|_{\frac{\sigma-1}{\sigma-\sigma^-(t)},\Omega} \|u\varphi\|_{\frac{\sigma-1}{\sigma^-(t)-1},\Omega} \leq C\, \|u\varphi\|_{\frac{\sigma-1}{\sigma^-(t)-1},\Omega}\ ,\ C = C(|\Omega|, \sigma^\pm).$$

Finally we come at the ordinary differential inequality

$$\mu''(t) \geq -\lambda\mu + \alpha(t) \min\left[\mu^{\sigma^-(t)-1}, \mu^{\sigma^+(t)-1}\right] := h(t,\mu),$$

with constant parameter λ. It's very easy verify that

$$h(t,\mu) \geq h(\mu) := -\lambda\mu + \alpha_0 \min\left[\mu^{\sigma^--1}, \mu^{\sigma^+-1}\right]$$

and in this case we arrive to ordinary differential inequality (5.39)

$$\mu''(t) \geq -\lambda\mu + \alpha_0 \min\left[\mu^{\sigma^--1}, \mu^{\sigma^+-1}\right] = h(\mu),$$

with constants parameters $\lambda,\ \alpha_0, \sigma^-, \sigma^+$. Next it's enough to apply the Lemma 1.

Theorem 4. *Let u be a nonnegative solution to problem (3.1)-(3.3). Let conditions (3.6), (5.47), (5.43) hold and*

$$\mu_0 = \mu(0) = \int_\Omega u_0(x)\phi(x)dx = a_0 \geq 1,$$

$$\mu_0' = \mu'(0) = \int_\Omega u_1(x)\phi(x)dx = a_1 > 0,$$

$$h_0(0,\mu_0) \geq \lambda - \alpha_0 = C_1 > 0,$$

$$\int_{\mu_0}^\infty \frac{d\mu}{\sqrt{\int_{\mu_0}^\mu s^{\epsilon(\ln\ \frac{s}{\gamma})+1} ds}} < \infty.$$

Then, there exists a finite time $T_{\max} < \infty$ such that

$$\sup_{0\leq t\leq T_{\max}} \|u(x,t)\|_{\infty,\ \Omega} = \infty.$$

PROOF: Applying the arguments of previous Theorem we arrive at ODI (5.40)

$$\mu''(t) \geq -\lambda\mu + \alpha_0 \min\left[\mu^{\sigma^-(t)-1}, \mu^{\sigma^+(t)-1}\right] := h_0(t,\mu)$$

with constant parameters $\lambda,\ \alpha_0$ and variable exponents $\sigma^{\pm}(t)$. Next we can to apply the Lemma 2

Remark 5. *It is should be noted that the condition (5.47) can be satisfied and for functions $b(x,t) \geq 0$. Let's assume for simplicity that the eigenvalue function $\varphi(x,t) \leq M < \infty$ and*

$$\left(\int\limits_{\Omega} \left(\frac{1}{b(\cdot,t)}\right)^{\frac{1}{\sigma^-(t)-2}} dx\right)^{\sigma^-(t)-2} \leq C_b < \infty.$$

In fact in this case we have

$$\alpha(t) = \left(\int\limits_{\Omega} \left(b(\cdot,t)\varphi^{2-\sigma(\cdot,t)}(\cdot)\right)^{\frac{1}{2-\sigma^-(t)}} dx\right)^{2-\sigma^-(t)} \geq$$

$$\geq \frac{1}{C_b \sup_{x,t} M^{\sigma(x,t)-2}} = \alpha_0 > 0.$$

5.2.1. Regional blow up.

The conclusions about the blow-up of solutions of problem (3.1)-(3.3) remain true if instead of the whole domain Ω we restrict the study to a subdomain. Let us assume that there exists a subdomain $\Omega' = D,\ \overline{D} \subset \Omega$, *meas* $\Omega > 0$, such that there exists nonnegative eigenvalue λ and nonnegative eigenvalue $\varphi \geq 0$, satisfying

$$-\Delta\varphi = \lambda\varphi \text{ in } D,\ \varphi = 0, \text{ on } \partial D.$$

