FINITE-TIME BLOWUP FOR A COMPLEX GINZBURG-LANDAU EQUATION WITH LINEAR DRIVING

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Abstract. In this paper, we consider the complex Ginzburg–Landau equation $u_t = e^{i\theta}[\Delta u + |u|^\alpha u] + \gamma u$ on $\mathbb{R}^N$, where $\alpha > 0$, $\gamma \in \mathbb{R}$ and $-\pi/2 < \theta < \pi/2$. By convexity arguments we prove that, under certain conditions on α, θ, γ, a class of solutions with negative initial energy blows up in finite time.

1. Introduction

In this article, we consider the complex Ginzburg-Landau equation

$$\begin{cases} u_t = e^{i\theta}[\Delta u + |u|^\alpha u] + \gamma u, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

in $\mathbb{R}^N$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\alpha > 0$ and $\gamma \in \mathbb{R}$[1], and we look for conditions on the initial value u_0 and the parameters θ, α and γ that ensure finite-time blowup of the solution. Equation (1.1) is a particular case of the more general complex Ginzburg–Landau equation

$$u_t = e^{i\theta}\Delta u + e^{i\phi}|u|^\alpha u + \gamma u, \tag{1.2}$$

which is used to model such phenomena as superconductivity, chemical turbulence, and various types of fluid flows; see [2] and the references cited therein. Note that the solutions of equation (1.1) satisfy certain energy identities (see Section 2), which are not shared by the solutions of (1.2).

Equation (1.1) with $\theta = 0$ is nonlinear heat equation

$$\begin{cases} u_t - \Delta u = |u|^\alpha u + \gamma u, \\ u(0) = u_0, \end{cases} \tag{1.3}$$

while for $\theta = \pm\pi/2$, (1.1) reduces to the nonlinear Schrödinger equation

$$\begin{cases} u_t = \pm i(\Delta u + |u|^\alpha u) + \gamma u, \\ u(0) = u_0. \end{cases} \tag{1.4}$$

In particular, (1.1) is "intermediate" between the nonlinear heat and Schrödinger equations.

We recall (see Section 2) that the Cauchy problem (1.1) is locally well-posed in $H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, where $C_0(\mathbb{R}^N)$ is the space of continuous functions $\mathbb{R}^N \to \mathbb{C}$ which vanish at infinity, equipped with the sup norm. In particular, given any $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, there exists a unique solution u of (1.1) defined on a

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[1]In principle, one could let γ be complex, but the imaginary part can be eliminated by the change of variable $v(t,x) = e^{-it\Im\gamma}u(t,x)$.

maximal interval $[0, T_{\max})$, i.e., $u \in C([0, T_{\max}), H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N))$. If the maximal existence time $T_{\max}$ is finite, then the solution blows up at $T_{\max}$ in $C_0(\mathbb{R}^N)$.

The effect of the driving term γu can be easily seen on the ODE associated with (1.1), i.e.,

$$v' = e^{i\theta} |v|^\alpha v + \gamma v. \tag{1.5}$$

The solution of (1.5) with the initial condition $v(0) = v_0 \in \mathbb{C}$, is given by

$$v(t) = e^{\gamma t} \Bigl[1 - \frac{e^{\alpha \gamma t} - 1}{\gamma} |v_0|^\alpha \cos\theta \Bigr]^{-\frac{1}{\alpha}(1 + i \tan\theta)} v_0, \tag{1.6}$$

as long as this formula makes sense. (The term $\frac{e^{\alpha\gamma t} - 1}{\gamma}$ must be replaced by αt if $\gamma = 0$.) If $\gamma = 0$, then we see that for every $v_0 \neq 0$, the solution blows up in finite time. The same conclusion holds if $\gamma > 0$. On the other hand, when $\gamma < 0$, whether or not v blows up depends on the size of $|v_0|$. More precisely, if $|v_0| > \frac{-\gamma}{\cos\theta}$, then v blows up in finite time, whereas if $|v_0| \le \frac{-\gamma}{\cos\theta}$, then v is global.

When $\gamma = 0$, finite-time blowup for equation (1.1) is known to occur under a negative energy condition. More precisely, let the energy E be defined by

$$E(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 - \frac{1}{\alpha + 2} \int_{\mathbb{R}^N} |w|^{\alpha + 2}, \tag{1.7}$$

for $w \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. If $\theta = 0$, then it follows from Levine [4] that the solution of the nonlinear heat equation (1.3) blows up in finite time if $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ satisfies $E(u_0) < 0$. For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, negative energy initial values also yield finite-time blowup for the equation (1.1), see [1]. If $\theta = \pm\frac{\pi}{2}$, then the solution of the nonlinear Schrödinger equation (1.4) blows up in finite time provided $\frac{4}{N} \le \alpha < \frac{4}{N-2}$ and the initial value $u_0 \in H^1(\mathbb{R}^N)$ satisfies $E(u_0) < 0$ and $|\cdot| u_0 \in L^2(\mathbb{R}^N)$. (See Zakharov [11] and Glassey [3].)

