TOPOLOGICAL SENSITIVITY ANALYSIS FOR HIGH ORDER ELLIPTIC OPERATORS
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Abstract. The topological derivative is defined as the first term of the asymptotic expansion of a given shape functional with respect to a small parameter that measures the size of a singular domain perturbation. It has applications in many different fields such as shape and topology optimization, inverse problems, image processing and mechanical modeling including synthesis and/or optimal design of microstructures, fracture mechanics sensitivity analysis and damage evolution modeling. The topological derivative has been fully developed for a wide range of second order differential operators. In this paper we deal with the topological asymptotic expansion of a class of shape functionals associated with elliptic differential operators of order $2m$, $m \geq 1$. The general structure of the polarization tensor is derived and the concept of degenerate polarization tensor is introduced. We provide full mathematical justifications for the derived formulas, including precise estimates of remainders.

1. Introduction

The topological derivative measures the sensitivity of a given shape functional with respect to infinitesimal singular domain perturbations, such as the insertion of holes, inclusions, source-terms or even cracks [12, 14, 16, 25]. Specifically, if the shape functional is denoted by $J(\Omega)$ and the domain obtained after a perturbation of size $\varepsilon$ localized around a point $z$ is denoted by $\Omega_\varepsilon$, it is defined by

$$DTJ(\Omega) = \lim_{\varepsilon \to 0} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varphi(\varepsilon)},$$

for some appropriate scaling function $\varphi(\varepsilon)$. This notion has proved to be a powerful tool for the treatment of different problems such as topology optimization, inverse analysis and image processing (see e.g. [7, 11, 17, 18, 19]), and has become a subject of intensive research. There are also some applications in the multi-scale constitutive modeling context [8], fracture mechanics sensitivity analysis [27] and damage evolution modeling [2]. All these problems share in common to be governed by partial differential equations (PDE’s), and the type of PDE obviously impacts drastically on the mathematical analysis involved. Concerning the theoretical development of the topological asymptotic analysis, the reader may refer to the papers [4, 13, 22], among others.

According to the literature, the topological derivative concept has been fully developed for a wide range of second order equations, while a forth order equation is addressed in [6]. In this paper, the topological asymptotic expansion of a class of shape functionals associated with an elliptic differential operator of order $2m$, with $m \geq 1$, is derived. The topologically perturbed domain is obtained when an arbitrary shaped hole is introduced inside the initial domain. Then, the resulting void is filled with a phase whose material properties present a contrast with the original ones. The main ingredient arising in the asymptotic formula is the so-called Pólya-Szegő polarization tensor [24] (see also [3]), of which we derive the general structure for the operators under consideration. We also introduce the concept of degenerate polarization tensor, in the sense that it is independent of the shape of the topological perturbation and, at the same time, its entries do not remain bounded when the contrast on the material properties goes to zero. In this particular case it is remarkable that the polarization tensor can be easily obtained in its closed form. We show that this phenomenon of degeneracy occurs when the operator satisfies a particular algebraic property which is easy to check, a typical example being the bi-Laplacian. Finally, we provide full mathematical justifications for the derived formulas with precise estimates for the remainders of the topological asymptotic expansion.

Key words and phrases. topological sensitivity analysis, topological derivative, high order elliptic operators, polarization tensor, degenerate polarization tensor.
The paper is organized as follows. Some notation and preliminary statements are introduced in Section 2. Basic properties of the boundary value problems under consideration are collected in Section 3. The topological asymptotic expansion for a class of shape functionals is derived in its general form in Section 4, and the concept of degenerate polarization tensor is introduced in Section 5. The appropriate estimates of remainders are provided in Section 6. Some particular cases of differential operators, including degenerate cases, are presented in Section 7 together with a set of examples with analytical solution. The extension of the obtained results to elliptic systems is discussed in Section 8. Some concluding remarks and perspectives are given in Section 9. In Appendix A, the notion of collectively compact operators used throughout the analysis is recalled. Finally, the weighted and quotient Sobolev spaces needed for the formulation of appropriate exterior problems, appearing in particular in the construction of the polarization tensor, are described in Appendix B.

2. Preliminaries and notation

Let Ω be an open and bounded subset of \( \mathbb{R}^n \) \((n \in \mathbb{N}^*)\) and \( \hat{x} \in \Omega \) be fixed. Given an open, bounded and smooth subset \( \omega \) of \( \mathbb{R}^n \) containing the origin, we define for every \( \varepsilon > 0 \) the set

\[ \omega_\varepsilon(\hat{x}) = \hat{x} + \varepsilon \omega. \]

Let \( \rho_0 \in L^\infty(\Omega) \) be a given function which takes a constant value \( \hat{\rho}_0 \) in a neighborhood of \( \hat{x} \) and such that \( \text{essinf}_{\Omega} \rho_0 > 0 \). Moreover, given a constant \( \hat{\rho}_1 > 0 \), we set for all \( \varepsilon \geq 0 \)

\[ \rho_\varepsilon(x) = \begin{cases} \rho_0(x) & \text{if } x \in \Omega \setminus \omega_\varepsilon, \\ \hat{\rho}_1 & \text{if } x \in \Omega \cap \omega_\varepsilon. \end{cases} \tag{2.1} \]

For all multi-index \( \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n \) and any \( \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n \), we denote by

\[ |\alpha| = \sum_{i=1}^n \alpha_i, \quad |\xi| = \left( \sum_{i=1}^n \xi_i^2 \right)^{1/2}, \quad \xi^\alpha = \prod_{i=1}^n \xi_i^{\alpha_i}, \]

the length of \( \alpha \), the norm of \( \xi \) and the \( \alpha \)-power of \( \xi \), respectively. To avoid any ambiguity, all multi-indices will be denoted by the letters \( \alpha, \beta \) or \( \gamma \). The derivative of order \( \alpha \) of a distribution \( u \) is defined by

\[ D^\alpha u = \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n} u. \]

Let \( m \in \mathbb{N}^* \). We consider a family of real constant coefficients \( (a_{\alpha\beta})_{|\alpha|=|\beta|=m} \) satisfying the following properties.

- Symmetry: it holds for every \( \alpha, \beta \)
  \[ a_{\alpha\beta} = a_{\beta\alpha}. \tag{2.2} \]

- Positivity: for any family of real numbers \( (y_\alpha)_{|\alpha|=m} \) it holds
  \[ \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} y_\alpha y_\beta \geq 0. \tag{2.3} \]

- Uniform ellipticity: there exists \( \kappa > 0 \) such that
  \[ \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} |\xi^\alpha + \beta|^2 \geq \kappa |\xi|^{2m} \quad \forall \xi \in \mathbb{R}^n. \tag{2.4} \]

We define the homogeneous operator \( A_\varepsilon : H_0^m(\Omega) \to H^{-m}(\Omega) \) by

\[ (A_\varepsilon u, v)_{H^{-m}(\Omega), H_0^m(\Omega)} = \sum_{|\alpha|=|\beta|=m} \int_\Omega \rho_\varepsilon a_{\alpha\beta} D^\alpha u D^\beta v \, dx \quad \forall u, v \in H_0^m(\Omega). \tag{2.5} \]

We will later argue that \( A_\varepsilon \) is invertible (see Corollary 3.2).

We further consider coefficients \( (b_{\alpha\beta, \varepsilon}) \) defined for all \( \varepsilon \geq 0 \) and \( \alpha, \beta \) such that \( |\alpha| \leq m \) and \( |\beta| \leq m - 1 \). We assume that, for \( \varepsilon \) small enough,

\[ b_{\alpha\beta, \varepsilon} - b_{\alpha\beta, 0} = q_{\alpha\beta} \chi_{\omega_\varepsilon}. \]
for some coefficients $g_{\alpha\beta}$, and with $\chi_{\omega_\varepsilon}$ the characteristic function of $\omega_\varepsilon$. We define the operator $B_\varepsilon : H^0_0(\Omega) \to H^{-m}(\Omega)$ by

$$\langle B_\varepsilon u, v \rangle_{H^{-m}(\Omega), H^m_0(\Omega)} = \sum_{|\alpha| \leq m} \int_{\Omega} h_{\alpha\beta, \varepsilon} D^\alpha u D^\beta v \, dx \quad \forall u, v \in H^m_0(\Omega).$$

We assume that, for all $\varepsilon$ sufficiently small, the operator $A_\varepsilon + B_\varepsilon$ and its adjoint $A_\varepsilon + B_\varepsilon^*$ are injective. We will infer (see Proposition 3.3) that $A_\varepsilon + B_\varepsilon$ is invertible, as well as its adjoint (the proof is the same). Henceforth $\varepsilon$ will always be implicitly assumed to be small enough.

Given a source $f \in H^{-m}(\Omega)$ we denote for every $\varepsilon > 0$ by $u_\varepsilon \in H^m_0(\Omega)$ the unique solution of

$$(A_\varepsilon + B_\varepsilon)u_\varepsilon = f.$$  

The goal of this paper is to analyze the asymptotic behavior of a shape functional of the form $j(\varepsilon) = J_\varepsilon(u_\varepsilon)$ when $\varepsilon \to 0$.

### 3. Well-posedness

The space $H^m(\Omega)$ is endowed with the standard norm $\| \cdot \|_{H^m(\Omega)}$ and the associated seminorm $| \cdot |_{H^m(\Omega)}$ defined by

$$\| u \|_{H^m(\Omega)}^2 = \sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^2(\Omega)}^2, \quad | u |_{H^m(\Omega)}^2 = \sum_{|\alpha| = m} \| D^\alpha u \|_{L^2(\Omega)}^2.$$

The expression (2.5) defines a symmetric and continuous bilinear form on $H^m_0(\Omega)$. The coercivity is based on the Lemma below. We will use the standard notation $\hat{u}$ for the Fourier-Plancherel transform of $u \in L^2(\mathbb{R}^n)$, $\overline{\cdot}$ for the complex conjugate of $z$ and $|z|$ for the modulus of $z$.

**Lemma 3.1.** There exists $c > 0$ independent of $\varepsilon$ such that

$$\langle A_\varepsilon u, u \rangle_{H^{-m}(\Omega), H^m_0(\Omega)} \geq c\| u \|_{H^m_0(\Omega)}^2 \quad \forall u \in H^m_0(\Omega).$$

**Proof.** By density, we can assume that $u$ belongs to $D(\Omega)$, the set of compactly supported and infinitely differentiable functions defined in $\Omega$. Next we extend $u$ by zero outside $\Omega$. Due to the positivity assumption, the function $\sum_{|\alpha| = |\beta| = m} a_{\alpha\beta} D^\alpha u D^\beta u$ is nonnegative. Thus, with $\underline{\rho} = \min(\text{essinf}_{\Omega} \rho_0, \rho_1)$, we have

$$\langle A_\varepsilon u, u \rangle_{H^{-m}(\Omega), H^m_0(\Omega)} \geq \underline{\rho} \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta} \int_{\Omega} D^\alpha u D^\beta u \, dx.$$

Passing to the Fourier transform, we have by the Parseval equality

$$\langle A_\varepsilon u, u \rangle_{H^{-m}(\Omega), H^m_0(\Omega)} \geq \underline{\rho} \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta} \int_{\mathbb{R}^n} \overline{\hat{u}}^\alpha \hat{u}^\beta \, d\xi = \underline{\rho} \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta} \int_{\mathbb{R}^n} \xi^{\alpha+\beta} |\hat{u}|^2 \, d\xi.$$

The uniform ellipticity assumption yields

$$\langle A_\varepsilon u, u \rangle_{H^{-m}(\Omega), H^m_0(\Omega)} \geq \underline{\rho} \varepsilon \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{u}|^2 \, d\xi.$$

The expansion of $|\xi|^{2m} = \sum_{i=1}^n \xi_i^{2m}$ results in an expression of the form

$$|\xi|^{2m} = \sum_{|\alpha| = m} c_\alpha \xi^{2\alpha}$$

for some coefficients $c_\alpha \geq \varepsilon > 0$. This entails

$$\langle A_\varepsilon u, u \rangle_{H^{-m}(\Omega), H^m_0(\Omega)} \geq \underline{\rho} \varepsilon \sum_{|\alpha| = m} c_\alpha \int_{\mathbb{R}^n} |\xi^{\alpha} \hat{u}|^2 \, d\xi \geq \underline{\rho} \varepsilon \sum_{|\alpha| = m} \int_{\mathbb{R}^n} |\xi^{\alpha} \hat{u}|^2 \, d\xi.$$

Using again the Parseval equality leads to

$$\langle A_\varepsilon u, u \rangle_{H^{-m}(\Omega), H^m_0(\Omega)} \geq \underline{\rho} \varepsilon \sum_{|\alpha| = m} \int_{\Omega} |D^\alpha u|^2 \, dx = \underline{\rho} \varepsilon \| u \|_{H^m_0(\Omega)}^2.$$  

\[\square\]

By the Lax-Milgram theorem and the Poincaré inequality in $H^m_0(\Omega)$ we infer the following.
Corollary 3.2. For all \( f \in H^{-m}(\Omega) \) and all \( \varepsilon \) there exists a unique \( u \in H^0_0(\Omega) \) such that
\[
(A_{\varepsilon} u, \eta)_{H^{-m}(\Omega),H^m_0(\Omega)} = (f, \eta) \quad \forall \eta \in H^m_0(\Omega).
\]
Moreover, there exists a constant \( c \) independent of \( \varepsilon \) such that \( \|u\|_{H^m(\Omega)} \leq c \|f\|_{H^{-m}(\Omega)} \).

