ON A PARABOLIC REACTION-DIFFUSION SYSTEM WITH NONLOCAL DRIFTS

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Abstract. In this work, we consider a parabolic system of reaction-diffusion equations coupled through interacting nonlocal drifts. For this system, we prove local and global existence of smooth solutions, and uniqueness as well. Moreover, we establish some blow-up results and, in particular, we give a criteria of simultaneous blow-up. Finally, through the use of invariant regions, we obtain some criteria of exponential convergence toward a homogeneous state. In this article, we improve the results obtained by Chipot and Lovat [6], Corrêa, Menezes and Ferreira [11], and Chipot, Valente and Caffarelli [10] to ours case of coupled systems, in what they concern to the existence, uniqueness and exponential decay of solutions. Our system of reaction-diffusion equations captures some features of the semi-empirical $k-\epsilon$ model for turbulent flow and of the multi-component Keller-Segel model for cell migration due to random motion and chemotaxis.

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1. Introduction

Reaction-diffusion equations are widely used for the description and explanation of various physical, chemical and biological phenomena. In the strict sense of the mathematical term, a reaction-diffusion equation is an equation that comprises a reaction term and a diffusion term in the form

$$u_t = \operatorname{div}(D\nabla u) + f(u). \tag{1.1}$$

Here, $u = u(x,t)$ is a state variable that describes an unknown quantity at position x and time t. Typical examples, are the temperature of a material, the concentration of a substance and the density of a population. The reaction is expressed by the function $f(u)$ and describes processes that really change the present u, like production or consumption of heat, the disintegration of a chemical substance or a population growth law. The classical approach to the diffusion term $\operatorname{div}(D\nabla u)$ is via conservation of mass and the Fick law, a mathematical formalism that generalizes the Fourier law for heat conduction. Another way to introduce diffusion, and that is often used in many biological processes, is by considering the random walk of the diffusing particles (see *e.g.* Murray [17]). The letter D is called the diffusivity and is an important indicative of the diffusion mobility. For many problems in the applications, a first approximation of the phenomenon under study consists in considering a constant diffusivity, case in which D is most known as the diffusivity coefficient. However, if we want to capture the whole problem, one should be aware that the diffusivity will depend on the space and time position and moreover on the state of the material, or on the concentration or density of the substance or population. Thus, in the general case, the diffusivity D may depend on (x,t) and yet on u or even on ∇u. For instance, in many problems of mathematical biology there is an increase

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of diffusion due to population pressure. In this case, the environment is of prime importance and one easily imagine that the diffusion mobility, besides the natural dependence on (x, t), may also depend on the own density u. For this kind of problems, the diffusion is nonlinear and equation (1.1) reads as

$$u_t = \operatorname{div}(D(u)\nabla u) + f(u). \tag{1.2}$$

To the mathematical analysis of many problems, it is better to replace the operator $\operatorname{div}(D(u)\nabla\cdot)$ in (1.2) by a simpler Laplace operator. This can usually be done by making use of the Kirchhoff transform (see *e.g.* Carslaw and Jaeger [4]), which applied to (1.2) transforms this equation into

$$u_t - a(u)\,\Delta\, u = f(u). \tag{1.3}$$

This procedure allows us to replace the nonlinearity of the diffusion term by another nonlinearity, now expressed as a drift function $a(u)$ multiplied to the Laplace operator. These type of equations also arise in some contexts naturally, as is the case of the Fokker-Planck equation in which the diffusivity coefficient D is related to the properties of the underlying random motion of particles (see *e.g.* Grzybowski [12]). An interesting feature of the reaction-diffusion equation (1.3), is the local character of its drift that occurs at the same time as its nonlinearity. With this respect, we observe that the drift $a(u)$ in equation (1.3) is essentially a local function in the sense that, for each time $t \geq 0$, it depends simultaneous on the point x and on the pointwise density $u(x, t)$. Nevertheless, this consideration is incompatible with the physic notion of measure, since we are not able to measure the pointwise diffusivity of a pointwise density. One possibility to overcome this difficulty, consists in considering a ball $B(x, \epsilon)$ centered at x with radius ϵ and to replace $a(x, t, u)$ by $a\left(x, t, \frac{1}{|B(x,\epsilon)|}\int_{B(x,\epsilon)} u(y, t)\, dy\right)$ (see *e.g.* Lovat [16]). In the sequel, we shall say that the drift function a is nonlocal, because a depends on a kind of an average, over a neighborhood of x, involving u somehow, instead. Therefore the correspondingly problem of reaction-diffusion considered here is nonlocal in the sense that the drift function a is determined by a global quantity, as the total mass, total flux or total energy. For many concrete problems in the applications, there are two or more quantities interacting through a system of reaction-diffusion equations. These systems involve constituents that are locally transformed into each other by reactions and transported in space by diffusion. In particular, systems of reaction-diffusion equations allow for much more complex behavior than a scalar reaction-diffusion equation does. Consequently, it is very natural the generalization of (1.3) to the following system of reaction-diffusion equations

$$\begin{cases} u_t - a_1(p(u), q(v))\Delta\, u = f(u, v) \\ v_t - a_2(r(u), s(v))\Delta\, v = g(u, v). \end{cases} \tag{1.4}$$

Here, p, r, f and q, s, g are functions expressing interacting diffusions or interacting reactions in each species u and v, respectively. It is worth noting that the reaction-diffusion equations in (1.4) are coupled, not only in the reaction functions f and g, but also, and this is the novelty of our work, through the nonlocal drift functions a_1 and a_2. Examples of applications of (1.4) with constant drifts a_1 and a_2, can be found in many models of population dynamics, combustion theory, chemical kinetics and cell dynamics (see *e.g.* Murray [17] and Grzybowski [12]). Models with nonlinear diffusion are not so common, but we can find an interesting association in the semi-empirical $k-\epsilon$ model, the first two equation model suggested for predicting the behavior of turbulent flows. In this model, initially proposed by Kolmogorov and later developed by Jones and Spalding, the evolution of turbulent kinetics energy k and of the rate of dissipation of energy ϵ are modeled by a particular case of the system (1.4). See *e.g.* Bertsch, Dal Passo and Kersner [2] and the references therein to the works by Kolmogorov, Jones and Launder, and particularly by Barenblatt. Another example of a reaction-diffusion system with nonlinear diffusion that arises in the applications, is the multi-component Keller-Segel model. This model is widely used in mathematical biology for studying cell migration due to random motion and chemotaxis. The reaction-diffusion system (1.4) also captures some features of the Keller-Segel model, although the full problem involves cross-diffusion between the cell population and the chemoattractant concentration. See *e.g.* Byrne and Owen [3] and the references therein, in particular to the works by Patlak and by Keller and Segel.