If $\gamma > 0$, obvious modifications of the arguments used when $\gamma = 0$ provide similar results. In particular, if the initial value $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ satisfies $E(u_0) < 0$, then the corresponding solution of (1.3) blows up in finite time. Moreover, if $\frac{4}{N} \le \alpha < \frac{4}{N-2}$ and the initial value $u_0 \in H^1(\mathbb{R}^N)$ satisfies $E(u_0) < 0$ and $|\cdot| u_0 \in L^2(\mathbb{R}^N)$, then the solution of (1.4) blows up in finite time. The situation is similar for general $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, and a simple modification of the argument of [1] shows finite-time blowup for initial values with negative energy. More precisely, we have the following result.

Theorem 1.1. *Assume*

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \tag{1.8}$$

$\alpha > 0$ *and* $\gamma > 0$. *Let* $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ *and let* $u \in C([0, T_{\max}), H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N))$ *be the corresponding maximal solution of* (1.1). *If* $E(u_0) < 0$, *where* E *is defined by* (1.7), *then* u *blows up in finite time, i.e.,* $T_{\max} < \infty$.

When $\gamma < 0$, the situation is more delicate. For the nonlinear heat equation (1.3), Levine's calculations [4] can be adapted in order to show that if the initial value $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ satisfies $E_\gamma(u_0) < 0$, where

$$E_\gamma(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 - \frac{1}{\alpha + 2} \int_{\mathbb{R}^N} |w|^{\alpha + 2} - \frac{\gamma}{2} \int_{\mathbb{R}^N} |w|^2,$$

then the solution of (1.3) blows up in finite time. (See [8, Theorem 17.6].) For the nonlinear Schrödinger equation (1.4), Glassey's proof [3] is not immediately applicable. Sufficient conditions for finite-time blowup were obtained by M. Tsutsumi [9] (see also [7]) by a delicate modification of the variance argument of [3]. It follows in particular from the calculations in [9, 7] that if $\frac{4}{N} < \alpha < \frac{4}{N-2}$ and the initial value $u_0 \in H^1(\mathbb{R}^N)$ satisfies

$$E(u_0) + \frac{\alpha\gamma}{N\alpha - 4} \mathrm{Im} \int_{\mathbb{R}^N} u_0 (x \cdot \nabla \overline{u_0}) + \frac{\alpha^2\gamma^2}{(N\alpha-4)^2} \int_{\mathbb{R}^N} |x|^2 |u_0|^2 < 0,$$

then the solution of (1.4) blows up in finite time. Note that the above condition becomes stronger and stronger as $\alpha \downarrow \frac{4}{N}$, and that no energy-type sufficient condition is known for blowup if $\alpha = \frac{4}{N}$. (The case $\alpha = \frac{4}{N}$ is studied in [6] by a very different method.)

For the equation (1.1) with $\gamma < 0$, we have the following result.

Theorem 1.2. *Assume*

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}, \tag{1.9}$$

$\alpha > 0$ and $\gamma < 0$. Let $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ and let $u \in C([0, T_{\max}), H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N))$ be the corresponding maximal solution of (1.1). Suppose further that

$$(\alpha + 2)\cos(2\theta) + 2(1 - \cos\theta) \ge 2\cos\theta. \tag{1.10}$$

If

$$E(u_0) + \xi \int_{\mathbb{R}^N} |u_0|^2 < 0, \tag{1.11}$$

where E is defined by (1.7) and

$$\xi = -\frac{\gamma}{\cos\theta} \max\Big\{ \frac{1}{\alpha}, \frac{(\alpha + 2)\cos(2\theta) + 2(1 - \cos\theta)}{2} \Big\}, \tag{1.12}$$

then u blows up in finite time, i.e., $T_{\max} < \infty$.

Theorem 1.2 calls for several comments. First, assume α and θ satisfy (1.9)-(1.10) and let $\psi \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $\psi \neq 0$. It follows that $u_0 = \kappa\psi$, with $k \in \mathbb{C}$, satisfies (1.11) provided $|\kappa|$ is sufficiently large.

Next, assumption (1.9) means that equation (1.1) is not (formally) close to the nonlinear Schrödinger equation (1.4). Assuming (1.9), we see that (1.10) is satisfied for a fixed θ if α is sufficiently large. Alternatively, (1.10) is satisfied for a fixed $\alpha > 0$ if $|\theta|$ is sufficiently small.