Proposition 3.3. For all \( f \in H^{-m}(\Omega) \) and all \( \varepsilon \) there exists a unique \( u \in H^0_0(\Omega) \) such that
\[
((A_{\varepsilon} + B_{\varepsilon}) u, \eta)_{H^{-m}(\Omega),H^m_0(\Omega)} = (f, \eta) \quad \forall \eta \in H^m_0(\Omega).
\]
Moreover, there exists a constant \( c \) independent of \( \varepsilon \) such that \( \|u\|_{H^m(\Omega)} \leq c \|f\|_{H^{-m}(\Omega)} \).

Proof. Since \( A_{\varepsilon} \) is invertible, we have
\[
(A_{\varepsilon} + B_{\varepsilon}) u = f \iff (I + B_{\varepsilon} A_{\varepsilon}^{-1}) A_{\varepsilon} u = f,
\]
where \( I \) stands for the identity operator of \( H^{-m}(\Omega) \). By Corollary 3.2, \( A_{\varepsilon}^{-1} : H^{-m}(\Omega) \to H^m_0(\Omega) \) is uniformly bounded. Next, the operator \( B_{\varepsilon} \) can be decomposed as \( B_{\varepsilon} = J B_{\varepsilon} \) with \( J \) the canonical embedding of \( H^{1-m}(\Omega) \) into \( H^{-m}(\Omega) \) and \( B_{\varepsilon} \) the operator defined algebraically like \( B_{\varepsilon} \), but acting from \( H^m(\Omega) \) into \( H^{1-m}(\Omega) \). By construction, \( B_{\varepsilon} \) is uniformly bounded and, by combination of the Rellich and Schauder theorems, \( J \) is compact. It follows that the family of operators \( \{B_{\varepsilon} A_{\varepsilon}^{-1} : H^{-m}(\Omega) \to H^{1-m}(\Omega), \varepsilon \geq 0\} \) is collectively compact (see Appendix A).

In order apply Theorem A.1, let us prove that it is also pointwise sequentially compact. Let \( (\varepsilon_k) \) be a bounded sequence of nonnegative numbers. By the Bolzano-Weierstrass theorem there exists \( \varepsilon_\infty \geq 0 \) such that, for a non-relabeled subsequence, \( \varepsilon_k \to \varepsilon_\infty \). Let now \( \varphi \in H^{-m}(\Omega) \) be arbitrary and define \( \psi_k = A_{\varepsilon_k}^{-1} \varphi \in H^m_0(\Omega) \). Then we have \( \rho_{\varepsilon_k} \to \rho_{\varepsilon_\infty} \) almost everywhere, which implies by a standard argument (see e.g. Theorem 16.4.1 of [10]) that \( \psi_k \to \psi_\infty := A_{\varepsilon_\infty}^{-1} \varphi \) weakly in \( H^m_0(\Omega) \). We now write for any \( \eta \in H^m_0(\Omega) \):
\[
(B_{\varepsilon_k} A_{\varepsilon_k}^{-1} \varphi, \eta)_{H^{-m}(\Omega),H^m_0(\Omega)} = (B_{\varepsilon_k}^* \eta, A_{\varepsilon_k}^{-1} \varphi)_{H^{-m}(\Omega),H^m_0(\Omega)}.
\]
Lebesgue’s dominated convergence theorem yields that \( B_{\varepsilon_k} \eta \to B_{\varepsilon_\infty} \eta \) strongly in \( H^{-m}(\Omega) \). As a product of weakly and strongly convergent sequences we infer:
\[
(B_{\varepsilon_k} A_{\varepsilon_k}^{-1} \varphi, \eta)_{H^{-m}(\Omega),H^m_0(\Omega)} \to (B_{\varepsilon_\infty} \eta, A_{\varepsilon_\infty}^{-1} \varphi)_{H^{-m}(\Omega),H^m_0(\Omega)} = (B_{\varepsilon_\infty} A_{\varepsilon_\infty}^{-1} \varphi, \eta)_{H^{-m}(\Omega),H^m_0(\Omega)}.
\]
This means that \( B_{\varepsilon_k} A_{\varepsilon_k}^{-1} \varphi \to B_{\varepsilon_\infty} A_{\varepsilon_\infty}^{-1} \varphi \) weakly in \( H^{-m}(\Omega) \), but the convergence is actually strong by compactness of the sequence. We have thus proved that \( B_{\varepsilon_k} A_{\varepsilon_k}^{-1} \to B_{\varepsilon_\infty} A_{\varepsilon_\infty}^{-1} \) pointwise.

By the Fredholm alternative, the operator \( I + B_{\varepsilon} A_{\varepsilon}^{-1} \) is invertible for each \( \varepsilon \geq 0 \), since it is injective by assumption. Therefore, by virtue of Theorem A.1, the operators \( (I + B_{\varepsilon} A_{\varepsilon}^{-1})^{-1} \) are uniformly bounded. Writing that \( u = A_{\varepsilon}^{-1} (I + B_{\varepsilon} A_{\varepsilon}^{-1})^{-1} f \) and using again Corollary 3.2 provides the desired uniform bound.

We will later need the following variant of Lemma 3.1. The proof, which is very similar, is left to the reader.

Lemma 3.4. Let \( \rho \) be a positive constant. There exists \( c > 0 \) such that, whenever \( \rho \in L^\infty(\mathbb{R}^n) \) is essentially bounded from below by \( \rho \) and \( D^\alpha u \in L^2(\mathbb{R}^n) \) for every \( \alpha \) with \( |\alpha| = m \), we have
\[
\sum_{|\alpha| = |\beta| = m} \int_{\mathbb{R}^n} \rho \alpha \beta D^\alpha u D^\beta u \, dx \geq c |u|^2_{H^m(\mathbb{R}^n)}.
\]

As opposed to the previous case where the domain \( \Omega \) was bounded, in Lemma 3.4 the seminorm \( |u|^2_{H^m(\mathbb{R}^n)} \) is not a norm and the Poincaré inequality does not hold. Hence, in order to prove the existence and uniqueness of a solution in \( \mathbb{R}^n \), the Lax-Milgram Lemma cannot be applied directly. To address this issue, we introduce in Appendix B a weighted space \( W^m(\mathbb{R}^n) \) (cf. Eq. (B.1)) and its quotient space \( W^m(\mathbb{R}^n)/\mathcal{P}_{m-1} \) where \( \mathcal{P}_{m-1} \) is the space of polynomials of degree not greater than \( m - 1 \). We have the following extension of the Poincaré inequality (cf. Corollary B.5).

Lemma 3.5. There exists \( c > 0 \) such that, for all \( u \in W^m(\mathbb{R}^n) \),
\[
\|u\|_{W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}} \leq c |u|_{H^m(\mathbb{R}^n)}.
\]

The combination of Lemmas 3.4 and 3.5 will lead to useful existence and uniqueness results for problems defined in \( \mathbb{R}^n \). We recall that the approach with quotient spaces is due to Deny and Lions as reported by Ciarlet in [23] (see e.g., Theorem 14.1 as applied to the Finite Element Method).
4. Derivation of the general formula

4.1. A preliminary abstract theorem: asymptotic expansion of a cost function. The following theorem provides a general framework for the sensitivity analysis of a cost function associated with a constraint in variational form. It has been introduced in [4], however we give here a short proof for completeness.

**Theorem 4.1.** Let $\mathcal{V}$ be a vector space and $I$ be a real interval containing 0. For all $\varepsilon \in I$ consider a vector $u_\varepsilon \in \mathcal{V}$ such that:

$$ a_\varepsilon(u_\varepsilon, \eta) = \langle f_\varepsilon, \eta \rangle \quad \forall \eta \in \mathcal{V}, \tag{4.1} $$

where $a_\varepsilon$ and $f_\varepsilon$ are bilinear form on $\mathcal{V} \times \mathcal{V}$ and a linear functional on $\mathcal{V}$, respectively. Consider also a functional $J_\varepsilon : \mathcal{V} \to \mathbb{R}$ and a linear functional $g_\varepsilon \in \mathcal{V}'$. Suppose that the following hypotheses hold.

1. For all $\varepsilon$, there exists $v_\varepsilon \in \mathcal{V}$ such that

$$ a_\varepsilon(\eta, v_\varepsilon) = -\langle g_\varepsilon, \eta \rangle \quad \forall \eta \in \mathcal{V}. \tag{4.2} $$

2. There exist real numbers $\delta a$, $\delta f$ and a function $\varepsilon \mapsto \varphi(\varepsilon) \in \mathbb{R}$ such that, when $\varepsilon \to 0$,

$$ (a_\varepsilon - a_0)(u_0, v_\varepsilon) = \varphi(\varepsilon)\delta a + o(\varphi(\varepsilon)), \tag{4.3} $$

$$ \langle f_\varepsilon - f_0, v_\varepsilon \rangle = \varphi(\varepsilon)\delta f + o(\varphi(\varepsilon)). \tag{4.4} $$

3. There exist real numbers $\delta J_1, \delta J_2$ such that

$$ J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_0) + \varphi(\varepsilon)\delta J_1 + \varphi(\varepsilon)\delta J_2 + o(\varphi(\varepsilon)), \tag{4.5} $$

$$ J_\varepsilon(u_0) = 0. \tag{4.6} $$

Then we have

$$ J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = \varphi(\varepsilon)(\delta a - \delta f + \delta J_1 + \delta J_2) + o(\varphi(\varepsilon)). \tag{4.7} $$

**Proof.** From (4.5) and (4.6), we obtain

$$ J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = (a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o(\varphi(\varepsilon))). $$

Taking into account (4.2) and the fact that $u_\varepsilon - u_0 \in \mathcal{V}$, we get

$$ J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = (a_\varepsilon(u_\varepsilon, v_\varepsilon) + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o(\varphi(\varepsilon))) = (a_\varepsilon(u_\varepsilon - u_0)(u_0, v_\varepsilon) + a_\varepsilon(u_0, v_\varepsilon) + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o(\varphi(\varepsilon))). $$

Taking (4.1) yields

$$ J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = (a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + \langle f_\varepsilon - f_0, v_\varepsilon \rangle + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o(\varphi(\varepsilon))). $$

Finally, using the hypotheses (4.3) and (4.4), we arrive at

$$ J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = \varphi(\varepsilon)(\delta a + \delta f) + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o(\varphi(\varepsilon)), $$

which leads to the result. \( \square \)

4.2. Preliminary definitions. In the present context the function spaces are $\mathcal{V} = H^m_0(\Omega)$ and $\mathcal{V}' = H^{-m}(\Omega)$, and the bilinear form $a_\varepsilon$ is defined by

$$ a_\varepsilon(u, v) = \langle (A_\varepsilon + B_\varepsilon)u, v \rangle_{H^{-m}(\Omega), H^m_0(\Omega)}. $$

We consider a family of functionals $J_\varepsilon : H^m_0(\Omega) \to \mathbb{R}$ satisfying Eqs. (4.5) and (4.6) for some $g_\varepsilon = g \in H^{-m}(\Omega)$, independent of $\varepsilon$, and $\varphi(\varepsilon) = \varepsilon^n$. A class of such functionals will be given in Section 4.5. Moreover we assume for simplicity that $f_\varepsilon = f \in H^{-m}(\Omega)$ is also independent of $\varepsilon$, from which Eq. (4.4) is straightforwardly satisfied with $\delta f = 0$.