With the motivation introduced above, we are now in conditions to state the main problem that shall be studied in this work. Let Ω be a bounded domain (convex open subset) of $\mathbb{R}^n$, with a compact boundary denoted by $\partial\Omega$. The letter n states for the dimension of the considered space domain and we assume that $n\in\mathbb{N}$, although the dimensions of physical interest are $n=1,\ 2,\ 3$. In this work, we consider the following parabolic problem for a system of reaction-diffusion equations with nonlocal drifts:

$$\begin{cases} u_t - a_1(p(u),q(v))\Delta u = f(u,v) & \text{in } \Omega\times(0,\infty) \\ v_t - a_2(r(u),s(v))\Delta v = g(u,v) & \text{in } \Omega\times(0,\infty) \\ u=u_0\geq 0, \quad v=v_0\geq 0 & \text{in } \Omega\,, \quad \text{when } t=0; \end{cases} \tag{1.5}$$

where $p,\ q,\ r,\ s$ are assumed to be continuous mappings from $C^1(\mathbb{R}^n)$ into $\mathbb{R}$. Examples of these functions, would be $\int_\Omega u\,dx$ and $\int_\Omega |\nabla u|^{\gamma-2}\,dx$ for suitable values of $\gamma\geq 1$. The functions $a_i:\mathbb{R}^2\to\mathbb{R}$, $i=1,2$ are assumed to be continuous and satisfy to

$$0<m_i\leq a_i(s)\leq M_i<\infty, \tag{1.6}$$

for some positive constants m_i and M_i, with $i=1,2$. Condition (1.6) expresses the fact that we shall consider uniformly parabolic systems (1.5). On the reaction functions $f,\ g:\mathbb{R}^2\to\mathbb{R}$, we assume they are locally Lipschitz-continuous, *i.e.* for each open subset $U\subset\mathbb{R}^2$

$$\begin{aligned} &\exists\, M_f: |f(u_1,v_1)-f(u_2,v_2)|\leq M_f\,|(u_1,v_1)-(u_2,v_2)| \quad \text{for all } (u_1,v_1),\ (u_2,v_2)\in U\,, \\ &\exists\, M_g: |g(u_1,v_1)-g(u_2,v_2)|\leq M_g\,|(u_1,v_1)-(u_2,v_2)| \quad \text{for all } (u_1,v_1),\ (u_2,v_2)\in U\,. \end{aligned} \tag{1.7}$$

To the best of our knowledge, the first works on the mathematical analysis of diffusion equations, with nonlocal drifts as mentioned above, are due to Chipot and his collaborators [16, 6, 7, 8, 9, 10, 20]. However, it should be noted that Lions [15] has earlier proposed an abstract framework to handle hyperbolic problems with similar nonlocal drifts, previously and separately introduced by Dickey and Pohožaev (see the exact references in [15]). Hereupon, we should mention that we do not intend to be exhaustive in the references, but just to point out the most important connections of earlier works to our article. To begin with, in [16, 6] has been proved the existence and uniqueness of local and global solutions to the following parabolic diffusion problem:

$$\begin{cases} u_t - a(l(u))\Delta\, u = f & \text{in } \Omega\times(0,T) \\ u=u_0 \quad \text{in } \Omega & \text{when } t=0. \end{cases} \tag{1.8}$$

Here, Ω was assumed to be a bounded open subset in $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$, T is an arbitrary time, the drift a is some function from $\mathbb{R}$ into $(0,+\infty)$ and l is a continuous mapping from $L^2(\Omega)$ into $\mathbb{R}$. The authors have worked on different problems for distinct drifts, but always depending on $\int_\Omega u\,dx$, and under different boundary conditions: Dirichlet, Neumann and mixed boundary conditions. In [7], the same authors have studied the case when $u(\cdot,t)\in V\subset H^1(\Omega)$ for all $t\in(0,T)$ and, besides existence and uniqueness results, they have analyzed the asymptotic behavior of the solutions as well. This question, was deeply investigated in [9] and, in particular, these authors have proved the convergence of the solutions to a steady state. Again the authors [16, 6, 7], have considered, in [8], a class of nonlocal elliptic and parabolic problems related to (1.8), now with homogeneous Dirichlet boundary conditions, for which they have proved existence and uniqueness results. The analysis of problem (1.8), considered with a nonlocal drift depending on the Dirichlet integral $\int_\Omega |\nabla u|^2\,dx$, was carried out in [10, 20]. The asymptotic behavior of the solutions to the problem (1.8), considered with a nonlocal drift written as a kernel, *i.e.* $l(u)=\int_\Omega g(x)u(x,t)\,dx$, where g is a given function in $L^2(\Omega)$, has been performed in [20] too. Reaction-diffusion analogues of the parabolic problem (1.8) were considered by the authors [1, 11] in the following form

$$\begin{cases} u_t - a(l(u))\Delta\, u = f(u) & \text{in } \Omega\times(0,T) \\ u=u_0 \quad \text{in } \Omega & \text{when } t=0. \end{cases} \tag{1.9}$$

In [1], was considered the problem (1.9) in a rather general Banach space and the authors have worked on the case of $a(u)=\left(\int_\Omega u\,dx\right)^{-1}$. This assumption, led them to an equivalent reaction-diffusion problem with a nonlocal drift, now multiplied by the reaction term $f(u)$. For these problems, the authors have established local existence and uniqueness results and, in addition, they have found conditions on the initial data in order to obtain time properties of finite extinction or persistency of the solutions. In [11], the authors have extended the results of [16, 6, 7, 8, 9, 10] to the case of the reaction-diffusion problem (1.9). In particular, they have considered both stationary and evolution situations, where the nonlinearity appears not only in the nonlocal drift term $a(l(u))$, but also on the right-hand side in which one has the nonlinear function $f(u)$.

In this paper we are mainly interested in local and global existence results for the reaction-diffusion system (1.5), and with the uniqueness as well. The proofs of existence and uniqueness of local and global smooth solutions, shall be performed, by using semigroup techniques, at Section 2. Parabolic nonlocal problems can present several equilibria, which makes the study of the asymptotic behavior of the solutions to the system (1.5) very challenging. Therefore, it is also our aim in this work, not only to analyze the blow-up phenomena, but also the convergence of the solutions towards a homogenous state. The issue of blow-up, shall be discussed at Section 3. Finally, by using invariant regions, we give, at Section 4, sufficient conditions for the convergence towards a homogeneous state, and for the time exponential decay. To the best of our knowledge, our results are the first for nonlocal coupled system of reaction-diffusion. It is worth to observe that all of our results apply for more general systems of reaction-diffusion, as is the case of

$$\begin{cases} u_t - a_1(p_1(u), \dots p_i(u), q_1(v), \dots, q_j(v))\Delta\, u = f(u,v) & \text{in } \Omega\times(0,T) \\ v_t - a_2(r_1(u), \dots r_k(u), s_1(v), \dots, s_l(v))\Delta\, u = g(u,v) & \text{in } \Omega\times(0,T), \end{cases}$$

where $i\,,\ j\,,\ k\,,\ l > 1$ are fixed integers, not necessarily with the same value. Moreover, the results here contained complement some of the results established in [14, 5].