The assumptions of Theorem 1.2 are probably not optimal, since letting $\gamma \downarrow 0$ yields the "natural" condition $E(u_0) < 0$, but also the structural conditions (1.9)-(1.10). In particular, Theorem 1.2 does not include the result of [1]. On the other hand, note that if $\gamma < 0$, then there does not exist any map $F : H^1 \cap C_0 \to \mathbb{R}$ such that if $F(u_0) < 0$, then the solution of (1.1) blows up in finite time for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. (At least if $\alpha < 4/N$.) Indeed, given any $\alpha < 4/N$ and any $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, it follows from Remark 2.6 that the solution of (1.1) is global provided θ is sufficiently close to $\pm\frac{\pi}{2}$. This is in sharp contrast with the case $\gamma = 0$, where negative energy yields finite-time blowup for every $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Note that finite-time blowup of certain solutions (in L^∞, not necessarily of finite energy) of (1.2) is proved in [5] under the structural assumptions $-\frac{\pi}{2} \le \theta, \phi \le \frac{\pi}{2}$ and $\tan^2\phi + (\alpha+2)\tan\theta\tan\phi < \alpha+1$. For the equation (1.1), the last assumption reduces to $\tan^2\theta < \frac{\alpha+1}{\alpha+3}$, i.e.,

$$(\alpha + 2)\cos^2\theta > \frac{\alpha + 3}{2}. \tag{1.13}$$

(Note that $\tan^2\theta < 1$ so that in particular θ satisfies (1.9).) On the other hand, condition (1.10) is equivalent to

$$(\alpha + 2)\cos^2\theta \ge \frac{\alpha}{2} + 2\cos\theta. \tag{1.14}$$

Conditions (1.13) and (1.14) are not comparable. In particular, condition (1.13) is stronger if θ is close to $\pm\frac{\pi}{4}$, whereas condition (1.14) is stronger if θ is close to 0.

The rest of this paper is organized as follows. In Section 2, we recall some simple properties of the Cauchy problem (1.1). In Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2.

2. The Cauchy problem

Assume (1.8). It is well known that the operator $e^{i\theta}\Delta$ with domain $H^2(\mathbb{R}^N)$ generates an analytic semigroup of contractions $(\boldsymbol{\mathcal{T}}_\theta(t))_{t\geq 0}$ on $L^2(\mathbb{R}^N)$. Moreover,

$$\|\boldsymbol{\mathcal{T}}_\theta(t)\psi\|_{L^r} \leq (\cos\theta)^{-\frac{N}{2}(1-\frac{1}{p}+\frac{1}{r})} t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})}\|\psi\|_{L^p}, \tag{2.1}$$

for $1\leq p\leq r\leq\infty$ and θ satisfying (1.8) and $(\boldsymbol{\mathcal{T}}_\theta(t))_{t\geq 0}$ is a bounded C_0 semigroup on $L^p(\mathbb{R}^N)$ for $1\leq p<\infty$ and on $C_0(\mathbb{R}^N)$. Moreover, equation (1.1) can be written in the equivalent integral form

$$u(t) = \boldsymbol{\mathcal{T}}_\theta(t)u_0 + \int_0^t \boldsymbol{\mathcal{T}}_\theta(t-s)[e^{i\theta}|u(s)|^\alpha u(s) + \gamma u(s)]\,ds. \tag{2.2}$$

It is immediate by applying a contraction mapping argument to (2.2) that the Cauchy problem (1.1) is locally well posed in $C_0(\mathbb{R}^N)$. Moreover, it is easy to see using the estimates (2.1) that $C_0(\mathbb{R}^N)\cap H^1(\mathbb{R}^N)$ is preserved under the action of (1.1). More precisely, we have the following result.

Proposition 2.1. *Suppose (1.8), $\alpha>0$ and $\gamma\in\mathbb{R}$. Given any $u_0\in C_0(\mathbb{R}^N)\cap H^1(\mathbb{R}^N)$, there exist $T>0$ and a unique function $u\in C([0,T],C_0(\mathbb{R}^N)\cap H^1(\mathbb{R}^N))\cap C((0,T),H^2(\mathbb{R}^N))\cap C^1((0,T),L^2(\mathbb{R}^N))$ which satisfies (1.1) for all $t\in(0,T)$ and such that $u(0)=u_0$. Moreover, u can be extended to a maximal interval $[0,T_{\max})$, and if $T_{\max}<\infty$, then $\|u(t)\|_{L^\infty}\to\infty$ as $t\uparrow T_{\max}$.*

Remark 2.2. Let $u_0\in C_0(\mathbb{R}^N)\cap H^1(\mathbb{R}^N)$ and u the corresponding solution of (1.1) defined on the maximal interval $[0,T_{\max})$, and given by Proposition 2.1. If, in addition, $\alpha<4/N$, then (1.1) is locally well posed in $L^2(\mathbb{R}^N)$ (see [10]). It is not difficult to show using the estimates (2.1) that the maximal existence times in $H^1(\mathbb{R}^N)\cap C_0(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ are the same; and so if $T_{\max}<\infty$, then $\|u(t)\|_{L^2}\to\infty$ as $t\uparrow T_{\max}$.

