ANALYSIS OF THE EXISTENCE FOR THE STEADY NAVIER-STOKES EQUATIONS WITH ANISOTROPIC DIFFUSION

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Abstract. The boundary-value problem for the generalized Navier-Stokes equations with anisotropic diffusion is considered in this work. For this problem we prove the existence of weak solutions in the sense that solutions and test functions are considered in the same admissible function space. We prove also the existence of very weak solutions, *i.e.*, solutions for which the test functions have more regularity. By exploiting several examples we show, in the case of dimension 3, that these existence results improve its isotropic versions in almost all directions or for particular choices of all the diffusion coefficients.

Key words and phrases: Anisotropic diffusion, generalized Navier-Stokes, steady problem, existence, weak solutions, very weak solutions.

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1. Introduction

The aim of this work is to analyze the existence of weak solutions for the boundary-value problem posed by the generalized Navier-Stokes equations with anisotropic diffusion. The problem is considered in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$, and the governing equations are:

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega; \tag{1.1}$$

$$\mathbf{div}(\mathbf{u} \otimes \mathbf{u}) = \mathbf{f} - \nabla p + \sum_{i=1}^{N} D_i \left(|D_i \mathbf{u}|^{q_i - 2} D_i \mathbf{u} \right) \quad \text{in} \quad \Omega; \tag{1.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on} \quad \partial\Omega. \tag{1.3}$$

Here, the vector function $\mathbf{u} = (u_1, \cdots, u_N)$ and the scalar function p are the unknowns of the problem. The vector function $\mathbf{f} = (f_1, \cdots, f_N)$ is a given problem data, $D_i \mathbf{u} = (\partial_i u_1, \ldots, \partial_i u_N)$ and $\partial_i u_j = \frac{\partial u_j}{\partial x_i}$. The exponents q_i are assumed to be constants with possible distinct values but such that $1 < q_i < \infty$ for any $i = 1, \ldots, N$.

The isotropic version of the problem (1.1)-(1.3) has been studied in the context of Mathematical Fluid Mechanics during the last 50 years and the simplest case corresponds to the following (isotropic) diffusion term (see *e.g.* [15, 17, 18]):

$$\mathcal{A}(\mathbf{u}) = \mathbf{div} \left(|\mathbf{D}(\mathbf{u})|^{q-2} \mathbf{D}(\mathbf{u}) \right), \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2} \left(\partial_i u_j + \partial_j u_i \right)_{i,\, j=1,\ldots,N}, \quad 1 < q < \infty. \tag{1.4}$$

In this scope, $\mathbf{u}$ is the velocity field, p stands for the pressure divided by the constant density, $\mathbf{f}$ is the external forces field and q is the power-law index that characterizes the flow.

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Using the same notation of (1.4), we can write the anisotropic diffusion term present in (1.2) as

$$\mathbf{A}(\mathbf{u})=\sum_{i=1}^{N}\mathcal{A}_i(\mathbf{u})\,,\quad \text{where}\quad \mathcal{A}_i(\mathbf{u})=D_i\left(|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right). \tag{1.5}$$

Letting $\{\mathbf{e}_1,\dots,\mathbf{e}_N\}$ be the canonical basis of $\mathbb{R}^N$, we can write the anisotropic diffusion in the following meaningful formulation

$$\mathcal{A}_i(\mathbf{u})=\mathbf{div}\left(|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\otimes\mathbf{e}_i\right)\Rightarrow\mathbf{A}(\mathbf{u})=\mathbf{div}\left(\sum_{i=1}^{N}|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\otimes\mathbf{e}_i\right). \tag{1.6}$$

Existence results for the isotropic version of the problem (1.1)-(1.4) started with the work [14] for $N=3$ and with [17] for a general dimension N. Later on these results were improved firstly in the works [10, 22] and finally in [11, 8], with many other works, that for reasons of lack of space we can not cite, published between the first and the last.

With respect to the existence results for elliptic or parabolic equations with different types of anisotropic diffusion there is an increase of interest in studying such problems in recent years. We address the reader to [9, 26] and the references cited therein for existence and regularity results on anisotropic quasilinear elliptic equations. For existence and uniqueness results on parabolic problems with anisotropic diffusion, we address the reader to [2, 24] and the references cited therein. See also [17, Exemple 1.7.1] were the existence of weak solutions is proved for a parabolic problem with anisotropic diffusion.

In the literature we have not found any reference to Navier-Stokes equations with anisotropic diffusion as we consider in problem (1.1)-(1.4). The denomination anisotropic Navier-Stokes equations is used in the literature always for the classical 3D Navier-Stokes problem where anisotropy means that the vertical viscosity is distinct from the horizontal one (see *e.g.* [5, 6]). There is also in the literature other types of Navier-Stokes equations where anisotropy is considered in other terms (see *e.g.* [3]). In the present work, although we can find some features of our problem on the applications, our motivation is purely mathematical. In particular, we shall see that, by considering an anisotropic diffusion, we can considerably decrease the lower bounds of the diffusion exponents. Therefore our work improves the results of [14, 17] and of [11, 8] in almost all directions or for particular choices of all the diffusion coefficients. This can be potentially useful in the aforementioned works where the vertical and horizontal viscosities are distinct.

The article is organized as follows. Section 2 is dedicated to introduce the main concepts of the anisotropic function spaces we are going to work with. The existence of weak solutions to the problem (1.1)-(1.4) in the spirit of the works [14, 17] is established in Section 3. In Section 4 we will provide an extensive list of examples, in the case of $N=3$, for several situations where the anisotropic existence results improve the isotropic ones. The existence of very weak solutions to the problem (1.1)-(1.4), *i.e.*, solutions in the spirit of the work [11], is proved in Section 5. For this kind of weak solutions, we also provide several examples where the anisotropic existence results improve the isotropic ones, now in Section 6.

The notation used throughout this article and the main notions of the considered (isotropic) function spaces are largely standard in the literature of Partial Differential Equations and in Mathematical Fluid Mechanics as well. We address the reader to the monographs [15, 17, 12, 18] for any question related to that matter.

2. Anisotropic spaces

In order to introduce the notions of weak solutions we shall consider in this work, let us recall the well-known function spaces of Mathematical Fluid Mechanics. Given $r,\ q$ such that $1<r,\ q<\infty$, we set

$$\mathcal{V}:=\{\mathbf{v}\in\mathbf{C}_0^\infty(\Omega):\operatorname{div}\mathbf{v}=0\}, \tag{2.1}$$

$$\mathbf{H}_r:=\text{closure of }\mathcal{V}\text{ in the norm }\|\cdot\|_{\mathbf{L}^r(\Omega)}, \tag{2.2}$$

$$\mathbf{V}_q:=\text{closure of }\mathcal{V}\text{ in the norm }\|\cdot\|_{\mathbf{W}^{1,q}(\Omega)}. \tag{2.3}$$

Due to the possible presence of different exponents q_i for distinct directions, we need to consider the weak solutions to the problem (1.1)-(1.3) in some anisotropic Sobolev space. We define the vector $\mathbf{q}$ in $\mathbb{R}^N$, whose components are the exponents of the anisotropic diffusion introduced in the (1.2), by

$$\mathbf{q} := (q_1, \cdots, q_N), \quad 1 < q_i < \infty \quad \forall\, i \in \{1, \ldots, N\}\,, \tag{2.4}$$

and let

$$\beta := \max_{i=1,\ldots,N} q_i \quad \text{and} \quad \alpha := \min_{i=1,\ldots,N} q_i\,. \tag{2.5}$$

To avoid any confusion that $\mathbf{q}$ is in fact a vector, in the rest of our work we will emphasize this meaning by writing an arrow over $\mathbf{q}$: $\overrightarrow{\mathbf{q}}$. We define the following anisotropic Sobolev space

$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) = \left\{\mathbf{v} \in \mathbf{W}^{1,1}(\Omega) : \quad D_i\mathbf{v} \in \mathbf{L}^{q_i}(\Omega) \quad \forall\, i = 1, \ldots, N\right\}\,,$$

which is a separable Banach space for the norm

$$\|\mathbf{v}\|_{\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega)} = \|\mathbf{v}\|_{\mathbf{L}^1(\Omega)} + \sum_{i=1}^{N} \|D_i\mathbf{v}\|_{\mathbf{L}^{q_i}(\Omega)}\,.$$

An important limitation of the anisotropic Sobolev space $\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega)$, is that, for bounded domains Ω, the validity of Sobolev imbeddings is restricted to rectangular domains. The reason is that the Extension Theorem, which is used in the isotropic case, combines the derivatives in a such way that the extended function no more belongs to the anisotropic Sobolev space (see *e.g.* [13]). In fact, for rectangular domains Ω, the following imbedding is continuous (*cf.* [21, Theorem 1])

$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow \mathbf{L}^s(\Omega) \quad \text{for any } s : \begin{cases} 1 \le s \le \overline{q}^{\,*}\,, & \sum_{j=1}^N \frac{1}{q_j} > 1 \\ 1 \le s < \infty\,, & \sum_{j=1}^N \frac{1}{q_j} \le 1 \end{cases}\,, \tag{2.6}$$

where $\overline{q}^{\,*}$ denotes the Sobolev conjugate of $\overline{q}$, the harmonic mean of $q_1, \ldots, q_N$:

$$\overline{q}^{\,*} := \frac{N\overline{q}}{N-\overline{q}}, \quad \overline{q} := \frac{N}{\sum_{j=1}^N \frac{1}{q_j}}, \quad \sum_{j=1}^{N} \frac{1}{q_j} > 1 \quad \Rightarrow \quad \overline{q}^{\,*} = \frac{N}{\sum_{j=1}^N \frac{1}{q_j} - 1}\,. \tag{2.7}$$

Moreover, the imbedding (2.6) is compact (*cf.* [21, Theorem 2]), and we denote this fact by

$$\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^s(\Omega) \quad \text{for any } q : 1 \le s < \overline{q}^{\,*}. \tag{2.8}$$

In some situations it is possible to remove the restrictions on the shape's domain and to enlarge the interval of s for the validity of (2.6) and (2.8). Let us see this fact by defining

$$\mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) := \text{closure of } \mathbf{C}_0^\infty(\Omega) \text{ in the anisotropic norm of } \mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega)\,.$$

In this case, we have (*cf.* [9, Theorem 1]), in the interesting case of $\sum_{j=1}^N \frac{1}{q_j} > 1$, that

$$\mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow \mathbf{L}^s(\Omega) \quad \text{for any } s : 1 \le s \le q_a^* \tag{2.9}$$

and

$$\mathbf{W}_0^{1,\overrightarrow{\mathbf{q}}}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^s(\Omega), \quad \text{for any } q : 1 \le s < q_a^*\,, \tag{2.10}$$

where q_a^* is the critical exponent defined by

$$q_a^* := \max\{\overline{q}^{\,*}, \beta\}\,. \tag{2.11}$$

Remark 2.1. *Note that*

$$q_i > 1 \Rightarrow \overline{q}^{\,*} > \frac{Nq_i}{(N-2)q_i + 1} \quad \forall\, i = 1, \ldots, N$$

and in particular, for $N = 2$, $\overline{q}^{\,*} > 2q_i$ for all $i = 1, \ldots, N$. Thus, for $N = 2$, $\overline{q}^{\,*} > \beta$ and therefore $q_a^* = \overline{q}^{\,*}$. But, if $N > 2$, it may well happen that $\beta > \overline{q}^{\,*}$. In fact, for $N > 2$,

$$\beta > \overline{q}^{\,*} \Leftrightarrow \overline{q} < \frac{\beta N}{\beta + N} \Leftrightarrow \sum_{j=1,\ q_j \neq \beta}^{N} \frac{1}{q_j} > 1 + \frac{N-1}{\beta}\,.$$

This means that, in typical situations when $N > 2$, $q_a^* = \beta$ if, by one hand, all the components q_i of $\overrightarrow{\mathbf{q}}$ but β are not too far apart and must belong to the lowest ranges of their admissible interval stated in (2.4). On the other hand, β must be too far apart from the other components. For instance, when $N = 3$, $\overrightarrow{\mathbf{q}} = \left(\frac{10}{7}, \frac{20}{7}, \beta\right)$, we have $\beta > \overline{q}^*$ if and only $\beta > 40$.

Let us now define the anisotropic analogue of $\mathbf{V}_q$ by

$$\mathbf{V}_{\overrightarrow{\mathbf{q}}} := \text{closure of } \mathcal{V} \text{ in the anisotropic norm of } \mathbf{W}^{1,\overrightarrow{\mathbf{q}}}(\Omega)\,. \tag{2.12}$$

Due to Poincaré's inequality, an equivalent norm in $\mathbf{V}_{\overrightarrow{\mathbf{q}}}$ is defined by

$$\|\mathbf{v}\|_{\mathbf{V}_{\overrightarrow{\mathbf{q}}}} = \sum_{i=1}^{N} \|D_i\mathbf{v}\|_{\mathbf{L}^{q_i}(\Omega)}$$

Moreover, defining for any $i = 1, \ldots, N$

$$\mathbf{V}_{q_i} := \text{closure of } \mathcal{V} \text{ in the (isotropic) norm } \|\cdot\|_{\mathbf{W}^{1,q_i}(\Omega)}\,,$$

another equivalent norm in $\mathbf{V}_{\overrightarrow{\mathbf{q}}}$ is defined by

$$\|\mathbf{v}\|_{\mathbf{V}_{\overrightarrow{\mathbf{q}}}} = \sum_{i=1}^{N} \|\mathbf{v}\|_{\mathbf{V}_{q_i}}\,.$$

In the sequel we will denote by $\mathbf{V}'_q$, $\mathbf{V}'_{q_i}$ and $\mathbf{V}'_{\overrightarrow{\mathbf{q}}}$ the dual spaces of $\mathbf{V}_q$, $\mathbf{V}_{q_i}$ and $\mathbf{V}_{\overrightarrow{\mathbf{q}}}$, respectively.