Under the above assumptions, it is clear that Eqs. (4.1)-(4.6) are all satisfied provided that Eq. (4.3) holds true. Therefore, it suffices to prove Eq. (4.3) to obtain the general expression of the topological asymptotic expansion of $J_\varepsilon(u_\varepsilon)$. This is the purpose of the subsequent derivations.

We recall that the background state $u_0 \in H^m_0(\Omega)$ and the background adjoint state $v_0 \in H^m_0(\Omega)$ are respectively defined as the solutions of

$$ (A_0 + B_0^*)u_0 = f \quad \text{and} \quad (A_0 + B_0^*)v_0 = -g, \tag{4.8} $$

with $B_0^*$ the adjoint operator of $B_0$. Moreover, Eq. (4.2) can be rewritten as

$$ (A_\varepsilon + B_\varepsilon^*)v_\varepsilon = -g. $$
To begin with we introduce the variation
\[ \tilde{v}_\varepsilon := v_\varepsilon - v_0, \] (4.9)
and, in order to prove Eq. (4.3), we define the quantity
\[ V_\varepsilon(\varepsilon) := (a_\varepsilon - a_0)(u_0, v_\varepsilon). \] (4.10)
To proceed with the analysis of the asymptotic behavior of \( V_\varepsilon(\varepsilon) \) we shall use the spaces \( W^m(\mathbb{R}^n) \) and \( W^m(\mathbb{R}^n)/P_{m-1} \) defined in Appendix B. In the course of the analysis we will need some auxiliary functions. They are defined thereafter.

First, we define \( h_\varepsilon \in W^m(\mathbb{R}^n)/P_{m-1} \) as the solution of
\[
\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{p}_\varepsilon a_{\alpha\beta} D^\alpha h_\varepsilon D^\beta \eta \, dx = -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} D^\alpha v_0 D^\beta \eta \, dx \tag{4.11}
\]
for all \( \eta \in W^m(\mathbb{R}^n)/P_{m-1} \), with
\[
\hat{\rho}_\varepsilon(x) = \begin{cases} 
\hat{\rho}_0 & \text{if } x \in \mathbb{R}^n \setminus \omega_\varepsilon, \\
\hat{\rho}_1 & \text{if } x \in \omega_\varepsilon.
\end{cases} \tag{4.12}
\]
We next set
\[ H_\varepsilon(y) = \varepsilon^{-m} h_\varepsilon(\hat{x} + \varepsilon y). \] (4.13)
Then, we define for each \( \gamma \) with \( |\gamma| = m \) the function \( \Psi_\gamma \in W^m(\mathbb{R}^n)/P_{m-1} \) solution of
\[
\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{p}_\alpha a_{\alpha\beta} D^\alpha \Psi_\gamma(y) D^\beta \Phi(y) \, dy = -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\beta|=m} \int_\omega a_{\alpha\beta} D^\beta \Phi(y) \, dy \tag{4.14}
\]
for all \( \Phi \in W^m(\mathbb{R}^n)/P_{m-1} \), with
\[
\hat{\rho}(y) = \begin{cases} 
\hat{\rho}_0 & \text{if } y \in \mathbb{R}^n \setminus \omega, \\
\hat{\rho}_1 & \text{if } y \in \omega.
\end{cases} \tag{4.15}
\]
Note that the existence and uniqueness of the solutions of (4.11) and (4.14) is a consequence of Lemma 3.4 and Lemma 3.5. We set
\[ H = \sum_{|\gamma|=m} D^\gamma v_0(\hat{x}) \Psi_\gamma. \] (4.16)
Finally we define the polarization tensor \( (p_{\alpha\beta}) \) by its entries
\[ p_{\alpha\beta} = |\omega| (r - 1) a_{\alpha\beta} + k_{\alpha\beta}, \] (4.17)
with
\[ r = \frac{\hat{\rho}_1}{\hat{\rho}_0} \]
the contrast and
\[
k_{\alpha\gamma} = (\frac{\hat{\rho}_1}{\hat{\rho}_0} - 1) \sum_{|\beta|=m} a_{\alpha\beta} \int_\omega D^\beta \Psi_\gamma(y) \, dy. \tag{4.18}
\]

### 4.3. Asymptotic expansion of the bilinear form.

**Lemma 4.2.** For \( \varepsilon \) sufficiently small, the following expression of (4.10) holds true:
\[
V_\varepsilon(\varepsilon) = \varepsilon^n \hat{\rho}_0 \sum_{|\alpha|=|\beta|=m} p_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + \varepsilon^n |\omega| \sum_{|\alpha| \leq m} q_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + \sum_{i=1}^5 \mathcal{E}_i(\varepsilon), \tag{4.19}
\]
with the remainders
\[
\mathcal{E}_i(\varepsilon) = (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} \left( (D^\alpha u_0 D^\beta v_0 - D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x})) \right) \, dx
\]
\[
+ \sum_{|\alpha| \leq m} \int_\omega a_{\alpha\beta} \left( (D^\alpha u_0 D^\beta v_0 - D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x})) \right) \, dx,
\]
\[ \mathcal{E}_2(\varepsilon) = (\tilde{\rho}_1 - \tilde{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} (D^\alpha u_0 - D^\alpha u_0(\tilde{x})) D^\beta \tilde{v}_\varepsilon \, dx, \]

\[ \mathcal{E}_3(\varepsilon) = \sum_{|\alpha| \leq m \atop |\beta| \leq m-1} \int_{\omega} q_{\alpha\beta} D^\alpha u_0 D^\beta \tilde{v}_\varepsilon \, dx, \]

\[ \mathcal{E}_4(\varepsilon) = (\tilde{\rho}_1 - \tilde{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha u_0(\tilde{x}) D^\beta (\tilde{v}_\varepsilon - h_\varepsilon)(y) \, dy, \]

\[ \mathcal{E}_5(\varepsilon) = \varepsilon^n (\tilde{\rho}_1 - \tilde{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha u_0(\tilde{x}). D^\beta (H_\varepsilon - H)(y) \, dy. \]

**Proof.** We have by definition

\[ V_\varepsilon(\varepsilon) = \sum_{|\alpha|=|\beta|=m} \int_{\Omega} (\rho_\varepsilon - \rho_0) a_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon \, dx + \sum_{|\alpha| \leq m} \int_{\omega} (a_{\alpha\beta,\varepsilon} - a_{\alpha\beta,0}) D^\alpha u_0 D^\beta v_\varepsilon \, dx, \]

hence, for \( \varepsilon \) small enough,

\[ V_\varepsilon(\varepsilon) = \sum_{|\alpha|=|\beta|=m} \int_{\Omega} (\tilde{\rho}_1 - \tilde{\rho}_0) a_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon \, dx + \sum_{|\alpha| \leq m} \int_{\omega} q_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon \, dx. \]

We make the splitting \( V_\varepsilon(\varepsilon) = V_\varepsilon^1(\varepsilon) + V_\varepsilon^2(\varepsilon) \) with

\[ V_\varepsilon^1(\varepsilon) = (\tilde{\rho}_1 - \tilde{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon \, dx + \sum_{|\alpha| \leq m} \int_{\omega} q_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon \, dx, \]

\[ V_\varepsilon^2(\varepsilon) = (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon \, dx + \sum_{|\alpha| \leq m} \int_{\omega} q_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon \, dx. \]

- **First approximation.** With the help of the splitting \( D^\alpha u_0 D^\beta v_\varepsilon = D^\alpha u_0(\tilde{x}) D^\beta v_\varepsilon + [D^\alpha u_0 - D^\alpha u_0(\tilde{x})] D^\beta v_\varepsilon(\tilde{x}) \) we obviously get

\[ V_\varepsilon^1(\varepsilon) = |\omega| (\tilde{\rho}_1 - \tilde{\rho}_0) \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\alpha u_0(\tilde{x}) D^\beta v_\varepsilon + |\omega| \sum_{|\alpha| \leq m} q_{\alpha\beta} D^\alpha u_0(\tilde{x}) D^\beta v_\varepsilon + \mathcal{E}_1(\varepsilon). \quad (4.20) \]

- **Second and third approximation.** Similarly we have

\[ V_\varepsilon^2(\varepsilon) = \sum_{|\alpha|=|\beta|=m} \int_{\Omega} (\hat{\rho}_1 - \hat{\rho}_0) a_{\alpha\beta} D^\alpha u_0(\tilde{x}) D^\beta \tilde{v}_\varepsilon \, dx + \mathcal{E}_2(\varepsilon) + \mathcal{E}_4(\varepsilon). \quad (4.21) \]

- **Fourth approximation.** We now approximate \( \tilde{v}_\varepsilon \). We have for any \( \eta \in H^m_0(\Omega): \)

\[ a_\varepsilon(\eta, \tilde{v}_\varepsilon) = a_\varepsilon(\eta, v_\varepsilon) - a_\varepsilon(\eta, v_0) = -(g, \eta) - (a_\varepsilon - a_0)(\eta, v_0) - a_0(\eta, v_0) = -(a_\varepsilon - a_0)(\eta, v_0). \quad (4.22) \]

For \( \varepsilon \) small enough, we therefore have

\[ \sum_{|\alpha|=|\beta|=m} \int_{\Omega} \rho_\varepsilon a_{\alpha\beta} D^\alpha \eta D^\beta \tilde{v}_\varepsilon \, dx + \sum_{|\alpha| \leq m \atop |\beta| \leq m-1} \int_{\Omega} h_{\alpha\beta,\varepsilon} D^\alpha \eta D^\beta \tilde{v}_\varepsilon \, dx \]

\[ = -(\tilde{\rho}_1 - \tilde{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha \eta D^\beta v_\varepsilon \, dx - \sum_{|\alpha| \leq m \atop |\beta| \leq m-1} \int_{\omega} q_{\alpha\beta} D^\alpha \eta D^\beta v_\varepsilon \, dx, \quad (4.23) \]

which suggests to approximate \( \tilde{v}_\varepsilon \) in (4.21) by \( h_\varepsilon \), solution of (4.11). We arrive at

\[ V_\varepsilon^2(\varepsilon) = (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha u_0(\tilde{x}) D^\beta h_\varepsilon \, dx + \mathcal{E}_2(\varepsilon) + \mathcal{E}_3(\varepsilon) + \mathcal{E}_4(\varepsilon). \]
By the change of variable \( x = \hat{x} + \varepsilon y \) this can be rewritten

\[
V^2_a(\varepsilon) = \varepsilon^n (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta H_\varepsilon(y) dy + \mathcal{E}_2(\varepsilon) + \mathcal{E}_3(\varepsilon) + \mathcal{E}_4(\varepsilon). \tag{4.24}
\]

- **Fifth approximation.** On one hand, plugging \( h_\varepsilon(x) = \varepsilon^m H_\varepsilon(\varepsilon^{-1}(x - \hat{x})) \) and \( \eta(x) = \varepsilon^m \phi_\varepsilon(\varepsilon^{-1}(x - \hat{x})) \) in \eqref{eq:4.11} yields after the change of variable \( x = \hat{x} + \varepsilon y \) in both integrals of \eqref{eq:4.11}

\[
\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}(y)a_{\alpha\beta} D^\alpha H_\varepsilon(y) D^\beta \phi_\varepsilon(y) dy = -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} D^\alpha v_0(\hat{x} + \varepsilon y) D^\beta \phi_\varepsilon(y) dy
\]

for every \( \phi_\varepsilon \in W^m(\mathbb{R}^n)/P_{m-1} \), with \( \hat{\rho} \) defined by \eqref{eq:4.15}. On the other hand, combining \eqref{eq:4.14} and \eqref{eq:4.16} results in

\[
\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}(y)a_{\alpha\beta} D^\alpha H(y) D^\beta \Phi(y) dy = -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} D^\alpha v_0(\hat{x}) D^\beta \Phi(y) dy \tag{4.26}
\]

for every \( \Phi \in W^m(\mathbb{R}^n)/P_{m-1} \). On replacing \( H_\varepsilon \) by \( H + (H_\varepsilon - H) \) in \eqref{eq:4.24} we obtain

\[
V^2_a(\varepsilon) = \varepsilon^n (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta H(y) dy + \sum_{i=2}^5 \mathcal{E}_i(\varepsilon). \tag{4.27}
\]

Plugging \eqref{eq:4.16} into the above yields

\[
V^2_a(\varepsilon) = \varepsilon^n (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) D^\gamma \phi_\varepsilon(y) dy + \sum_{i=2}^5 \mathcal{E}_i(\varepsilon).
\]

After rearrangement we obtain

\[
V^2_a(\varepsilon) = \varepsilon^n \hat{\rho}_0 \sum_{|\alpha|=|\beta|=m} k_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + \sum_{i=2}^5 \mathcal{E}_i(\varepsilon), \tag{4.28}
\]

with \( k_{\alpha\beta} \) defined by \eqref{eq:4.18}. Altogether we arrive at

\[
V_a(\varepsilon) = \varepsilon^n \sum_{|\alpha|=|\beta|=m} (|\omega| (\hat{\rho}_1 - \hat{\rho}_0) a_{\alpha\beta} + \hat{\rho}_0 k_{\alpha\beta}) D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + \sum_{i=1}^5 \mathcal{E}_i(\varepsilon).
\]

The expression \eqref{eq:4.17} of the polarization tensor leads to \eqref{eq:4.19}. \( \Box \)

The following lemma provides the appropriate estimates for the remainders \( \mathcal{E}_i(\varepsilon) \). This is the core of the analysis. The proof is deferred to Section 6.