We collect below the energy identities that we use in the next sections.

Proposition 2.3. *Suppose (1.8), $\alpha>0$ and $\gamma\in\mathbb{R}$. If $u_0\in C_0(\mathbb{R}^N)\cap H^1(\mathbb{R}^N)$ and and u is the corresponding solution of (1.1) defined on the maximal interval $[0,T_{\max})$, then the following properties hold.*

(i) *Set*

$$I(w) = \int_{\mathbb{R}^N}|\nabla w|^2 - \int_{\mathbb{R}^N}|w|^{\alpha+2}, \tag{2.3}$$

for $w\in C_0(\mathbb{R}^N)\cap H^1(\mathbb{R}^N)$. It follows that

$$\int_{\mathbb{R}^N}\overline{u}u_t = \gamma\int_{\mathbb{R}^N}|u|^2 - e^{i\theta}I(u). \tag{2.4}$$

In particular,

$$|I(u)| = \Big|\int_{\mathbb{R}^N}\overline{u}u_t - \gamma\int_{\mathbb{R}^N}|u|^2\Big|, \tag{2.5}$$

$$|I(u(t))|^2 = \Big|\int_{\mathbb{R}^N}\overline{u}u_t\Big|^2 + \gamma^2\Big(\int_{\mathbb{R}^N}|u|^2\Big)^2 - \gamma\Big(\int_{\mathbb{R}^N}|u|^2\Big)\frac{d}{dt}\int_{\mathbb{R}^N}|u|^2, \tag{2.6}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^N}|u|^2 = 2\gamma\int_{\mathbb{R}^N}|u|^2 - 2\cos\theta\ I(u), \tag{2.7}$$

for all $0 < t < T_{\max}$.

(ii) *If* E *is defined by* (1.7)*, then*

$$\frac{d}{dt}E(u(t)) = -\cos\theta \int_{\mathbb{R}^N} |u_t|^2 + \gamma^2 \cos\theta \int_{\mathbb{R}^N} |u|^2 - \gamma\cos(2\theta) I(u), \tag{2.8}$$

and

$$\frac{d}{dt}\Big[E(u(t)) - \frac{\gamma}{2}\cos\theta \int_{\mathbb{R}^N} |u|^2\Big] = -\cos\theta \int_{\mathbb{R}^N} |u_t|^2 + \gamma \sin^2\theta \; I(u), \tag{2.9}$$

for all $0 < t < T_{\max}$.

Proof. Multiplying equation (1.1) by $\overline{u}$ and integrating by parts on $\mathbb{R}^N$ yields (2.4), and identities (2.5), (2.6) and (2.7) are immediate consequences. Multiplying equation (1.1) by $e^{-i\theta}\overline{u}_t$, integrating by parts on $\mathbb{R}^N$ and taking the real part, we obtain

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= -\cos\theta \int_{\mathbb{R}^N} |u_t|^2 + \gamma \Re\Big(e^{i\theta} \int_{\mathbb{R}^N} \overline{u} u_t\Big) \\ &= -\cos\theta \int_{\mathbb{R}^N} |u_t|^2 + \gamma\Big(\cos\theta \; \Re \int_{\mathbb{R}^N} \overline{u} u_t - \sin\theta \; \Im \int_{\mathbb{R}^N} \overline{u} u_t\Big).\end{aligned} \tag{2.10}$$

Furthermore, it follows from (2.4) that

$$\Re\Big(e^{i\theta} \int_{\mathbb{R}^N} \overline{u} u_t\Big) = \gamma \cos\theta \int_{\mathbb{R}^N} |u|^2 - \cos(2\theta) I(u). \tag{2.11}$$

Identity (2.8) follows from the first identity in (2.10), and (2.11). On the other hand, taking the imaginary part of (2.4), we obtain

$$\Im \int_{\mathbb{R}^N} \overline{u} u_t = -\sin\theta \; I(u). \tag{2.12}$$

Identity (2.9) now follows from the second identity in (2.10), and (2.12). □

Remark 2.4. Note that

$$I(w) = (\alpha+2)E(w) - \frac{\alpha}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \le (\alpha+2)E(w). \tag{2.13}$$

We conclude this section with a global existence property for sufficiently small initial values in the case $\gamma < 0$.

