

On the Regularity of the Free Boundary for Quasilinear Obstacle Problems

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Abstract

We extend the regularity of the free boundary of the obstacle problem to a class of heterogeneous quasilinear degenerate elliptic operators which includes the $p(x)$ -Laplacian, in particular. Under the assumption of Lipschitz continuity of the order of the power growth $p(x)$, we use the growth rate of the solution near the free boundary to obtain its porosity, which implies it is of Lebesgue measure zero. Establishing the growth rate of the gradient of the solution, we show the finiteness of the $(n - 1)$ -dimensional Hausdorff measure of the free boundary, which yields, in particular, that up to a negligible singular set the free boundary is the union of at most a countable family of C^1 hypersurfaces.

1 Introduction

In [1] Caffarelli remarks that the quadratic growth of the solution of the free boundary of the obstacle problem for the Laplacian implies an estimate of the $(n - 1)$ -dimensional Hausdorff measure of the free boundary and a stability property. This result has a simple generalization to second order linear elliptic operators with Lipschitz continuous coefficients, as observed by one of the authors in [14], page 221, which allows the extension of those properties to $C^{1,1}$ solutions of the obstacle problem for certain quasilinear operators of minimal surfaces type (see Theorem 7:5.1 of [14], page 221).

Hausdorff measure estimates were obtained for homogeneous nonlinear operators of the p -obstacle problem ($2 < p < \infty$) by Lee and Shahgholian [11], and

for general potential operators by Monneau [12], in a special case corresponding to an obstacle problem arising in superconductor modelling with convex energy, and by three of the authors in [4] to the so called A-obstacle in Orlicz-Sobolev spaces, that includes a large class of degenerate and singular elliptic operators. As it is well-known from geometric measure theory, the importance of the estimate on the $(n - 1)$ -dimensional Hausdorff measure of the free boundary, by a result of Federer, implies that the non-coincidence set $\{u > 0\}$ is then a set of locally finite perimeter. As an important consequence, by a well-known result of De Giorgi (see [8], page 54), the free boundary $\partial\{u > 0\}$ may be written, up to a possible singular set of $\|\nabla\chi_{\{u>0\}}\|$ -measure zero, as a countable union of C^1 hypersurfaces. The main result of our work is the extension of these properties to a more general class of heterogeneous quasilinear degenerate elliptic operators which includes the $p(x)$ -Laplacian.

On the other hand, it was shown by Karp, Kilpeläinen, Petrosyan and Shahgholian [10], for the p -obstacle problem ($1 < p < \infty$), that the free boundary is porous with a certain constant $\delta > 0$, that is, there exists $r_0 > 0$ such that for each $x \in \partial\{u > 0\}$ and $0 < r < r_0$, there exists a point y such that $B_{\delta r}(y) \subset B_r(x) \setminus \partial\{u > 0\}$. Since a porous set in \mathbb{R}^n has Hausdorff dimension strictly smaller than n (see [13] or [18]), it follows that the free boundary has Lebesgue measure zero, which allows us to write the solution of the obstacle problem as an a.e. solution of a quasilinear elliptic equation in the whole domain involving the characteristic function $\chi_{\{u>0\}}$ of the non-coincidence set (see Theorem 3.1 below, that extends earlier results [2] and [3], respectively, for the A-obstacle and $p(x)$ -obstacle problems). This property is important to show, under general non-degenerate assumptions on the data, a stability of the non-coincidence set in Lebesgue measure, as observed, for instance, in [4], [15] and [16].

Let Ω be a bounded open connected subset of \mathbb{R}^n , $n \geq 2$, $f \in L^\infty(\Omega)$ and $g \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, $g \geq 0$. We consider the quasilinear obstacle problem ($A(x)$ -obstacle problem) with a zero obstacle:

$$\begin{cases} Au := \operatorname{div}(a(x, \nabla u)) = f(x) & \text{in } \{u > 0\}, \\ u \geq 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where we denote by $\{u > 0\} := \{x \in \Omega : u(x) > 0\}$ the non-coincidence set. The weak formulation of this problem is given by the following variational inequality

$$(P) \begin{cases} \text{Find } u \in K_g \text{ such that :} \\ \int_{\Omega} \left(a(x, \nabla u) \cdot \nabla(v - u) + f(x)(v - u) \right) dx \geq 0 \quad \forall v \in K_g, \end{cases}$$

where $K_g = \{v \in W^{1,p(\cdot)}(\Omega) : v - g \in W_0^{1,p(\cdot)}(\Omega), v \geq 0 \text{ a.e. in } \Omega\}$, p is a measurable real valued function defined in Ω and satisfying for some positive numbers p_- and p_+

$$1 < p_- \leq p(x) \leq p_+ < \infty \quad \text{a.e. } x \in \Omega. \quad (1.1)$$

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, where $W^{1,p(\cdot)}(\Omega)$ is the variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \nabla u \in (L^{p(\cdot)}(\Omega))^n \right\}$$

and $L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \rho(u) = \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}$

is equipped with the Luxembour norm

$$\|u\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \rho(u/\lambda) \leq 1 \right\}.$$

$W^{1,p(\cdot)}$ is equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}} = \|u\|_{L^{p(\cdot)}} + \|\nabla u\|_{L^{p(\cdot)}},$$

where

$$\|\nabla u\|_{L^{p(\cdot)}} = \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p(\cdot)}}.$$

By $B_r(x)$ we shall denote the open ball in \mathbb{R}^n with center x and radius r . The conjugate of $p(x)$, defined by $\frac{p(x)}{p(x)-1}$, will be denoted by $q(x)$. If the center of a ball is not mentioned, then it is the origin.

We first give some classical properties of the solution below. In section 2, we establish the growth rate of a class of functions. In section 3, we obtain the exact growth rate of the solution of the problem (P) near the free boundary, from which we deduce its porosity. In section 4, we give the growth rate of the gradient of the solution. In section 5, we prove the finiteness of the $(n-1)$ -Hausdorff measure of the free boundary for the particular homogeneous case of p -Laplacian type operators. Finally, in section 6, we establish the finiteness of $(n-1)$ -Hausdorff measure of the free boundary. These results extend those for the Laplacian [1], for the p -Laplacian [10], [11] ($p > 2$), for the $p(x)$ -Laplacian [3] (porosity of the free boundary) and for the A -Laplacian [2], [4].