Let us introduce the function

$$\mu(t) = \int\limits_{D} u\varphi dx$$

Let us multiply the equation (3.1) by de function φ and integration over D and and using the last relation, we obtain

$$\mu''(t) = \int\limits_{D} u_{tt}\varphi dx = \int\limits_{D} u\Delta\varphi dx - \int\limits_{\partial D} u\frac{\partial\varphi}{\partial n}dx + \int\limits_{D} bu^{\sigma(x,t)}\varphi dx$$

$$= -\lambda\mu - \int\limits_{\partial D} udx + \int\limits_{D} bu^{\sigma(x,t)}\varphi dx,$$

where $\overrightarrow{n}$ denotes the outward normal to ∂D. Taking into account the properties $0 \le u(x,t)$ and

$$\varphi \ge 0 \text{ in } D, \ \ \varphi|_\Gamma = 0 \to \left.\frac{\partial \varphi}{\partial n}\right|_{\partial D} \le 0,$$

we have that

$$-\int\limits_{\partial D} u \frac{\partial \varphi}{\partial n} dx \ge 0.$$

The differential inequality for $\mu(t)$ takes on the form

$$\mu''(t) \ge -\lambda\mu + \int\limits_D b u^{\sigma(x,t)} \varphi dx.$$

Next we can repeat the above-mentioned arguments for the domain D.

6. Quasilinear equations of general form

Let us consider a nonnegative solution $u(x,t)$ of the problem

$$\left\{ \begin{array}{l} u_{tt} = Lu + f(x,t,u) \quad \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \ u_t(x,0) = u_1(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{array} \right. \tag{6.49}$$

where L is the linear elliptic operator

$$Lu = D_i \left(a_{ij}(x) D_j u + a_i(x) u\right) + a_0(x) u, \text{ where } D_j = \frac{\partial}{\partial x_j},$$

and the right-hand side has the form

$$f(x,t,u) = \sum_{k=1}^{N} b_k(x,t) u^{\sigma_k(x,t)-1} + \sum_{i=N}^{K} c_i(x,t) \int_\Omega d_i(s,t) u^{\sigma_i(s,t)-1} ds,$$

with $b_k \ge 0$, $c_i \ge 0$, $d_i \ge 0$. Let us denote by L^* the conjugate operator

$$\left\{ \begin{array}{l} L^*\phi \equiv -D_j \left(a_{ij}(x) D_i \phi\right) - a_i(x) D_i \phi + a_0(x)\phi, \\ (u, L^*\phi)_\Omega = (\phi, Lu)_\Omega \quad \forall\, u,\, \phi \in W_0^{1,2}(\Omega), \end{array} \right.$$

and assume that there exist a nonnegative function $\phi \ge 0$ and a positive number $\lambda > 0$ such that

$$L^*\phi = -\lambda\phi \text{ in } \Omega, \qquad \phi = 0 \text{ on } \Gamma.$$

Multiplying (6.48) by u and integrating over Ω we arrive at the relation

$$\mu''(t) = \int\limits_D u_{tt} \varphi dx = -\lambda\mu + I_1 + I_2,$$

where

$$I_1 = \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k(x,t)-1} \right) \phi \, dx,$$

$$I_2 = \int_\Omega \left(\sum_{i=N}^{K} c_i(x,t) \int_\Omega d_i(s,t) u^{\sigma_i(s,t)-1} ds \right) \phi \, dx.$$

I_1, I_2 are estimated from below in the following way:

$$I_1 \geq \left(\sum_{k=1}^{N} \int_\Omega \left(b_k \phi^{2-\sigma_k} \right)^{\frac{1}{2-\sigma_k^-}} dx \right)^{2-\sigma_k^-} \min \left[\mu^{\sigma_k^-(t)-1}, \mu^{\sigma_k^+(t)-1} \right] =$$
$$= \alpha_k(t) \min \left[\mu^{\sigma_k^-(t)-1}, \mu^{\sigma_k^+(t)-1} \right], \; k = 1, ..., N,$$

and

$$I_2 \geq \sum_{i=N+1}^{K} \int_\Omega c_i(x,t) \phi(x,t) \min \left[\mu^{\sigma_k^-(t)-1}, \mu^{\sigma_k^+(t)-1} \right] \left(\int_\Omega d_i^{\frac{1}{2-\sigma_i^-(t)}}(s,t) \phi^{\frac{\sigma_i-1}{\sigma_i-2}}(s,t) ds \right)^{2-\sigma_i^-} dx =$$

$$= \sum_{i=N+1}^{Q} \alpha_i(t) \min \left[\mu^{\sigma_k^-(t)-1}, \mu^{\sigma_k^+(t)-1} \right], k = N+1, \ldots, K.$$