Proposition 2.5. *There exists a constant* $K > 0$ *with the following property. Given* $\gamma < 0$, $0 < \alpha < \frac{4}{N}$, *and* $C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, *let* $u \in C([0, T_{\max}), H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N))$ *be the corresponding, maximal solution of* (1.1)*. If*

$$\|u_0\|_{L^2} \le \left[\frac{4|\gamma|}{(4-N\alpha)(\frac{N\alpha}{4}K)^{\frac{N\alpha}{4-N\alpha}} K\cos\theta}\right]^{\frac{4-N\alpha}{4\alpha}}, \tag{2.14}$$

then u *is global, i.e.,* $T_{\max} = \infty$.

Proof. For $0 \le t < T_{\max}$, set

$$f(t) = \int_{\mathbb{R}^N} |u(t,x)|^2 dx. \tag{2.15}$$

Recall that (by Sobolev's or Gagliardo-Nirenberg's inequality) that there exists a constant K such that

$$\|w\|_{L^{\alpha+2}}^{\alpha+2} \le K \|\nabla w\|_{L^2}^{\frac{N\alpha}{2}} \|w\|_{L^2}^{\frac{4-(N-2)\alpha}{2}}, \tag{2.16}$$

for all $0 \le \alpha \le \frac{4}{N}$ and all $w \in H^1(\mathbb{R}^N)$. Applying the elementary inequality $xy \le \frac{\varepsilon^p x^p}{p} + \frac{y^{p'}}{p'\varepsilon^{p'}}$ with $p = \frac{4}{N\alpha}$, we deduce from (2.16) that

$$\|w\|_{L^{\alpha+2}}^{\alpha+2} \le K\Big(\frac{N\alpha}{4}\varepsilon^{\frac{4}{N\alpha}} \|\nabla w\|_{L^2}^2 + \frac{4-N\alpha}{4}\varepsilon^{-\frac{4}{4-N\alpha}} \|w\|_{L^2}^{\frac{2[4-(N-2)\alpha]}{4-N\alpha}}\Big). \tag{2.17}$$

It now follows from (2.15), (2.7) and (2.17) that

$$\frac{df}{dt} \le 2\gamma f + \cos\theta\ K\varepsilon^{-\frac{4}{4-N\alpha}} \frac{4-N\alpha}{2} f^{\frac{4-(N-2)\alpha}{4-N\alpha}} - 2\cos\theta\Big[1-\varepsilon^{\frac{4}{N\alpha}}\frac{N\alpha}{4}K\Big]\|\nabla u\|_{L^2}^2.$$

Letting $\varepsilon = (\frac{N\alpha}{4}K)^{-\frac{N\alpha}{4}}$, we deduce that

$$\frac{df}{dt} \le 2\gamma f + \cos\theta\ K\Big(\frac{N\alpha}{4}K\Big)^{\frac{N\alpha}{4-N\alpha}} \frac{4-N\alpha}{2} f^{\frac{4-(N-2)\alpha}{4-N\alpha}}. \tag{2.18}$$

This is an inequality of the form $f' + af \le bf^{1+\nu}$. If $f(0)^\nu \le \frac{a}{b}$, then this implies $f(t) \le e^{-at}(f(0)^{-\nu} - \frac{b}{a})^{-\frac{1}{\nu}}$. Therefore, it follows from (2.18) and (2.14) that $\sup_{0\le t<T_{\max}} \|u(t)\|_{L^2} < \infty$. Applying Remark 2.2, we conclude that $T_{\max} = \infty$. □

Remark 2.6. For a fixed α, condition (2.14) becomes better and better as $\theta \to \pm\frac{\pi}{2}$. (This is not too surprising. Indeed, for the limiting nonlinear Schrödinger equation, global existence holds for every initial value.) More precisely, the right-hand side of (2.14) goes to ∞ as $\theta \to \pm\frac{\pi}{2}$. In particular, if we fix $\alpha < \frac{4}{N}$, $\gamma < 0$ and an initial value u_0, then the solution of (1.1) is global if θ is sufficiently close to $\pm\frac{\pi}{2}$.

3. Proof of Theorem 1.1

We follow the argument of [1] after an appropriate change of variables. Set

$$v(t) = e^{-\gamma t}u(t), \tag{3.1}$$

for $0 \le t < T_{\max}$. (Note that $v_t = e^{i\theta}[\Delta v + e^{\alpha\gamma t}|v|^\alpha v]$.) Set

$$\widetilde{f}(t) = \int_{\mathbb{R}^N} |v|^2 = e^{-2\gamma t}\int_{\mathbb{R}^N} |u|^2, \tag{3.2}$$

$$\widetilde{\jmath}(t) = \int_{\mathbb{R}^N} |\nabla v|^2 - e^{\alpha\gamma t}\int_{\mathbb{R}^N} |v|^{\alpha+2} = e^{-2\gamma t}I(u), \tag{3.3}$$

and

$$\widetilde{e}(t) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - \frac{e^{\alpha\gamma t}}{\alpha+2}\int_{\mathbb{R}^N} |v|^{\alpha+2} = e^{-2\gamma t}E(u). \tag{3.4}$$