We assume that the function $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is such that $a(x, 0) = 0$ for a.e. $x \in \Omega$, and satisfies the structural assumptions for some positive constants c_0, c_1, c_2 , namely [6]

$$\sum_{i,j=1}^n \frac{\partial a_i}{\partial \eta_j}(x, \eta) \xi_j \xi_j \geq c_0 |\eta|^{p(x)-2} |\xi|^2, \quad (1.2)$$

$$\sum_{i,j=1}^n \left| \frac{\partial a_i}{\partial \eta_j}(x, \eta) \right| \leq c_1 |\eta|^{p(x)-2} \quad (1.3)$$

for a.e. $x \in \Omega$, a.e. $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}^n \setminus \{0\}$ and for all $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, and

$$|a(x_1, \eta) - a(x_2, \eta)| \leq c_2 |x_1 - x_2| (|\eta|^{p(x_1)-1} + |\eta|^{p(x_2)-1}) |\ln |\eta|| \quad (1.4)$$

for all $x_1, x_2 \in \Omega$, $\eta \in \mathbb{R}^n \setminus \{0\}$.

Remark 1.1. Assumptions (1.2), (1.3) imply [5], [17], for some positive constants c_3, c_4

$$a(x, \xi) \cdot \xi \geq c_3 |\xi|^{p(x)} \quad \text{and} \quad |a(x, \xi)| \leq c_4 |\xi|^{p(x)-1}.$$

First, we recall the following existence and uniqueness result [7], [16].

Proposition 1.1. Assume that $f \in L^{q(\cdot)}(\Omega)$ and $g \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Then there exists a unique solution u to the problem (P).

For the proof of the following proposition, we refer to [3], Proposition 1.2.

Proposition 1.2. If u is the solution of (P) then

- i) $f \geq 0$ in $\Omega \implies 0 \leq u \leq \|g\|_{L^\infty}$ in Ω .
- ii) $Au = f$ in $\mathcal{D}'(\{u > 0\})$.
- iii) $f \chi_{\{u > 0\}} \leq Au \leq f$ a.e. in Ω .

Remark 1.2. Equation ii) and inequalities iii) of Proposition 1.2 were established in [16], in the framework of entropy solutions, under the condition:

$$\text{ess inf}_{x \in \Omega} (q_1(x) - (p(x) - 1)) > 0, \text{ where } q_1(x) = \frac{q_0(x)p(x)}{q_0(x)+1} \text{ and } q_0(x) = \frac{np(x)}{n-p(x)} \frac{p-1}{p}.$$

Remark 1.3. If $f \geq 0$ in Ω and $f \in L_{loc}^\infty(\Omega)$, we know from Proposition 1.2 that u is bounded and Au is locally bounded in Ω . Moreover, if $p(x)$ is Lipschitz continuous, and $a(x, \xi)$ satisfies (1.2)-(1.4), then we have [6], $u \in C_{loc}^{1,\alpha}(\Omega)$, for some $\alpha \in (0, 1)$.

2 A class of functions on the unit ball

In all what follows, we assume that p is Lipschitz continuous, that is, there exists a positive constant L such that

$$|p(x) - p(y)| \leq L|x - y| \quad \forall x, y \in \Omega. \quad (2.1)$$

In this section, we study a family \mathcal{F}_A of solutions of problems defined on the unit ball B_1 . More precisely, $u \in \mathcal{F}_A$ if it satisfies :

$$\begin{cases} u \in W^{1,p(\cdot)}(B_1), & u(0) = 0, \\ 0 \leq u \leq 1 & \text{in } B_1, & \|Au\|_{L^\infty(B_1)} \leq 1. \end{cases}$$

Condition $u(0) = 0$ makes sense, since from [6] we know that $u \in C_{loc}^{1,\alpha}(B_1)$, for some $\alpha \in (0, 1)$. In particular, there exist two positive constants $\alpha = \alpha(n, c_0, c_1, c_2, p_-, p_+, L)$ and $C = C(n, c_0, c_1, c_2, p_-, p_+, L)$ such that

$$\|u\|_{C^{1,\alpha}(\bar{B}_{3/4})} \leq C, \quad \forall u \in \mathcal{F}_A. \quad (2.2)$$

The following theorem gives a growth rate of the elements in the class \mathcal{F}_A .

Theorem 2.1. *There exists a positive constant $C_0 = C_0(n, c_0, c_1, c_2, p_-, p_+, L)$ such that, for every $u \in \mathcal{F}_A$, we have*

$$0 \leq u(x) \leq C_0|x|^{q_0}, \quad \forall x \in B_1,$$

where $q_0 = \frac{p_0}{p_0 - 1}$ is the conjugate of $p_0 = p(0)$.

Let us first introduce some notations. For a nonnegative bounded function u , we define the quantity $S(r, u) = \sup_{x \in B_r} u(x)$. We also define, for each $u \in \mathcal{F}_A$, the set

$$\mathbb{M}(u) = \{j \in \mathbb{N} : 2^{q_0} S(2^{-j-1}, u) \geq S(2^{-j}, u)\}.$$

Then we have

Lemma 2.1. *If $\mathbb{M}(u) \neq \emptyset$, then there exists a constant \tilde{c}_0 depending only on $n, c_0, c_1, c_2, p_-, p_+$ and L such that*

$$S(2^{-j-1}, u) \leq \tilde{c}_0(2^{-j})^{q_0}, \quad \forall u \in \mathcal{F}_A, \quad \forall j \in \mathbb{M}(u).$$

Proof. Arguing by contradiction, we assume that $\forall k \in \mathbb{N}$ there exists $u_k \in \mathcal{F}_A$ and $j_k \in \mathbb{M}(u_k)$ such that

$$S(2^{-j_k-1}, u_k) \geq k(2^{-j_k})^{q_0}. \quad (2.3)$$

Consider the function

$$v_k(x) = \frac{u_k(2^{-j_k}x)}{S(2^{-j_k-1}, u_k)}$$

defined in B_1 . By definition of v_k and $\mathbb{M}(u_k)$, we have

$$\left\{ \begin{array}{l} 0 \leq v_k \leq \frac{S(2^{-j_k}, u_k)}{S(2^{-j_k-1}, u_k)} \leq 2^{q_0} \quad \text{in } B_1, \\ \sup_{x \in \bar{B}_{1/2}} v_k(x) = 1, \quad v_k(0) = 0. \end{array} \right.$$

Now, let $p_k(x) = p(2^{-j_k}x)$, $s_k = \frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)}$, and define for $(x, \xi) \in B_1 \times \mathbb{R}^n$

$$a^k(x, \xi) := s_k^{p_k(x)-1} a(2^{-j_k}x, \frac{1}{s_k}\xi). \quad (2.4)$$

We claim that

$$|A_k v_k(x)| := |\operatorname{div}(a^k(x, \nabla v_k(x)))| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.5)$$