2. Local Existence of smooth solutions

In this section, we give sufficient conditions on the initial data in order to prove the existence of smooth solutions. With this end in mind, we use semigroups techniques, the main difficulty being the fact that the diffusion coefficients are not constants. A suitable change of the time variable will allow us to overcome this difficulty. As the reader will observe, different scales in the time variable needs to be consider.

For the convenience of the reader, we first remember the following results from the semigroups theory.

Theorem 2.1. *Let A be a sectorial operator with $Re(\sigma(A)) > \delta > 0$, where δ is a real constant. Then the following assertions hold:*

(1) $A^\alpha e^{-tA} = e^{-tA}A^\alpha$ *on* $D(A^\alpha)$ *for every* $t > 0$ *and* $\alpha \geq 0$*;*

(2) *For every $\alpha \geq 0$, there exists a constant C_α such that*

$$\left\|A^\alpha e^{-tA}\right\| \leq C_\alpha t^{-\alpha} e^{-\delta\, t} \quad \textit{for all } t > 0, \tag{2.1}$$

and, if $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$, then

$$\left\|(e^{-tA} - I)x\right\| \leq \frac{C_{1-\alpha}}{\alpha} t^\alpha \left\|A^\alpha x\right\| \quad \textit{for all } t > 0\,, \tag{2.2}$$

where C_α is bounded for α in any compact interval of $(0,\infty)$ and is still bounded as $\alpha \to +0$.

Proof. We address the reader to *e.g.* Pazy [18, pp. 69-75] for the proofs of these properties. □

For $\alpha \geq 0$, we define the space $X_\alpha = D(A^\alpha)$ which is Banach space for the norm

$$\|x\|_\alpha = \|A^\alpha x\|\,, \qquad x \in D(A^\alpha).$$

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. Then, for any $0 \leq \alpha \leq 1$*

$$X_\alpha \quad \subset \quad W^{k,q}(\Omega) \quad \textit{for} \quad k - \frac{n}{q} < 2\alpha - \frac{n}{p} \quad \textit{and} \quad q \geq p, \tag{2.3}$$

$$X_\alpha \quad \subset \quad C^v(\overline{\Omega}) \quad \textit{for} \quad 0 \leq v \leq 2\alpha - \frac{n}{p}. \tag{2.4}$$

and the imbeddings are continuous.

Proof. See *e.g.* Pazy [18, Theorem 8.4.3]. □

Remark 2.1. *By a smooth boundary $\partial\Omega$, we mean, in this work, a boundary that makes possible the validity of the continuous imbeddings (2.3)-(2.4). In particular, the validity of (2.3) requires that $\partial\Omega$ is only Lipschitz-continuous, whereas for (2.4) we need to assume that $\partial\Omega$ is of class C^v (see e.g. Pazy* [18, Theorems 7.1.1, 7.1.2]*).*

Now, we take α and $p \in (1, \infty)$ such that

$$1 < 2\alpha - \frac{n}{p}. \tag{2.5}$$

By (2.4) of Theorem 2.1,

$$X_\alpha \subset C^v(\overline{\Omega}) \subset C^1(\overline{\Omega}) \qquad \text{for} \quad 0 \le v < 2\alpha - \frac{n}{p} \tag{2.6}$$

and the embedding is continuous.

In order to formulate the equation for u in an equivalent integral form, we first make the change of variables

$$\tau(t) = \int_0^t a_1(\tau^*) d\tau^* \tag{2.7}$$

where $a_1(s) := a_1(p(s), q(s))$. Due to (1.6) and since p and q are continuous forms, this transformation defines a bijective mapping with smooth inverse that we denote by $t(\tau)$. Then, we get the following equation for u:

$$u_\tau(x, t(\tau)) - \Delta u(x, t(\tau)) = \frac{f(u(x, t(\tau)), v(x, t(\tau))}{a_1(t(\tau))}.$$

In a like manner we get that the second equation in (1.5) is equivalent to:

$$v_{\overline{\tau}}(x, t(\overline{\tau})) - \Delta v(x, t(\overline{\tau})) = \frac{g(u(x, t(\overline{\tau})), v(x, t(\overline{\tau}))}{a_2(t(\overline{\tau}))}.$$

where $\overline{\tau}$ is giving by the relation

$$\overline{\tau}(t) = \int_0^t a_2(\tau_*) d\tau_*. \tag{2.8}$$

Let X denote the Lebesgue space $L^p(\Omega)$ $(1 < p < \infty)$ with usual norm. We consider the operator A_p defined by

$$A_p u = -\Delta u + u \quad \text{for} \quad u \in D(A_p) = \left\{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}. \tag{2.9}$$

A_p is a closed linear operator, with a densely defined domain $D(A_p)$. Moreover, A_p is a sectorial operator in X whose spectrum $\sigma(A_p)$ is independent of p and $Re(\sigma(A_p)) > 0$. Hence, we can define the fractional power of A_p. Let $X_\alpha = D(A_p^\alpha)$ with norm defined by

$$\|x\|_\alpha = \left\| A_p^\alpha x \right\|_{L^p}, \quad x \in D(A_p^\alpha). \tag{2.10}$$

System (1.5) can be written in the form

$$\begin{cases} \dfrac{du}{d\tau} + A_p u = F(u, v) \\[2ex] \dfrac{dv}{d\overline{\tau}} + A_p v = G(u, v) \\[2ex] u(x, t(0)) = u_0, \qquad v(x, t(0)) = v_0\,, \end{cases} \tag{2.11}$$

where

$$F((x, t(\tau)), v(x, t(\tau))) := \frac{f(u(x, t(\tau)), v(x, t(\tau))}{a_1(t(\tau))} + u(x, t(\tau))$$

and

$$G(u(x, t(\overline{\tau})), v(x, t(\overline{\tau}))) := \frac{g(u(x, t(\overline{\tau})), v(x, t(\overline{\tau}))}{a_2(t(\overline{\tau}))} + v(x, t(\overline{\tau})).$$

In order to show the local existence of solutions to this system, we first prove the existence of mild solutions $u,\ v$ by using the Banach fixed point theorem. This result, allows us to prove the existence of unique mild solutions $u,\ v$ on $[0,T]$ for some $T>0$ such that $(u,v)\in C\left([0,T]\,;X_\alpha\times X_\alpha\right)$. We prove then that $F(u,v)$ and $G(u,v)$ are locally Hölder continuous on $[0,T]$ from which follows that u and v are solutions on $[0,T]$ to our system. Finally, we show that these solutions are solutions in the classical sense.