It follows from (2.4), (3.1) and (3.3) that

$$\int_{\mathbb{R}^N} \overline{v}v_t = -e^{i\theta}\widetilde{\jmath}(t), \tag{3.5}$$

so that by (3.2)

$$\frac{d\widetilde{f}}{dt} = 2\Re\int_{\mathbb{R}^N} \overline{v}v_t = -2\cos\theta\ \widetilde{\jmath}(t). \tag{3.6}$$

Moreover, it follows from (3.4), (2.8), (3.1), (3.6) and (3.3) that

$$\frac{d\widetilde{e}}{dt} = -\cos\theta\int_{\mathbb{R}^N} |v_t|^2 - e^{\alpha\gamma t}\frac{\alpha\gamma}{\alpha+2}\int_{\mathbb{R}^N} |v|^{\alpha+2} \le -\cos\theta\int_{\mathbb{R}^N} |v_t|^2. \tag{3.7}$$

Since $\widetilde{e}(0) = E(u_0) < 0$, we deduce from (3.7) that

$$\widetilde{e}(t) < 0, \tag{3.8}$$

for all $0 \le t < T_{\max}$. It follows from (3.7), Cauchy-Schwarz's inequality, (3.5) and (3.6) that

$$\begin{aligned} -\widetilde{f}\frac{d\widetilde{e}}{dt} &\ge \cos\theta\int |v|^2\int |v_t|^2 \ge \cos\theta\Big|\int \overline{v}v_t\Big|^2 \\ &= \cos\theta\ \widetilde{\jmath}^2 = \frac{1}{2}(-\widetilde{\jmath})\frac{d\widetilde{f}}{dt}. \end{aligned} \tag{3.9}$$

On the other hand, note that

$$\widetilde{\jmath} = (\alpha+2)\widetilde{e} - \frac{\alpha}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \le (\alpha+2)\widetilde{e}. \tag{3.10}$$

It follows from (3.8) and (3.10) that $\widetilde{\jmath} < 0$, so that by (3.7), $\frac{d\widetilde{f}}{dt} > 0$; and so, we deduce from (3.9)-(3.10) that

$$-\widetilde{f}\frac{d\widetilde{e}}{dt} \ge -\frac{\alpha+2}{2}\widetilde{e}\frac{d\widetilde{f}}{dt}. \tag{3.11}$$

Therefore,

$$\frac{d}{dt}[-\widetilde{e}\widetilde{f}^{-\frac{\alpha+2}{2}}] \ge 0, \tag{3.12}$$

so that

$$-\widetilde{e}(t) \ge [-\widetilde{e}(0)]\widetilde{f}(0)^{-\frac{\alpha+2}{2}}\widetilde{f}(t)^{\frac{\alpha+2}{2}} = (-E(u_0))\|u_0\|_{L^2}^{-(\alpha+2)}\widetilde{f}(t)^{\frac{\alpha+2}{2}}. \tag{3.13}$$

It now follows from (3.6), (3.10) and (3.13) that

$$\begin{aligned}\frac{d\widetilde{f}}{dt} &= -2\cos\theta\,\widetilde{\jmath} \ge -2(\alpha+2)\cos\theta\,\widetilde{e}\\ &\ge 2(\alpha+2)\cos\theta\,(-E(u_0))\|u_0\|_{L^2}^{-(\alpha+2)}\widetilde{f}^{\frac{\alpha+2}{2}},\end{aligned} \tag{3.14}$$

which implies

$$\frac{d}{dt}\{\alpha(\alpha+2)\cos\theta\,(-E(u_0))\|u_0\|_{L^2}^{-(\alpha+2)}t + \widetilde{f}^{-\frac{\alpha}{2}}\} \le 0. \tag{3.15}$$

Since (3.15) holds for all $0 \le t < T_{\max}$, we deduce that

$$T_{\max} \le \frac{\|u_0\|_{L^2}^2}{\alpha(\alpha+2)\cos\theta\,(-E(u_0))} < \infty. \tag{3.16}$$

This completes the proof.

4. Proof of Theorem 1.2

Consider u_0 as in the statement and u the corresponding solution of (1.1) defined on the maximal interval $[0, T_{\max})$. We first show that a certain energy of u remains negative as long as u exists. Then, we use this property in order to derive a differential inequality which shows that u cannot be global.