Then one can easily verify that

$$\begin{aligned} A_k v_k(x) &= 2^{-j_k} s_k^{p_k(x)-1} (A u_k)(2^{-j_k} x) \\ &\quad + 2^{-j_k} (\ln(s_k)) s_k^{p_k(x)-1} a(2^{-j_k} x, \nabla u_k(2^{-j_k} x)) \nabla p(2^{-j_k} x). \end{aligned}$$

Using the structural assumptions (second inequality in Remark 1.1) and the fact that $u_k \in \mathcal{F}_A$, and $|\nabla p|_{L^\infty(\Omega)} \leq L$ (by (2.1)), this leads to

$$|A_k v_k(x)| \leq 2^{-j_k} s_k^{p_k(x)-1} + c_4 L 2^{-j_k} |\ln(s_k)| s_k^{p_k(x)-1} |\nabla u_k(2^{-j_k} x)|^{p_k(x)-1}.$$

Since $u_k \geq 0$ in B_1 , $u_k(0) = 0$, and $u_k \in C^1(\overline{B}_{3/4})$, we have $\nabla u_k(0) = 0$. Combining this result and (2.2), we get

$$\forall k \in \mathbb{N}, \quad \forall x \in B_1 \quad |\nabla u_k(2^{-j_k} x)| \leq C(2^{-j_k})^\alpha.$$

It follows that

$$|A_k v_k(x)| \leq 2^{-j_k} s_k^{p_k(x)-1} (1 + c_4 L(C)^{p_k(x)-1} |\ln(s_k)| (2^{-j_k})^{\alpha(p_k(x)-1)}). \quad (2.6)$$

Note that $S(2^{-j_k-1}, u_k) = u_k(z_k)$, for some $z_k \in \overline{B}_{2^{-j_k-1}}$. Since $u_k(0) = 0$ and $u_k \in C^1(\overline{B}_{3/4})$, we deduce that

$$S(2^{-j_k-1}, u_k) \leq C|z_k| \leq C2^{-j_k-1}.$$

Consequently, we obtain

$$s_k = \frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)} \geq \frac{2^{-j_k}}{C2^{-j_k-1}} = \frac{2}{C} = \mu.$$

We recall from [3] (see also the proof of (4.11) and (4.13)) that there exist positive constants $\tilde{c}_1 = \tilde{c}_1(\alpha, p_0, \mu)$ and $\tilde{c}_2 = \tilde{c}_2(\alpha, L, p_0, \mu)$ such that

$$|\ln(s_k)| (2^{-j_k})^{\alpha(p_k(x)-1)} \leq \frac{\tilde{c}_1}{k^{\alpha(p_0-1)^2}} \quad \text{and} \quad 2^{-j_k} s_k^{p_k(x)-1} \leq \frac{\tilde{c}_2}{k^{p_0-1}}, \quad \forall k \in \mathbb{N},$$

which together with (2.6) gives (2.5).

Lemma 2.2. *With the notation above, $a^k(x, \xi)$ defined in (2.4) satisfies all structural conditions (with the same constants as $a(x, \xi)$). Moreover, we have uniformly in $(x, \xi) \in B_1 \times B_M$, for any $M > 0$*

$$\left| \frac{\partial a_i^k}{\partial x_j} \right| \leq L_k \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.7)$$

Proof. It is easy to see that

$$\begin{aligned}
\sum_{i,j=1}^n \frac{\partial a_i^k}{\partial \eta_j}(x, \eta) \xi_j \xi_j &= \sum_{i,j=1}^n s_k^{p_k(x)-1} \frac{1}{s_k} \frac{\partial a_i}{\partial \eta_j}(2^{-j_k} x, \frac{1}{s_k} \eta) \xi_i \xi_j \\
&\geq c_0 s_k^{p_k(x)-2} \left| \frac{\eta}{s_k} \right|^{p_k(x)-2} |\xi|^2 \\
&= c_0 |\eta|^{p_k(x)-2} |\xi|^2.
\end{aligned}$$

$$\begin{aligned}
\sum_{i,j=1}^n \left| \frac{\partial a_j^k}{\partial \eta_j}(x, \eta) \right| &= \sum_{i,j=1}^n s_k^{p_k(x)-1} \frac{1}{s_k} \left| \frac{\partial a_i}{\partial \eta_j}(2^{-j_k} x, \frac{1}{s_k} \eta) \right| \\
&\leq c_1 s_k^{p_k(x)-2} \left| \frac{\eta}{s_k} \right|^{p_k(x)-2} \\
&= c_1 |\eta|^{p_k(x)-2}.
\end{aligned}$$

Now, to prove (2.7), we use the second inequality in Remark 1.1 and (1.4)

$$\begin{aligned}
\left| \frac{\partial a_i^k}{\partial x_j} \right| &= \left| \frac{\partial}{\partial x_j} \left(s_k^{p_k(x)-1} a_i(2^{-j_k} x, \frac{1}{s_k} \xi) \right) \right| \\
&\leq |\nabla(s_k^{p_k(x)-1})| \left| a_i(2^{-j_k} x, \frac{1}{s_k} \xi) \right| \\
&\quad + 2^{-j_k} s_k^{p_k(x)-1} \left| \frac{\partial a_i}{\partial x_j}(2^{-j_k} x, \frac{1}{s_k} \xi) \right| \\
&\leq c_4 L 2^{-j_k} s_k^{p_k(x)-1} |\ln(s_k)| \left| \frac{\xi}{s_k} \right|^{p_k(x)-1} \\
&\quad + 2c_2 2^{-j_k} s_k^{p_k(x)-1} \left| \frac{\xi}{s_k} \right|^{p_k(x)-1} \left| \ln \left| \frac{\xi}{s_k} \right| \right| \\
&= \left(c_4 L 2^{-j_k} |\ln(s_k)| + 2c_2 2^{-j_k} \left| \ln \left| \frac{\xi}{s_k} \right| \right| \right) |\xi|^{p_k(x)-1} =: L_k
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2^{-j_k} |\xi|^{p_k(x)-1} \left| \ln \left| \frac{\xi}{s_k} \right| \right| &= 2^{-j_k} |\xi|^{p_k(x)-1} |\ln(|\xi|) - \ln(s_k)| \\
&\leq 2^{-j_k} |\xi|^{p_k(x)-1} |\ln(|\xi|)| \\
&\quad + 2^{-j_k} |\ln(s_k)| |\xi|^{p_k(x)-1}
\end{aligned}$$

The first term uniformly goes to zero (for $(x, \xi) \in B_1 \times B_M$, for any $M > 0$) when $k \rightarrow \infty$. Since $2^{-j_k} |\ln(s_k)| \rightarrow 0$ as $k \rightarrow 0$ ([3]), so does the second term. \square

Therefore, the pointwise limit of $a^k(x, \xi)$ does not depend on x :

$$a^k(x, \xi) \rightarrow \tilde{a}(\xi),$$

where \tilde{a} is a vector field satisfying the same structural assumptions (1.2), (1.3), with $p(x)$ replaced by $p_0 = p(0)$.