Gathering these formulas we arrive at the nonlinear ODI of the type

$$\mu''(t) \geq -\lambda \mu + \sum_{i=1}^{K} \alpha_i(t) \min \left[\mu^{\sigma_k^-(t)-1}, \mu^{\sigma_k^+(t)-1} \right].$$

Now, repeating the same argument used in theorems 3 and 4, we can state the main theorem of this section.

Assume that $b_k(x,t)$, $\sigma_k(x,t)$, $c_i(x,t)$ and $d_i(x,t), \sigma_i(x,t)$, coefficients and exponents the function $f(x,t,u)$ satisfies the conditions

and

$$0 \leq b^-(t) \leq b_k(x,t) \leq b^+(t) < \infty, \tag{6.50}$$

$$2 < \sigma_k^- \leq \sigma_k(x,t) \leq \sigma_k^+ < \infty, \; for \; k = 1, ..., N, \tag{6.51}$$

$$0 \leq c^-(t) \leq c_i(x,t) \leq c^+(t) < \infty, \tag{6.52}$$

$$0 \leq d^-(t) \leq d_i(x,t) \leq d^+(t) < \infty, \tag{6.53}$$

$$2 < \sigma_i^- \leq \sigma_i(x,t) \leq \sigma_i^+ < \infty, \; for \; i = N, ..., k. \tag{6.54}$$

$$\alpha_k(t) = \left(\int_\Omega \left(b_k(\cdot,t) \varphi^{2-\sigma_k^-(\cdot,t)}(\cdot) \right)^{\frac{1}{2-\sigma_k^-(t)}} dx \right)^{2-\sigma_k^-(t)} \geq \alpha_0 = const > 0. \tag{6.55}$$

$$\alpha_i(t) = \left(\int_\Omega d_i^{\overline{\frac{1}{2-\sigma_i^-(t)}}}(s,t) \phi^{\frac{\sigma_i-1}{\sigma_i-2}}(s,t) ds \right)^{2-\sigma_i^-} dx \geq \alpha_1 = const > 0. \qquad (6.56)$$

Theorem 5. *Let u be a nonnegative strong solution to problem (6.49), with $b_k(x,t)$, $\sigma_k(x,t), c_i(x,t), d_i(x,t), \sigma_i(x,t), \alpha_k(t)$ and $\alpha_i(t)$ satisfying the conditions (6.50)-(6.56). Then, there exists a finite time $T_{\max} < \infty$ such that*

$$\sup_{0 \leq t \leq T \max} \|u(x,t)\|_{\infty,\ \Omega} = +\infty.$$

Remark 6. *The above arguments are applicable to equations of higher order. Let us consider the problem*

$$\begin{cases} u_{tt} = Lu + f(x,t,u) \quad \text{in } Q_T, \\ B_k u = 0 \text{ on } \Gamma_T,\ k = 1, \ldots, m, \\ u(x,0) = u_0(x) \text{ in } \Omega, \end{cases}$$

where L is an operator of order $2m$, with the coefficients independent of t and such that eigenvalue problem for the conjugate operator

$$L^* \phi = \lambda \phi \text{ in } \Omega, \qquad B_k^* \phi = 0 \text{ on } \Gamma,\ k = 1, \ldots, m$$

has a nonnegative solution ϕ with positive λ. Proceeding in the standard way we reduce the study of sufficient conditions of the blow-up behavior to the analysis of the ordinary differential inequality for the function $\mu(t) = \int_\Omega u(x,t)\, \phi(x)\, dx$.

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