Lemma 2.2 (Existence and uniqueness of a mild solution)**.** *For any $u_0,v_0\in X_\alpha$, the system (2.11) has a unique mild solution (u,v) on the interval $[0,T]$ for some $T>0$.*

Proof. We have to prove the existence of solutions to the system

$$\begin{aligned} u(\tau) &= e^{-\tau A_p}u_0+\int_0^\tau e^{-(\tau-s)A_p}F((x,t(\tau)),v(x,t(\tau))ds,\\ v(\overline{\tau}) &= e^{-\overline{\tau}A_p}v_0+\int_0^{\overline{\tau}} e^{-(\overline{\tau}-s)A_p}G(u(x,t(\overline{\tau})),v(x,t(\overline{\tau})))ds. \end{aligned} \tag{2.12}$$

Let us define

$$\begin{aligned} \Psi_1(u,v) &= e^{-\tau A_p}u_0+\int_0^\tau e^{-(\tau-s)A_p}F((x,t(\tau)),v(x,t(\tau))ds,\\ \Psi_2(u,v) &= e^{-\overline{\tau}A_p}v_0+\int_0^{\overline{\tau}} e^{-(\overline{\tau}-s)A_p}G(u(x,t(\overline{\tau})),v(x,t(\overline{\tau})))ds, \end{aligned}$$

and $\Psi=(\Psi_1,\Psi_2)$ with the domain $E_{\Psi,\varrho}$ given by

$$(u,v)\in E_{\Psi,\varrho}\Leftrightarrow (u(0),v(0))=(u_0,v_0)\quad\text{and}\quad [(u,v)-(u_0,v_0)]_\Psi\le\varrho,$$

where

$$[(u,v)]_\Psi=\sup_{0\le\tau\le T}\|u(\tau)\|_\alpha+\sup_{0\le\overline{\tau}\le T}\|v(\overline{\tau})\|_\alpha\,.$$

We now verify the hypotheses of the Banach fixed point theorem as follows.

Step 1: Ψ is a mapping from $E_{\Psi,\rho}$ into itself.
For $(u,v)\in C([0,T]\,;X_\alpha\times X_\alpha))$, we observe that

$$A_p^\alpha\Psi_1(u,v)=e^{-tA_p}A_p^\alpha u_0+\int_0^\tau A_p^\alpha e^{-(\tau-s)A_p}F(u(s),v(s))ds \tag{2.13}$$

By (2.1), we have

$$\begin{aligned} &\left\|A_p^\alpha\Psi_1(u,v)(t)-A_p^\alpha u_0\right\|_{L^p}\\ &\le\left\|(e^{-\tau A_p}-I)A_p^\alpha u_0\right\|_{L^p}+\int_0^\tau\left\|A_p^\alpha e^{-(\tau-s)A_p}\right\|_{L^p}\|F(u(s),v(s))\|_{L^p}\,ds\\ &\le\left\|(e^{-\tau A_p}-I)A_p^\alpha u_0\right\|_{L^p}+C_\alpha\int_0^\tau(\tau-s)^{-\alpha}e^{-\alpha(t-s)}\|F(u(s),v(s))\|_{L^p}\,ds. \end{aligned} \tag{2.14}$$

Similar results follow for Ψ_2, since $\lim_{t\to+0}\left\|(e^{-tA_p}-I)A_p^\alpha\varphi\right\|_{L^p}=0$.
Taking into account that for a general continuous function h on a bounded domain Ω, we have that

$$\|h\|_{L^p}=\left(\int_\Omega|h|^p\,dx\right)^{1/p}\le C\,\|h\|_{L^\infty}\le C\,\|h\|_{C^\nu(\Omega)}$$

and the local Lipschitz continuity of f (cf. (1.7)), we obtain that

$$\begin{aligned} \|F(u,v)\|_{L^p} &\le \|F(u,v)-F(0,0)\|_{L^p}+\|F(0,0)\|_{L^p}\\ &\le C\,\||u|+|v|\|_{L^p}+\|F(0,0)\|_{L^p}\\ &= C(\|u\|_{L^p}+\|v\|_{L^p}+1)\\ &\le C(\|u\|_{C^\nu(\Omega)}+\|v\|_{C^\nu(\Omega)}+1) \end{aligned}$$

Then it follows by (2.4) that,

$$\|F(u,v)\|_{L^p} \leq C(\|u\|_\alpha + \|v\|_\alpha + 1) \tag{2.15}$$

and similarly

$$\|G(u,v)\|_{L^p} \leq C(\|u\|_\alpha + \|v\|_\alpha + 1), \tag{2.16}$$

provided that $0 \leq \upsilon \leq 2\alpha - \frac{n}{p}$. Choosing T small enough and taking into account the estimates (2.14), (2.15) and (2.16), we see that $\Psi = (\Psi_1, \Psi_2)$ is a mapping from $E_{\Psi,\upsilon}$ into $E_{\Psi,\upsilon}$.

Step 2: We show that Ψ is a contraction:
By (2.13),

$$\begin{aligned}
&\left\|A_p^\alpha \Psi_1(u,v) - A_p^\alpha \Psi_1(\overline{u},\overline{v})\right\|_{L^p} \\
\leq\ & \left\|\left(e^{-\tau A_p} A_p^\alpha \varphi + \int_0^\tau A_p^\alpha e^{-(\tau-s)A_p} F(u(s),v(s))ds\right)\right. \\
&\left. - \left(e^{-\tau A_p} A_p^\alpha \varphi + \int_0^\tau A_p^\alpha e^{-(\tau-s)A_p} F(\overline{u}(s),\overline{v}(s))ds\right)\right\|_{L^p} \\
=\ & \left\|\int_0^\tau \left(A_p^\alpha e^{-(\tau-s)A_p} F(u(s),v(s)) - A_p^\alpha e^{-(\tau-s)A_p} F(\overline{u}(s),\overline{v}(s))\right) ds\right\|_{L^p} \\
\leq\ & \int_0^\tau \left\|\left(A_p^\alpha e^{-(\tau-s)A_p} F(u(s),v(s))) - A_p^\alpha e^{-(\tau-s)A_p} F(\overline{u}(s),\overline{v}(s))\right)\right\|_{L^p} ds
\end{aligned}$$

Factorizing and using the general property (2.1), we get that

$$\begin{aligned}
&\left\|A_p^\alpha \Psi_1(u_1,u_2) - A_p^\alpha \Psi_1(\overline{u}_1,\overline{u}_2)\right\|_{L^p} \\
\leq\ & \int_0^\tau \left\|A_p^\alpha e^{-(\tau-s)A_p}(F(u(s),v(s)) - F(\overline{u}(s),\overline{v}(s))\right\|_{L^p} ds \\
\leq\ & \int_0^\tau \left\|A_p^\alpha e^{-(\tau-s)A_p}\right\| \|(F(u(s),v(s)) - F(\overline{u}(s),\overline{v}(s)\|_{L^p} ds \\
\leq\ & C_\alpha \int_0^\tau (\tau-s)^\alpha e^{-\delta(\tau-s)} \|F(u(s),v(s)) - F(\overline{u}(s),\overline{v}(s))\|_{L^p} ds
\end{aligned}$$

Now, using (1.6) and (1.7), we obtain

$$\begin{aligned}
&\|F(u(s),v(s)) - F(\overline{u}(s),\overline{v}(s))\|_{L^p} \\
=\ & \|f(u(s),v(s)) + u(s) - f(\overline{u}(s),\overline{v}(s)) - \overline{u}(s)\|_{L^p} \\
\leq\ & \|f(u(s),v(s)) - f(\overline{u}(s),\overline{v}(s)) + (u(s) - \overline{u}(s))\|_{L^p} \\
\leq\ & (L_f + 1)\left(\|(u - \overline{u}\|_{L^p} + \|v - \overline{v}\|_{L^p}\right).
\end{aligned}$$

Arguing in a completely analogous way, we derived similar inequalities for G instead of F. Then, it follows that Ψ is a contraction provided that if T is taken small enough. Hence, there exists a unique fixed point (u,v) of Ψ which is the mild solution of (2.11) on $[0,T]$. □

Lemma 2.3. *Let $f \in L^1([0,T];X)$ be locally Hölder-continuous on $(0,T]$. Then, for any $u_0 \in X_\alpha$ the mild solution of*

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = f(t), & \text{for } t > 0\,, \\[2ex] u(0) = u_0 \end{cases} \tag{2.17}$$

is a classical solution.