Conclusion of proof of Lemma 2.1. Now, by taking into account the uniform bound of v_k , (2.5), and the fact that p_k satisfies (1.1) and (2.1) with the same constants, we deduce [6] that there exist two positive constants δ and C , independent of k , such that $v_k \in C^{1,\delta}(\overline{B}_{3/4})$ and $\|v_k\|_{C^{1,\delta}(\overline{B}_{3/4})} \leq C$, for all $k \geq k_0$. It follows then from the Ascoli-Arzelà's theorem that there exists a subsequence, still denoted by v_k , and a function $v \in C^{1,\delta'}(\overline{B}_{3/4})$ such that $v_k \rightarrow v$ in $C^{1,\delta'}(\overline{B}_{3/4})$, for any $\delta' \in (0, \delta)$. Moreover, it is clear that v satisfies (in the weak sense)

$$\left\{ \begin{array}{l} \operatorname{div}(\tilde{a}(\nabla v)) = 0 \quad \text{in } B_{3/4}, \quad v \geq 0 \quad \text{in } B_{3/4}, \\ \sup_{x \in B_{1/2}} v(x) = 1, \quad v(0) = 0. \end{array} \right.$$

By the strong maximum principle (see [9], for instance) we have necessarily $v \equiv 0$ in $B_{3/4}$, which is in contradiction with $\sup_{x \in B_{1/2}} v(x) = 1$. □

Proof of Theorem 2.1. The theorem is proved by induction. Using Lemma 2.2, the proof follows step by step as the one of Theorem 2.1 of [3] (see also the proof of Theorem 4.1 below). □

3 Porosity of the free boundary

In all what follows, we assume that there exist positive constants λ, Λ such that

$$0 < \lambda \leq f \leq \Lambda < \infty, \quad \text{a.e. in } \Omega. \quad (3.1)$$

The following lemma and Theorem 2.1 give the exact growth rate of the solution of the problem (P) near the free boundary. This extends a result established in [1] for the Laplacian, and generalized in [10] for the p -Laplacian (see also [2] and [3] for the A -Laplacian and $p(x)$ -Laplacian respectively).

Lemma 3.1. *Suppose that $u \in W^{1,p(\cdot)}(\Omega)$ is a nonnegative continuous function satisfying*

$$Au = f \quad \text{in} \quad \mathcal{D}'(\{u > 0\}).$$

Then there exists $r_ > 0$ such that for each $y \in \overline{\{u > 0\}}$ and $r \in (0, r_*)$ satisfying $B_r(y) \subset \Omega$, we have for an appropriate constant $C(y) > 0$*

$$\sup_{\partial B_r(y)} u \geq C(y) r^{\frac{p(y)}{p(y)-1}} + u(y).$$

Proof. It is enough to prove the result for $y \in \{u > 0\}$. For each y we consider the function defined by

$$v(x) := v(x, y) := C(y)|x - y|^{\frac{p(y)}{p(y)-1}},$$

where $C(y)$ is to be chosen later.

We claim that there exists $r_* > 0$ such that

$$\forall r \in (0, r_*), \quad \forall y \in \Omega, \quad \forall x \in B_r(y) \subset \Omega \quad Av \leq \lambda. \quad (3.2)$$

To prove (3.2), we compute $\nabla_x v$ and divergence of $a(x, \nabla_x v)$:

$$\begin{aligned} \operatorname{div}(a(x, \nabla v)) &= \operatorname{div}(a(x, C(y)q(y)|x - y|^{q(y)-2}(x - y))) \\ &= \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}(x, w) + \sum_{i,j=1}^n \frac{\partial a_i}{\partial \eta_j}(x, w) \cdot \frac{\partial w_j}{\partial x_i}(x) \\ &= \sum_{i=1}^n \frac{\partial a_i}{\partial x_i} + C(y)q(y)|x - y|^{q(y)-2} \sum_{i,j=1}^n \left(\delta_{ij} \right. \\ &\quad \left. + (q(y) - 2) \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} \right) \frac{\partial a_i}{\partial \eta_j}, \end{aligned}$$

where $w(x) := C(y)q(y)|x - y|^{q(y)-2}(x - y)$.

Therefore, using the structural assumptions (1.3), (1.4), we get

$$\begin{aligned} |\operatorname{div}(a(x, \nabla v))| &\leq 2c_2|w|^{p(x)-1} |\ln |w|| \\ &\quad + c_1 \max(1, q(y) - 1) (C(y)q(y))^{p(x)-1} |x - y|^{(q(y)-1)(p(x)-2)+q(y)-2} \\ &=: S_1 + S_2. \end{aligned}$$

To estimate S_1 , we write

$$\begin{aligned} S_1 &= 2c_2|w|^{p(x)-1} |\ln(|w|)| \\ &= 2c_2(C(y)q(y))^{p(x)-1} |x - y|^{(p(x)-1)(q(y)-1)} |\ln(C(y)q(y)) + (q(y) - 1) \ln|x - y|| \\ &\leq 2c_2(q(y))^{p(x)-1} (C(y))^{p(x)-1} |x - y|^{(p(x)-1)(q(y)-1)} |\ln(C(y)q(y))| \\ &\quad + 2c_2(q(y) - 1) (C(y)q(y))^{p(x)-1} |x - y|^{(p(x)-1)(q(y)-1)} |\ln(|x - y|)| \end{aligned}$$

Since $r \ln r \rightarrow 0$, when $r \rightarrow 0$, then S_1 can be made as small as we wish, if x is close to y , and $C(y)$ is small enough. To estimate S_2 , we first observe that

$$|x - y|^{(q(y)-1)(p(x)-2)+q(y)-2} = |x - y|^{\frac{p(x)-p(y)}{p(y)-1}}$$

and for $|x - y| < r < \frac{1}{e}$, we have

$$|x - y|^{\frac{p(x)-p(y)}{p(y)-1}} = e^{\frac{p(x)-p(y)}{p(y)-1} \ln(|x-y|)} \leq e^{\frac{L}{p-1} |x-y| |\ln(|x-y|)|} \leq e^{\frac{L}{p-1} r |\ln(r)|},$$

and since

$$\begin{aligned} S_2 &= c_1 \max(1, q(y) - 1) (C(y)q(y))^{p(x)-1} |x - y|^{\frac{p(x)-p(y)}{p(y)-1}} \\ &\leq c_1 \max(1, q(y) - 1) (C(y)q(y))^{p(x)-1} e^{\frac{L}{p-1} r |\ln(r)|}, \end{aligned}$$

S_2 also can be made small, if r and $C(y)$ are small enough. It is clear now that (3.2) holds.