3. Weak formulation

First we observe that, according to what is customary in Mathematical Fluid Mechanics, the determination of the pressure p is not a problem. In fact, after we determine the velocity field $\mathbf{u}$, we can recover the pressure by applying de Rham's theorem (see *e.g.* [15, 17, 12, 18]). Therefore our main concern in this work will be to establish the existence results for the unknown field $\mathbf{u}$. With this in mind, we are now in conditions to define the notions of weak solutions we shall consider and to state the main results of this work.

Definition 3.1. *Let $N \geq 2$ and $\overrightarrow{\mathbf{q}} = (q_1, \ldots, q_N)$, with $1 < q_i < \infty$ for any $i = 1, \ldots, N$. Assume that $\mathbf{f} \in \mathbf{V}'_{\overrightarrow{\mathbf{q}}}$. A vector field $\mathbf{u}$ is a weak solution to the problem (1.1)-(1.3), if:*

1. *$\mathbf{u} \in \mathbf{V}_{\overrightarrow{\mathbf{q}}}$;*
2. *For every $\mathbf{v} \in \mathbf{V}_{\overrightarrow{\mathbf{q}}} \cap \mathbf{L}^{\theta}(\Omega)$*

$$\sum_{i=1}^{N} \int_{\Omega} |D_i\mathbf{u}|^{q_i-2} D_i\mathbf{u} \cdot D_i\mathbf{v}\, d\mathbf{x} + \int_{\Omega} \mathbf{div}(\mathbf{u} \otimes \mathbf{u}) \cdot \mathbf{v}\, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}\, d\,\mathbf{x}, \tag{3.1}$$

where

$$\frac{1}{q_a^*} + \frac{1}{\alpha} + \frac{1}{\theta} = 1. \tag{3.2}$$

Remark 3.1. *The need of the test function $\mathbf{v} \in \mathbf{L}^{\theta}(\Omega)$, with θ satisfying to (3.2), is necessary for the boundedness of the convective integral. In fact, by Hölder's inequality together with the anisotropic Sobolev imbedding inequality (2.9), we have*

$$\left|\int_{\Omega} \mathbf{u} \otimes \mathbf{u} : \nabla\,\mathbf{v}\, d\mathbf{x}\right| = \left|-\int_{\Omega} [(\mathbf{u} \cdot \nabla)\,\mathbf{u}] \cdot \mathbf{v}\, d\mathbf{x}\right|$$
$$\leq \|\mathbf{u}\|_{\mathbf{L}^{q_a^*}(\Omega)} \|\nabla\,\mathbf{u}\|_{\mathbf{L}^{\alpha}(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^{\theta}(\Omega)} \leq C\|\mathbf{u}\|^2_{\mathbf{V}_{\overrightarrow{\mathbf{q}}}} \|\mathbf{v}\|_{\mathbf{L}^{\theta}(\Omega)} < \infty,$$

provided that (3.2) holds, Ω is bounded, $\mathbf{u} \in \mathbf{V}_{\overrightarrow{\mathbf{q}}}$ and $\mathbf{v} \in \mathbf{L}^{\theta}(\Omega)$. If $\theta \leq q_a^*$, then, by (2.9), $\mathbf{V}_{\overrightarrow{\mathbf{q}}} \hookrightarrow \mathbf{L}^{\theta}(\Omega)$ and therefore it is enough to consider (3.1) for every $\mathbf{v} \in \mathbf{V}_{\overrightarrow{\mathbf{q}}}$.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume that $\mathbf{f} \in \mathbf{V}'_{\overrightarrow{\mathbf{q}}}$. If*

$$\alpha \geq \left(\frac{q_a^*}{2}\right)', \tag{3.3}$$

then there exists, at least, a weak solution to the problem (1.1)-(1.3) in the sense of Definition 3.1.

The analysis of the relation (3.3), with the indication of several examples, will be done in the final part of this section and at Section 4.

Proof. The proof of Theorem 3.1 will take us the rest of this section.

3.1. **Step 1: Existence of approximative solutions.** For the smallest integer $s > 1 + \frac{N}{2}$, we define

$$\mathbf{V}^s := \text{closure of } \mathcal{V} \text{ in } \mathbf{W}^{s,2}(\Omega)\,.$$

Let $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ be a set of non-trivial solutions $\mathbf{v}_j$ of the following spectral problem associated to the eigenvalues $\lambda_j > 0$:

$$\sum_{|\gamma|=s}\int_\Omega \mathrm{D}^\gamma\mathbf{v}_j\cdot\mathrm{D}^\gamma\varphi\,d\mathbf{x} = \lambda_j\int_\Omega \mathbf{v}_j\cdot\varphi\,d\mathbf{x}\quad \forall\,\varphi\in\mathbf{V}^s\,.$$

The family $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ is orthogonal in $\mathbf{H}_2$ and can be chosen as being orthonormal in $\mathbf{V}^s$ (see *e.g.* [18, p. 290]). Given $m \in \mathbb{N}$, let us also consider the correspondingly m-dimensional space, say $\mathbf{V}^m$, spanned by $\mathbf{v}_1$, $\dots$, $\mathbf{v}_m$. For each $m \in \mathbb{N}$, we search for an approximative solution $\mathbf{u}^m$ of (3.1) in the form

$$\mathbf{u}^m = \sum_{k=1}^m c_k^m\mathbf{v}_k\,, \tag{3.4}$$

where $\mathbf{v}_k \in \mathbf{V}^m$. This function is found by solving the following system of m nonlinear algebraic equations with respect to the m unknowns $c_1^m, \dots, c_m^m$, obtained from (3.1):

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega |D_i\mathbf{u}^m|^{q_i-2}D_i\mathbf{u}^m\cdot D_i\mathbf{v}_k\,d\mathbf{x} + \int_\Omega \mathbf{div}(\mathbf{u}^m\otimes\mathbf{u}^m)\cdot\mathbf{v}_k\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}_k\,d\,\mathbf{x},\\ &\text{for } k = 1,\dots,m.\end{aligned} \tag{3.5}$$

The existence of a solution $\mathbf{u}^m$ to (3.5) can be proved on the basis of Riesz and Brouwer's theorems (see *e.g.* [17, Lemme 1.4.3]).

3.2. **Step 2: A priori estimates.** Multiplying (3.5) by c_k^m and adding up the resulting equation from $k = 1$ until $k = m$, we obtain

$$\sum_{i=1}^N \|D_i\mathbf{u}^m\|^{q_i}_{\mathbf{L}^{q_i}(\Omega)} = \int_\Omega \mathbf{f}\cdot\mathbf{u}^m\,d\,\mathbf{x}. \tag{3.6}$$

By the application of Hölder's inequality together with the anisotropic Sobolev inequality (2.9) and Cauchy's inequality, one obtains from (3.6)

$$\sum_{i=1}^N \|D_i\mathbf{u}^m\|^{q_i}_{\mathbf{L}^{q_i}(\Omega)} \leq C\sum_{i=1}^N \|\mathbf{f}\|^{q_i'}_{\mathbf{V}'_{\overrightarrow{\mathbf{q}}}}\,. \tag{3.7}$$

The estimate (3.7) and the assumption that $\mathbf{f} \in \mathbf{V}'_{\overrightarrow{\mathbf{q}}}$ yield

$$\|\mathbf{u}^m\|_{\mathbf{V}_{q_i}} \leq C \qquad \text{for all } i = 1,\dots,N, \tag{3.8}$$

$$\|\mathbf{u}^m\|_{\mathbf{V}_{\overrightarrow{\mathbf{q}}}} \leq C \tag{3.9}$$

and, consequently,

$$\|D_i\left(|D_i\mathbf{u}^m|^{q_i-2}D_i\mathbf{u}^m\right)\|_{\mathbf{V}'_{q_i}} \leq C \qquad \text{for all } i = 1,\dots,N, \tag{3.10}$$

$$\left\| \sum_{i=1}^{N} D_i \left(|D_i \mathbf{u}^m|^{q_i-2} D_i \mathbf{u}^m \right) \right\|_{\mathbf{V}'_{\overrightarrow{\mathbf{q}}}} \leq C\,. \tag{3.11}$$

Then, due to (3.8) and (3.9), by one hand, and to (3.10) and (3.11), on the other, it follows that, by means of reflexivity, there exists $\mathbf{u} \in \mathbf{V}_{\overrightarrow{\mathbf{q}}}$, $\mathbf{S}_i \in \mathbf{V}'_{q_i}$, with $i = 1, \ldots, N$, and $\mathbf{S} \in \mathbf{V}'_{\overrightarrow{\mathbf{q}}}$ such that

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{weakly in } \mathbf{V}_{q_i}, \quad \text{as } m \to \infty,$$

for all $i = 1, \ldots, N$,

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{weakly in } \mathbf{V}_{\overrightarrow{\mathbf{q}}}, \quad \text{as } m \to \infty$$

and

$$D_i \left(|D_i \mathbf{u}^m|^{q_i-2} D_i \mathbf{u}^m \right) \longrightarrow \mathbf{S}_i \quad \text{weakly in } \mathbf{V}'_{q_i}, \quad \text{as } m \to \infty, \tag{3.12}$$

for all $i = 1, \ldots, N$,

$$\sum_{i=1}^{N} D_i \left(|D_i \mathbf{u}^m|^{q_i-2} D_i \mathbf{u}^m \right) \longrightarrow \mathbf{S} \quad \text{weakly in } \mathbf{V}'_{\overrightarrow{\mathbf{q}}}, \quad \text{as } m \to \infty. \tag{3.13}$$

Obviously, $\mathbf{S} = \mathbf{S}_1 + \cdots + \mathbf{S}_N$. Next, we shall pass to the limit $m \to \infty$ in the equation (3.5).

3.3. **Step 3: Convergence of the convective term.** By the usual Sobolev compact imbedding and by the anisotropic Sobolev compact imbedding (2.8), one obtains from (3.8) or (3.9) that, respectively,

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{strongly in } \mathbf{L}^{\gamma_i}(\Omega), \quad \text{as } m \to \infty, \quad \text{for } 1 \leq \gamma_i < q_i^* \tag{3.14}$$

for all $i = 1, \ldots, N$, or

$$\mathbf{u}^m \longrightarrow \mathbf{u} \quad \text{strongly in } \mathbf{L}^{\gamma}(\Omega), \quad \text{as } m \to \infty, \quad \text{for } 1 \leq \gamma < q_a^*. \tag{3.15}$$

On the other hand, for every $i = 1, \ldots, N$

$$u_i \mathbf{u}^m \quad \text{is bounded in } \mathbf{L}^{q_i'}(\Omega) \text{ provided } 2q_i' \leq q_i^* \tag{3.16}$$

for all $i = 1, \ldots, N$, and

$$\mathbf{u}^m \otimes \mathbf{u}^m \quad \text{is bounded in } \mathbf{L}^{\alpha'}(\Omega) \text{ provided } 2\alpha' \leq q_a^*. \tag{3.17}$$

At this point, we observe that $\alpha \geq \frac{3N}{N+2}$ implies $2q_i' \leq q_i^*$ for all $i = 1, \ldots, N$, and $\alpha \geq \left(\frac{q_a^*}{2}\right)' \Leftrightarrow 2\alpha' \leq q_a^*$. Thus (3.16) or (3.17) is verified, if, respectively,

$$\alpha \geq \frac{3N}{N+2} \qquad \text{or} \qquad \alpha \geq \left(\frac{q_a^*}{2}\right)'. \tag{3.18}$$

Moreover, (3.14) and (3.16) from one hand, and (3.15) and (3.17) on the other, yield that

$$u_i^m \mathbf{u}^m \longrightarrow u_i \mathbf{u} \quad \text{weakly in } \mathbf{L}^{q_i'}(\Omega), \quad \text{as } m \to \infty, \tag{3.19}$$

for all $i = 1, \ldots, N$, and

$$\mathbf{u}^m \otimes \mathbf{u}^m \longrightarrow \mathbf{u} \otimes \mathbf{u} \quad \text{weakly in } \mathbf{L}^{\alpha'}(\Omega), \quad \text{as } m \to \infty\,. \tag{3.20}$$

Note that, due to the choice of s, $\mathbf{V}^s \hookrightarrow \mathbf{V}_{\overrightarrow{\mathbf{q}}} \hookrightarrow \mathbf{V}_\alpha$ for any $\overrightarrow{\mathbf{q}} = (q_1, \ldots, q_N)$, with $q_i > 1$ for all $i = 1, \ldots, N$. As a consequence of (3.12) and (3.19) or (3.20), we can pass to the limit $m \to \infty$ in the integral identity (3.5) to obtain

$$-\sum_{i=1}^{N} \int_\Omega \mathbf{S}_i \cdot \mathbf{v}_k \, d\mathbf{x} - \int_\Omega \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}_k \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_k \, d\mathbf{x} \tag{3.21}$$

for a fixed k.