Proof. See Pazy [18, Corollary 4.3.3]. □

Lemma 2.4 (Regularity of the mild solution (u,v))**.** *For any $u_0, v_0 \in X_\alpha$ the mild solution of (2.11) is a classical solution.*

Proof. We will show that the function $\tau \longmapsto F(u(\tau), v(\tau))$ is locally Hölder-continuous. Then Lemma 2.3 implies that the mild solution $u(\tau)$ of (2.11) is a classical solution of the first equation in (1.5).
With the relation

$$A_p^\alpha u(\tau) = e^{-\tau A_p} A_p^\alpha \varphi + \int_0^\tau A_p^\alpha e^{-(\tau-s)A_p} F(u(s), v(s)) ds$$

for $0 < \tau < \tau + h < T$, we have that

$$\begin{aligned}
& A_p^\alpha (u(\tau+h) - u(\tau)) \\
= \; & \left(e^{-(\tau+h)A_p} A_p^\alpha u_0 + \int_0^{\tau+h} A_p^\alpha e^{-(\tau+h-s)A_p} F(u(s), v(s)) ds \right) \\
& - \left(e^{-\tau A_p} A_p^\alpha u_0 + \int_0^\tau A_p^\alpha e^{-(\tau-s)A_p} F(u(s), v(s)) ds \right) \\
= \; & (e^{-hA_p} - I) e^{-\tau A_p} A_p^\alpha u_0 + \\
& \left(\int_0^\tau A_p^\alpha e^{-(\tau+h-s)A_p} F(u(s), v(s)) ds - \int_0^\tau A_p^\alpha e^{-(\tau-s)A_p} F(u(s), v(s)) ds \right) \\
& + \int_\tau^{\tau+h} A_p^\alpha e^{-(\tau+h-s)A_p} F(u(s), v(s)) ds \\
= \; & (e^{-hA_p} - I) e^{-\tau A_p} A_p^\alpha u_0 + \int_0^\tau \left(e^{-hA_p} - I \right) A_p^\alpha e^{-(\tau-s)A_p} F(u(s), v(s)) ds \\
& + \int_\tau^{\tau+h} A_p^\alpha e^{-(\tau+h-s)A_p} F(u(s), v(s)) ds \\
= \; & I + II + III
\end{aligned}$$

Let $0 < \gamma < 1 - \alpha$. In order to estimate I, we use (2.1)-(2.2) and, as a consequence, we have that

$$\|I\|_{L^p} \leq \frac{C_{1-\gamma}}{\gamma} h^\gamma \left\| A_p^\gamma e^{-\tau A_p} A^\alpha \varphi \right\|_{L^p} \leq \frac{C_{1-\gamma} C(\gamma)}{\gamma} h^\gamma \tau^{-\gamma} \|\varphi\|_\alpha .$$

For the estimate of the expression II, we define $M := \sup_{0 \leq \tau \leq T} \|f_1(u(\tau), v(\tau))\|_{L^p}$ and we use again (2.1)-(2.2) in the following way,

$$\begin{aligned}
\|II\| & \leq \frac{C_{1-\gamma}}{\gamma} h^\gamma \int_0^\tau \left\| A_p^{\alpha+\gamma} e^{-(\tau-s)A_p} F(u(s), v(s))) \right\|_{L^p} ds \\
& \leq \frac{C_{1-\gamma}}{\gamma} C_{\gamma+\alpha} h^\gamma M \int_0^\tau (\tau - s)^{-(\gamma+\alpha)} ds \\
& = \frac{C_{1-\gamma} C_{\gamma+\alpha}}{\gamma(1-\gamma-\alpha)} M h^\gamma \tau^{1-(\gamma+\alpha)}
\end{aligned}$$

Similarly, we obtain that

$$\|III\| \leq C_\alpha M \int_\tau^{\tau+h} (\tau + h - s)^{-\alpha} ds = \frac{C_\alpha M}{1-\alpha} h^{1-\alpha}.$$

Hence

$$\|u(\tau+h) - u(\tau)\|_\alpha \leq K_1(\tau) h^\gamma \tag{2.18}$$

where $K_1(\tau) = C(\tau^{-\gamma} + 1) \in C((0,T)) \cap L^1(0,T)$.
We can repeat the same arguing, now with v instead of u, to get

$$\|v(\overline{\tau} + h) - v(\overline{\tau})\|_\alpha \leq K_2(\overline{\tau}) h^\gamma, \tag{2.19}$$

where $0 < \gamma < \alpha$ and $K_2(\overline{\tau}) = C(\overline{\tau}^{-\gamma} + 1) \in C((0,T)) \cap L^1(0,T)$. From equation (2.7) and (2.8) we deduce that

$$\frac{d\tau}{d\overline{\tau}} = \frac{a_1}{a_2}.$$

Due to the assumptions made on the drifts a_1 and a_2 (see (1.6)), it follows that $\overline{\tau} \longmapsto \tau(\overline{\tau})$ and $\tau \longmapsto \overline{\tau}(\tau)$ are C^1 functions with local inverses, which in turn are also of class C^1. Since Hölder-continuity is preserved under the composition of functions, we use (2.15)-(2.16) together with (2.18)-(2.19) to conclude that $(0,T) \ni \tau \to F(u(\tau), v(\tau)) \in X$ and $(0,T) \ni \overline{\tau} \to G(u(\overline{\tau}), v(\overline{\tau})) \in X$ are locally-Hölder continuous. In consequence, Lemma 2.3 implies that the mild solution (u,v) is a classical solution to the system (1.5). □

At this point we need to remember the following result.

Lemma 2.5. *If u is a solution of*

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = h(u(t)), & \textit{for } t > 0, \\ \\ u(0) = u_0 \end{cases}$$

where $h \in L^1(\mathbb{R})$, then $\frac{du}{dt} : (0,T) \to X_\beta$ is locally Hölder-continuous for any $\beta \in (0,1)$.