Now let $\epsilon > 0$ and consider the following function $u_\epsilon(x) = u(x) - (1 - \epsilon)u(y)$. We have from (3.1)-(3.2)

$$Au_\epsilon = Au = f \geq \lambda \geq Av \quad \text{in} \quad B_r(y) \cap \{u > 0\}.$$

Moreover,

$$u_\epsilon = -(1 - \epsilon)u(y) \leq 0 \leq v \quad \text{on} \quad (\partial\{u > 0\}) \cap B_r(y).$$

If we also have

$$u_\epsilon \leq v \quad \text{on} \quad (\partial B_r(y)) \cap \{u > 0\},$$

then we get by the weak maximum principle

$$u_\epsilon \leq v \quad \text{in} \quad B_r(y) \cap \{u > 0\}.$$

But $u_\epsilon(y) = \epsilon u(y) > 0 = v(y)$, which constitutes a contradiction.

So there exists $z \in (\partial B_r(y)) \cap \{u > 0\}$ such that $u_\epsilon(z) > v(z)$. Since v is radial, we get

$$\begin{aligned} \sup_{\partial B_r(y)} (u - (1 - \epsilon)u(y)) &= \sup_{\partial B_r(y)} u_\epsilon \geq \sup_{\partial B_r(y) \cap \{u > 0\}} u_\epsilon \geq u_\epsilon(z) \\ &> v(z) = C(y)r^{\frac{p(y)}{p(y)-1}}. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get

$$\sup_{\overline{B_r(y)}} u \geq \sup_{\partial B_r(y)} u \geq C(y)r^{\frac{p(y)}{p(y)-1}} + u(y).$$

□

We shall denote by u the solution of the problem (P) . The main result of this section is the porosity of the free boundary $(\partial\{u > 0\}) \cap \Omega$.

We recall that a set $E \subset \mathbb{R}^n$ is called porous with porosity δ , if there is an $r_0 > 0$ such that

$$\forall x \in E, \quad \forall r \in (0, r_0), \quad \exists y \in \mathbb{R}^n \quad \text{such that} \quad B_{\delta r}(y) \subset B_r(x) \setminus E.$$

A porous set of porosity δ has Hausdorff dimension not exceeding $n - c\delta^n$, where $c = c(n) > 0$ is a constant depending only on n . In particular, a porous set has Lebesgue measure zero (see [13] or [18] for instance).

Theorem 3.1. *Let r_* be as in Lemma 3.1, $R \in (0, r_*)$ and $x_0 \in \Omega$ such that $\overline{B_{4R}(x_0)} \subset \Omega$. Then $(\partial\{u > 0\}) \cap B_R(x_0)$ is porous with porosity constant depending only on $n, p_-, p_+, L, c_0, c_1, c_2, \lambda, \Lambda, R$, and $\|g\|_{L^\infty}$. As an immediate consequence, we have*

$$Au = f\chi_{\{u>0\}} \quad \text{a.e. in } \Omega.$$

We need a lemma.

Lemma 3.2. *Let $R > 0$ and $x_0 \in \Omega$ such that $\overline{B_{4R}(x_0)} \subset \Omega$. We consider, for $y_0 \in \overline{B_{2R}(x_0)} \cap \{u = 0\}$ and $M > 0$, the functions defined in $\overline{B_1}$ by*

$$\bar{a}(z, \xi) = a(y_0 + Rz, M\xi), \quad \bar{u}(z) = \frac{u(y_0 + Rz)}{MR}. \quad (3.3)$$

Then we have $\bar{u} \in \mathcal{F}_{\bar{A}}$, for all $R \leq R_0 = \frac{1}{\Lambda}$ and $M \geq M_0 = \frac{\|g\|_{L^\infty}}{R}$, where \bar{A} is the operator corresponding to \bar{a} .

Proof. First, note that \bar{a} and \bar{u} are well defined, since we have $\overline{B_R(y_0)} \subset \overline{B_{3R}(x_0)} \subset \Omega$. Moreover, we have $\bar{u}(0) = \frac{u(y_0)}{MR} = 0$, and for $M \geq \frac{\|g\|_{L^\infty}}{R}$, we have $0 \leq \bar{u} \leq 1$ in B_1 .

Note that $\bar{a}(z, \xi)$ satisfies to all structural conditions (not necessarily with the same constants as for a) with $\bar{p}(z) := p(y_0 + Rz)$ instead of p .

Next, one can easily verify that \bar{u} satisfies

$$\begin{aligned} \bar{A}\bar{u} &:= \operatorname{div}(\bar{a}(z, \nabla\bar{u}(z))) \\ &= \operatorname{div}(a(y_0 + Rz, \nabla u(y_0 + Rz))) \\ &= R(Au)(y_0 + Rz) \leq R\Lambda \leq 1 \end{aligned}$$

if $R \leq R_0 = \frac{1}{\Lambda}$, and we conclude that $\bar{u} \in \mathcal{F}_{\bar{A}}$ for all $M \geq M_0$ and $R \leq R_0$. \square

Proof of Theorem 3.1. Now, to prove the theorem, we argue as in [3]. Let r_* be as in Lemma 3.1 and $R_* = \min(r_*, R_0)$. Let then $R \in (0, R_*)$ be such that $\overline{B_{4R}(x_0)} \subset \Omega$, and let $x \in E = \partial\{u > 0\} \cap \overline{B_R(x_0)}$. For each $0 < r < R$, we have $\overline{B_r(x)} \subset B_{2R}(x_0) \subset \Omega$. Let $y \in \partial B_r(x)$ such that $u(y) = \sup_{\partial B_r(x)} u$. Then we

have by Lemma 3.1

$$u(y) \geq C'_0 r^{\frac{p(x)}{p(x)-1}} + u(x) = C'_0 r^{\frac{p(x)}{p(x)-1}}. \quad (3.4)$$

Hence $y \in B_{2R}(x_0) \cap \{u > 0\}$. We denote by $d(y) = \operatorname{dist}(y, \overline{B_{2R}(x_0)} \cap \{u = 0\})$ the distance from y to the set $\overline{B_{2R}(x_0)} \cap \{u = 0\}$.