3.4. **Convergence of the anisotropic diffusion term.** We firstly observe that the diffusion term is the sum of N possible different diffusion terms which are strictly monotonous. Therefore our anisotropic diffusion term is strictly monotonous once that

$$\sum_{i=1}^{N}\langle|D_i\xi|^{q_i-2}D_i\xi-|D_i\eta|^{q_i-2}D_i\eta, D_i\xi-D_i\eta\rangle_{\mathbf{L}^{q_i'}(\Omega)\times\mathbf{L}^{q_i}(\Omega)}>0$$

for every $\xi,\ \eta\in\mathbf{V}_{\vec{\mathbf{q}}}$, with $\xi\neq\eta$. Consequently, appealing to this strict monotonicity property, we can use the Minty trick (*cf.* [17, pp. 212-215]) to identify, in the limit integral equation (3.21),

$$\mathbf{S}_i=D_i\left(|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right)$$

for all $i=1,\dots,N$. Thus, we obtain, from (3.21),

$$\sum_{i=1}^{N}\int_\Omega|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\cdot D_i\mathbf{v}_k\,d\mathbf{x}-\int_\Omega\mathbf{u}\otimes\mathbf{u}:\nabla\mathbf{v}_k\,d\mathbf{x}=\int_\Omega\mathbf{f}\cdot\mathbf{v}_k\,d\mathbf{x}\tag{3.22}$$

for a fixed k. Finally, and to conclude the proof, we observe that, since $\mathbf{v}_k\in\mathbf{V}^m\hookrightarrow\mathbf{V}_{\vec{\mathbf{q}}}$, we can pass to the limit $k\to\infty$ in equation (3.22) to obtain (3.1) for all $\mathbf{v}\in\mathbf{V}_{\vec{\mathbf{q}}}$.

3.5. **Analysis of the relation (3.18).** In this part, we are mainly interested in the study of the conditions (3.18), essential for the existence of weak solutions to the problem (1.1)-(1.3). Since the improvements of the existence results for this problem, consist in to decrease the lower bound of the range of α, and once that (3.18) gives us two different possibilities, we may therefore consider

$$\alpha\geq\min\left\{\frac{3N}{N+2},\left(\frac{q_a^*}{2}\right)'\right\}.\tag{3.23}$$

The minimum in (3.23) reads

$$\min\left\{\frac{3N}{N+2},\left(\frac{q_a^*}{2}\right)'\right\}=\begin{cases}\dfrac{3N}{N+2} & \text{if}\quad q_a^*<\dfrac{3N}{N-1}\ \textbf{: Case 1}\\[2ex] \left(\dfrac{q_a^*}{2}\right)' & \text{if}\quad q_a^*\geq\dfrac{3N}{N-1}\ \textbf{: Case 2}.\end{cases}\tag{3.24}$$

Next we will prove that the Case 1 is incompatible with (3.23). From (3.23) and according to the definition of q_a^* (see (2.11) and Remark 2.1), we should have, in this case,

$$\alpha\geq\frac{3N}{N+2}:\tag{3.25}$$

Case 1.1 if

$$\beta<\frac{3N}{N-1}\tag{3.26}$$

and if

$$N\neq2\quad\text{and}\quad\beta>\overline{q}^*\Leftrightarrow\sum_{i\neq N}\frac{1}{q_i}>1+\frac{N-1}{\beta};\tag{3.27}$$

or
Case 1.2 if

$$\overline{q}^*<\frac{3N}{N-1}\Leftrightarrow\overline{q}<\frac{3N}{N+2}\Leftrightarrow\sum_i\frac{1}{q_i}>\frac{N+2}{3}\tag{3.28}$$

and if

$$N=2\quad\text{or}\quad\overline{q}^*\geq\beta\Leftrightarrow\sum_{i\neq N}\frac{1}{q_i}\leq1+\frac{N-1}{\beta}.\tag{3.29}$$

A simple analysis of the Cases 1.1 and 1.2, shows us that none of these cases is possible to occur. The Case 1.1 cannot happen, since (3.25) and (3.26) are incompatible with (3.27). In fact,

$$\alpha \geq \frac{3N}{N+2} \Rightarrow \sum_i \frac{1}{q_i} \leq \frac{N+2}{3} < 1 + \frac{N}{\beta} \Leftarrow \beta < \frac{3N}{N-1}\,.$$

With respect to the Case 1.2, we see that (3.25) and (3.28) are incompatible, because

$$\overline{q}^* < \frac{3N}{N-1} \Rightarrow \frac{N+2}{3} < \sum_i \frac{1}{q_i} \leq \frac{N+2}{3} \Leftarrow \alpha \geq \frac{3N}{N+2}\,.$$

In view of this analysis, we see that the minimum of (3.24) cannot be attained at $\alpha = \frac{3N}{N+2}$. Therefore the lower bound for α which provides us with an existence result, in the sense of Definition 3.1, is given by (3.3). □

Remark 3.2. *Theorem 3.1 still holds if we instead assume that* $\mathbf{f} \in \mathbf{L}^{(q_a^*)'}(\Omega)$, *where* $(q_a^*)'$ *is the Hölder conjugate of* q_a^*. *In this case it is enough to replace (3.7) by*

$$\sum_{i=1}^{N} \|D_i \mathbf{u}^m\|^{q_i}_{\mathbf{L}^{q_i}(\Omega)} \leq C \sum_{i=1}^{N} \|\mathbf{f}\|^{q_i'}_{\mathbf{L}^{(q_a^*)'}}\,.$$

4. Analysis of the extent of Theorem 3.1

In this section we analyze what is the real meaning of the condition (3.18), fundamental to prove Theorem 3.1. We start by observing that if $q_i = q$ for all $i = 1, \ldots, N$, then (3.3) is equivalent to the well-known relation

$$q \geq \frac{3N}{N+2} \tag{4.1}$$

and, to the best of our knowledge, there is no existence result, in the sense of the isotropic version of Definition 3.1, for lower values of q (see *e.g.* [17]). As we shall see in the sequel, condition (3.18) allows us to obtain an existence result with some components of $\overrightarrow{\mathbf{q}}$ considerably lower than those provided by the isotropic condition (4.1).

To realize what condition (3.18) means, we analyze, next, the Case 2 of (3.24) written in the proof of Theorem 3.1. In order to simplify the exposition, in this section, we assume that

$$\beta = q_N \geq q_{N-1} \geq \cdots \geq q_2 \geq q_1 = \alpha\,.$$

From the definition of q_a^* (see (2.11) and Remark 2.1), we shall have, from (3.23), several cases.

4.1. Case 2.1.

$$\alpha \geq \left(\frac{\beta}{2}\right)' \Leftrightarrow \beta \geq 2\alpha' \tag{4.2}$$

if

$$\beta \geq \frac{3N}{N-1} \tag{4.3}$$

and if

$$N \neq 2 \quad \text{and} \quad \beta > \overline{q}^* \Leftrightarrow \sum_{i\neq N} \frac{1}{q_i} > 1 + \frac{N-1}{\beta}\,; \tag{4.4}$$

The simultaneous satisfaction of conditions (4.2)-(4.4) allows us to determine a precise range for β whenever $N > 2$:

$$\beta \geq \max\left\{2\alpha', \frac{3N}{N-1}, \frac{N-1}{\sum_{i\neq N}\frac{1}{q_i} - 1}, q_{N-1}\right\}\,. \tag{4.5}$$

The range of β does not degenerate, if

$$\sum_{i\neq N}\frac{1}{q_i}>1\Leftrightarrow\sum_{i\neq N,\ N-1}\frac{1}{q_i}>1-\frac{1}{q_{N-1}}\,. \tag{4.6}$$

Example 4.1. *For $N=3$, we have*

$$\frac{1}{\alpha}>\frac{1}{q_2'}\quad\textit{and}\quad q_2\geq\alpha\quad\Rightarrow\quad\alpha\leq q_2<\alpha'\quad\textit{and}\quad\alpha<2\,.$$

For instance, the following choice of α gives us the ranges of q_2 and β in the Case 2.1:

$$\alpha=\frac{3}{2}\quad\Rightarrow\quad\frac{3}{2}\leq q_2<3\quad\textit{and}\quad\beta\geq 6\,.$$

4.2. **Case 2.2.** The other possibility which can occurs in Case 2 of (3.24) is studied now:

$$\alpha\geq\left(\frac{\overline{q}^*}{2}\right)'\Leftrightarrow\overline{q}^*\geq 2\alpha'\Leftrightarrow\sum_{i}\frac{1}{q_i}\leq 1+\frac{N}{2\alpha'} \tag{4.7}$$

if

$$\overline{q}^*\geq\frac{3N}{N-1}\Leftrightarrow\overline{q}\geq\frac{3N}{N+2}\Leftrightarrow\sum_{i}\frac{1}{q_i}\leq\frac{N+2}{3} \tag{4.8}$$

and if

$$N=2\quad\text{or}\quad\overline{q}^*\geq\beta\Leftrightarrow\sum_{i\neq N}\frac{1}{q_i}\leq 1+\frac{N-1}{\beta}\,. \tag{4.9}$$

The analysis of (4.7)-(4.9) shows us that if $N=2$, then

$$\beta\geq\max\left\{\frac{\alpha}{2(\alpha-1)},\alpha\right\}\,. \tag{4.10}$$

Example 4.2. *In the case of $N=2$, the following choices of α give the ranges of validity for β:*

$$\alpha=\frac{4}{3}\quad\Rightarrow\quad\beta\geq 2\,,\qquad\alpha=\frac{5}{3}\quad\Rightarrow\quad\beta\geq\frac{5}{3}\,.$$

If $N\neq 2$, then (4.7)-(4.9) imply that

$$\beta\in\left[\max\left\{\left(1+\frac{N}{2\alpha'}-\sum_{i\neq N}\frac{1}{q_i}\right)^{-1},\left(\frac{N+2}{3}-\sum_{i\neq N}\frac{1}{q_i}\right)^{-1},q_{N-1}\right\},\frac{N-1}{\sum_{i\neq N}\frac{1}{q_i}-1}\right] \tag{4.11}$$

and (4.6) is required as well. In order to study the different possibilities given by (4.11), let us denote the maximum referred there by $\max\{\cdot,\cdot,\cdot\}$. Now several situations can happen.

4.2.1. *Case 2.2-a.* Conjugating the possibility of $\max\{\cdot,\cdot,\cdot\}=q_{N-1}$ with the requirement that the interval in (4.11) must be non-empty, we obtain, for $N>2$,

$$\sum_{i\neq N,\ N-1}\frac{1}{q_i}\in\left(1-\frac{1}{q_{N-1}},\min\left\{1+\frac{N}{2\alpha'}-\frac{2}{q_{N-1}},\frac{N+2}{3}-\frac{2}{q_{N-1}},1+\frac{N-2}{q_{N-1}}\right\}\right]\,. \tag{4.12}$$

We have in this case three different situations.

(1) The first situation gives

$$\alpha\leq\frac{3N}{N+2},\ q_{N-1}\leq 2\alpha'\ \Rightarrow\sum_{i\neq N,\ N-1}\frac{1}{q_i}\in\left(1-\frac{1}{q_{N-1}},1+\frac{N}{2\alpha'}-\frac{2}{q_{N-1}}\right] \tag{4.13}$$

and this interval is non-empty if $q_{N-1}>\frac{2\alpha'}{N}$.

Example 4.3. *If $N = 3$, we have*

$$1 < \alpha \le \frac{9}{5}, \quad \frac{2\alpha'}{3} < q_2 \le 2\alpha', \quad q_2 \ge \alpha \quad \textit{and} \quad 1 - \frac{1}{q_2} < \frac{1}{\alpha} \le 1 + \frac{3}{2\alpha'} - \frac{2}{q_2}$$

implying that

$$1 < \alpha < 2\,, \quad \frac{4\alpha'}{5} \le q_2 < \alpha' \quad \textit{and} \quad q_2 \le \beta \le \frac{2}{\frac{1}{\alpha} + \frac{1}{q_2} - 1}\,.$$

In particular,

$$\alpha = \frac{4}{3}\,, \quad q_2 = \frac{18}{5} \quad \Rightarrow \quad \frac{18}{5} \le \beta \le 72\,.$$

(2) The second possibility is given by

$$\alpha \ge \frac{3N}{N+2},\; q_{N-1} \le \frac{3N}{N-1} \;\Rightarrow \sum_{i\neq N,\ N-1} \frac{1}{q_i} \in \left(1 - \frac{1}{q_{N-1}}, \frac{N+2}{3} - \frac{2}{q_{N-1}}\right] \tag{4.14}$$

and the last interval is non-empty, if $q_{N-1} > \frac{3}{N-1}$ which is always true for $N > 3$.

Example 4.4. *If $N = 3$, we have*

$$\alpha \ge \frac{9}{5}, \quad \frac{3}{2} < q_2 \le \frac{9}{2}, \quad q_2 \ge \alpha \quad \textit{and} \quad 1 - \frac{1}{q_2} < \frac{1}{\alpha} \le \frac{5}{3} - \frac{2}{q_2}$$

implying that

$$\frac{9}{5} \le \alpha < 2\,, \quad \alpha \le q_2 < \frac{9}{2} \quad \textit{and} \quad q_2 \le \beta \le \frac{2}{\frac{1}{\alpha} + \frac{1}{q_2} - 1}\,.$$

In particular,

$$\alpha = \frac{11}{6}\,, \quad q_2 = 2 \quad \Rightarrow \quad 2 \le \beta \le 22\,.$$

(3) The third situation is more difficult to achieve, because

$$q_{N-1} \ge \max\left\{\frac{3N}{N-1}, 2\alpha'\right\} \;\Rightarrow \sum_{i\neq N,\ N-1} \frac{1}{q_i} \in \left(1 - \frac{1}{q_{N-1}}, 1 + \frac{N-2}{q_{N-1}}\right] \tag{4.15}$$

and this interval is non-empty for all $N > 1$. For instance, when $N = 3$, (4.15) is impossible, because $q_2 \ge 2\alpha'$ implies $1 - \frac{1}{q_2} > \frac{1}{\alpha}$.