**Lemma 4.3.** Suppose that \( f, g \) are of regularity \( H^s \) in a neighborhood of \( \hat{x} \) with \( s > \max(0, \frac{d}{2} + 1 - m) \). Then the remainders \( \mathcal{E}_i(\varepsilon) \) in Lemma 4.2 satisfy \( |\mathcal{E}_i(\varepsilon)| = o(\varepsilon^n) \) for each \( i = 1, 2, 3, 4, 5 \).

4.4. **Asymptotic expansion of the cost function.**

**Theorem 4.4.** For every \( \varepsilon \) sufficiently small let \( u_\varepsilon \) be the solution of \eqref{eq:2.6}. Suppose that the cost function \( J_\varepsilon \) is such that \eqref{eq:4.5} and \eqref{eq:4.6} hold true for \( \varphi(\varepsilon) = \varepsilon^n \) and \( q_\varepsilon = g \) independent of \( \varepsilon \), and that \( f, g \) are of regularity \( H^s \) in a neighborhood of \( \hat{x} \) with \( s > \max(0, \frac{d}{2} + 1 - m) \). Then we have

\[
J_\varepsilon(u_\varepsilon) - J_0(u_0)
= \varepsilon^n \left[ \hat{\rho}_0 \sum_{|\alpha|=|\beta|=m} p_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + |\omega| \sum_{|\alpha| \leq m} q_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + \delta J_1 + \delta J_2 \right] + o(\varepsilon^n), \tag{4.29}
\]

with the entries of the polarization tensor \( (p_{\alpha\beta})_{|\alpha|=|\beta|=m} \) given by \eqref{eq:4.17}, and \( u_0, v_0 \) solutions of \eqref{eq:4.8}. \( \Box \)
Lemma 6.1 implies that \( \| i \), i.e., generate when its entries do not remain bounded when the contrast with respect to the nucleation of a Neumann hole, and it is observed for instance in dimension holes is not defined, see Remark 4.6. This situation occurs when the cost functional is discontinuous.

Definition 5.1 (Degenerate polarization tensor) \( \zeta \) explicit expression of the polarization tensor in this case. To this aim we introduce the family of.

We already have (4.6) with (4.26) – (4.28) which notably includes least square type functionals.

Theorem 4.7. Suppose that the cost function satisfies

\[
J_\varepsilon(u + h) - J_\varepsilon(u) = \langle g, h \rangle_{H^{-m}(\Omega), H^{-m}(\Omega)} + O(\|h\|^2) \tag{4.30}
\]

for all \( u, h \in H^m(\Omega) \), where either \( \|h\| = \|h\|_{H^m(\Omega), N} \), with \( N \) an arbitrary neighborhood of \( \tilde{x} \), or \( \|h\| = \|h\|_{H^{-m+1}(\Omega)} \). Then (4.5) and (4.6) hold true for \( \varphi(\varepsilon) = \varepsilon^n \) and \( \delta J_1 = \delta J_2 = 0 \).

Proof. We already have (4.6) with \( \delta J_2 = 0 \), since \( J_\varepsilon \) is independent of \( \varepsilon \). Then (4.30) yields

\[
J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) - \langle g, u_\varepsilon - u_0 \rangle = O(\|u_\varepsilon - u_0\|^2). \tag{4.31}
\]

Lemma 6.1 implies that \( \|v_\varepsilon - v_0\| = O(\varepsilon^n/2) \). Obviously, a similar estimate holds for the direct state, i.e., \( \|u_\varepsilon - u_0\| = O(\varepsilon^n/2) \). Together with (4.31), we derive (4.5) with \( \delta J_1 = 0 \).

5. A CLASS OF DEGENERATE PROBLEMS

5.1. Degenerate expression of the polarization tensor.

Definition 5.1 (Degenerate polarization tensor). We say that the polarization tensor (4.17) is degenerate when its entries do not remain bounded when the contrast \( r \) tends to zero.

In particular, when the polarization tensor is degenerate, the topological sensitivity for Neumann holes is not defined, see Remark 4.6. This situation occurs when the cost functional is discontinuous with respect to the nucleation of a Neumann hole, and it is observed for instance in dimension \( n = 1 \) for the Laplacian (see Section 7.2). We will see that it can also occur in higher dimension, but for higher order operators.

The goal of this section is to give a sufficient condition of degeneracy, as well as to provide an explicit expression of the polarization tensor in this case. To this aim we introduce the family of piecewise constant functions \( \zeta_{\alpha\gamma} : \mathbb{R}^m \rightarrow \mathbb{R} \) defined by

\[
\zeta_{\alpha\gamma}(x) = \begin{cases} 
-\hat{\rho}_1 - \hat{\rho}_0 & \text{if } x \in \omega, \\
0 & \text{if } x \in \mathbb{R}^m \setminus \omega,
\end{cases} \tag{5.1}
\]

with \( \delta_{\alpha\gamma} = 1 \) if \( \alpha = \gamma \), \( \delta_{\alpha\gamma} = 0 \) otherwise. We have for all \( \Phi \in W^m(\mathbb{R}^m)/P_{m-1}^m \):

\[
\sum_{|\alpha| = |\beta| = m} \int_{\mathbb{R}^m} \hat{\rho}_a \zeta_{\alpha\gamma} D^\beta \Phi(y) \, dy = -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\beta| = m} \int_{\omega} a_{\alpha\beta} D^\beta \Phi(y) \, dy. \tag{5.2}
\]

In the definition of the polarization tensor Eq. (4.14) appeared as a critical step. Therefore the following assumption is made.
Assumption 5.2. For any multi-index $\gamma$ with $|\gamma| = m$, there exists a function $\Psi_\gamma \in W^m(\mathbb{R}^n)/P_{m-1}$ satisfying
\[
\sum_{|\alpha| = m} a_{\alpha \beta} D^\alpha \Psi_\gamma = \sum_{|\alpha| = m} a_{\alpha \beta} \zeta_{\alpha \gamma} \quad \forall \beta, |\beta| = m.
\] (5.3)

It immediately stems from Assumption 5.2, using (5.2), that $\Psi_\gamma$ solves (4.14). Hence (4.18) results in:
\[
k_{\alpha \gamma} = \left( \frac{\hat{\rho}_1}{\rho_0} - 1 \right) \int_\omega \sum_{|\beta| = m} a_{\alpha \beta} D^\beta \Psi_\gamma(y) \, dy = \left( \frac{\hat{\rho}_1}{\rho_0} - 1 \right) \int_\omega \sum_{|\beta| = m} a_{\alpha \beta} \zeta_{\beta \gamma}
\]
\[
= -\left( \frac{\hat{\rho}_1}{\rho_0} - 1 \right) \sum_{|\beta| = m} a_{\alpha \beta} \frac{\hat{\rho}_1 - \hat{\rho}_0}{\rho_1} |\omega| \delta_{\beta \gamma} = -a_{\alpha \gamma} \frac{(\hat{\rho}_1 - \hat{\rho}_0)^2}{\rho_1 \rho_0} |\omega|,
\] (5.4)

with $|\omega|$ the $n$-dimensional Lebesgue measure of $\omega$. Moreover, plugging (5.4) into (4.17) provides the following closed formula for the polarization tensor.

Proposition 5.3 (Degenerate polarization tensor). If Assumption 5.2 is fulfilled, then we have
\[
p_{\alpha \gamma} = |\omega|(1 - \frac{1}{p}) a_{\alpha \gamma}.
\] (5.5)

We also note that, in addition to be degenerate in the sense of Definition 5.1, the polarization tensor (5.5) is independent of the shape of $\omega$.

5.2. Characterization of a degenerate problem. We shall now give sufficient conditions for Assumption 5.2 to be satisfied. To do so, set $q = \mathbb{N} \setminus \{1\} \times \{m\}$ and define the linear map
\[
\Lambda : \mathbb{R}^q \rightarrow \mathbb{R}^q
\]
\[
(U_\alpha)_{|\alpha| = m} \mapsto (V_\beta)_{|\beta| = m}
\]
such that
\[
V_\beta = \sum_{|\alpha| = m} a_{\alpha \beta} U_\alpha.
\]

We recall the following general result from [20], Theorem 7.1.20.

Theorem 5.4. If $P$ is a homogeneous elliptic (i.e. $P(\xi) = 0 \Rightarrow \xi = 0$) polynomial of degree $p$ in $\mathbb{R}^n$, then the differential operator $P(D)$ has a fundamental solution of the form
\[
E = E_0 - Q(x) \log |x|,
\] (5.6)

where $E_0$ is homogeneous of degree $p - n$, $C^\infty$ and analytic in $\mathbb{R}^n \setminus \{0\}$ and $Q$ is a polynomial which is identically 0 when $n > p$ and is homogeneous of order $p - n$ when $n \leq p$. (cf. [20] for its explicit expression).

Corollary 5.5. Let $E$ be the fundamental solution introduced in Theorem 5.4 and $\alpha \in \mathbb{N}^n$ be such that $|\alpha| = k$. For all $R > 1$ there exists $c > 0$ such that, for all $x \in \mathbb{R}^n$ with $|x| > R$,
\[
\text{if } n > p, \quad |D^\alpha E(x)| \leq c|x|^{p-n-k},
\]
\[
\text{if } n \leq p, \quad |D^\alpha E(x)| \leq c|x|^{p-n-k} \log |x|.
\]

Proof. We first concentrate on the term $E_0$ of the decomposition (5.6). For all $(r, v) \in \mathbb{R}_+ \times \mathbb{R}^n$ we have
\[
E_0(rv) = r^{p-n} E_0(v).
\]

Differentiating $k$ times with respect to $v$ in the direction $(\delta v_1, ..., \delta v_k)$ gives
\[
d^k E_0(rv)(r\delta v_1, ..., r\delta v_k) = r^{p-n} d^k E_0(v)(\delta v_1, ..., \delta v_k),
\]

whereby
\[
\|d^k E_0(rv)\| = r^{p-n-k}\|d^k E_0(v)\|.
\]

Choosing now $x = rv$, with $|v| = 1$ yields
\[
\|d^k E_0(x)\| = |x|^{p-n-k}\|d^k E_0(v)\| \leq c|x|^{p-n-k},
\]

which in turn implies
\[
|D^\alpha E_0(x)| \leq c|x|^{p-n-k} \quad \forall |\alpha| = k.
\]
This provides the result for \( n > p \). We now assume that \( n \leq p \). Similarly to the previous calculation we obtain
\[
|D^n Q(x)| \leq c|x|^{p-n-k} \quad \forall |\alpha| = k.
\] (5.7)
Denoting by \( \tilde{E}(x) = Q(x) \log |x| \), we have, for some coefficients \( c_{\alpha\beta} \geq 0 \),
\[
D^n \tilde{E}(x) = D^n Q(x) \log |x| + \sum_{|\alpha|+|\beta|=|\gamma| \atop |\beta| \geq 1} c_{\alpha\beta} D^\alpha Q(x) D^\beta (\log |x|).
\]
Using (5.7) we get whenever \( |\gamma| = k \) and \(|x| > R\):
\[
|D^n \tilde{E}(x)| \leq c|x|^{p-n-k} \log |x| + c \sum_{|\alpha|+|\beta|=k \atop |\beta| \geq 1} |x|^{p-n-|\alpha|} |x|^{-|\beta|} \leq c|x|^{p-n-k} \log |x|.
\]
Thus we get the results. \( \Box \)

We now state and prove one of the main results of our work, which allows to easily determine whether the polarization tensor associated to an elliptic problem of order \( 2m \) is degenerate in the sense of Definition 5.1.