From Lemma 2.1 and Lemma 3.2, there exists a constant C_0 such that

$$u(y) \leq C_0 (d(y))^{\frac{p(y_0)}{p(y_0)-1}}. \quad (3.5)$$

We deduce from (3.4)-(3.5) that

$$C'_0 r^{\frac{p(x)}{p(x)-1}} \leq u(y) \leq C_0 (d(y))^{\frac{p(y_0)}{p(y_0)-1}}, \quad (3.6)$$

which, by using the Lipschitz continuity of $p(x)$, leads to (see the proof of Theorem 3.1 in [3])

$$d(y) \geq \delta r,$$

where $\delta > 0$ is some constant smaller than one and depends only on $n, p_-, p_+, L, c_0, c_1, c_2, \lambda, \Lambda, R$, and $\|g\|_{L^\infty}$.

Let now $y^* \in [x, y]$ such that $|y - y^*| = \delta r/2$. Then we have ([3])

$$B_{\frac{\delta}{2}r}(y^*) \subset B_{\delta r}(y) \cap B_r(x).$$

Moreover, we have

$$B_{\delta r}(y) \cap B_r(x) \subset \{u > 0\},$$

since $B_{\delta r}(y) \subset B_{d(y)}(y) \subset \{u > 0\}$ and $d(y) \geq \delta r$.

Hence we have

$$B_{\frac{\delta}{2}r}(y^*) \subset B_{\delta r}(y) \cap B_r(x) \subset B_r(x) \setminus \partial\{u > 0\} \subset B_r(x) \setminus E.$$

□

Note that as a consequence of Theorem 2.1 and Lemma 3.2, we obtain the growth rate of the solution u of (P) near the free boundary.

Proposition 3.1. *Let $R_0 > 0$ be as in Lemma 3.2, $R \in (0, R_0)$ and $x_0 \in \Omega$ such that $u(x_0) = 0$ and $\overline{B_{4R}(x_0)} \subset \Omega$. Then there exists a positive constant \tilde{C}_0 depending only on $n, p_-, p_+, L, \Lambda, c_0, c_1, c_2$, and $\|g\|_{L^\infty}$ such that we have*

$$u(x) \leq \tilde{C}_0 |x - x_0|^{\frac{p(x_0)}{p(x_0)-1}} \quad \forall x \in B_R(x_0).$$

Proof. Let R and x_0 be as in the proposition. Consider the functions $\bar{a}(y, \xi)$ and $\bar{u}(y)$ defined in Lemma 3.2, for $M > 0$. By Lemma 3.2, there exists M_0 such that for all $M \geq M_0$ we have $\bar{u} \in \mathcal{F}_{\bar{A}}$. Applying Theorem 2.1 for $M = M_0$ and $R = R_0$, we obtain for a positive constant $C_0 > 0$ depending only on $n, p_-, p_+, L, c_0, c_1, c_2$

$$\bar{u}(y) \leq C_0 |y|^{\frac{\bar{p}(0)}{\bar{p}(0)-1}} \quad \forall y \in B_1.$$

Taking $y = \frac{|x-x_0|}{R_0}$ for $x \in B_R(x_0)$, we get

$$u(x) \leq \frac{C_0 M_0 R_0}{R_0^{\frac{p(x_0)}{p(x_0)-1}}} |x - x_0|^{\frac{p(x_0)}{p(x_0)-1}} = \frac{C_0 \|g\|_{L^\infty}}{R_0^{\frac{p(x_0)}{p(x_0)-1}}} |x - x_0|^{\frac{p(x_0)}{p(x_0)-1}} = \tilde{C}_0 |x - x_0|^{\frac{p(x_0)}{p(x_0)-1}}.$$

□

4 Growth rate of the gradient of the solution

The main result in this section is a growth rate near a free boundary of the gradient of the solution of the problem (P) , which generalizes results in [1] and [11] (see also [4] for the A-Laplace obstacle problem). This result will be used in section 6 to establish that the $(n-1)$ -Hausdorff measure of the free boundary is finite.

Theorem 4.1. *Let $R_0 > 0$ be as in Lemma 3.2, $R \in (0, R_0)$ and $x_0 \in \Omega$ such that $u(x_0) = 0$ and $\overline{B_{4R}(x_0)} \subset \Omega$. Then there exists a positive constant \tilde{C}_1 depending only on $n, p_-, p_+, L, \Lambda, c_0, c_1, c_2$, and $\|g\|_{L^\infty}$ such that we have*

$$|\nabla u(x)| \leq \tilde{C}_1 |x - x_0|^{\frac{1}{p(x_0)-1}} \quad \forall x \in B_R(x_0).$$

To prove Theorem 4.1, we shall need the following theorem which gives a growth rate of the gradient of the elements of the class \mathcal{F}_A .

Theorem 4.2. *There exists a positive constant $C_1 = C_1(n, p_-, p_+, L, c_0, c_1, c_2)$ such that for every $u \in \mathcal{F}_A$, we have*

$$|\nabla u(x)| \leq C_1 |x|^{\frac{1}{p(0)-1}} \quad \forall x \in B_1.$$

For $u \in \mathcal{F}_A$, we consider the set

$$\mathbb{P}(u) = \{j \in \mathbb{N} / 2^{\frac{1}{p(0)-1}} S(2^{-j-1}, |\nabla u|) \geq S(2^{-j}, |\nabla u|)\}.$$

To prove Theorem 4.2, we need the following lemma:

Lemma 4.1. *If $\mathbb{P}(u) \neq \emptyset$, then there exists a constant \tilde{c}_1 depending only on $n, p_-, p_+, L, c_0, c_1, c_2$ such that*

$$S(2^{-j-1}, |\nabla u|) \leq \tilde{c}_1 (2^{-j})^{\frac{1}{p_0-1}} \quad \forall u \in \mathcal{F}_A, \quad \forall j \in \mathbb{P}(u).$$