4.2.2. *Case 2.2-b.* If $\max\{\cdot,\cdot,\cdot\} = \left(1 + \frac{N}{2\alpha'} - \sum_{i\neq N} \frac{1}{q_i}\right)^{-1}$, then $\alpha \le \frac{3N}{N+2}$ and

$$\sum_{i\neq N,\ N-1} \frac{1}{q_i} \in \left(\max\left\{1 - \frac{1}{q_{N-1}}, 1 + \frac{N}{2\alpha'} - \frac{2}{q_{N-1}}\right\}, 1 + \frac{N-1}{2\alpha'} - \frac{1}{q_{N-1}}\right]\,.$$

Now, there are two possibilities.

(1) The first one gives

$$q_{N-1} \le \frac{2\alpha'}{N} \;\Rightarrow \sum_{i\neq N,\ N-1} \frac{1}{q_i} \in \left(1 - \frac{1}{q_{N-1}}, 1 + \frac{N-1}{2\alpha'} - \frac{1}{q_{N-1}}\right] \tag{4.16}$$

and this interval is non-empty for all N.

Example 4.5. *If $N = 3$, we have*

$$\alpha \le \frac{9}{5}, \quad q_2 \le \frac{2\alpha'}{3}, \quad q_2 \ge \alpha \quad \textit{and} \quad 1 - \frac{1}{q_2} < \frac{1}{\alpha} \le 1 + \frac{1}{\alpha'} - \frac{1}{q_2}$$

implying that

$$1 < \alpha \le \frac{9}{5}\,, \quad \max\left\{\frac{\alpha'}{2}, \alpha\right\} \le q_2 \le \frac{2\alpha'}{3}$$

and

$$\max\left\{\left(1+\frac{3}{2\alpha'}-\frac{1}{\alpha}-\frac{1}{q_2}\right)^{-1}, q_2\right\} \leq \beta \leq \frac{2}{\frac{1}{\alpha}+\frac{1}{q_2}-1}\,.$$

In particular,

$$\alpha = \frac{3}{2}\,, \quad q_2 = \frac{5}{2} \quad \Rightarrow \quad \frac{30}{13} \leq \beta \leq 30\,.$$

(2) For the second possibility,

$$q_{N-1} \geq \frac{2\alpha'}{N} \;\Rightarrow\; \sum_{i\neq N,\ N-1}\frac{1}{q_i} \in \left(1+\frac{N}{2\alpha'}-\frac{2}{q_{N-1}}, 1+\frac{N-1}{2\alpha'}-\frac{1}{q_{N-1}}\right] \tag{4.17}$$

and this interval is non-empty if $q_{N-1} < 2\alpha'$.

Example 4.6. *If $N = 3$, we have*

$$\alpha \leq \frac{9}{5}, \quad \frac{2\alpha'}{3} \leq q_2 < 2\alpha', \quad q_2 \geq \alpha \quad \textit{and} \quad 1+\frac{3}{2\alpha'}-\frac{2}{q_{N-1}} < \frac{1}{\alpha} \leq 1+\frac{1}{\alpha'}-\frac{1}{q_2}$$

implying that

$$1 < \alpha \leq \frac{9}{5}\,, \quad \max\left\{\frac{\alpha'}{2}, \alpha\right\} \leq q_2 < \frac{4\alpha'}{5}$$

and

$$\max\left\{\left(1+\frac{3}{2\alpha'}-\frac{1}{\alpha}-\frac{1}{q_2}\right)^{-1}, q_2\right\} \leq \beta \leq \frac{2}{\frac{1}{\alpha}+\frac{1}{q_2}-1}\,.$$

In particular,

$$\alpha = \frac{6}{5}\,, \quad q_2 = 4 \quad \Rightarrow \quad 6 \leq \beta \leq 24\,.$$

4.2.3. *Case 2.2-c.* If $\max\{\cdot,\cdot,\cdot\} = \left(\frac{N+2}{3}-\sum_{i\neq N}\frac{1}{q_i}\right)^{-1}$, then $\alpha \geq \frac{3N}{N+2}$,

$$\sum_{i\neq N,\ N-1}\frac{1}{q_i} \in \left[\frac{N+2}{3}-\frac{2}{q_{N-1}}, \frac{N^2+N+1}{3N}-\frac{1}{q_{N-1}}\right]$$

and this interval is non-empty if $q_{N-1} < \frac{3N}{N-1}$. This case is substantial more restrictive in the sense that the first two terms of $\overrightarrow{\mathbf{q}}$ need to be much more closer.

Example 4.7. *If $N = 3$,*

$$\alpha \geq \frac{9}{5}, \quad q_2 < \frac{9}{2}, \quad q_2 \geq \alpha \quad \textit{and} \quad \frac{5}{3}-\frac{2}{q_2} \leq \frac{1}{\alpha} \leq \frac{13}{9}-\frac{1}{q_2}$$

implying that the only possible choices are

$$q_2 = \alpha = \frac{9}{5} \quad \textit{and} \quad \frac{9}{5} \leq \beta \leq 18\,.$$

5. Very weak formulation

In this section we are interested in weak solutions that will allow us to establish an existence result for lower bounds of α which are smaller than those provided by (3.3). For this purpose, we shall need to consider weaker solutions in the sense that the test functions have more regularity. With this in mind, we introduce below the notion of very weak solution we shall consider in this work.

Definition 5.1. *Let $N \geq 2$ and $\overrightarrow{\mathbf{q}} = (q_1, \dots, q_N)$, with $1 < q_i < \infty$ for any $i = 1, \dots, N$. Assume that $\mathbf{f} \in \mathbf{L}^1(\Omega)$. A vector field $\mathbf{u}$ is a very weak solution to the problem (1.1)-(1.3), if:*

(1) $\mathbf{u} \in \mathbf{V}_{\overrightarrow{\mathbf{q}}}$;

(2) *For every* $\varphi \in \mathcal{V}$

$$\sum_{i=1}^{N} \int_{\Omega} |D_i \mathbf{u}|^{q_i-2} D_i \mathbf{u} \cdot D_i \varphi \, d\mathbf{x} - \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : \nabla \varphi \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \varphi \, d\mathbf{x}.$$

The main result of this work is written in what follows. We establish here the existence of very weak solutions for the problem (1.1)-(1.3).

Theorem 5.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$, $N \geq 2$, *with a Lipschitz-continuous boundary* $\partial\Omega$. *Assume that* $\mathbf{f} \in \mathbf{V}'_{\overrightarrow{\mathbf{q}}}$. *If*

$$q_a^* > 2, \tag{5.1}$$

then there exists, at least, a weak solution to the problem (1.1)-(1.3) in the sense of Definition 5.1.

Remark 5.1. *Condition (5.1) is needed to use the compact imbedding* $\mathbf{V}_{\overrightarrow{\mathbf{q}}} \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$ *valid for* $2 < q_a^*$ (cf. *(2.10)-(2.11)). This condition reads as*

$$q_a^* = \overline{q}^* \quad \text{and} \quad \overline{q} > \frac{2N}{N+2}, \tag{5.2}$$

or

$$q_a^* = \beta \quad \text{and} \quad \beta > 2\,. \tag{5.3}$$

Proof. The proof of Theorem 5.1 shall be split into several parts.

5.1. **The regularized problem.** For any $\epsilon > 0$, $\alpha > 1$ and $\sigma \geq 2\alpha'$, let us consider the following regularized problem:

$$\operatorname{div} \mathbf{u}_\epsilon = 0 \quad \text{in} \quad \Omega, \tag{5.1.1}$$

$$\mathbf{div}(\mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \epsilon |\mathbf{u}_\epsilon|^{\sigma-2} \mathbf{u}_\epsilon = \mathbf{f} - \nabla p_\epsilon + \sum_{i=1}^{N} D_i \left(|D_i \mathbf{u}_\epsilon|^{q_i-2} D_i \mathbf{u}_\epsilon \right) \quad \text{in} \quad \Omega, \tag{5.1.2}$$

$$\mathbf{u}_\epsilon = \mathbf{0} \qquad \text{on} \quad \partial\Omega. \tag{5.1.3}$$

A vector function $\mathbf{u}_\epsilon \in \mathbf{V}_{\overrightarrow{\mathbf{q}}} \cap \mathbf{H}_\sigma$ is a very weak solution to the problem (5.1.1)-(5.1.3), if, for every $\varphi \in \mathcal{V}$,

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega} |D_i \mathbf{u}_\epsilon|^{q_i-2} D_i \mathbf{u}_\epsilon \cdot D_i \varphi \, d\mathbf{x} - \int_{\Omega} \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon : \nabla \varphi \, d\mathbf{x} + \epsilon \int_{\Omega} |\mathbf{u}_\epsilon|^{\sigma-2} \mathbf{u}_\epsilon \cdot \varphi \, d\mathbf{x} \\ &= \int_{\Omega} \mathbf{f} \cdot \varphi \, d\mathbf{x}\,. \end{aligned} \tag{5.1.4}$$

Remark 5.2. *Note that* $\mathbf{V}_{\overrightarrow{\mathbf{q}}} \hookrightarrow \mathbf{H}_\sigma$ *for* $\sigma \leq q_a^*$ *and, in this case, it is enough to consider* $\mathbf{u}_\epsilon \in \mathbf{V}_{\overrightarrow{\mathbf{q}}}$.

Proposition 5.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$, $N \geq 2$, *with a Lipschitz-continuous boundary* $\partial\Omega$. *Assume that* $\alpha > 1$ *and* $\sigma \geq 2\alpha'$. *Then, for each* $\epsilon > 0$, *there exists a weak solution* $\mathbf{u}_\epsilon \in \mathbf{V}_{\overrightarrow{\mathbf{q}}} \cap \mathbf{H}_\sigma$ *to the problem (5.1.1)-(5.1.3) in the sense of (5.1.4). In addition, every weak solution satisfies to the following energy equality:*

$$\sum_{i=1}^{N} \int_{\Omega} |D_i \mathbf{u}_\epsilon|^{q_i} \, d\mathbf{x} + \epsilon \int_{\Omega} |\mathbf{u}_\epsilon|^{\sigma} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_\epsilon \, d\mathbf{x}. \tag{5.1.5}$$

Proof. The proof of existence of a very weak solution $\mathbf{u}_\epsilon \in \mathbf{V}_{\overrightarrow{\mathbf{q}}} \cap \mathbf{H}_\sigma$ to the problem (5.1.1)-(5.1.3) follows by the same arguments used in the proof of Theorem 3.1 and by observing that $\mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}$ is bounded in $\mathbf{L}^1(\Omega)$ provided $\sigma \geq 2\alpha'$.

The energy relation (5.1.5) follows by testing (5.1.2) by a very weak solution and integrating over Ω, and observing that now the convective term is zero. □

5.2. **Existence of approximative solutions.** Let $\mathbf{u}_\epsilon \in \mathbf{V}_{\vec{\mathbf{q}}} \cap \mathbf{H}_\sigma$ be a very weak solution to the problem (5.1.1)-(5.1.3). From Proposition 5.1 (*cf.* (5.1.5)), using Hölder's and Young's inequalities, we can prove that

$$\sum_{i=1}^{N} \int_\Omega |D_i \mathbf{u}_\epsilon|^{q_i}\, d\mathbf{x} + \epsilon \int_\Omega |\mathbf{u}_\epsilon|^\sigma\, d\mathbf{x} \le C, \tag{5.2.1}$$

where, by the assumption that $\mathbf{f} \in \mathbf{V}'_{\vec{\mathbf{q}}}$, C is a positive constant. As a consequence of (5.2.1), it follows that

$$\|\mathbf{u}_\epsilon\|_{\mathbf{V}_{\vec{\mathbf{q}}}} \le C \tag{5.2.2}$$

and, for every $i \in \{1, \dots, N\}$,

$$\left\| |D_i \mathbf{u}_\epsilon|^{q_i - 2} D_i \mathbf{u}_\epsilon \right\|_{\mathbf{L}^{q_i'}(\Omega)} \le C. \tag{5.2.3}$$

On the other hand, by using (5.2.2) and the anisotropic Sobolev inequality, we have

$$\|\mathbf{u}_\epsilon\|_{\mathbf{L}^{q_a^*}(\Omega)} \le C, \tag{5.2.4}$$

$$\|\mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon\|_{\mathbf{L}^{\frac{q_a^*}{2}}(\Omega)} \le C. \tag{5.2.5}$$

Note that the constants in (5.2.1)-(5.2.5) are distinct and do not depend on ϵ. From (5.2.2)-(5.2.5) and by means of reflexivity, there exists a sequence of positive numbers ϵ_m such that $\epsilon_m \to 0$, as $m \to \infty$, and

$$\mathbf{u}_{\epsilon_m} \to \mathbf{u} \quad \text{weakly in } \mathbf{V}_{\vec{\mathbf{q}}}, \quad \text{as } m \to \infty, \tag{5.2.6}$$

for every $i = 1, \dots, N$

$$|D_i \mathbf{u}_{\epsilon_m}|^{q_i - 2} D_i \mathbf{u}_{\epsilon_m} \to \mathbf{S}_i \quad \text{weakly in } \mathbf{L}^{q_i'}(\Omega), \quad \text{as } m \to \infty, \tag{5.2.7}$$

$$\mathbf{u}_{\epsilon_m} \to \mathbf{u} \quad \text{weakly in } \mathbf{L}^{q_a^*}(\Omega), \quad \text{as } m \to \infty. \tag{5.2.8}$$

$$\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} \to \mathbf{G} \quad \text{weakly in } \mathbf{L}^{\frac{q_a^*}{2}}(\Omega), \quad \text{as } m \to \infty. \tag{5.2.9}$$

Let us prove that also

$$\epsilon_m |\mathbf{u}_{\epsilon_m}|^{\sigma - 2} \mathbf{u}_{\epsilon_m} \to 0 \quad \text{weakly in} \quad \mathbf{L}^{\sigma'}(\Omega), \quad \text{as} \quad m \to \infty. \tag{5.2.10}$$

Indeed, for $\varphi \in \mathbf{V}_{\vec{\mathbf{q}}} \cap \mathbf{H}_\sigma$, we have that

$$\left| \int_\Omega \epsilon_m |\mathbf{u}_{\epsilon_m}|^{\sigma-2} \mathbf{u}_{\epsilon_m} \cdot \varphi\, d\mathbf{x} \right| \le$$
$$C_1 \epsilon_m^{\frac{1}{\sigma}} \left(\int_\Omega \epsilon_m |\mathbf{u}_{\epsilon_m}|^\sigma\, d\mathbf{x} \right)^{\frac{1}{\sigma'}} \|\varphi\|_{\mathbf{L}^\sigma(\Omega)} \le C_2 \epsilon_m^{\frac{1}{\sigma}} \to 0, \text{ as } m \to \infty.$$

Note that the boundedness of the last integral term follows from (5.2.1).