**Theorem 5.6.** Suppose that \( \text{rank}(\Lambda) = 1 \). Then Assumption 5.2 is fulfilled. In consequence the polarization tensor admits the expression (5.5), hence it is degenerate.

**Proof.** Let \( V = (V_\beta)_{|\beta|=m} \in \text{im}(\Lambda) \) be such that \( V_\beta \neq 0 \) for some \( \beta \). Since \( \text{dim}(\text{im} \Lambda) = 1 \) we have \( \text{im} \Lambda = \text{span}(V) \) and
\[
\Lambda((D^\alpha \Psi)_{|\alpha|=m}) = V \iff \sum_{|\alpha|=m} a_{\alpha\beta} D^\alpha \Psi = V_\beta.
\] (5.8)
For \( V \) defined by \( V_\beta = \sum_{|\alpha|=m} a_{\alpha\beta} \xi_\alpha \gamma \), \( \gamma \) fixed, a solution to the rightmost equality of (5.8) is given by
\[
\Psi_\gamma = E \ast V_\beta,
\]
where \( E \) is the fundamental solution of the operator \( \sum_{|\alpha|=m} a_{\alpha\beta} D^\alpha \). Let us show that the polynomial \( P(\xi) = \sum_{|\alpha|=m} a_{\alpha\beta} \xi^\alpha \) associated to this operator is elliptic. Thus, assume that \( P(\xi) = 0 \). For \( |\alpha| = m \) we set \( U_\alpha = \xi^\alpha \), and we define \( U = (U_\alpha)_{|\alpha|=m} \). We have \( [\Lambda(U)]_{|\beta|=1} = P(\xi) = 0 \), and since \( \Lambda(U) \in \text{span}(V) \) (i.e., \( \Lambda(U) = \lambda V \) for some \( \lambda \)) with \( V_\beta \neq 0 \), we infer \( \Lambda(U) = 0 \). Therefore, \( \sum_{|\alpha|=m} a_{\alpha\beta} \xi^\alpha = 0 \) for every \( \beta \) with \( |\beta| = m \). Multiplying by \( \xi^\beta \) and summing over \( |\beta| = m \), this implies \( \sum_{|\alpha|=m} a_{\alpha\beta} \xi^\alpha + \beta = 0 \). By the uniform ellipticity assumption (2.4) we derive \( \xi = 0 \). Therefore, the fundamental solution of the operator \( P(D) = \sum_{|\alpha|=m} a_{\alpha\beta} D^\alpha \) satisfies Theorem 5.4. Using Corollary 5.5 with \( p = m \), it is easily checked that \( E \in W^m(\mathbb{R}^n) \) (defined in Appendix B). Hence \( \Psi_\gamma \in W^m(\mathbb{R}^n) \) as well, and the proof is achieved. \( \Box \)

### 6. Estimation of the remainders

This section is devoted to the proof of Lemma 4.3. We will use the letter \( c \) to denote a generic positive constant independent of \( \varepsilon \).

#### 6.1. Preliminary estimates.

As in the proof of Lemma 4.2, we set
\[
\tilde{v}_\varepsilon = v_\varepsilon - v_0 \in H^m_0(\Omega).
\]
We introduce the difference
\[
econe = \tilde{v}_\varepsilon - h_\varepsilon \in H^m(\Omega)/P_{m-1},
\]
where \( h_\varepsilon \in W^m(\mathbb{R}^n)/P_{m-1} \) solves (4.11). Moreover, we set
\[
\hat{\rho}_\varepsilon(x) = \begin{cases} 
\rho_\varepsilon(x) & \text{if } x \in \Omega, \\
\hat{\rho}_0 & \text{if } x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
Lemma 6.1. Let \( a > 0 \) be such that the open ball of centre \( \hat{x} \) and radius \( a \), denoted by \( B_a \), is contained in \( \Omega \), and choose an arbitrary \( \delta' \in (0, 1/2) \). For \( \varepsilon \) small enough we have that
\[
|h_\varepsilon|_{H^m(\mathbb{R}^n \setminus B_a)} \leq \varepsilon^{n-\delta'}, \quad ||\bar{v}_\varepsilon||_{H^m(\Omega \setminus B_a)} \leq \varepsilon^{n/2+\delta''}, \quad ||\tilde{v}_\varepsilon||_{H^{m-1}(\Omega \setminus B_a)} \leq \varepsilon^{n/2+\delta''}, \quad ||\bar{v}_\varepsilon||_{H^{m-1}(\Omega)} \leq \varepsilon^{n/2+\delta''} \tag{6.1}
\]
for some \( \delta'' > 0 \).

Proof. The proof is divided into five steps.

- **First step:** estimation of \( |h_\varepsilon|_{H^m(\mathbb{R}^n)} \). By elliptic regularity applied to (4.25) (see Lemmas 3.4 and 3.5) we get \( ||H_\varepsilon||_{W^m(\mathbb{R}^n)/P_{m-1}} \leq c \), hence in particular
\[
|H_\varepsilon|_{H^m(\mathbb{R}^n)} \leq c. \tag{6.3}
\]

A change of variable results in
\[
|h_\varepsilon|_{H^m(\mathbb{R}^n)} \leq \varepsilon^{n/2}. \tag{6.4}
\]

- **Second step:** estimation of \( |h_\varepsilon|_{H^m(\mathbb{R}^n \setminus B_a)} \). We derive from (4.25) that, for all \( \phi_\varepsilon \in W^m(\mathbb{R}^n)/P_{m-1} \),
\[
\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \rho_0 a_{\alpha\beta} D^\alpha H_\varepsilon(y) D^\beta \phi_\varepsilon(y) \, dy \tag{6.5}
\]

\[
= \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \rho_0 a_{\alpha\beta} D^\alpha H_\varepsilon(y) D^\beta \phi_\varepsilon(y) \, dy + \sum_{|\alpha|=|\beta|=m} \int_{\omega} (\tilde{v}_\varepsilon - \hat{v}_1) a_{\alpha\beta} D^\alpha H_\varepsilon(y) D^\beta \phi_\varepsilon(y) \, dy
\]

\[
= (\tilde{v}_\varepsilon - \hat{v}_1) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} (D^\alpha v_\varepsilon(\hat{x} + \varepsilon y) + D^\alpha H_\varepsilon(y)) D^\beta \phi_\varepsilon(y) \, dy.
\]

We define the distribution \( T_\varepsilon \in D'(\mathbb{R}^n) \) by
\[
\langle T_\varepsilon, \eta \rangle = (\tilde{v}_\varepsilon - \hat{v}_1) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} (D^\alpha v_\varepsilon(\hat{x} + \varepsilon y) + D^\alpha H_\varepsilon(y)) D^\beta \eta(y) \, dy.
\]

We therefore have, in the sense of distributions,
\[
(-1)^m \sum_{|\alpha|=|\beta|=m} \rho_0 a_{\alpha\beta} D^{\alpha + \beta} H_\varepsilon = T_\varepsilon. \tag{6.7}
\]

We call \( E \) the fundamental solution of the differential operator \((-1)^m \sum_{|\alpha|=|\beta|=m} \rho_0 a_{\alpha\beta} D^{\alpha + \beta}\), whereby a solution of (6.7) is given by \( H_\varepsilon^* = T_\varepsilon \ast E \). By elliptic regularity, since clearly \( T_\varepsilon \in H^{-m}(\mathbb{R}^n) \), we have \( H_\varepsilon^* \in H^m_{loc}(\mathbb{R}^n) \). In addition, if \( \text{dist}(x, \omega) > 0 \) we have the expressions
\[
H_\varepsilon^*(x) = (\tilde{v}_\varepsilon - \hat{v}_1) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} (D^\alpha v_\varepsilon(\hat{x} + \varepsilon y) + D^\alpha H_\varepsilon(y)) \, dy.
\]

and
\[
D^\gamma H_\varepsilon^*(x) = (\tilde{v}_\varepsilon - \hat{v}_1) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} (D^\alpha v_\varepsilon(\hat{x} + \varepsilon y) + D^\alpha H_\varepsilon(y)) D^{\gamma + \beta} E(x - y) \, dy. \tag{6.8}
\]

The Cauchy-Schwarz inequality applied to (6.8) implies
\[
|D^\gamma H_\varepsilon^*(x)| \leq c \sum_{|\beta|=m} \left( \int_{\omega} |D^{\gamma + \beta} E(x - y)|^2 \, dy \right)^{1/2}.
\]

By Corollary 5.5 with \( p = 2m \) we infer that, for any \( R > 2 \), \( \delta' \in (0, 1/2) \) and \( |\gamma| = m - k \) such that \( \omega \subset B(0, R/2) \),
\[
|D^\gamma H_\varepsilon^*(x)| \leq c |x|^{k-n+\gamma + \delta'} \quad \forall |x| > R. \tag{6.9}
\]

This implies in particular that \( H_\varepsilon^* \in W^m(\mathbb{R}^n) \), and by uniqueness that \( H_\varepsilon^* \) is a representative for \( H_\varepsilon \). Recalling that \( h_\varepsilon(x) = \varepsilon^m H_\varepsilon(\varepsilon^{-1}(x - \hat{x})) \), we select the representative \( h_\varepsilon^*(x) = \varepsilon^m H_\varepsilon^*(\varepsilon^{-1}(x - \hat{x})) \), hence
\[
D^\gamma h_\varepsilon^*(x) = \varepsilon^k D^\gamma H_\varepsilon^*(\varepsilon^{-1}(x - \hat{x})), \quad |\gamma| = m - k. \tag{6.10}
\]
From (6.9) we derive
\[ |D^\gamma h_\varepsilon(x)| \leq c\varepsilon^k |\varepsilon^{-1}(x - \tilde{x})|^{k-n+\delta'} = c\varepsilon^{-\delta'} |x - \tilde{x}|^{k-n+\delta'}, \quad \forall |x| > \varepsilon R, \quad |\gamma| = m - k. \tag{6.11} \]
Therefore, choosing an arbitrary \( a > 0 \), we obtain
\[ \|h_\varepsilon^{*}\|_{W^m(\mathbb{R}^n \setminus B_a)} \leq c\varepsilon^{-\delta'} \tag{6.12} \]
for any \( \varepsilon \) small enough. In particular this yields \( |h_\varepsilon|_{H^m(\mathbb{R}^n \setminus B_a)} \leq c\varepsilon^{-\delta'} \).