Proof. Arguing by contradiction, we assume that $\forall k \in \mathbb{N}$ there exists $u_k \in \mathcal{F}_A$ and $j_k \in \mathbb{P}(u_k)$ such that

$$S(2^{-j_k-1}, |\nabla u_k|) \geq k (2^{-j_k})^{\frac{1}{p_0-1}}. \quad (4.1)$$

Consider

$$v_k(x) = \frac{u_k(2^{-j_k}x)}{2^{-j_k} S(2^{-j_k-1}, |\nabla u_k|)}$$

defined in B_1 . We have by definition of v_k and $\mathbb{P}(u_k)$, and Theorem 2.1

$$0 \leq v_k \leq \frac{C_0 |2^{-j_k}x|^{\frac{p_0}{p_0-1}}}{k 2^{-j_k} (2^{-j_k})^{\frac{1}{p_0-1}}} \leq \frac{C_0}{k} \quad \text{in } B_1 \quad (4.2)$$

$$\sup_{x \in B_{1/2}} |\nabla v_k(x)| = 1, \quad v_k(0) = 0 \quad (4.3)$$

$$\sup_{x \in B_1} |\nabla v_k(x)| = \frac{\sup_{x \in B_1} |\nabla u_k(2^{-j_k}x)|}{S(2^{-j_k-1}, |\nabla u_k|)} = \frac{S(2^{-j_k}, |\nabla u_k|)}{S(2^{-j_k-1}, |\nabla u_k|)} \leq 2^{\frac{1}{p_0-1}}. \quad (4.4)$$

Now let $p_k(x) = p(2^{-j_k}x)$, $t_k = \frac{1}{S(2^{-j_k-1}, |\nabla u_k|)}$ and

$$a^k(x, \xi) = t_k^{p_k(x)-1} a(2^{-j_k}x, \frac{1}{t_k}\xi).$$

There exists $k_0 \in \mathbb{N}$ and a positive constant $C = C(n, p_-, p_+, c_0, c_1, c_2, L)$ independent of k such that

$$|(A_k v_k)(x)| := |\operatorname{div}(a^k(x, \nabla v_k(x)))| \leq \frac{C}{k^{p_0-1}} \leq 1 \quad \text{for all } k \geq k_0. \quad (4.5)$$

Indeed,

$$\begin{aligned} (A_k v_k)(x) &= \operatorname{div}(a^k(x, \nabla v_k(x))) \\ &= \operatorname{div}(t_k^{p_k(x)-1} a(2^{-j_k}x, \nabla u_k(2^{-j_k}x))) \\ &= \nabla(t_k^{p_k(x)-1}) a(2^{-j_k}x, \nabla u_k(2^{-j_k}x)) \\ &\quad + 2^{-j_k} t_k^{p_k(x)-1} \left(\operatorname{div}(a(\cdot, \nabla u_k(\cdot))) \right) (2^{-j_k}x) \end{aligned}$$

Using the fact that $u_k \in \mathcal{F}_A$ and $|\nabla p|_{L^\infty} \leq L$ (by (2.1)) and structural assumptions (second inequality in Remark 1.1), we get

$$|(A_k v_k)(x)| \leq 2^{-j_k} t_k^{p_k(x)-1} + c_4 L 2^{-j_k} |\ln(t_k)| t_k^{p_k(x)-1} |\nabla u_k(2^{-j_k}x)|^{p_k(x)-1}.$$

Moreover, we have

$$|\nabla u_k(2^{-j_k}x)| \leq S(2^{-j_k}, |\nabla u_k|) \leq 2^{\frac{1}{p_0-1}} S(2^{-j_k-1}, |\nabla u_k|) = \frac{2^{\frac{1}{p_0-1}}}{t_k}.$$

It follows that

$$|A_k v_k(x)| \leq 2^{-j_k} t_k^{p_k(x)-1} + c_4 L 2^{-j_k} |\ln(t_k)| (2^{\frac{1}{p_0-1}})^{p_k(x)-1}. \quad (4.6)$$

Note that we have by (2.2), $S(2^{-j_k-1}, |\nabla u_k|) \leq S(2^{-1}, |\nabla u_k|) \leq C$. Consequently, we obtain

$$t_k = \frac{1}{S(2^{-j_k-1}, u_k)} \geq C^{-1} \quad \text{for all } k \geq 1. \quad (4.7)$$

Estimate of $2^{-j_k} |\ln(t_k)| (2^{\frac{1}{p_0-1}})^{p_k(x)-1}$:

First we have

$$(2^{\frac{1}{p_0-1}})^{p_k(x)-1} \leq 2^{\frac{p_+-1}{p_--1}}. \quad (4.8)$$

Next, using the fact that \ln is a continuous function in $[C^{-1}, \infty)$ and satisfies

$$\lim_{t \rightarrow \infty} \frac{\ln t}{t^{p_0-1}} = 0,$$

we deduce that there exists a positive constant $\bar{c}_1 = \bar{c}_1(p_0, C)$ such that

$$|\ln(t)| \leq \bar{c}_1 t^{p_0-1} \quad \forall t \geq C^{-1}.$$

In particular, we obtain

$$|\ln(t_k)| \leq \bar{c}_1 t_k^{p_0-1} \quad \forall k \in \mathbb{N}. \quad (4.9)$$

Moreover, we have from (4.1)

$$2^{-j_k} t_k^{p_0-1} \leq \frac{1}{k^{p_0-1}}. \quad (4.10)$$

We deduce from (4.8)-(4.10) that

$$2^{-j_k} |\ln(t_k)| \left(2^{\frac{1}{p_0-1}}\right)^{p_k(x)-1} \leq \bar{c}_1 2^{\frac{p_+-1}{p_0-1}} 2^{-j_k} t_k^{p_0-1} \leq \frac{\bar{c}_1 2^{\frac{p_+-1}{p_0-1}}}{k^{p_0-1}}. \quad (4.11)$$

Estimate of $2^{-j_k} t_k^{p_k(x)-1}$: We first write

$$2^{-j_k} t_k^{p_k(x)-1} = 2^{-j_k} t_k^{p_0-1} t_k^{p_k(x)-p_0}.$$

From (2.1), (4.9) and (4.10), we deduce that

$$\begin{aligned} t_k^{p_k(x)-p_0} &= e^{(p(2^{-j_k} x) - p(0)) \ln(t_k)} \leq e^{L 2^{-j_k} |\ln(t_k)|} \leq e^{\bar{c}_1 L 2^{-j_k} t_k^{p_0-1}} \\ &\leq e^{\frac{\bar{c}_1 L}{k^{p_0-1}}} \leq e^{\bar{c}_1 L} = \bar{c}_2(n, p_-, p_+, L, c_0, c_1, c_2). \end{aligned} \quad (4.12)$$

We deduce then from (4.10) and (4.12) that we have

$$2^{-j_k} t_k^{p_k(x)-1} \leq \frac{\bar{c}_2}{k^{p_0-1}} \quad \forall k \in \mathbb{N}. \quad (4.13)$$

Hence, we conclude from (4.6), (4.11) and (4.13) that (4.5) holds.