It is convenient to set

$$\rho = -\gamma > 0, \tag{4.1}$$

and

$$\eta = \rho\frac{(\alpha+2)\cos(2\theta) + 2(1-\cos\theta)}{2\cos\theta} \ge \rho > 0, \tag{4.2}$$

where the first inequality follows from (1.10). Moreover, let

$$\begin{aligned}e(t) &= E(u(t)),\\ j(t) &= I(u(t)),\\ f(t) &= \|u(t)\|_{L^2}^2,\end{aligned}$$

where E and I are defined by (1.7) and (2.3), respectively, and

$$e^\star(t) = e(t) + \eta f(t), \tag{4.3}$$

for $0 \le t < T_{\max}$. We first claim that

$$e(t) \le e^\star(t) < 0, \tag{4.4}$$

for all $0 \le t < T_{\max}$. Indeed, the first inequality in (4.4) follows from (4.3). Moreover, since $\cos(2\theta) > 0$ by (1.9), it follows from (2.13) that

$$\rho\cos(2\theta)j(t) \le \rho(\alpha+2)\cos(2\theta)e(t);$$

and so, we deduce from (2.8) that

$$\frac{de}{dt} \le -\cos\theta\,\|u_t\|_{L^2}^2 + \rho^2\cos\theta\,f + \rho(\alpha+2)\cos(2\theta)e,$$

i.e., using (4.3),

$$\frac{de}{dt} \le -\cos\theta\, \|u_t(t)\|_{L^2}^2 + \rho(\alpha+2)\cos(2\theta)e^\star + [\rho^2\cos\theta - \eta\rho(\alpha+2)\cos(2\theta)]f. \quad (4.5)$$

Note also that by (2.7) and (2.5)

$$\frac{df}{dt} \le -2\rho f + 2\cos\theta\, |j| \le -2\rho(1-\cos\theta)f + 2\cos\theta\Big|\int_{\mathbb{R}^N} \overline{u}u_t\Big|. \quad (4.6)$$

Since

$$2\Big|\int_{\mathbb{R}^N} \overline{u}u_t\Big| \le 2\|u_t\|_{L^2} f^{\frac{1}{2}} \le \frac{1}{\eta}\|u_t\|_{L^2}^2 + \eta f,$$

we deduce from (4.5) and (4.6) that

$$\frac{de^\star}{dt} \le \rho(\alpha+2)\cos(2\theta)e^\star + Af, \quad (4.7)$$

where $A = (\rho^2+\eta^2)\cos\theta - \eta\rho[(\alpha+2)\cos(2\theta) - 2(1-\cos\theta)]$. Note that by (4.2)

$$A = (\rho^2+\eta^2)\cos\theta - 2\eta^2\cos\theta = (\rho^2-\eta^2)\cos\theta \le 0;$$

and so we deduce from (4.7) that

$$\frac{de^\star}{dt} \le \rho(\alpha+2)\cos(2\theta)e^\star.$$

Therefore, $e^\star(t) \le e^{t\rho(\alpha+2)\cos(2\theta)}e^\star(0) < 0$, which proves the claim (4.4).

We now use the energy inequality (4.4) to obtain a differential inequality on f. Observe that by (2.7),

$$-\frac{\alpha+2}{2}e\Big[\frac{df}{dt} + 2\rho f\Big] = (\alpha+2)\cos\theta\; ej.$$

Since $0 > (\alpha+2)e \ge j$ by (4.4) and (2.13), we deduce that

$$-\frac{\alpha+2}{2}e\Big[\frac{df}{dt} + 2\rho f\Big] \le \cos\theta\; j^2. \quad (4.8)$$

Note that by (2.6),

$$j^2 = \Big|\int_{\mathbb{R}^N} \overline{u}u_t\Big|^2 + \rho^2 f^2 + \rho f\frac{df}{dt} \le f\|u_t\|_{L^2}^2 + \rho^2 f^2 + \rho f\frac{df}{dt}.$$

Therefore, it follows from (4.8) that

$$\cos\theta f\|u_t\|_{L^2}^2 \ge -\frac{\alpha+2}{2}e(t)\Big[\frac{df}{dt} + 2\rho f\Big] - \rho^2\cos\theta f^2 - \rho\cos\theta f\frac{df}{dt}. \quad (4.9)$$

On the other hand, multiplying (2.9) by f we obtain

$$-f\frac{de}{dt} - \frac{\rho}{2}\cos\theta\; f\frac{df}{dt} - \rho\sin^2\theta\; fj = \cos\theta\; \|u_t\|_{L^2}^2 f.$$

Applying (4.9), we deduce that

$$\begin{aligned} -f\frac{de}{dt} - \frac{\rho}{2}\cos\theta\; f\frac{df}{dt} - \rho\sin^2\theta\; fj \\ \ge -\frac{\alpha+2}{2}e\Big[\frac{df}{dt} + 2\rho f\Big] - \rho^2\cos\theta\; f^2 - \rho\cos\theta\; f\frac{df}{dt}, \end{aligned}$$

i.e.,

$$-f\frac{de}{dt} + \frac{\alpha+2}{2}e\frac{df}{dt} + \frac{\rho}{2}\cos\theta\; f\frac{df}{dt} \ge \rho\sin^2\theta\; fj - \rho(\alpha+2)ef - \rho^2\cos\theta\; f^2. \quad (4.10)$$