**Third step: estimation of** \( \|h_\varepsilon\|_{H^{m-1}(\Omega)} \). From (6.9) we obtain, for \( |\gamma| = m - k \),
\[ \|D^\gamma h_\varepsilon^{*}\|_{L^2(C(\varepsilon^{-1}R,0))} \leq c\varepsilon^k |x| - k - \delta' + c \tag{6.13} \]
and
\[ \|D^\gamma h_\varepsilon^{*}\|_{L^2(C(\varepsilon R,2R))} \leq c, \]
where \( C(a,b) \) stands for the ring of radii \( a \) and \( b \). The above inequality together with (6.3) yields, thanks to the Poincaré inequality, that
\[ \|h_\varepsilon^{*}\|_{H^m(B_R)} \leq c. \tag{6.14} \]
Combining (6.13) and (6.14) we arrive at
\[ \|D^\gamma h_\varepsilon^{*}\|_{L^2(B_{\varepsilon^{-1}R})} \leq c + c\varepsilon^k |x| - k - \delta'. \]
A change of variables provides
\[ \|D^\gamma h_\varepsilon^{*}\|_{L^2(\Omega)} \leq c\varepsilon^k + c\varepsilon^{-\delta'}. \]
For \( k \geq 1 \), as \( \delta' \in (0,n/2) \), the right hand side of the above inequality is of order \( O(\varepsilon^{\frac{n}{2} + \delta'}) \) for some \( \delta'' > 0 \). It follows that
\[ \|h_\varepsilon^{*}\|_{H^{m-1}(\Omega)} \leq c\varepsilon^{\frac{n}{2} - \delta''}. \tag{6.15} \]

**Fourth step: estimation of** \( \|\varepsilon|E|_{H^m(\Omega)} \). We set
\[ \langle \tilde{A}_\varepsilon u, v \rangle_{H^{-m}(\Omega), H^m(\Omega)} = \sum_{|\alpha| = |\beta| = m} \int_\Omega \hat{\rho}_\varepsilon a_{\alpha\beta} D^\alpha u D^\beta v dx \]
so that, in view of (4.11) applied to the selected representative \( h_\varepsilon^{*} \) and for a test function \( \eta \in H^m_0(\Omega) \) extended by 0, we have
\[ \tilde{A}_\varepsilon h_\varepsilon^{*} = -(\tilde{A}_\varepsilon - \hat{A}_\varepsilon) v_0. \]
Recalling that
\[ (A_\varepsilon + B_\varepsilon) v_\varepsilon = (A_\varepsilon + B_\varepsilon) v_0 = -g, \]
we find
\[ (A_\varepsilon + B_\varepsilon) \varepsilon = -(A_\varepsilon - A_0 + B_\varepsilon - B_0) v_0. \]
This entails, for \( \varepsilon |E|_{H^m(\Omega)} \)
\[ (A_\varepsilon + B_\varepsilon) \varepsilon = -(A_\varepsilon - A_0 + B_\varepsilon - B_0) v_0. \tag{6.16} \]
In addition it holds \( \varepsilon |E|_{\Omega} \) on \( \partial\Omega \). By Proposition 3.3 and classical arguments of elliptic regularity and trace theory we infer that
\[ \|\varepsilon |E|_{H^m(\Omega)} \|_{H^{-m}(\Omega)} \leq c(\|S_\varepsilon\|_{H^{-m}(\Omega)} + \|h_\varepsilon^{*}\|_{W^m(\mathbb{R}^n \setminus B_a)}). \]
Yet for every \( \eta \in H^m_0(\Omega) \) we have
\[ \langle S_\varepsilon, \eta \rangle_{H^{-m}(\Omega), H^m_0(\Omega)} = - \sum_{|\alpha| = |\beta| = m} \int_\Omega (\rho_\varepsilon - \rho_0) a_{\alpha\beta} D^\alpha h_\varepsilon^{*} D^\beta \eta dx - \sum_{|\alpha| \leq m} \int_\Omega b_{\alpha\beta\varepsilon} D^\alpha \eta D^\beta h_\varepsilon^{*} dx + \sum_{|\alpha| = m} \int_\Omega q_{\alpha\beta\varepsilon} D^\alpha \eta D^\beta h_\varepsilon^{*} dx. \]
Using (6.12), (6.15), and the fact that \( \rho_\varepsilon - \rho_0 = \hat{\rho}_\varepsilon - \hat{\rho}_0 \) for every \( \varepsilon \) small enough, we get
\[ \langle S_\varepsilon, \eta \rangle_{H^{-m}(\Omega), H^m_0(\Omega)} \leq c\varepsilon^{\frac{n}{2} + \delta''} \|\eta\|_{H^m(\Omega)}. \]
Using once more (6.12) we arrive at
\[ \|e^*\|_{H^n(\Omega)} \leq c\varepsilon^\frac{5}{2} + \delta^n, \] (6.17)

- **Fifth step: estimates on \( \tilde{\nu}_\varepsilon \).** From (6.1) and (6.12) we derive \( \|\tilde{\nu}_\varepsilon\|_{H^n(\Omega \setminus \Omega_0)} \leq c\varepsilon^\frac{5}{2} + \delta^n \). The Poincaré inequality entails \( \|\tilde{\nu}_\varepsilon\|_{H^n(\Omega \setminus \Omega_0)} \leq c\varepsilon^\frac{5}{2} + \delta^n \). Likewise, (6.4) yields \( \|\tilde{\nu}_\varepsilon\|_{H^n(\Omega)} \leq c\varepsilon^\frac{5}{2} \). We also derive from (6.15) and (6.17)
\[ \|\tilde{\nu}_\varepsilon\|_{H^{n-1}(\Omega)} \leq c\varepsilon^\frac{5}{2} + \delta^n. \] (6.18)

All the estimates are now proven. \( \square \)

We are now in position to estimate the remainders \( \mathcal{E}_i(\varepsilon), i = 1, 2, 3, 4, 5 \) of Lemma 4.2.

6.2. **First remainder.** Due to the assumed regularity of \( f \) and \( g \), it follows that \( D^\alpha u_0 \) and \( D^\alpha v_0 \) are \( C^1 \) in a neighborhood of \( \hat{x} \). By the mean value inequality we arrive at
\[ |\mathcal{E}_1(\varepsilon)| \leq c\varepsilon^{n+1}. \]

6.3. **Second remainder.** The Cauchy-Schwarz inequality entails
\[ \mathcal{E}_2(\varepsilon) \leq c\varepsilon \sqrt{|\omega|} |\tilde{\nu}_\varepsilon|_{H^m(\Omega)}. \]

Using Lemma 6.1 we straightforwardly get
\[ |\mathcal{E}_2(\varepsilon)| \leq c\varepsilon^{n+1}. \]

6.4. **Third remainder.** The Cauchy-Schwarz inequality yields
\[ |\mathcal{E}_3(\varepsilon)| \leq c\|1\|_{L^2(\omega)} |\tilde{\nu}_\varepsilon|_{H^{n-1}(\Omega)}. \]

From Lemma 6.1 we infer
\[ |\mathcal{E}_3(\varepsilon)| \leq c\varepsilon^{n+\delta^n}. \]

6.5. **Fourth remainder.** The Cauchy-Schwarz inequality yields
\[ |\mathcal{E}_4(\varepsilon)| \leq c\|1\|_{L^2(\omega)} |\varepsilon|_{H^m(\Omega)}. \]

Then Lemma 6.1 entails
\[ |\mathcal{E}_4(\varepsilon)| \leq c\varepsilon^{n+\delta^n}. \]

6.6. **Fifth remainder.** We begin by observing that, subtracting (4.26) from (4.25), one gets for any \( \phi \in W^m(\mathbb{R}^n) \)
\[
\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}(y) a_{\alpha\beta} D^\alpha (H_\varepsilon - H)(y) D^\beta \phi \, dy =
- \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} (\hat{\rho}_1 - \hat{\rho}_0) a_{\alpha\beta} (D^\alpha v_0(\hat{x} + \varepsilon y) - D^\alpha v_0(\hat{x})) D^\beta \phi(y) \, dy.
\]

Applying the Cauchy-Schwarz inequality to the right hand side and using the \( C^1 \) regularity of \( v_0 \) we get
\[
\left| \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}(y) a_{\alpha\beta} D^\alpha (H_\varepsilon - H)(y) D^\beta \phi \, dy \right| \leq c\varepsilon |\phi|_{H^m(\mathbb{R}^n)}. \]

By elliptic regularity (see Lemma 3.4) we infer that
\[ |H_\varepsilon - H|_{H^m(\mathbb{R}^n)} \leq c\varepsilon. \]

This implies by the Cauchy-Schwarz inequality that
\[ |\mathcal{E}_5(\varepsilon)| \leq c\varepsilon^{n+1}. \]
7. Selected applications

7.1. Examples of operators. In this section we review some classical elliptic operators. As the polarization tensor only depends on the principal symbol, we restrict our presentation to homogeneous operators, i.e., we assume that $b_{\alpha\beta\varepsilon\kappa} \equiv 0$. In order to check the uniform ellipticity condition, we set

$$P(\xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \xi^{\alpha+\beta}. $$

7.1.1. In dimension $n = 1$. We have $|\alpha| = m \Rightarrow \alpha = (m)$, hence $q = 1$ and rank($\Lambda$) = 1 for every $m \geq 1$. This case is always degenerate and the topological asymptotic expansion for $\omega$ being the interval $(-1, 1)$ is given by

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon \left[ 2\rho_0(1 + \frac{1}{r}) \frac{d^n u_0}{dx^n}(\bar{x}) \frac{d^n v_0}{dx^n}(\bar{x}) + \delta J_1 + \delta J_2 \right] + o(\varepsilon).$$

7.1.2. Laplacian in dimension $n \geq 2$. We have $m = 1$, hence $q = n$. Let $(e_i)_{i=1,...,n}$ be the canonical basis of $\mathbb{R}^n$. The bilinear form is

$$a_\varepsilon(u, v) = \int_\Omega \rho_\varepsilon \nabla u \nabla v dx = \sum_{i=1}^n \int_\Omega \rho_\varepsilon D^\varepsilon u D^\varepsilon v dx,$n

hence $P(\xi) = |\xi|^2$. In the basis formed by the vectors $(e_i)_{i=1,...,n}$ the matrix of $\Lambda$ is the identity matrix of order $n$, hence rank($\Lambda$) = $n$. This case is not degenerate. The polarization tensor is explicitly known for ellipses and ellipsoids, see e.g. [3, 4].

7.1.3. Bi-Laplacian. For this operator we have $m = 2$ and the bilinear form is

$$a_\varepsilon(u, v) = \int_\Omega \rho_\varepsilon \Delta u \Delta v dx.$$

This yields $P(\xi) = |\xi|^4$. Let us first focus for simplicity on the dimension $n = 2$. Ordering the family $(\alpha \in \mathbb{N}^2, |\alpha| = 2)$ as $((2, 0), (0, 2), (1, 1))$ the matrix of $\Lambda$ in the canonical basis of $\mathbb{R}^3$ is

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

It is immediately checked that $A \geq 0$ and rank($\Lambda$) = rank($A$) = 1. This case is thus degenerate. The same thing occurs in any dimension $n$, with, using a similar ordering, a matrix having as only nonzero coefficients an $n \times n$ upper left block of ones. Therefore we have in any dimension the topological asymptotic expansion:

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon^n \left[ \rho_0|\omega|(1 - \frac{1}{r}) \Delta u_0(\bar{x}) \Delta v_0(\bar{x}) + \delta J_1 + \delta J_2 \right] + o(\varepsilon^n).$$

7.1.4. Kirchhoff plate model. For this fourth order operator ($m = 2$) in dimension $n = 2$, the bilinear form is

$$a_\varepsilon(u, v) = k \int_\Omega \rho_\varepsilon (\lambda \Delta u \Delta v + 2\mu \nabla u : \nabla v) dx,$$

where $k = \tau^3/12$, $\tau > 0$ is the thickness of the plate, $\lambda, \mu \geq 0$ are the Lamé coefficients. This entails $P(\xi) = k(\lambda + 2\mu)|\xi|^4$, which is uniformly elliptic provided that either $\lambda > 0$ or $\mu > 0$. In the same basis as in the previous case, the matrix of $\Lambda$ is

$$A = k \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 4\mu \end{pmatrix} \geq 0. $$

We find $\det A = 16k^3\mu^2(\lambda + \mu)$, hence rank($A$) = 3 provided that $\mu > 0$. The problem is thus non-degenerate. The polarization tensor for a circular inclusion has been obtained in [6].

7.2.1. One-dimensional problem. We consider the compliance functional associated to a second order one-dimensional equation, where, for \(0 < a < 1/2\), the source term \(f\) has the form:

\[
f(x) = \begin{cases} 
2, & 0 < x < a, \\
0, & a \leq x < 1.
\end{cases}
\]

The cost functional associated to the unperturbed problem reads

\[
J_0(u_0) = \int_0^1 f u_0 = 2 \int_0^a u_0,
\]

with \(u_0\) solution to

\[
\begin{cases}
-u_0''(x) = 2, & 0 < x < a, \\
-u_0''(x) = 0, & a \leq x < 1,
\end{cases}
\]

\(u_0(0) = u_0(1) = 0\).