Now we observe that, due to (5.2.6), the application of the anisotropic Sobolev compact imbedding implies

$$\mathbf{u}_{\epsilon_m} \to \mathbf{u} \quad \text{strongly in } \mathbf{L}^\gamma(\Omega), \quad \text{as } m \to \infty, \quad \text{for any } \gamma : 1 \le \gamma < q_a^*. \tag{5.2.11}$$

As a consequence of (5.2.11),

$$\mathbf{u}_{\epsilon_m} \to \mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{as } m \to \infty \tag{5.2.12}$$

and

$$\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} \to \mathbf{u} \otimes \mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{as } m \to \infty. \tag{5.2.13}$$

Then (5.2.9) and (5.2.13) allow us to use a well known result (see *e.g.* [17, Lemme 1.1.3]) to prove that

$$\mathbf{G} = \mathbf{u} \otimes \mathbf{u}\,. \tag{5.2.14}$$

Moreover, for any $\varphi \in \mathcal{V}$

$$\int_\Omega \mathbf{div}\,(\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m}) \cdot \varphi\, d\mathbf{x} = -\int_\Omega \mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} : \nabla\varphi\, d\mathbf{x} \leq C\|\mathbf{u}_{\epsilon_m}\|^2_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{u}_{\epsilon_m}\|^2_{\mathbf{V}_{\vec{\mathbf{q}}}} \tag{5.2.15}$$

provided $2 \leq q_a^*$, which is fulfilled due to assumption (5.1). Finally, using the convergence results (5.2.6)-(5.2.10) and observing (5.2.14)-(5.2.15), we can pass to the limit $m \to \infty$ in the following integral identity, which results from (5.1.4),

$$\sum_{i=1}^{N} \int_\Omega |D_i \mathbf{u}_{\epsilon_m}|^{q_i-2} D_i \mathbf{u}_{\epsilon_m} \cdot D_i\varphi\, d\mathbf{x} - \int_\Omega \mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} : \nabla\,\varphi\, d\mathbf{x} + \epsilon_m \int_\Omega |\mathbf{u}_{\epsilon_m}|^{\sigma-2}\mathbf{u}_{\epsilon_m} \cdot \varphi\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \varphi\, d\,\mathbf{x}\,, \tag{5.2.16}$$

valid for all $\varphi \in \mathcal{V}$, to obtain

$$\sum_{i=1}^{N} \int_\Omega \mathbf{S}_i \cdot D_i\varphi\, d\mathbf{x} - \int_\Omega \mathbf{u} \otimes \mathbf{u} : \nabla\varphi\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \varphi\, d\mathbf{x} \quad \forall\, \varphi \in \mathcal{V}. \tag{5.2.17}$$

5.3. **Determination of the pressure.** Since we shall use test functions which are not divergence free, we first have to determine the approximative pressure from the weak formulation (5.2.16). First, let us set

$$\mathbf{Q}_{\epsilon_m} := -\sum_{i=1}^{N} D_i \left(|D_i \mathbf{u}_{\epsilon_m}|^{q_i-2} D_i \mathbf{u}_{\epsilon_m}\right) + \mathbf{div}\,(\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m}) + \epsilon_m |\mathbf{u}_{\epsilon_m}|^{\sigma-2}\mathbf{u}_{\epsilon_m} - \mathbf{f}\,. \tag{5.3.1}$$

Using the assumption that $\mathbf{f} \in \mathbf{V}'_{\vec{\mathbf{q}}}$ and (5.1) together with the results (5.2.3), (5.2.5) and (5.2.10), we can prove that

$$\mathbf{Q}_{\epsilon_m} \in \mathbf{W}^{-1,r'}(\Omega), \tag{5.3.2}$$

where r can be taken in such a way that

$$\max\left\{\sigma, \beta, \frac{q_a^*}{q_a^*-2}\right\} =: r_0 \leq r < \infty\,. \tag{5.3.3}$$

Note that $q_a^* > 2$ by assumption (5.1) and $r_0 = \max\{\sigma, \beta\}$ if $q_a^* = \overline{q}^*$ and $\sum_{j=1}^{N} \frac{1}{q_j} \leq 1$. Then we define a linear functional

$$\Pi_{\epsilon_m} : \mathbf{W}_0^{1,r}(\Omega) \to \mathbf{W}^{-1,r'}(\Omega) \tag{5.3.4}$$

by

$$\langle \Pi_{\epsilon_m}, \varphi \rangle_{\mathbf{W}^{-1,r'}(\Omega) \times \mathbf{W}_0^{1,r}(\Omega)} := \int_\Omega \mathbf{Q}_{\epsilon_m} \cdot \varphi\, d\mathbf{x}. \tag{5.3.5}$$

Using (5.3.4)-(5.3.5), we can prove, owing to (5.3.2), that exists a positive constant C independent of m such that

$$\|\Pi_{\epsilon_m}\|_{\mathbf{W}^{-1,r'}(\Omega)} \leq C. \tag{5.3.6}$$

Moreover, since $\mathcal{V}$ is dense in $\mathbf{W}_0^{1,r}(\Omega)$, we can see, due to (5.2.16), (5.3.1) and (5.3.5), that

$$\langle \Pi_{\epsilon_m}, \varphi \rangle_{\mathbf{W}^{-1,r'}(\Omega) \times \mathbf{W}_0^{1,r}(\Omega)} = 0 \quad \forall\, \varphi \in \mathbf{V}_r. \tag{5.3.7}$$

By virtue of (5.3.4)-(5.3.7), we can apply a version of de Rham's Theorem due to Bogovskiĭ and Pileckas (see *e.g.* [12, Theorems III.3.1 and III.5.2]) to prove the existence of a unique function

$$p_{\epsilon_m} \in \mathbf{L}^r(\Omega), \qquad \text{with} \quad \int_\Omega p_{\epsilon_m} d\mathbf{x} = 0, \tag{5.3.8}$$

such that

$$\langle \Pi_{\epsilon_m}, \varphi \rangle_{\mathbf{W}^{-1,r}(\Omega)\times \mathbf{W}_0^{1,r}(\Omega)} = \int_\Omega p_{\epsilon_m} \,\mathrm{div}\varphi \, d\mathbf{x} \quad \forall\, \varphi \in \mathbf{W}_0^{1,r}(\Omega) \tag{5.3.9}$$

and

$$\|p_{\epsilon_m}\|_{\mathbf{L}^r(\Omega)} \le \|\Pi_{\epsilon_m}\|_{\mathbf{W}^{-1,r'}(\Omega)}. \tag{5.3.10}$$

Then, gathering the information of (5.2.16), (5.3.1), (5.3.5) and (5.3.9), we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega |D_i \mathbf{u}_{\epsilon_m}|^{q_i-2} D_i \mathbf{u}_{\epsilon_m} \cdot D_i \varphi \, d\mathbf{x} - \int_\Omega \mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} : \nabla\, \varphi \, d\mathbf{x} \\ &+ \epsilon_m \int_\Omega |\mathbf{u}_{\epsilon_m}|^{\sigma-2} \mathbf{u}_{\epsilon_m} \cdot \varphi \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \varphi \, d\,\mathbf{x} + \int_\Omega p_{\epsilon_m} \,\mathrm{div}\varphi \, d\mathbf{x} \end{aligned} \tag{5.3.11}$$

for all $\varphi \in \mathbf{W}_0^{1,r}(\Omega)$. On the other hand, due to (5.3.6) and (5.3.10) and by means of reflexivity, we get, passing to a subsequence, that

$$p_{\epsilon_m} \to p_0 \quad \text{weakly in} \quad \mathrm{L}^r(\Omega), \quad \text{as} \quad m \to \infty. \tag{5.3.12}$$

Next, passing to the limit $m \to \infty$ in the integral identity (5.3.11) by using the convergence results (5.2.7), (5.2.9) together with (5.2.14), using also (5.2.10) and (5.3.12), we obtain

$$\sum_{i=1}^N \int_\Omega \mathbf{S}_i \cdot D_i \varphi \, d\mathbf{x} - \int_\Omega \mathbf{u} \otimes \mathbf{u} : \nabla \varphi \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \varphi \, d\mathbf{x} + \int_\Omega p_0 \,\mathrm{div}\varphi \, d\mathbf{x} \tag{5.3.13}$$

for all $\varphi \in \mathbf{W}_0^{1,r}(\Omega)$.

On the other hand, proceeding analogously as we did for (5.3.4)-(5.3.6), we can define a linear functional

$$\Pi_0 : \mathbf{W}_0^{1,r}(\Omega) \to \mathbf{W}^{-1,r'}(\Omega) \tag{5.3.14}$$

by

$$\langle \Pi_0, \varphi \rangle_{\mathbf{W}^{-1,r'}(\Omega)\times \mathbf{W}_0^{1,r}(\Omega)} := \int_\Omega \mathbf{Q}_0 \cdot \varphi \, d\mathbf{x}, \tag{5.3.15}$$

where

$$\mathbf{Q}_0 := -\sum_{i=1}^N D_i \mathbf{S}_i + \mathbf{div}\,(\mathbf{u} \otimes \mathbf{u}) - \mathbf{f}\,,$$

such that (5.3.6)-(5.3.7) are verified with Π_0 and $\mathbf{Q}_0$ in the places of Π_{ϵ_m} and $\mathbf{Q}_{\epsilon_m}$. In consequence, by the same version of de Rham's Theorem aforementioned, there exists a unique function

$$\overline{p}_0 \in \mathbf{L}^r(\Omega), \qquad \text{with} \quad \int_\Omega \overline{p}_0 \, d\mathbf{x} = 0, \tag{5.3.16}$$

such that

$$\langle \Pi_0, \varphi \rangle_{\mathbf{W}^{-1,r'}(\Omega)\times \mathbf{W}_0^{1,r}(\Omega)} = \int_\Omega \overline{p}_0 \,\mathrm{div}\varphi \, d\mathbf{x} \quad \forall\, \varphi \in \mathbf{W}_0^{1,r}(\Omega) \tag{5.3.17}$$

and (5.3.10) is verified with $\overline{p}_0$ and Π_0 in the places of p_{ϵ_m} and Π_{ϵ_m}. Then gathering (5.3.15) and (5.3.17), we achieve to

$$\sum_{i=1}^N \int_\Omega \mathbf{S}_i \cdot D_i \varphi \, d\mathbf{x} - \int_\Omega \mathbf{u} \otimes \mathbf{u} : \nabla \varphi \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \varphi \, d\mathbf{x} + \int_\Omega \overline{p}_0 \,\mathrm{div}\varphi \, d\mathbf{x} \tag{5.3.18}$$

for all $\varphi \in \mathbf{W}_0^{1,r}(\Omega)$.

Finally, combining (5.3.13) and (5.3.18), and by means of uniqueness, we conclude that

$$\overline{p}_0 = p_0.$$

5.4. **Decomposition of the pressure.** The main idea in this section is the application of a method to locally decompose the pressure found in the previous section. For that, we shall use a lemma which is proved by using a direct decomposition of L^q, which in turn is equivalent to the weak L^q-solvability of the Dirichlet problem for the Bilaplacian in bounded domains with C^2 boundaries (*cf.* [23]). With this in mind, let ω be a fixed but arbitrary domain such that

$$\omega \subset\subset \Omega \quad \text{and} \quad \partial\omega \text{ is } C^2. \tag{5.4.1}$$

For that, we need to invoke the following results, whose proofs follow immediately from [27, Lemmas 2.3 and 2.4].

Lemma 5.1. *Let $1 < s < \infty$ and $k \in \mathbb{N}$.*

(1) *Then for every $v^* \in \left(\mathrm{W}_0^{k,s}(\omega)\right)'$ there exists a unique $v \in \mathrm{W}_0^{k,s}(\omega)$ such that*

$$\int_\omega \mathrm{D}^\alpha v\, \mathrm{D}^\alpha \phi\, d\mathbf{x} = \langle v^*, \phi\rangle_{\mathbf{W}^{-k,s'}(\omega)\times \mathbf{W}_0^{k,s}(\omega)} \quad \forall\ \phi \in \mathrm{C}_0^\infty(\omega), \quad |\alpha| = k.$$

(2) *In addition, if exists $h \in \mathrm{L}^{s'}(\omega)$ such that*

$$\langle v^*, \phi\rangle_{\mathbf{W}^{-k,s'}(\omega)\times \mathbf{W}_0^{k,s}(\omega)} = \int_\omega h\ \mathrm{D}^\alpha \phi\, d\mathbf{x} \quad \forall\ \phi \in \mathrm{C}_0^\infty(\omega), \quad |\alpha| = k,$$

then

$$\|\mathrm{D}^\alpha v\|_{\mathrm{L}^{s'}(\omega)} \leq C\|h\|_{\mathrm{L}^{s'}(\omega)},$$

where C is a positive constant depending on s, N, ω and on the Calderón-Zigmund inequality's constant.