Since

$$j = -\frac{\rho}{\cos\theta}f - \frac{1}{2\cos\theta}\frac{df}{dt},$$

by (2.7), it follows from (4.10) that

$$
\begin{aligned}
&- f\frac{de}{dt} + \frac{\alpha+2}{2}e\frac{df}{dt} + \frac{\rho}{2}\cos\theta\, f\frac{df}{dt} \\
&\quad \geq -\frac{\rho^2\sin^2\theta}{\cos\theta}f^2 - \frac{\rho\sin^2\theta}{2\cos\theta}f\frac{df}{dt} - \rho(\alpha+2)ef - \rho^2\cos\theta f^2 \\
&\quad = -\frac{\rho^2}{\cos\theta}f^2 - \frac{\rho\sin^2\theta}{2\cos\theta}f\frac{df}{dt} - \rho(\alpha+2)ef;
\end{aligned}
$$

and so,

$$
- f\frac{de}{dt} + \frac{\alpha+2}{2}e\frac{df}{dt} + \frac{\rho}{2\cos\theta}f\frac{df}{dt} \geq -\frac{\rho^2}{\cos\theta}f^2 - \rho(\alpha+2)ef. \tag{4.11}
$$

Since $e = e^\star - \eta f \leq -\eta f$ by (4.3) and (4.4), and $\eta \geq \rho$ by (4.2), we see that

$$
-\rho(\alpha+2)ef \geq \eta\rho(\alpha+2)f^2 \geq \rho^2(\alpha+2)f^2.
$$

Therefore, we deduce from (4.11) that

$$
- f\frac{de}{dt} + \frac{\alpha+2}{2}e\frac{df}{dt} + \frac{\rho}{2\cos\theta}f\frac{df}{dt} \geq \Big[\alpha+2-\frac{1}{\cos\theta}\Big]\rho^2 f^2. \tag{4.12}
$$

Furthermore, $\cos(2\theta) = 2\cos^2\theta - 1$, so that (1.10) is equivalent to

$$
(\alpha+2)\cos\theta \geq 2 + \frac{\alpha}{2\cos\theta};
$$

and so

$$
(\alpha+2)\cos\theta \geq 2. \tag{4.13}
$$

It follows from (4.12) and (4.13) that

$$
- f\frac{de}{dt} + \frac{\alpha+2}{2}e\frac{df}{dt} + \frac{\rho}{2\cos\theta}f\frac{df}{dt} \geq \frac{1}{\cos\theta}\rho^2 f^2 \geq 0. \tag{4.14}
$$

Multiplying (4.14) by $f^{-\frac{\alpha+4}{2}} > 0$ we obtain

$$
\frac{d}{dt}\Big[-f^{-\frac{\alpha+2}{2}}e - \frac{\rho}{\alpha\cos\theta}f^{-\frac{\alpha}{2}}\Big] \geq 0. \tag{4.15}
$$

Set

$$
\zeta = -f(0)^{-\frac{\alpha+2}{2}}e(0) - \frac{\rho}{\alpha\cos\theta}f(0)^{-\frac{\alpha}{2}}, \tag{4.16}
$$

and note that by (1.11)-(1.12),

$$
\zeta > 0. \tag{4.17}
$$

Integrating (4.15) on $(0,t)$, we obtain

$$
- f^{-\frac{\alpha+2}{2}}e - \frac{\rho}{\alpha\cos\theta}f^{-\frac{\alpha}{2}} \geq \zeta. \tag{4.18}
$$

Multiplying (4.18) by $f^{\frac{\alpha+2}{2}}$ yields

$$
- e \geq \zeta f^{\frac{\alpha+2}{2}} + \frac{\rho}{\alpha\cos\theta}f. \tag{4.19}
$$

On the other hand, it follows from (2.7) and (2.13) that

$$
\frac{df}{dt} \geq -2\rho f - 2(\alpha+2)\cos\theta\; e. \tag{4.20}
$$

We deduce from (4.20) and (4.19) that

$$
\frac{df}{dt} \geq \frac{4\rho}{\alpha}f + 2(\alpha+2)\cos\theta\;\zeta f^{\frac{\alpha+2}{2}} \geq 2(\alpha+2)\cos\theta\;\zeta f^{\frac{\alpha+2}{2}}. \tag{4.21}
$$

It follows easily from (4.17) that f cannot satisfy (4.21) for all $t > 0$, so that $T_{\max} < \infty$. This completes the proof.

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