Proof. See *e.g.* Henry [13, Theorem 3.4.4]. □

Consider the first equation in (1.5). Since

$$(0,T) \ni s \to F(u(s), v(s)) \in X$$

is locally Hölder continuous, Lemma 2.5 asserts that $\frac{du}{dt} : (0,T) \to X_\beta$ is locally Hölder-continuous for any $\beta \in (0,1)$. Hence, by Lemma 2.1, u and $\frac{du}{dt} : (0,T) \to X_\beta$ are locally Hölder-continuous in the space variables for every ν satisfying to $0 \le \nu \le 2\beta - 2/p$. Then, u and $\frac{du}{dt}$ are Hölder-continuous on $\overline{\Omega} \times [t_1, t_2]$ for any $0 < t_1 < t_2 < T$. Similarly, v and $d\frac{dv}{dt} : (0,T) \to C^\nu(\overline{\Omega})$ are locally Hölder for $0 \le \nu \le 2\beta - 2/p$, and, consequently, v and $\frac{dv}{dt}$ are Hölder continuous on $\overline{\Omega} \times [t_1, t_2]$ for any $0 < t_1 < t_2 < T$.

Finally, once that $F(u,v),\ G(u,v) : (0,T) \to C^\nu(\overline{\Omega})$ are locally Hölder-continuous for $0 \le \nu < 2\beta - 2/p$, by well known regularity results for parabolic equations, yields that $u(t),\ v(t) \in C^{2+\nu}(\overline{\Omega})$ for every $t \in (0,T)$, where $0 \le \min\{2\beta - 2/p, 1\}$. Thus, (u,v) is a classical solution on $\overline{\Omega} \times [0,T]$.

Therefore, we have proved the following theorem.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. Assume that conditions (1.6)-(1.7) are fulfilled. Then for any $u_0, v_0 \in X_\alpha$, there exists $T > 0$ such that the system (1.5) has a unique classical solution (u,v) on $\overline{\Omega} \times [0,T]$.*

Remark 2.2. *The result of Theorem 2.2 can be generalized to the case of $f = f(x,t,u,v)$ and $g = g(x,t,u,v)$, which is of particular importance in many problems of applications where the reactions also depend on the space position and on the moment of time.*

3. Existence of Blow-up

Criteria of blow-up for systems of parabolic equations are normally more difficult to find than for the scalar case. The following version of Jensen's inequality will allow us to develop some criteria of blow-up to our system of reaction-diffusion (1.5).

Lemma 3.1 (Jensen Inequality for two functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is convex. Then for every $u,\ v \in L^1(\Omega)$,*

$$f\left(\frac{1}{|\Omega|}\int_\Omega u\,dx, \frac{1}{|\Omega|}\int_\Omega v\,dx\right) \le \frac{1}{|\Omega|}\int_\Omega f(u,v)\,dx\,, \tag{3.1}$$

where $|\Omega|$ denotes the n-Lebebsgue measure of Ω.

Proof. Due to the convexity of f, for each $(x_1, x_2) \in \mathbb{R}^2$ there exists $(z_1, z_2) \in \mathbb{R}^2$ such that

$$f(y_1, y_2) \geq f(x_1, x_2) + z_1(y_1 - x_1) + z_2(y_2 - x_2)$$

holds for all $(y_1, y_2) \in \mathbb{R}^2$, *i.e.* the graph of f lies above its supporting hyperplane at (x_1, x_2). In this inequality, let us take $x_1 = \frac{1}{|\Omega|}\int_\Omega u\,dx$, $x_2 = \frac{1}{|\Omega|}\int_\Omega v\,dx$, $y_1 = u$ and $y_2 = v$. This yield

$$f(u,v) \geq f\left(\frac{1}{|\Omega|}\int_\Omega u\,dx, \frac{1}{|\Omega|}\int_\Omega v\,dx\right) + z_1\left(u - \frac{1}{|\Omega|}\int_\Omega u\,dx\right) + z_2\left(v - \frac{1}{|\Omega|}\int_\Omega v\,dx\right).$$

Then, integrating over Ω, with respect to x, and observing that the terms which are multiplying by z_1 and z_2 vanish, we immediately arrive at (3.1). □

As a first example of the usefulness of the last Lemma, we have the following blow-up result.

Theorem 3.1 (Blow-up under Neumann boundary conditions)**.** *Let (u, v) be a couple of classical solutions to the reaction-diffusion system (1.5) endowed with the following Neumann boundary conditions:*

$$\nabla u \cdot \overrightarrow{n} = 0\,, \quad \nabla v \cdot \overrightarrow{n} = 0 \quad \text{on } \partial\Omega\,, \tag{3.2}$$

where $\overrightarrow{n}$ denotes the outward unit normal to $\partial\Omega$. Assume that:

1. *f and g are convex functions;*
2. *$f(u,v)+g(u,v) \geq h(u+v)$ for all $(u,v) \in \mathbb{R}^2$ and for some function $h : \mathbb{R} \longmapsto \mathbb{R}$ such that $h(w) > 0$ for all $w \geq \overline{u_0 + v_0}$ and*

$$t_* := \int_{\overline{u_0+v_0}}^{\infty} \frac{1}{h(s)} ds < \infty, \quad \text{where} \quad \overline{u_0 + v_0} = \frac{1}{|\Omega|}\int u_0 + v_0\,dx\,. \tag{3.3}$$

Then the solution (u, v) to the reaction-diffusion system {(1.5), (3.2)} blows-up in the finite time t_*.

Remark 3.1. *Some functions satisfying condition 2 are, in the case of $u_0 + v_0 > 0$, $f(s,r) = |r|^p$ and $g(s,r) = |s|^q$ for suitable $p \geq q > 1$, or still more general $f(s,r) = a|r|^p + b|r|^q$ $g(s,r) = c|r|^{\overline{p}} + d|r|^{\overline{q}}$ where a, b, c, d are positive real constants and p, q, $\overline{p}$, $\overline{q} > 1$. In the case of $u_0 + v_0 \leq 0$, one should consider, for instance, examples of the form $f(u,v) = g(u,v) = h(u+v) = (1 + |u+v|)^p$, for some $p > 1$, or $f(u,v) = g(u,v) = h(u+v) = e^{u+v}$.*

Proof. Adding up the equations for u and v in (1.5), we obtain

$$(u+v)_t - a_1\Delta u - a_2\Delta v = f(u,v) + g(u,v).$$

Let us denote $\overline{u(t)} = \frac{1}{|\Omega|}\int u(x,t)\,dx$, and $\overline{v(t)} = \frac{1}{|\Omega|}\int v(x,t)\,dx$. Integrating the above equation over Ω, using (3.2) and invoking the nonlocal character of a_1 and a_2, we obtain

$$\frac{d\,\overline{u(t)+v(t)}}{dt} = \frac{1}{|\Omega|}\int_\Omega f(u,v)\,dx + \frac{1}{|\Omega|}\int_\Omega g(u,v)\,dx\,.$$

Then, Lemma 3.1 and assumption (2) yield

$$\frac{d\,\overline{u(t)+v(t)}}{dt} \geq f\left(\overline{u(t)}, \overline{v(t)}\right) + g\left(\overline{u(t)}, \overline{v(t)}\right) \geq h(\overline{u(t)+v(t)})\,.$$