To simplify the notation in the sequel, let us set

$$\mathrm{A}_k^r(\omega) := \{a \in \mathrm{L}^r(\omega) : a = \mathrm{D}^\alpha u, \quad |\alpha| = k, \quad u \in \mathrm{W}_0^{k,r}(\omega)\}, \qquad 1 < r < \infty.$$

Then, applying the first part of Lemma 5.1 with $k = 2$, taking, for each $i = 1, \dots, N$, $s = q_i$ first and then $s = \left(\frac{q_a^*}{2}\right)'$, attending to the definitions of $\mathrm{A}_2^{q_i'}(\omega)$ and $\mathrm{A}_2^{\frac{q_a^*}{2}}(\omega)$, and using (5.2.7), for each $i = 1, \dots, N$, by one hand and (5.2.9) and (5.2.14) on the other, we can infer the existence of unique functions

$$p_{\epsilon_m}^i \in \mathrm{A}_2^{q_i'}(\omega), \quad i = 1, \dots, N, \tag{5.4.2}$$

and the existence of a unique function

$$p_{\epsilon_m}^\otimes \in \mathrm{A}_2^{\frac{q_a^*}{2}}(\omega) \tag{5.4.3}$$

such that, for every $i = 1, \dots, N$,

$$\int_\omega p_{\epsilon_m}^i \triangle\phi\, d\mathbf{x} = \int_\omega \left[\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2} D_i\mathbf{u}_{\epsilon_m} - \mathbf{S}_i\right)\otimes \mathbf{e}_i\right] : \nabla^2\phi\, d\mathbf{x}, \tag{5.4.4}$$

and

$$\int_\omega p_{\epsilon_m}^\otimes \triangle\phi\, d\mathbf{x} = -\int_\omega (\mathbf{u}_{\epsilon_m}\otimes\mathbf{u}_{\epsilon_m} - \mathbf{u}\otimes\mathbf{u}) : \nabla^2\phi\, d\mathbf{x} \tag{5.4.5}$$

for all $\phi \in \mathrm{C}_0^\infty(\omega)$. By the first part of Lemma 5.1, now with $k = 1$ and $s = \sigma'$, we obtain by virtue of (5.2.10) and attending to the definition of $\mathrm{A}_1^{\sigma'}(\omega)$ that exists a unique function

$$p_{\epsilon_m}^\sigma \in \mathrm{A}_1^{\sigma'}(\omega) \tag{5.4.6}$$

such that

$$\int_\omega p_{\epsilon_m}^\sigma \triangle\phi\, d\mathbf{x} = \epsilon_m \int_\omega |\mathbf{u}_{\epsilon_m}|^{\sigma-2}\mathbf{u}_{\epsilon_m}\cdot\nabla\phi\, d\mathbf{x} \tag{5.4.7}$$

for all $\phi \in \mathrm{C}_0^\infty(\omega)$. The second part of Lemma 5.1 gives us the estimates of the local pressures found in (5.4.2)-(5.4.3) and (5.4.6). In fact, attending to (5.2.7) and (5.4.4) by one hand, to (5.2.9), (5.2.14) and (5.4.5) on the other, and to (5.2.10) and (5.4.7) on another one, Lemma 5.1 yields

$$\|p^i_{\epsilon_m}\|_{\mathrm{L}^{q_i'}(\omega)} \leq C_i \||D_i\mathbf{u}_{\epsilon_m}|^{q_i-2}D_i\mathbf{u}_{\epsilon_m} - \mathbf{S}_i\|_{\mathbf{L}^{q_i'}(\omega)} \quad \text{for each } i = 1, \dots, N, \tag{5.4.8}$$

$$\|p^{\otimes}_{\epsilon_m}\|_{\mathrm{L}^{\frac{q_a^*}{2}}(\omega)} \leq C_{\otimes}\|\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} - \mathbf{u}\otimes\mathbf{u}\|_{\mathbf{L}^{\frac{q_a^*}{2}}(\omega)}, \tag{5.4.9}$$

$$\|p^{\sigma}_{\epsilon_m}\|_{\mathrm{L}^{\sigma'}(\omega)} \leq C_{\sigma}\|\epsilon_m|\mathbf{u}_{\epsilon_m}|^{\sigma-2}\mathbf{u}_{\epsilon_m}\|_{\mathbf{L}^{\sigma'}(\omega)}, \tag{5.4.10}$$

where C_i, $C_\otimes$ and C_σ are positive constants depending on q_i, q_a^* and σ, respectively, on N, ω and on the Calderón-Zigmund inequality's constant.

On the other hand, combining (5.3.11) and (5.3.13), and using the definition of the distributive derivative, we obtain

$$\begin{aligned} &-\sum_{i=1}^{N} D_i\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2}D_i\mathbf{u}_{\epsilon_m} - \mathbf{S}_i\right) + \mathbf{div}\left(\mathbf{u}_{\epsilon_m}\otimes\mathbf{u}_{\epsilon_m} - \mathbf{u}\otimes\mathbf{u}\right) \\ &+ \epsilon_m|\mathbf{u}_{\epsilon_m}|^{\sigma-2}\mathbf{u}_{\epsilon_m} = -\nabla\left(p_{\epsilon_m} - p_0\right) \end{aligned} \qquad \text{in } \mathcal{D}'(\omega). \tag{5.4.11}$$

Then, testing (5.4.11) by $\nabla\phi$, with $\phi \in \mathrm{C}_0^\infty(\omega)$, integrating over ω and comparing the resulting equation with the one resulting from adding (5.4.4), (5.4.5) and (5.4.7), we obtain

$$p_{\epsilon_m} - p_0 = \sum_{i=1}^{N} p^i_{\epsilon_m} + p^{\otimes}_{\epsilon_m} + p^{\sigma}_{\epsilon_m}\,.$$

Inserting this into (5.4.11), it follows that

$$\begin{aligned} &-\sum_{i=1}^{N} D_i\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2}D_i\mathbf{u}_{\epsilon_m} - \mathbf{S}_i\right) + \mathbf{div}\left(\mathbf{u}_{\epsilon_m}\otimes\mathbf{u}_{\epsilon_m} - \mathbf{u}\otimes\mathbf{u}\right) \\ &+ \epsilon_m|\mathbf{u}_{\epsilon_m}|^{\sigma-2}\mathbf{u}_{\epsilon_m} = -\sum_{i=1}^{N}\nabla p^i_{\epsilon_m} - \nabla p^{\otimes}_{\epsilon_m} - \nabla p^{\sigma}_{\epsilon_m} \end{aligned} \qquad \text{in } \mathcal{D}'(\omega). \tag{5.4.12}$$

5.5. **The Lipschitz truncation.** To start this section, let us set

$$\mathbf{w}_{\epsilon_m} := (\mathbf{u}_{\epsilon_m} - \mathbf{u})\chi_\omega, \tag{5.5.1}$$

where χ_ω denotes the characteristic function of the set ω introduced in (5.4.1). Having in mind the extension of (5.4.12) to $\mathbb{R}^N$, here we shall consider that

$$\boldsymbol{\Upsilon}_{\epsilon_m} := \sum_{i=1}^{N}\boldsymbol{\Upsilon}^i_{\epsilon_m} + \boldsymbol{\Upsilon}^{\otimes}_{\epsilon_m} + \boldsymbol{\Upsilon}^{\sigma}_{\epsilon_m} \tag{5.5.2}$$

is extended from ω to $\mathbb{R}^N$ by zero, where

$$\boldsymbol{\Upsilon}^i_{\epsilon_m} := -\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2}D_i\mathbf{u}_{\epsilon_m} - \mathbf{S}_i\right)\otimes\mathbf{e}_i + p^i_{\epsilon_m}\mathbf{I}, \tag{5.5.3}$$

$$\boldsymbol{\Upsilon}^{\otimes}_{\epsilon_m} := \mathbf{u}_{\epsilon_m}\otimes\mathbf{u}_{\epsilon_m} - \mathbf{u}\otimes\mathbf{u} + p^{\otimes}_{\epsilon_m}\mathbf{I}, \tag{5.5.4}$$

$$\boldsymbol{\Upsilon}^{\sigma}_{\epsilon_m} := \epsilon_m|\mathbf{u}_{\epsilon_m}|^{\sigma-2}\mathbf{u}_{\epsilon_m} + \nabla p^{\sigma}_{\epsilon_m}, \tag{5.5.5}$$

and $\mathbf{I}$ denotes the identity tensor.

Now, due to the definition (5.5.1) and by virtue of (5.2.6) and (5.2.11), we have

$$\mathbf{w}_{\epsilon_m} \to \mathbf{0} \quad \text{weakly in } \mathbf{V}_{\overrightarrow{\mathbf{q}}}, \quad \text{as } m\to\infty, \tag{5.5.6}$$

$$\mathbf{w}_{\epsilon_m} \to \mathbf{0} \quad \text{strongly in } \mathbf{L}^{\gamma}(\mathbb{R}^N), \quad \text{as } m\to\infty, \quad \text{for any } \gamma : 1 \leq \gamma < q_a^*. \tag{5.5.7}$$

Moreover, due to (5.2.7) and (5.4.8) by one hand, to (5.2.9) together with (5.2.14) and (5.4.9) on the other, and to (5.2.10) and (5.4.10) on another, we have

$$\|\boldsymbol{\Upsilon}^i_{\epsilon_m}\|_{\mathbf{L}^{q_i'}(\mathbb{R}^N)} \leq C \quad \forall\ i = 1,\dots,N\,, \tag{5.5.8}$$

$$\|\boldsymbol{\Upsilon}^{\otimes}_{\epsilon_m}\|_{\mathbf{L}^{\frac{q_a^*}{2}}(\mathbb{R}^N)} \leq C\,, \tag{5.5.9}$$

$$\|\boldsymbol{\Upsilon}^{\sigma}_{\epsilon_m}\|_{\mathbf{L}^{\sigma'}(\mathbb{R}^N)} \leq C\,. \tag{5.5.10}$$

In addition to (5.5.9), we see that, due to (5.2.11) and (5.4.9),

$$\boldsymbol{\Upsilon}^{\otimes}_{\epsilon_m} \to 0 \quad \text{strongly in } \mathbf{L}^{\frac{\gamma}{2}}(\mathbb{R}^N), \quad \text{as } m \to \infty, \quad \text{for any } \gamma : 1 \leq \gamma < q_a^*. \tag{5.5.11}$$

Next, let us consider the Hardy-Littlewood maximal functions of $|\mathbf{w}_{\epsilon_m}|$ and $|D_i\mathbf{w}_{\epsilon_m}|$ defined, for each $i = 1,\dots,N$, by

$$\mathcal{M}(|\mathbf{w}_{\epsilon_m}|)(\mathbf{x}) := \sup_{0<R<\infty} \frac{1}{\mathcal{L}_N(B_R(\mathbf{x}))}\int_{B_R(\mathbf{x})} |\mathbf{w}_{\epsilon_m}(\mathbf{y})|\,d\mathbf{y},$$
$$\mathcal{M}(|D_i\mathbf{w}_{\epsilon_m}|)(\mathbf{x}) := \sup_{0<R<\infty} \frac{1}{\mathcal{L}_N(B_R(\mathbf{x}))}\int_{B_R(\mathbf{x})} |D_i\mathbf{w}_{\epsilon_m}(\mathbf{y})|\,d\mathbf{y};$$

where $B_R(\mathbf{x})$ denotes the ball of $\mathbb{R}^N$ centered at $\mathbf{x}$ and with radius $R > 0$, and $\mathcal{L}_N(\omega)$ is the N-dimensional Lebesgue measure of ω.

The following auxiliar result gives us the bound for the set of relative size of a function.

Lemma 5.2. *Let $\mathbf{g} \in \mathbf{L}^{\gamma}(\mathbb{R}^N)$, $N \in \mathbb{N}$, for some $\gamma \geq 1$. Then*

$$\forall\ j \in \mathbb{N}_0 \quad \exists\ \lambda_j \in \left[2^{2^j+1}, 2^{2^{j+1}}\right] \ :\ \mathcal{L}_N\left(\left\{\mathbf{x} \in \mathbb{R}^N : |\mathbf{g}(\mathbf{x})| > \lambda_j\right\}\right) \leq \|\mathbf{g}\|^{\gamma}_{\mathbf{L}^{\gamma}(\mathbb{R}^N)} 2^{-j}\lambda_j^{-\gamma}\,,$$

where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.