The expression of \(u_0\) can be easily shown to be

\[
u_0(x) = \begin{cases} 
-x^2 - a^2 x + 2ax, & 0 < x < a, \\
-a^2(x - 1), & a \leq x < 1.
\end{cases}
\]

The compliance associated to the perturbed problem reads

\[
J_\varepsilon(u_\varepsilon) = \int_0^1 f u_\varepsilon = 2 \int_0^a u_\varepsilon,
\]

with \(u_\varepsilon\) solution to

\[
\begin{cases}
-u_\varepsilon''(x) = 2, & 0 < x < a, \\
-u_\varepsilon''(x) = 0, & a \leq x < \frac{1}{2} - \varepsilon, \\
-u_\varepsilon''(x) = 0, & \frac{1}{2} + \varepsilon < x < 1,
\end{cases}
\]

\(u_\varepsilon(\frac{1}{2} - \varepsilon) = u_\varepsilon(\frac{1}{2} + \varepsilon) = 0, \quad u_\varepsilon(0) = u_\varepsilon(1) = 0\).

This means that the domain is topologically perturbed by the introduction of a hole of size \(2\varepsilon\), with homogeneous Neumann boundary condition. The explicit solution is given by

\[
u_\varepsilon(x) = \begin{cases} 
-x^2 - 2ax, & 0 < x < a, \\
-a^2, & a \leq x < \frac{1}{2} - \varepsilon, \\
0, & \frac{1}{2} + \varepsilon < x < 1.
\end{cases}
\]

The solutions \(u_0\) and \(u_\varepsilon\) are represented in Fig. 7.2.1 for \(a = 1/4\). As we have already mentioned, this is a degenerate case. In fact, in this simple example the difference between \(J_\varepsilon(u_\varepsilon)\) and \(J_0(u_0)\) is explicitly given by a jump independent of \(\varepsilon\), namely

\[
J_\varepsilon(u_\varepsilon) - J_0(u_0) = a^4.
\]

This means that the cost functional \(J_\varepsilon(u_\varepsilon)\) is not continuous with respect \(\varepsilon\). Hence, the topological derivative for a Neumann hole is not defined.
7.2.2. Bi-Laplacian operator. Let us consider three balls \( B_1, B_a, B_\varepsilon \in \mathbb{R}^2 \) with centers at the origin and radii 1, \( a \), and \( \varepsilon \), respectively, such that \( \varepsilon < a < 1 \). We consider the compliance functional associated to the bi-Laplacian operator with source term

\[
f(x) = \begin{cases} 
8 & \text{if } x \in B_1 \setminus B_a, \\
0 & \text{if } x \in B_a.
\end{cases}
\]

The cost functional associated to the unperturbed problem reads

\[
J_0(u_0) = \int_{B_1} f u_0 = 8 \int_{B_1 \setminus B_a} u_0,
\]

with \( u_0 \) solution to

\[
\begin{cases} 
\Delta^2 u_0 = 8 & \text{in } B_1 \setminus B_a, \\
\Delta^2 u_0 = 0 & \text{in } B_a, \\
\partial_n u_0 = 0 & \text{on } \partial B_1.
\end{cases}
\]

For the perturbed problem with an homogeneous Neumann condition on the boundary of a hole \( B_\varepsilon \), the cost functional is

\[
J_\varepsilon(u_\varepsilon) = \int_{B_1} f u_\varepsilon = 8 \int_{B_1 \setminus B_\varepsilon} u_\varepsilon,
\]

with \( u_\varepsilon \) solution to

\[
\begin{cases} 
\Delta^2 u_\varepsilon = 8 & \text{in } B_1 \setminus B_\varepsilon, \\
\Delta^2 u_\varepsilon = 0 & \text{in } B_a \setminus B_\varepsilon, \\
u_\varepsilon = 0 & \text{on } \partial B_1, \\
\Delta u_\varepsilon = 0 & \text{on } \partial B_\varepsilon.
\end{cases}
\]

Using a polar coordinate system \((r, \theta)\), we find analytical expressions for both \( u_0 \) and \( u_\varepsilon \) by separation of variables, as plotted in Fig. 2. Due to the axis-symmetry of the problems, their solutions can be written in terms of \( r \) only, as shown in Fig. 3. In this example the difference between \( J_\varepsilon(u_\varepsilon) \) and \( J_0(u_0) \) is again given by a jump independent of \( \varepsilon \), namely

\[
J_\varepsilon(u_\varepsilon) - J_0(u_0) = \pi \left(a^4 - 4a^2 \log a - 1 \right)^2.
\]

The cost functional \( J_\varepsilon(u_\varepsilon) \) is not continuous with respect to \( \varepsilon \) at \( \varepsilon = 0 \), which confirms that this case is degenerate.
8. Generalization to elliptic systems

We explain here how the previous results can be generalized to differential systems. We restrict ourselves to homogeneous differential operators merely for notational simplicity. We concentrate on the main changes, that is, the expression of the polarization tensor.

8.1. General case. We consider now a vector field \( u_\varepsilon = (u^1_\varepsilon, \ldots, u^N_\varepsilon) \in H^m_0(\Omega)^N \) solution of

\[
\sum_{ij} \sum_{\alpha,\beta} \int_{\Omega} \rho \alpha ij D^\alpha u^i_\varepsilon D^\beta \eta^j \, dx = \sum_i (f^i, \eta^i) \quad \forall \eta \in H^m_0(\Omega)^N.
\]

By convention Latin indices, written as superscripts, range over the set \( \{1, \ldots, N\} \), whereas Greek multi-indices are of length \( n \). The system coefficients \( (a_{ij}^{\alpha \beta}) \) are supposed to satisfy the following properties.

- Symmetry: it holds for every \( \alpha, \beta, i, j \)
  \[
a_{ij}^{\alpha \beta} = a_{ji}^{\beta \alpha}.
\]

- Positivity: for any family of real numbers \( (y^i) \) it holds
  \[
  \sum_{ij} \sum_{\alpha \beta} a_{ij}^{\alpha \beta} y^i y^j \geq 0.
\]

- Uniform ellipticity: there exists \( \kappa > 0 \) such that
  \[
  \sum_{ij} \sum_{\alpha \beta} a_{ij}^{\alpha \beta} \xi^i \xi^j \geq \kappa |\xi|^2 \sum_i |y^i|^2 \quad \forall (\xi, y) \in \mathbb{R}^n \times \mathbb{C}^N.
\]

Based on these assumptions the asymptotic analysis can be easily generalized, which is left to the reader. This leads to define the function \( H = (H^1, \ldots, H^N) \), instead of (4.26), as the solution of

\[
\sum_{ij} \sum_{\alpha \beta} \int_{\mathbb{R}^n} \tilde{\rho}(y) a_{ij}^{\alpha \beta} D^\alpha H^i(y) D^\beta \Phi^j(y) \, dy = - (\tilde{\rho}_1 - \tilde{\rho}_0) \sum_{ij} \sum_{\alpha \beta} \int_{\omega} a_{ij}^{\alpha \beta} D^\alpha v^i_0(x) D^\beta \Phi^j(y) \, dy
\]

for every family of functions \( \Phi^1, \ldots, \Phi^N \in W^m(\mathbb{R}^n)/P_{m-1} \). By linearity, we have

\[
H^i = \sum_l \sum_\gamma \gamma v^i_0(x) \Psi^i_l,
\]

with \( \Psi^i_l \in W^m(\mathbb{R}^n)/P_{m-1} \) solving, for each \( \gamma, l, \)

\[
\sum_{ij} \sum_{\alpha \beta} \int_{\mathbb{R}^n} \tilde{\rho} a_{ij}^{\alpha \beta} D^\alpha \Psi^i_l \gamma(y) D^\beta \Phi^j(y) \, dy = - (\tilde{\rho}_1 - \tilde{\rho}_0) \sum_j \sum_\beta \int_\omega a_{ij}^{\alpha \beta} D^\beta \Phi^j(y) \, dy
\]

for all \( \Phi^1, \ldots, \Phi^N \in W^m(\mathbb{R}^n)/P_{m-1} \). The polarization tensor \( (p_{ij}^{\alpha \beta}) \) is defined by

\[
p_{ij}^{\alpha \beta} = |\omega| (r - 1) a_{ij}^{\alpha \beta} + k_{ij}^{\alpha \beta},
\]

with

\[
k_{ij}^{\alpha \beta} = \left( \frac{\tilde{\rho}_1}{\tilde{\rho}_0} - 1 \right) \sum_j \sum_\beta \int_\omega D^\beta \Psi^j_l(y) \, dy.
\]

Theorem 8.1. Suppose that the cost function \( J_\varepsilon \) is such that (4.5) and (4.6) hold true for \( \varepsilon = \varepsilon^2 \) and \( g = g \) independent of \( \varepsilon \), and that \( f, g \) are of regularity \( H^s \) in a neighborhood of \( x, s > \max(0, \frac{2}{7} + 1 - m) \). Then we have

\[
J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon^n \left[ \tilde{\rho}_0 \sum_{ij} \sum_{\alpha \beta} p_{ij}^{\alpha \beta} D^\alpha u^i_0(x) D^\beta v^j_0(\hat{x}) + \delta J_1 + \delta J_2 \right] + o(\varepsilon^n).
\]
8.2. Degenerate case. We introduce the family of piecewise constant functions \( \zeta_{il}^{\alpha\gamma} : \mathbb{R}^n \to \mathbb{R} \) defined by

\[
\zeta_{il}^{\alpha\gamma}(x) = \begin{cases} 
-\hat{\rho}_1 - \frac{\hat{\rho}_0}{\hat{\rho}_1} \delta_{il}^{\alpha\gamma} & \text{if } x \in \omega, \\
0 & \text{if } x \in \mathbb{R}^n \setminus \omega,
\end{cases}
\]

with \( \delta_{il}^{\alpha\gamma} = 1 \) if \( \alpha = \gamma \) and \( i = l \), \( \delta_{il}^{\alpha\gamma} = 0 \) otherwise. Assumption 5.2 is modified as follows.

Assumption 8.2. For any \( \gamma, l \), there exist functions \( \Psi_{il}^{\alpha\gamma} \in W^m(\mathbb{R}^n) \) satisfying

\[
\sum_i \sum_\alpha a_{il}^{ij \alpha \beta} D^\alpha \Psi_{il}^{\alpha\gamma} = \sum_i \sum_\alpha a_{il}^{ij \alpha \beta} \zeta_{il}^{\alpha\gamma} \quad \forall \beta, j.
\]

We arrive at the following expression of the polarization tensor.

Proposition 8.3 (Degenerate polarization tensor). If Assumption 8.2 is fulfilled, then we have

\[
p_{il}^{\alpha\gamma} = |\omega|(1 - \frac{1}{r}) a_{il}^{\alpha\gamma}.
\]

9. Conclusion

In this work we have derived the general form of the topological asymptotic expansion for a wide range of linear elliptic operators of order \( 2m \). We have also identified a class of degenerate problems, for which the closed formulation of the polarization tensor has been obtained. We have given a simple algebraic criterion to recognize the degenerate cases, and we have shown that a typical example of degenerate operator is the bi-Laplacian. As a consequence, the physical models whose state equations obey a PDE involving the bi-Laplacian will exhibit peculiar nucleation properties. By nucleation it is here meant changes of the physical properties of the body by removing and adding infinitesimal quantities of different materials with a view to the minimization of a cost function, usually taken as the energy of the model. As an example, heterogeneities in an elastic continuum can be modeled as small strain gradient perturbations, which in the scalar setting would mean an energy comprising a term of the form \( \epsilon |\Delta u|^2 \), with \( u \) stands for the displacement and \( \epsilon \) a small parameter, and a state equation thereby involving \( \Delta^2 \) (see [15] where such a problem is treated in the framework of homogenization). Another example is provided by time-dependent phase-change models involving Cahn-Hilliard type equations; a recent application in geology which could also fit our setting has been numerically studied in [21]. Finally we mention that the dislocation problem [26] which involves the bi-Laplacian (as a simplified model for the incompatibility operator appearing in elasticity of dislocated elastic bodies) will be further analyzed with a view to the results developed in the present paper.

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Appendix A. Collectively compact operators

Let \( X \) be a Banach space and \( \mathcal{K} \) be a subset of \( \mathcal{L}(X) \), where \( \mathcal{L}(X) \) is the set of bounded linear operators from \( X \) into itself. We say that \( \mathcal{K} \) is collectively compact if the set \( \{Kx, x \in X, \|x\| \leq 1, K \in \mathcal{K}\} \) is relatively compact. The following result is a corollary of Theorem 1.6 of [9]. A proof can be found in [5].