Finally, integrating between 0 and $t > 0$ and using (3.3), we obtain

$$\begin{aligned} t &\leq \int_0^t \frac{1}{h(\overline{u(t)+v(t)})}\frac{d\,\overline{u(t)+v(t)}}{d\tau}\,d\tau \\ &= \int_{\overline{u_0+v_0}}^{\overline{u(t)+v(t)}} \frac{ds}{h(s)} \leq \int_{\overline{u_0+v_0}}^{\infty} \frac{1}{h(s)} ds < \infty\,. \end{aligned}$$

Then, from the Theory of Ordinary Differential Equations (ODE), we conclude that $\overline{u(t)+v(t)}$ will blow-up in a finite time provided that $h(w) > 0$ for all $w \geq u_0 + v_0$. □

Theorem 3.2 (Blow-up under Dirichlet boundary conditions)**.** *Let (u, v) be a couple of classical solutions to the reaction-diffusion system (1.5) endowed with the following Dirichlet boundary conditions:*

$$u = 0\,, \quad v = 0 \quad \text{on } \partial\Omega\,. \tag{3.4}$$

Assume that f is convex, $f(u, v) \geq f(u, 0)$ for all $(u, v) \in \mathbb{R}^2$, $f(w, 0) + \lambda_1 a_1 w > 0$ for all $w \geq \mu(0)$ and

$$t_* := \int_{\mu(0)}^{\infty} \frac{d\mu}{f(\mu, 0) + \lambda_1 a_1 \mu} < \infty\,, \quad \text{where} \quad \mu(t) := \frac{1}{|\Omega|} \int_\Omega u(x, t)\phi(x)\, dt\,, \tag{3.5}$$

and λ_1 is the first eigenvalue to the problem

$$\begin{cases} \Delta\phi = \lambda_1 \phi \quad \text{in } \Omega \\ \phi = 0 \quad \text{on } \partial\Omega\,, \quad \displaystyle\int_\Omega \phi\, dx = 1\,. \end{cases} \tag{3.6}$$

Then, the first component of the solution (u, v) to the reaction-diffusion system {(1.5), (3.4) blows-up in the finite time t_*.

Proof. Firstly, we observe the well-known fact that the eigenvalue problem (3.6) has a solution $\phi \in H_0^1(\Omega)$ such that $\phi > 0$ in Ω, and

$$\lambda_1 = \inf \left\{ \|\nabla\varphi\|_{L^2} : \varphi \in H_0^1(\Omega),\ \|\varphi\|_{L^2} = 1 \right\}.$$

Moreover, this solution is C^∞ in the interior of Ω and is $C^{k+\beta}$ $(k \geq 2)$ on the boundary if $\partial\Omega \in C^{k+\beta}$. Multiplying the first equation of (1.5) by ϕ, integrating over Ω and using (3.4) and (3.6) together with the nonlocal character of a_1, we obtain

$$\frac{d}{dt} \int_\Omega u\phi\, dx - \lambda_1 a_1 \int_\Omega u\phi\, dx = \int_\Omega f(u, v)\phi\, dx$$

Observing that $\int_\Omega \phi\, dx = 1$, we can use Jensen's inequality (3.1), to prove that

$$\frac{1}{|\Omega|} \int_\Omega f(u, v)\phi\, dx \geq f\left(\frac{1}{|\Omega|} \int_\Omega u\,\phi\, dx, \frac{1}{|\Omega|} \int_\Omega v\,\phi\, dx \right).$$

Substituting this into the previous equation and, in addition, using the hypothesis that $f(u, v) \geq f(u, 0)$ for all $(u, v) \in \mathbb{R}^2$, we get

$$\mu'(t) - \lambda_1 a_1 \mu(t) \geq f(\mu(t), 0).$$

It should be noted that $\mu(t)$ is well defined on the existence interval of the solution u. Then, integrating the last inequality between 0 and $t > 0$, and using the fact that $\mu(0) \geq 0$ and hypothesis (3.5), we obtain

$$\begin{aligned} t &\leq \int_0^t \frac{\mu'(\tau)}{f(\mu(\tau), 0) + \lambda_1 a_1 \mu(\tau)}\, d\tau = \int_{\mu(0)}^{\mu(t)} \frac{d\mu}{f(\mu, 0) + \lambda_1 a_1 \mu} \\ &\leq \int_{\mu(0)}^{\infty} \frac{d\mu}{f(\mu, 0) + \lambda_1 a_1 \mu} < \infty\,. \end{aligned}$$

Then, from the ODE Theory, we conclude that $\mu(t)$, and consequently u, will blow-up in a finite time provided that $f(w, 0) + \lambda_1 a_1 w > 0$ for all $w \geq \mu(0)$. □

Remark 3.2. *We observe that according to the proof of the last result, we had no need to use the boundary condition $v = 0$ on $\partial\Omega$. Therefore, we still have blow-up of the first component even if v is not prescribed at the boundary.*

We end up this section by giving a criteria of simultaneous blow-up.

Theorem 3.3 (Simultaneous blow-up)**.** *Let us consider the following system with cross reactions*

$$\begin{aligned} u_t - a_1 \Delta u &= f(v) \\ v_t - a_2 \Delta v &= f(u)\,, \end{aligned} \tag{3.7}$$

which is supplemented with initial data u_0 and v_0, and with the Neumann boundary conditions (3.2). We assume that:

(1) *f is a convex function;*

(2) *$f(w) > 0$ for all $w \geq \overline{\frac{u_0+v_0}{2}}$, and*

$$t_* := \int_{\overline{\frac{u_0+v_0}{2}}}^{\infty} \frac{ds}{f(s)} < \infty\,, \quad \text{where} \quad \overline{\frac{u_0+v_0}{2}} = \frac{1}{2\,|\Omega|}\int u_0 + v_0\, dx\,. \tag{3.8}$$

Then the solution (u,v) to the reaction-diffusion system $\{(3.7),(3.2)\}$ blows up simultaneously in the finite time t_*, i.e,

$$\lim_{t\to t_*^-} \sup \|u(\cdot,t)\|_{L^\infty} = \infty \text{ iff } \lim_{t\to t_*^-} \sup \|v(\cdot,t)\|_{L^\infty} = \infty.$$

Proof. Arguing as we did in the first part of the proof of Theorem 3.1, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\overline{\frac{u(t)+v(t)}{2}}\right) =& \frac{1}{2\,|\Omega|}\int \left(f(u)+f(v)\right)\, dx \\ \geq& \frac{1}{2} f(\overline{u(t)}) + \frac{1}{2} f(\overline{v(t)}) \geq f\left(\overline{\frac{u(t)+v(t)}{2}}\right). \end{aligned}$$

Recall that $\overline{u(t)} = \frac{1}{|\Omega|}\int u(x,t)\,dx$ and $\overline{v(t)} = \frac{1}{|\Omega|}\int v(x,t)\,dx$, and observe that in the last inequality we again have made use of the convexity of f. Then, integrating between 0 and $t > 0$ and using (3.8), we obtain

$$\begin{aligned} t \leq& \int_0^t \frac{1}{f\left(\overline{\frac{u(t)+v(t)}{2}}\right)} \frac{d\,\overline{\frac{u(t)+v(t)}{2}}}{d\tau}\, d\tau \\ =& \int_{\overline{\frac{u_0+v_0}{2}}}^{\overline{\frac{u(t)+v(t)}{2}}} \frac{ds}{f(s)} \leq \int_{\overline{\frac{u_0+v_0}{2}}}^{\infty} \frac{1}{f(s)} ds < \infty\,. \end{aligned}$$

Again, as in the proof of Theorem 3.1, we conclude that $\overline{\frac{u(t)+v(t)}{2}}$ will blow-up in the finite time t_* provided that $f(w) > 0$ for all $w \geq \overline{\frac{u_0+v_0}{2}}$.