Proof. The proof follows easily from [8, p. 217]. □

Using the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$ (see *e.g.* [25, Theorem I.1.1]), we obtain, by the application of Lemma 5.2, that for all $m \in \mathbb{N}$ and all $j \in \mathbb{N}_0$ there exists

$$\lambda_{m,j} \in \left[2^{2^j+1}, 2^{2^{j+1}}\right] \tag{5.5.12}$$

such that

$$\mathcal{L}_N\left(F_{m,j}\right) \leq 2^{-j}\lambda_{m,j}^{-\gamma}\,\|\mathbf{w}_{\epsilon_m}\|_{\mathbf{L}^{\gamma}(\mathbb{R}^N)} \quad \text{for any } \gamma\ :\ 1 \leq \gamma < q_a^*, \tag{5.5.13}$$

$$\mathcal{L}_N\left(G^i_{m,j}\right) \leq 2^{-j}\lambda_{m,j}^{-q_i}\,\|D_i\mathbf{w}_{\epsilon_m}\|_{\mathbf{L}^{q_i}(\mathbb{R}^N)} \quad \text{for any } i = 1,\dots,N, \tag{5.5.14}$$

where

$$F_{m,j} := \left\{\mathbf{x} \in \mathbb{R}^N : \mathcal{M}(|\mathbf{w}_{\epsilon_m}|)(\mathbf{x}) > \lambda_{m,j}\right\},$$
$$G^i_{m,j} := \left\{\mathbf{x} \in \mathbb{R}^N : \mathcal{M}(|D_i\mathbf{w}_{\epsilon_m}|)(\mathbf{x}) > \lambda_{m,j}\right\}.$$

For each $i \in \{1,\dots,N\}$, we set

$$R^i_{m,j} := F_{m,j} \cup G^i_{m,j} \cup \left\{\mathbf{x} \in \mathbb{R}^N :\ \mathbf{x} \text{ is not a Lebesgue point of } |\mathbf{w}_{\epsilon_m}|\right\}. \tag{5.5.15}$$

Due to (5.5.6)-(5.5.7), (5.5.12) and (5.5.13)-(5.5.15), we obtain for each $i \in \{1,\dots,N\}$

$$\limsup_{m\to\infty} \mathcal{L}_N\left(R^i_{m,j}\right) \leq C2^{-j}\limsup_{m\to\infty} \lambda_{m,j}^{-q_i}. \tag{5.5.16}$$

The following result will help us to approximate ours $W^{1,\vec{\mathbf{q}}}$-functions by Lipschitz ones.

Lemma 5.3. *Let $\omega \subset \mathbb{R}^N$ be an open bounded set with a Lipschitz-continuous boundary $\partial\omega$ and let i be arbitrarily given in $\{1,\dots,N\}$. Assume that $\mathbf{u} \in \mathbf{W}^{1,1}_0(\omega)$ such that $D_i\mathbf{u} \in \mathbf{L}^{s_i}(\omega)$, where $1 \leq s_i < \infty$. Then, for every $\kappa,\ \lambda > 0$, there exists $\mathbf{v} \in \mathbf{W}^{1,\infty}_0(\omega)$ such that*

$$\|\mathbf{v}\|_{\mathbf{L}^{\infty}(\omega)} \leq \kappa,$$
$$\|D_i\mathbf{v}\|_{\mathbf{L}^{\infty}(\omega)} \leq C_i\lambda, \quad C_i = C_i(N,\omega).$$

In addition,

$$\{\mathbf{x}\in\omega:\mathbf{v}(\mathbf{x})\neq\mathbf{u}(\mathbf{x})\}\subset$$
$$\omega\cap\big(\{\mathbf{x}\in\mathbb{R}^N:\mathcal{M}(|\mathbf{u}|)(\mathbf{x})>\kappa\}\cup\{\mathbf{x}\in\mathbb{R}^N:\mathcal{M}(|D_i\mathbf{u}|)(\mathbf{x})>\lambda\}\cup$$
$$\{\mathbf{x}\in\mathbb{R}^N:\ \mathbf{x}\ \textit{is not a Lebesgue point of}\ |\mathbf{u}|\}\big)\,.$$

Proof. Due to the continuous imbedding $\{\mathbf{u}\in\mathbf{W}^{1,1}_0(\omega):\mathbf{u}\in\mathbf{L}^{q_i}(\omega)\}\hookrightarrow\mathbf{W}^{1,1}_0(\omega)$, the proof follows from [1, Theorem 1] and [16, Proposition 2.2] (see also [19, Lemma 1.68]). □

Then, by Lemma 5.3 together with the definition of $\mathbf{w}_{\epsilon_m}$ (*f.* (5.5.1)), for each $i\in\{1,\dots,N\}$ there exists

$$\mathbf{z}^i_{m,j}\in\mathbf{W}^{1,\infty}(\mathbb{R}^N),\qquad \mathbf{z}^i_{m,j}=\begin{cases}\mathbf{w}_{\epsilon_m} & \text{in }\omega\setminus A^i_{m,j}\\ 0 & \mathbb{R}^N\setminus\omega\end{cases},$$

where

$$A^i_{m,j}:=\{\mathbf{x}\in\omega:\mathbf{z}^i_{m,j}(\mathbf{x})\neq\mathbf{w}_{\epsilon_m}(\mathbf{x})\},\tag{5.5.17}$$

such that

$$\|\mathbf{z}^i_{m,j}\|_{\mathbf{L}^\infty(\omega)}\le\lambda_{m,j},\tag{5.5.18}$$

$$\|D_i\mathbf{z}^i_{m,j}\|_{\mathbf{L}^\infty(\omega)}\le C_i\lambda_{m,j},\quad C_i=C_i(N,\omega).\tag{5.5.19}$$

Moreover, by the last statement of the above lemma and using the notations (5.5.13)-(5.5.15) and (5.5.17), for each $i\in\{1,\dots,N\}$

$$A^i_{m,j}\subset\omega\cap R^i_{m,j}.\tag{5.5.20}$$

As a consequence of (5.5.16) and (5.5.20),

$$\limsup_{m\to\infty}\mathcal{L}_N\left(A^i_{m,j}\right)\le C2^{-j}\limsup_{m\to\infty}\lambda^{-q_i}_{m,j}.\tag{5.5.21}$$

Now, using the functions $\mathbf{z}^i_{m,j}$, we can build a function $\mathbf{z}_{m,j}\in\mathbf{W}^{1,\infty}_0(\omega)$ such that

$$\mathbf{z}_{m,j}=\begin{cases}\mathbf{w}_{\epsilon_m} & \text{in }\bigcup_{l=1}^N\left(\omega\setminus A^l_{m,j}\right)=\omega\setminus\bigcap_{l=1}^N A^l_{m,j}\\ 0 & \text{in }\mathbb{R}^N\setminus\omega\,.\end{cases},\tag{5.5.22}$$

For this function so built, and due to (5.5.6), (5.5.18)-(5.5.19) and (5.5.21), we can prove that for any $j\in\mathbb{N}_0$

$$\mathbf{z}_{m,j}\to\mathbf{0}\quad\text{weakly in }\mathbf{W}^{1,\overrightarrow{\mathbf{q}}}_0(\omega),\quad\text{as }m\to\infty.\tag{5.5.23}$$

Then by the anisotropic Sobolev's compact imbedding, we get for any $j\in\mathbb{N}_0$

$$\mathbf{z}_{m,j}\to\mathbf{0}\text{ strongly in }\mathbf{L}^\gamma(\omega),\quad\text{as }m\to\infty,\quad\text{for any }\gamma:1\le\gamma<q^*_a.$$

Using this information, (5.5.18) and interpolation, we prove that for any $j\in\mathbb{N}_0$

$$\mathbf{z}_{m,j}\to\mathbf{0}\quad\text{strongly in }\mathbf{L}^s(\omega),\quad\text{as }m\to\infty,\quad\text{for any }s:1\le s<\infty.\tag{5.5.24}$$

Finally, as a consequence of (5.5.23) and (5.5.24), we obtain for any $j\in\mathbb{N}_0$

$$\mathbf{z}_{m,j}\to\mathbf{0}\quad\text{weakly in }\mathbf{W}^{1,\overrightarrow{\mathbf{s}}}_0(\omega),\quad\text{as }m\to\infty,\quad\text{for any }\overrightarrow{\mathbf{s}}:1\le s_i<\infty.\tag{5.5.25}$$

5.6. **Convergence of the approximated anisotropic diffusion term.** Let us first observe that, using the notations (5.5.2)-(5.5.5), we can infer from (5.4.12) that

$$\int_\omega \left(\sum_{i=1}^N \boldsymbol{\Upsilon}^i_{\epsilon_m} + \boldsymbol{\Upsilon}^{\otimes}_{\epsilon_m}\right) : \nabla\phi\, d\mathbf{x} = \int_\omega \boldsymbol{\Upsilon}^{\sigma}_{\epsilon_m} \cdot \phi\, d\mathbf{x} \qquad \forall\ \phi \in \mathcal{D}(\omega). \tag{5.6.1}$$

On the other hand, due to (5.5.8)-(5.5.9),

$$\sum_{i=1}^N \boldsymbol{\Upsilon}^i_{\epsilon_m} + \boldsymbol{\Upsilon}^{\otimes}_{\epsilon_m} + \boldsymbol{\Upsilon}^{\sigma}_{\epsilon_m} \in \mathbf{L}^r(\mathbb{R}^N), \quad r := \min\left\{\beta', \frac{q_a^*}{2}, \sigma'\right\}.$$

Due to the fact that $\mathcal{D}(\omega)$ is dense in $\mathbf{W}^{1,r'}_0(\omega)$, since there always exists an anisotropic space $\mathbf{W}^{1,\overrightarrow{\mathbf{s}}}_0(\omega)$ such that $\mathbf{W}^{1,r'}_0(\omega)$ is a closed subspace of it, and by virtue of (5.5.25), we can replace the test function in (5.6.1) by $\phi = \mathbf{z}_{m,j}$ for all $j \in \mathbb{N}_0$. Expanding the terms of (5.6.1) through the notations (5.5.2)-(5.5.5) and subtracting and adding the integral sum

$$\sum_{i=1}^N \int_\omega \left(|D_i\mathbf{u}|^{q_i-2} D_i\mathbf{u} \otimes \mathbf{e}_i\right) : \nabla\mathbf{z}_{m,j}\, d\mathbf{x}$$

to the left hand side of the resulting equation, we obtain for any $j \in \mathbb{N}_0$

$$\begin{aligned}
&\sum_{i=1}^N \int_\omega \left[\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2} D_i\mathbf{u}_{\epsilon_m} - |D_i\mathbf{u}|^{q_i-2} D_i\mathbf{u}\right) \otimes \mathbf{e}_i\right] : \nabla\mathbf{z}_{m,j}\, d\mathbf{x} \\
=&\sum_{i=1}^N \int_\omega \left[\left(\mathbf{S}_i - |D_i\mathbf{u}|^{q_i-2} D_i\mathbf{u}\right) \otimes \mathbf{e}_i\right] : \nabla\mathbf{z}_{m,j}\, d\mathbf{x} + \\
&\sum_{i=1}^N \int_\omega p^i_{\epsilon_m} \operatorname{div} \mathbf{z}_{m,j}\, d\mathbf{x} + \\
&\int_\omega \left(\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} - \mathbf{u} \otimes \mathbf{u} + p^{\otimes}_{\epsilon_m}\mathbf{I}\right) : \nabla\mathbf{z}_{m,j}\, d\mathbf{x} + \\
&\int_\omega \left(\epsilon_m |\mathbf{u}_{\epsilon_m}|^{\sigma-2}\mathbf{u}_{\epsilon_m} + \nabla p^{\sigma}_{\epsilon_m}\right) \cdot \nabla\mathbf{z}_{m,j}\, d\mathbf{x} \\
:=& J^1_{m,j} + J^2_{m,j} + J^3_{m,j} + J^4_{m,j}.
\end{aligned} \tag{5.6.2}$$

We claim that, for a fixed j,

$$\sum_{i=1}^N \int_\omega \left[\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2} D_i\mathbf{u}_{\epsilon_m} - |D_i\mathbf{u}|^{q_i-2} D_i\mathbf{u}\right) \otimes \mathbf{e}_i\right] : \nabla\mathbf{z}_{m,j}\, d\mathbf{x} \leq C 2^{-\frac{j}{\alpha}} \tag{5.6.3}$$

To prove this, we will carry out the passage to the limit $m \to \infty$ in the absolute values $|J^i_{m,j}|$, $i = 1,\ 2,\ 3,\ 4$.