Theorem A.1. Let \( \mathcal{K} \) be a collectively compact set of bounded linear operators of \( X \). Assume further that \( \mathcal{K} \) is pointwise sequentially compact, i.e., for every sequence \( (K_n) \) of \( \mathcal{K} \) there exists a subsequence \( (K_{n_p}) \) and \( K \in \mathcal{K} \) such that \( K_{n_p}x \to Kx \) for all \( x \in X \). If \( I - K \) is invertible for all \( K \in \mathcal{K} \), then

\[
\sup_{K \in \mathcal{K}} \|(I - K)^{-1}\| < \infty.
\]

(A.1)
APPENDIX B. WEIGHTED AND QUOTIENT SOBOLEV SPACES

In this appendix we define the functional spaces which provide existence theorems in $\mathbb{R}^n$. The main result is found in Corollary B.5 which is the restatement of Lemma 3.5. Before arriving at this result several preliminary lemmas must be proved.

Let $B_a$ be the open ball centered at the origin and of radius $a$. We will denote by $r = |x|$ the radial coordinate.

**Lemma B.1.** Let $a > 0$, $B'_a = \mathbb{R}^n \backslash \overline{B}_a$ and $q \in (-\infty, 1]$. If $2q + n \neq 0$, then it holds for all $u \in \mathcal{D}(B'_a)$
\[
\|r^q u\|_{L^2(B'_a)} \leq \frac{2}{|2q + n|} \|r^{q+1} \nabla u\|_{L^2(B'_a)}.
\]

**Proof.** Let $v \in S_n$, the unit sphere of $\mathbb{R}^n$. Integrating by parts yields
\[
\int_a^\infty r^{2q+n-1}u(rv)^2 dr = -\frac{2}{2q+n} \int_a^\infty r^{2q+n} u(rv) \nabla u(rv).v dr \leq \frac{2}{2q+n} \int_a^\infty r^{2q+n} |u(rv)||\nabla u(rv)| dr.
\]

We obtain by the Cauchy-Schwarz inequality
\[
\int_a^\infty r^{2q+n-1}u(rv)^2 dr \leq \frac{2}{2q+n} \left( \int_a^\infty r^{2q+n-1} |u(rv)|^2 dr \right)^{1/2} \left( \int_a^\infty r^{2q+n+1} |\nabla u(rv)|^2 dr \right)^{1/2}.
\]

This implies
\[
\int_a^\infty r^{2q+n-1}u(rv)^2 dr \leq \left( \frac{2}{2q+n} \right)^2 \int_a^\infty r^{2q+n+1} |\nabla u(rv)|^2 dr.
\]

Next, we have
\[
\|r^q u\|_{L^2(B'_a)}^2 = \int_{S_n} \int_a^\infty r^{2q+n-1} u(rv)^2 dr dv \leq \left( \frac{2}{2q+n} \right)^2 \int_{S_n} \int_a^\infty r^{2q+n+1} |\nabla u(rv)|^2 dr dv = \left( \frac{2}{2q+n} \right)^2 \|r^{q+1} \nabla u\|_{L^2(B'_a)}^2,
\]

which leads to the desired result. \qed

Let $\delta \in (0, 1/2)$ be fixed. For every $k \in \mathbb{N}$ we introduce the weight functions as follows:
\[
w_k(x) = (1 + |x|^2)^{\frac{p_k}{2}} \quad \text{(B.1)}
\]

with
\[
p_k = \begin{cases} 0 & \text{if } k = 0, \\ -k - \delta & \text{if } k \geq 1 \text{ and } n = 1, \\ 1 - \frac{n}{2} - k - \delta & \text{if } k \geq 1 \text{ and } n \geq 2. \end{cases}
\]

**Lemma B.2.** Let $a > 0$, $B'_a = \mathbb{R}^n \backslash B_a$, $m \in \mathbb{N}$. For all $k = 0, \ldots, m$ and every $u \in \mathcal{D}(B'_a)$, we have
\[
\sup_{|\alpha|=m-k} \|w_k D^\alpha u\|_{L^2(B'_a)} \leq c_k \sup_{|\alpha|=m} \|D^\alpha u\|_{L^2(B'_a)},
\]

where $c_k$ is a positive constant.

**Proof.** The result is obvious for $k = 0$, thus we assume that $k \geq 1$. We treat first the case $n \geq 2$. By induction from Lemma B.1, we infer for $|\beta| = m - k$,
\[
\|r^{1-\frac{\delta}{2}-k-\delta} D^\beta u\|_{L^2(B'_a)} \leq c \sup_{|\alpha|=m} \|r^{1-\frac{\delta}{2}-\delta} D^\alpha u\|_{L^2(B'_a)} \leq c a^{1-\frac{\delta}{2}-\delta} \sup_{|\alpha|=m} \|D^\alpha u\|_{L^2(B'_a)}.
\]

The desired estimate follows straightforwardly. Suppose now that $n = 1$. Again by induction from Lemma B.1, we obtain for $|\beta| = m - k$
\[
\|r^{-k-\delta} D^\beta u\|_{L^2(B'_a)} \leq c \sup_{|\alpha|=m} \|r^{-\delta} D^\alpha u\|_{L^2(B'_a)} \leq c a^{-\delta} \sup_{|\alpha|=m} \|D^\alpha u\|_{L^2(B'_a)},
\]

leading to the result. \qed
For any open subset \( A \) of \( \mathbb{R}^n \) we define the space
\[
W^m(A) = \left\{ u \in \mathcal{D}'(A) \mid \forall k = 0, \ldots, m, |\alpha| = m - k \Rightarrow w_k D^\alpha u \in L^2(A) \right\},
\]
where the weights are given by (B.1). It is endowed with the norm
\[
\|u\|_{W^m(A)} = \left[ \sum_{k=0}^{m} \sum_{|\alpha|=m-k} w_k \|D^\alpha u\|_{L^2(A)}^2 \right]^{1/2}.
\]
This norm is associated with an inner product \( \langle \cdot, \cdot \rangle_{W^m(A)} \), for which it is easily shown that \( W^m(A) \) is a Hilbert space.

We define \( W^m_0(A) \) as the closure of \( \mathcal{D}(A) \) in \( W^m(A) \). Let \( u \in W^m(B'_a) \) and \( \eta \) be a smooth function such that \( \eta = 1 \) in \( B_{2a} \) and \( \eta = 0 \) in \( B'_{3a} \). Then \( \eta u \in H^m(B'_a) \) and \( (1 - \eta)u \in W^m(\mathbb{R}^n) \). This allows us to define the normal trace of \( u \) on \( \partial B_a \) of order \( j \), \( j \leq m - 1 \), denoted by \( \partial^m_n u \). Also, one may prove by standard arguments (see e.g. \([1]\)) that \( \mathcal{D}(\mathbb{R}^n) \) is dense in \( W^m(\mathbb{R}^n) \). This implies the following result.

**Lemma B.3.** We have
\[
W^m_0(B'_a) = \left\{ u \in W^m(B'_a) \mid \partial^m_n u = 0 \ \forall j = 0, \ldots, m - 1 \right\}.
\]

**Proposition B.4.** Let \( \mathcal{H} \) be a closed subspace of \( W^m(\mathbb{R}^n) \) and \( \| \cdot \|_\mathcal{H} \) be a norm on \( \mathcal{H} \) such that, for some constants \( c_1, c_2 > 0 \), it holds
\[
c_1 \|u\|_{W^m(\mathbb{R}^n)} \leq \|u\|_\mathcal{H} \leq c_2 \|u\|_{W^m(\mathbb{R}^n)} \quad \forall u \in \mathcal{H}.
\]

Then, on the space \( \mathcal{H} \), the norms \( \| \cdot \|_\mathcal{H} \) and \( \| \cdot \|_{W^m(\mathbb{R}^n)} \) are equivalent.

**Proof.** We must show that there exists a constant \( c_3 \) such that
\[
\|u\|_{W^m(\mathbb{R}^n)} \leq c_3 \|u\|_\mathcal{H}.
\]
By contradiction, we assume that there exists a sequence \( (u_p) \in \mathcal{H} \) such that, for every \( p \),
\[
\|u_p\|_{W^m(\mathbb{R}^n)} = 1, \quad \|u_p\|_\mathcal{H} < \frac{1}{p}.
\]
Let \( \eta \) be defined as above. The Leibniz formula provides, for all \( v \in W^m(B'_a) \),
\[
|\eta v|_{H^m(B_{3a})} \leq \eta |v|_{H^m(B_{3a})} + c |v|_{H^m(B_{2a})},
\]
\[
|1 - \eta|v|_{H^m(B_{3a})} \leq |v|_{H^m(B_{2a})} + c |v|_{H^m(B_{1a})}.
\]
The embedding of \( H^m(B_{3a}) \) into \( H^{m-1}(B_{3a}) \) is compact and the sequence \( (u_p) \) is bounded in \( H^m(B_{3a}) \). We still denote by \( (u_p) \) a subsequence such that \( u_p \to w \) in \( H^{m-1}(B_{3a}) \). Using Lemma B.2, (B.5) and the assumptions we get
\[
\|(1 - \eta)(u_p - u_q)\|_{W^m(B'_{2a})} \leq c \|u_p - u_q\|_\mathcal{H} + c \|u_p - u_q\|_{H^{m-1}(B_{3a})}.
\]
Moreover, the Poincaré inequality in \( H^m_0(B_{3a}) \) together with (B.4) and the assumptions yield
\[
\|\eta(u_p - u_q)\|_{H^{m-1}(B_{3a})} \leq c \|u_p - u_q\|_\mathcal{H} + c \|u_p - u_q\|_{H^{m-1}(B_{3a})}.
\]
Therefore, \( (\eta u_p) \) and \( (1 - \eta)u_p \) are Cauchy sequences in \( H^m(B_{3a}) \) and \( W^m(B'_{2a}) \), respectively. Thus, there exist \( (v_1, v_2) \in H^m(B_{3a}) \times W^m(B'_{2a}) \) such that \( \eta u_p \to v_1 \) in \( H^m(B_{3a}) \) and \( (1 - \eta)u_p \to v_2 \) in \( W^m(B'_{2a}) \). After summation, we infer \( u_p \to v := v_1 + v_2 \) in \( W^m(\mathbb{R}^n) \). By assumption, this limit holds also in \( \mathcal{H} \). From (B.3) we obtain a contradiction. \( \square \)

Let \( P_{m-1} \) be the space of polynomials of degree not greater than \( m - 1 \). It is easily checked that \( P_{m-1} \) is a subspace of \( W^m(\mathbb{R}^n) \). The quotient space \( W^m(\mathbb{R}^n)/P_{m-1} \) is endowed with the norm
\[
\|u\|_{W^m(\mathbb{R}^n)/P_{m-1}} = \inf_{p \in P_{m-1}} \|u + p\|_{W^m(\mathbb{R}^n)},
\]
where \( u \) is an arbitrary representative of its class. Proposition B.4 implies that the seminorm \( |u|_{H^m(\mathbb{R}^n)} \) is an equivalent norm to \( \|u\|_{W^m(\mathbb{R}^n)/P_{m-1}} \). This will be made clear in the following corollary, which is a restatement of Lemma 3.5, and whose proof is now given.

**Corollary B.5.** There exists \( c > 0 \) such that, for all \( u \in W^m(\mathbb{R}^n) \),
\[
\|u\|_{W^m(\mathbb{R}^n)/P_{m-1}} \leq c |u|_{H^m(\mathbb{R}^n)}.
\]
Proof. By standard arguments of the calculus of variations the infimum in (B.6) is attained at a unique point. The Euler-Lagrange equation applied to the problem with squared norm reads for the minimizer $v := u + p$

$$\langle v, \tilde{p} \rangle_{W^m(\mathbb{R}^n)} = 0 \quad \forall \tilde{p} \in \mathcal{P}_{m-1},$$

which is equivalent to

$$\langle v, x^\alpha \rangle_{W^m(\mathbb{R}^n)} = 0 \quad \forall |\alpha| \leq m - 1. \quad (B.7)$$

Therefore, $W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ can be identified with the subspace of the functions $v \in W^m(\mathbb{R}^n)$ satisfying (B.7). In addition, the seminorm $|.|_{H^m(\mathbb{R}^n)}$ is a norm on this space. We then apply Proposition B.4 with $\mathcal{H} = W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ and $||u||_H := ||u||_{H^m(\mathbb{R}^n)}$.

Remark B.6. It appears from inspection of the proof that Corollary B.5 remains true if $\mathbb{R}^n$ is replaced by any open set containing the origin.

References


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