Now, in order to show that the blow-up is simultaneous, let us suppose that u is bounded on some interval, say $[0,T]$, *i.e.*

$$\sup_{0\leq t\leq T} \|u\|_{L^\infty} \leq C, \quad C = Const. > 0.$$

In consequence, v satisfies to the following differential inequality

$$v_t - a_2 \Delta v \leq K,$$

for some positive constant K. Next, we consider the problem

$$\begin{aligned} V_t - a_2\Delta V &= K \\ V(x,t) &= v(x,t) \ \text{ on } \partial\Omega \times [0,T] \\ V(x,0) &= v_0(x)\,, \end{aligned}$$

which defines V as a bounded supersolution for the equation in v, *i.e.* $v \leq V$. Thus we have proven that u bounded implies that v is bounded too. In a like manner, we have that v bounded implies that u is also bounded. In conclusion, if blow-up happens it should be simultaneously. □

4. Invariant Regions and convergence toward homogeneous states

The aim of the present section, is to give some criteria of convergence towards a homogeneous state. In the scalar case, if we take

$$a(t) = a\left(\int_{\mathbb{R}^n} |\nabla u|^2\, dx\right),$$

we can readily obtain a Lyapunov function and therefore the analysis of the asymptotic behavior follows as in [10]. For another kind of functionals and, in particular, for systems like (1.5), a Lyapunov functional is not feasible. Our approach will be first to give a criterium for the existence of invariant regions and then to exploit this to study the asymptotic behavior.

First we recall the definition of invariant region.

Definition 4.1. *Let $\Sigma \subset \mathbb{R}^2$ be a closed set and let (u,v) a solution for $0 \le t \le T \le \infty$ to the system (1.5) supplemented with initial and boundary conditions. If the boundary values as well as (u,v) are is Σ for all $x \in \Omega$ and all $0 \le t \le T$, then Σ is an invariant set for the solution (u,v).*

Theorem 4.1. *Let $\Sigma = [a,b] \times [c,d] \subset \mathbb{R}^2$ be a rectangle in the $uv-$plane and let $\overrightarrow{n}$ be the outward unit normal. If $(f(u,v), g(u,v)) \cdot \overrightarrow{n} < 0$ on $\partial\Sigma$, then Σ is an invariant set to the reaction-diffusion system (1.5).*

Proof. Let us assume that Σ is not an invariant region and suppose for example that $u(x_0,t_0) = b$ for some (x_0,t_0) with $u(x,t) < b$ for all $x \in \Omega$ and $0 < t < t_0$, and yet

$$u_t(x_0,t_0) \ge 0\,. \tag{4.1}$$

At $t = t_0$, the trajectory $u(x,t_0)$ touch for the first time the line $u = b$ and still without getting outside from Σ, and therefore with tangency. As a consequence, the function $u(x,t_0)$, regarded only as a function of x, must have a maximum at $x = x_0$. It follows that $\Delta u(x_0,t_0) \le 0$ and at (x_0,t_0) we have

$$u_t = a_1(p(u), q(v))\Delta u + f(u,v) \le f(u,v) = (f(u,v), g(u,v)) \cdot (1,0) < 0. \tag{4.2}$$

Eq. (4.2) contradicts (4.1) and therefore Σ must be an invariant region set. □

As we pointed out at the introductory section, nonlocal parabolic problems can have several stationary solutions associated. We can now use the invariant regions in order to decide when the asymptotic behavior is toward a homogenous state.

Following Smoller [19, Chapter 14], let X be a Banach space and for $(u,v) \in X \times X$ assume that the application

$$(u,v) \longmapsto (f(u,v), g(u,v))$$

is Fréchet differentiable, with Fréchet derivative df. The mapping $f \to df_u$ is continuous from X to $Hom(X)$. Let λ be the principal eigenvalue of $-\Delta$, $m := \min\{m_1, m_2\}$,

$$M := \max\left\{\sqrt{|\partial_1 f(u)|^2 + |\partial_2 f(u)|^2} : (u,v) \in \Sigma\right\}$$

and $\sigma := \lambda d - M$. In addition, for $(g,h) \in L^2(\Omega) \times L^2(\Omega)$ we denote

$$\|(g,h)\| := \|g\|_{L^2} + \|h\|_{L^2}\,.$$

Theorem 4.2. *Let us consider the system (1.5) supplemented with homogenous boundary conditions. Assume that (1.5) admits a bounded invariant region Σ and*

$$\{(u_0(x), v_0(x)) : x \in \Omega\} \subset \Sigma\,.$$

Then we have

$$u(x,t) \to \frac{1}{|\Omega|}\int u(x,t)dx \ \text{ as } t \to \infty \ \text{ and } v(x,t) \to \frac{1}{|\Omega|}\int v(x,t)dx \ \text{ as } t \to \infty\,,$$

where the convergence take place in the norm of $L^2(\Omega)$ and it is exponentially in time.

Proof. This result is a consequence from Theorem 4.1 and [19, Theorem 14.17]. Therefore we only give a sketch of the proof. We denote $\overline{u(t)} := \frac{1}{|\Omega|}\int u(x,t)\,dx$. By Poincaré's inequality, we have

$$\lambda \left\| u(\cdot,t) - \overline{u(t)} \right\|_{L^2} \le \|\nabla u(\cdot,t)\|_{L^2}\,.$$

In consequence, it is enough to prove that

$$\|\nabla u(\cdot,t)\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

With this end in mind, we define

$$E(t) := \frac{1}{2}\|\nabla u(\cdot,t)\|^2_{L^2} + \frac{1}{2}\|\nabla v(\cdot,t)\|^2_{L^2}\,.$$

Then arguing as in [19], we obtain

$$\frac{dE}{dt} \leq 2\,(2M - m\lambda)\,E(t),$$

which yields

$$E(t) \leq E(0)e^{2(2M-m\lambda)t}.$$

Taking into account that $2M - m\lambda < 0$, we conclude that $E(t) \to 0$, as $t \to \infty$. □

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