- $\limsup_{m\to\infty} |J^1_{m,j}| = 0$. By (5.5.25), with $\overrightarrow{\mathbf{s}} = \overrightarrow{\mathbf{q}}$, this is true once we can justify that $\mathbf{S}_i - |D_i\mathbf{u}|^{q_i-2} D_i\mathbf{u}$ is uniformly bounded in $\mathbf{L}^{q_i'}(\omega)$ for all $i = 1, \dots, N$. But this is an immediate consequence of (5.2.7).
- $\limsup_{m\to\infty} |J^2_{m,j}| \leq C 2^{-\frac{j}{\alpha}}$. In fact, by Hölder's inequality and (5.4.8) together with (5.2.3), and using the definition of $\mathbf{z}_{m,j}$ (*f.* (5.5.22)) together with the fact that $\operatorname{div}\mathbf{w}_{\epsilon_m} = 0$ in ω (*f.* (5.5.1)),

$$\begin{aligned}
|J^2_{m,j}| \leq& C_1 \sum_{i=1}^N \|\operatorname{div} \mathbf{z}_{m,j}\|_{\mathbf{L}^{q_i}\left(\bigcap_{l=1}^N A^l_{m,j}\right)} \leq C_1 \sum_{i=1}^N \sum_{k=1}^N \|D_k\mathbf{z}_{m,j}\|_{\mathbf{L}^{q_i}\left(\bigcap_{l=1}^N A^l_{m,j}\right)} \\
\leq& C_1 \sum_{i=1}^N \sum_{k=1}^N \|D_k\mathbf{z}_{m,j}\|_{\mathbf{L}^\infty(\omega)} \mathcal{L}_N \left(\bigcap_{l=1}^N A^l_{m,j}\right)^{\frac{1}{q_i}} \leq C_1 \sum_{i=1}^N \sum_{k=1}^N \|D_k\mathbf{z}_{m,j}\|_{\mathbf{L}^\infty(\omega)} \mathcal{L}_N \left(A^i_{m,j}\right)^{\frac{1}{q_i}}.
\end{aligned}$$

Then, by the application of (5.5.19) and (5.5.21), it follows

$$\limsup_{m\to\infty} |J^2_{m,j}| \leq C_1 \sum_{i=1}^{N}\sum_{k=1}^{N} C_k \limsup_{m\to\infty} \lambda_{m,j} \left(C_2 2^{-j}\lambda_{m,j}^{-q_i}\right)^{\frac{1}{q_i}} \leq C_3 2^{-\frac{j}{\alpha}}. \tag{5.6.4}$$

- $\limsup_{m\to\infty}|J^3_{m,j}| = 0$. Using Hölder's inequality and the notation (5.5.4), we have

$$\begin{aligned}|J^3_{m,j}| \leq& \|\boldsymbol{\Upsilon}^{\otimes}_{\epsilon_m}\|_{\mathbf{L}^1(\omega)}\|\nabla \mathbf{z}_{m,j}\|_{\mathbf{L}^\infty(\omega)}\\ \leq& C_1\|\boldsymbol{\Upsilon}^{\otimes}_{\epsilon_m}\|_{\mathbf{L}^1(\omega)}\sum_{k=1}^{N}\|D_k\mathbf{z}_{m,j}\|_{\mathbf{L}^\infty(\omega)} \to 0, \quad \text{as } m\to\infty.\end{aligned}$$

The last inequality and the conclusion follow, respectively, from (5.5.19) and (5.5.11) with $\gamma = 2$, observing that we have used here the fact that $2 < q^*_a$, which in turn is assured by our main assumption (5.1).

- $\limsup_{m\to\infty}|J^4_{m,j}| = 0$. By (5.5.25), for instance with $s_i = \sigma$ for all $i = 1,\dots,N$, this is true once we can justify that $\epsilon_m|\mathbf{u}_{\epsilon_m}|^{\sigma-2}\mathbf{u}_{\epsilon_m} + \nabla p^\sigma_{\epsilon_m}$ is uniformly bounded in $\mathbf{L}^{\sigma'}(\omega)$. But this is an immediate consequence of (5.5.10).

Gathering the estimates above we just have proven (5.6.3).

We proceed with the proof by using an argument exploited in [7, Theorem 5] (see also the references cited in [7, Remark 4]). Firstly, observing the definition of $\mathbf{z}_{m,j}$ (*f.* (5.5.22)), we have

$$\sum_{i=1}^{N}\int_\omega \left[\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2}D_i\mathbf{u}_{\epsilon_m} - |D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right)\otimes \mathbf{e}_i\right] : \nabla \mathbf{z}_{m,j}\,d\mathbf{x} := I_{m,j} + II_{m,j}\,, \tag{5.6.5}$$

where

$$\begin{aligned}I_{m,j} :=& \sum_{i=1}^{N}\int_{\omega\setminus A_{m,j}} \left[\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2}D_i\mathbf{u}_{\epsilon_m} - |D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right)\otimes \mathbf{e}_i\right] : \nabla \mathbf{z}_{m,j}\,d\mathbf{x},\\ II_{m,j} :=& \sum_{i=1}^{N}\int_{A_{m,j}} \left[\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2}D_i\mathbf{u}_{\epsilon_m} - |D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right)\otimes \mathbf{e}_i\right] : \nabla \mathbf{z}_{m,j}\,d\mathbf{x}.\end{aligned}$$

Then (5.6.3) and (5.6.5) imply that

$$\limsup_{m\to\infty} I_{m,j} \leq C\left(\limsup_{m\to\infty}|II_{m,j}| + 2^{-\frac{j}{\alpha}}\right). \tag{5.6.6}$$

For the term $II_{m,j}$, we use the same arguing as we did for (5.6.4) to prove that for any $j\in\mathbb{N}_0$

$$\limsup_{m\to\infty}|II_{m,j}| \leq C2^{-\frac{j}{\alpha}}. \tag{5.6.7}$$

As a consequence of (5.6.6) and (5.6.7), we have for any $j\in\mathbb{N}_0$

$$\limsup_{m\to\infty} I_{m,j} \leq C2^{-\frac{j}{\alpha}}. \tag{5.6.8}$$

Now, by Hölder's inequality and having in mind the definition of $\mathbf{z}_{m,j}$ (*f.* (5.5.22)), we obtain for any $\theta\in(0,1)$

$$\begin{aligned}\sum_{i=1}^{N}\int_\omega |g^i_{\epsilon_m}|^\theta\,d\mathbf{x} \leq& C\sum_{i=1}^{N}\left(\int_{\omega\setminus A_{m,j}}\left|g^i_{\epsilon_m}\right|\,d\mathbf{x}\right)^\theta \mathcal{L}_N(\omega\setminus A_{m,j})^{1-\theta}\\ &+ C\sum_{i=1}^{N}\left(\int_{A_{m,j}}\left|g^i_{\epsilon_m}\right|\,d\mathbf{x}\right)^\theta \mathcal{L}_N(A_{m,j})^{1-\theta}\,,\end{aligned} \tag{5.6.9}$$

where, for each $i = 1,\dots,N$,

$$g^i_{\epsilon_m} := \left[\left(|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2}D_i\mathbf{u}_{\epsilon_m} - |D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\right)\otimes \mathbf{e}_i\right] : \nabla \mathbf{z}_{m,j}\,.$$

Arguing as we did to prove (5.6.7)-(5.6.8) and using (5.5.21), it follows from (5.6.9) that

$$\limsup_{m\to\infty}\sum_{i=1}^{N}\int_{\omega}|\mathbf{g}^i_{\epsilon_m}|^{\theta}\,d\mathbf{x}\leq C2^{-\theta\frac{j}{\alpha}}\,. \tag{5.6.10}$$

Since $\theta\in(0,1)$ and $j\in\mathbb{N}_0$ is arbitrary, $2^{-\theta\frac{j}{\alpha}}\to 0$, as $j\to\infty$. This and (5.6.10) imply that for any $\theta\in(0,1)$

$$\limsup_{m\to\infty}\int_{\omega}|g^i_{\epsilon_m}|^{\theta}\,d\mathbf{x}=0\qquad\text{for all } i=1,\dots,N.$$

Then, passing to a subsequence,

$$g^i_{\epsilon_m}\to 0\quad\text{a.e. in }\omega,\quad\text{as } m\to\infty,\qquad\text{for all } i=1,\dots,N. \tag{5.6.11}$$

Applying [7, Lemma 6] together with (5.6.11) (see also [17, Lemme 2.2.2]), we obtain for every $i=1,\dots,N$

$$|D_i\mathbf{u}_{\epsilon_m}|^{q_i-2}D_i\mathbf{u}_{\epsilon_m}\to|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}\quad\text{a.e. in }\omega,\quad\text{as } m\to\infty. \tag{5.6.12}$$

Finally, (5.2.3) and (5.6.12) allow us to use Vitali's theorem together with (5.2.7) in order to conclude that $\mathbf{S}_i=|D_i\mathbf{u}|^{q_i-2}D_i\mathbf{u}$ for all $i=1,\dots,N$. □

Remark 5.3. *We can drop the condition that Ω has a Lipschitz-continuous boundary $\partial\Omega$. In this case, we need to consider at Section 5.3 a fixed but arbitrary subdomain ω' of Ω such that*

$$\omega'\subset\subset\Omega\quad\textit{and}\quad\partial\omega'\textit{ is Lipschitz}. \tag{5.6.13}$$

*This is motivated by the application of a version of de Rham's Theorem (*cf. [4] *and* [20]*) used in our proof and which always assume Lipschitz-continuous boundaries. With this modification, the domain ω stated in (5.4.1) should be considered in a way such that $\omega\subset\subset\omega'\subset\subset\Omega$ and $\partial\omega$ is C^2.*

6. Analysis of the extent of Theorem 5.1

In the final section of this work, we will perform an analysis similar to that one we have made at Section 4, but now for the case of very weak solutions. Here we also start by observing that if $q_i=q$ for all $i=1,\dots,N$, then (5.1) is equivalent to the relation

$$q>\frac{2N}{N+2} \tag{6.1}$$

and, to the best of our knowledge, there is no existence result, now in the sense of the isotropic version of Definition 5.1, for lower values of q (see [11, 8]). Also here we shall see that condition (5.1) will allow us to consider an existence result with some components of $\overrightarrow{\mathbf{q}}$ considerably lower than those provided by (6.1). In order to simplify the exposition, we also assume in this section that

$$\beta=q_N\geq q_{N-1}\geq\cdots\geq q_2\geq q_1=\alpha\,.$$

By the definition of the critical exponent q^*_a (see (2.11)) and recalling what have been observed in Remark 5.1, condition (5.1) can be written in the following equivalent form:

$$\begin{cases}\beta>2 & \text{if}\quad \beta>\overline{q}^*\ \textbf{: Case 1}\\[2mm] \overline{q}>\dfrac{2N}{N+2} & \text{if}\quad \beta\leq\overline{q}^*\Leftrightarrow\overline{q}\geq\dfrac{\beta N}{\beta+N}\Leftrightarrow\displaystyle\sum_{i\neq N}\frac{1}{q_i}\leq 1+\frac{N-1}{\beta}\ \textbf{: Case 2}\,.\end{cases} \tag{6.2}$$

Case 1. Recalling Remark 2.1 this first case reads

$$\beta > 2 \quad \text{and} \quad \overline{q} < \frac{\beta\, N}{\beta + N} \Leftrightarrow \sum_{i \neq N} \frac{1}{q_i} > 1 + \frac{N-1}{\beta}\,.$$

Example 6.1. *For* $N = 3$*,*

$$\beta > 2 \quad \textit{and} \quad \sum_{i \neq N} \frac{1}{q_i} > 1 + \frac{2}{\beta}\,.$$

For instance, the following choice of β *gives us the link between* α *and* q_2*:*

$$\beta = \frac{5}{2} \quad \Rightarrow \quad \frac{1}{\alpha} + \frac{1}{q_2} > \frac{9}{5}\,.$$

In particular, if $q_2 = \alpha$*, then*

$$\beta = \frac{5}{2} \quad \Rightarrow \quad 1 < \alpha = q_2 < \frac{10}{9}\,.$$

The previous example shows us it is possible that $q_i \searrow 1$ for all $i \neq N$, which considerably decreases these components, when comparing with (6.1).

Case 2. Remark 2.1 allows us also to read, now the second case, in the following form:

$$\overline{q} > \frac{2N}{N+2} \Leftrightarrow \sum_{i \neq N} \frac{1}{q_i} < 1 + \frac{N}{2} - \frac{1}{\beta} \quad \text{and} \quad \overline{q} \geq \frac{\beta\, N}{\beta + N} \Leftrightarrow \sum_{i \neq N} \frac{1}{q_i} \leq 1 + \frac{N-1}{\beta}\,.$$

In consequence, this case occurs if

$$\sum_{i \neq N} \frac{1}{q_i} < \min\left\{1 + \frac{N}{2} - \frac{1}{\beta}, 1 + \frac{N-1}{\beta}\right\} = \begin{cases} 1 + \dfrac{N}{2} - \dfrac{1}{\beta} & \text{if} \quad \beta < 2 : \textbf{Case 2.1} \\ 1 + \dfrac{N-1}{\beta} & \text{if} \quad \beta \geq 2 : \textbf{Case 2.2}\,. \end{cases}$$

Case 2.1. In this case, we have

$$\beta < 2 \quad \text{and} \quad \sum_{i \neq N} \frac{1}{q_i} < 1 + \frac{N}{2} - \frac{1}{\beta}\,.$$

Example 6.2. *For* $N = 3$*,*

$$\beta < 2 \quad \textit{and} \quad \sum_{i \neq N} \frac{1}{q_i} < \frac{5}{2} - \frac{1}{\beta}\,.$$

For instance, the following choice of β *gives us the link between* α *and* q_2*:*

$$\beta = \frac{3}{2} \quad \Rightarrow \quad \frac{1}{\alpha} + \frac{1}{q_2} < \frac{11}{6}\,.$$

In particular, if $q_2 = \alpha$*, then*

$$\beta = \frac{3}{2} \quad \Rightarrow \quad \frac{12}{11} < \alpha = q_2 \leq \frac{3}{2}\,.$$

In this example we have similar results to the example of Case 1, but, in addition, all the exponents q_i are closer and less than 2. Another difference here is that we have a threshold higher than 1 from which the exponents are not decreasing over.

Case 2.2. For this case,

$$\beta \geq 2 \quad \text{and} \quad \sum_{i\neq N} \frac{1}{q_i} < 1 + \frac{N-1}{\beta}\,.$$

Example 6.3. *For $N = 3$,*

$$\beta \geq 2 \quad \textit{and} \quad \sum_{i\neq N} \frac{1}{q_i} < 1 + \frac{2}{\beta}\,.$$

For instance, the following choice of β gives us the link between α and q_2:

$$\beta = \frac{5}{2} \quad \Rightarrow \quad \frac{1}{\alpha} + \frac{1}{q_2} < \frac{9}{5}\,.$$

In particular, if $q_2 = \alpha$, then

$$\beta = \frac{5}{2} \quad \Rightarrow \quad \frac{10}{9} < \alpha = q_2 \leq \frac{5}{2}\,.$$

This example shows that, in this case, the exponents q_i although remain close, they do not approach as close to 1 as in the previous cases. Moreover it is also possible that all them may be above 2.

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