Conclusion: Now, having in mind Lemma 2.2, taking into account the uniform bounds (4.2) of v_k , and the fact that p_k satisfies (1.1) and (2.1) with the same constants, we deduce [6] that there exist two positive constants $\delta = \delta(n, p_-, p_+, L, c_0, c_1, c_2)$ and $C = C(n, p_-, p_+, L, c_0, c_1, c_2)$ such that for all $k \geq k_0$, $v_k \in C^{1,\delta}(\bar{B}_{3/4})$ and $\|v_k\|_{C^{1,\delta}\bar{B}_{3/4}} \leq C$. It follows then from Ascoli-Arzelà's theorem that there exists a subsequence, still denoted by v_k , and a function $v \in C^{1,\delta'}(\bar{B}_{3/4})$ such that $v_k \rightarrow v$ in $C^{1,\delta'}(\bar{B}_{3/4})$, for any $\delta' \in (0, \delta)$. Moreover, it is clear from (4.2)-(4.4) that v satisfies

$$\begin{aligned}
A_0 v &:= \operatorname{div}(\tilde{a}(\nabla v)) = 0 \quad \text{in } B_{3/4}, & v &\geq 0 \quad \text{in } B_{3/4}, \\
\sup_{x \in B_{1/2}} |\nabla v(x)| &= 1, & v(0) &= 0.
\end{aligned}$$

By the strong maximum principle [9], we should have $v \equiv 0$ in $B_{3/4}$, which is in contradiction with $\sup_{x \in B_{1/2}} |\nabla v(x)| = 1$. \square

Proof of Theorem 4.2. Note that it is enough to prove the estimate for $|x| < 1/2$. Let $u \in \mathcal{F}_A$ and $x \in B_{1/2} \setminus \{0\}$. Then there exists $j \in \mathbb{N}$ such that $2^{-j-1} \leq |x| \leq 2^{-j}$ and we have

$$|\nabla u(x)| \leq \sup_{y \in \overline{B}_{2^{-j}}} |\nabla u(y)| = S(2^{-j}, |\nabla u|). \quad (4.14)$$

We know that $S(1/2, |\nabla u|) \leq C = C(n, p_-, p_+, L, c_0, c_1, c_2)$. We shall prove by induction that we have for $c'_1 = \max(\tilde{c}_1, C2^{\frac{1}{p_0-1}})$

$$S(2^{-j}, |\nabla u|) \leq c'_1 (2^{-j})^{\frac{1}{p_0-1}} \quad \forall j \in \mathbb{N}. \quad (4.15)$$

For $j = 1$, we have

$$S(2^{-1}, |\nabla u|) = S(1/2, |\nabla u|) \leq C = C2^{\frac{1}{p_0-1}} (2^{-1})^{\frac{1}{p_0-1}} \leq c'_1 (2^{-1})^{\frac{1}{p_0-1}}.$$

Let $j \geq 2$. Assume that $S(2^{-j}, |\nabla u|) \leq c'_1 (2^{-j})^{\frac{1}{p_0-1}}$. We distinguish two cases:

- If $j \in \mathbb{P}(u)$, we have by Lemma 4.1,

$$\begin{aligned}
S(2^{-(j+1)}, |\nabla u|) &= S(2^{-j-1}, |\nabla u|) \\
&\leq \tilde{c}_1 (2^{-j})^{\frac{1}{p_0-1}} \\
&= \tilde{c}_1 2^{\frac{1}{p_0-1}} (2^{-(j+1)})^{\frac{1}{p_0-1}} \\
&\leq c'_1 (2^{-(j+1)})^{\frac{1}{p_0-1}}.
\end{aligned}$$

- If $j \notin \mathbb{P}(u)$, we have $S(2^{-(j+1)}, |\nabla u|) = S(2^{-j-1}, |\nabla u|) < 2^{\frac{-1}{p_0-1}} S(2^{-j}, |\nabla u|)$. By the induction assumption, we get

$$S(2^{-(j+1)}, |\nabla u|) \leq 2^{\frac{-1}{p_0-1}} c'_1 (2^{-j})^{\frac{1}{p_0-1}} = c'_1 (2^{-(j+1)})^{\frac{1}{p_0-1}}.$$

We conclude from (4.14)-(4.15) that

$$|\nabla u(x)| \leq S(2^{-j}, |\nabla u|) \leq c'_1 (2^{-j})^{1/(p_0-1)} \leq c'_1 (2|x|)^{\frac{1}{p_0-1}} = C_1 |x|^{\frac{1}{p_0-1}}.$$

\square

Proof of Theorem 4.1. Let R and x_0 be as in the statement of Theorem 4.1. For $M > 0$, consider the functions $\bar{a}(y, \xi)$ and $\bar{u}(y)$ introduced in Lemma 3.2.

Note that $\bar{a}(y, \xi)$ satisfies to all structural conditions (not necessarily with the same constants as for a) with $\bar{p}(y) := p(x_0 + Ry)$ instead of p , and \bar{p} satisfies (1.1) (with the same constants) and (2.1) (with the constant LR).

By Lemma 3.2, for all $M \geq M_0$ we have $\bar{u} \in \mathcal{F}_{\bar{A}}$. Applying Theorem 4.2 for $M = M_0$ and $R = R_0$, we obtain for a positive constant C_1 , depending only on $n, p_-, p_+, c_0, c_1, c_2$ and LR_0 , that

$$|\nabla \bar{u}(y)| \leq C_1 |y|^{\frac{1}{\bar{p}_0 - 1}}, \quad \forall y \in B_1. \quad (4.16)$$

Taking $y = \frac{|x - x_0|}{R_0}$ for $x \in B_R(x_0)$, we get by (4.16)

$$|\nabla u(x)| \leq \frac{M_0 C_1}{R_0^{\frac{1}{\bar{p}_0 - 1}}} |x - x_0|^{\frac{1}{p(x_0) - 1}} = \frac{C_1 \|g\|_{L^\infty}}{R_0^{\frac{p(x_0)}{p(x_0) - 1}}} |x - x_0|^{\frac{1}{\bar{p}_0 - 1}} = \tilde{C}_1 |x - x_0|^{\frac{1}{p(x_0) - 1}} = .$$

□

5 The homogeneous case of p -Laplacian type

We consider a vector valued function $a : \Omega \rightarrow \mathbb{R}^n$, satisfying $a(0) = 0$, and the structural assumptions for all $\xi \in \mathbb{R}^n$ and $\eta \in \Omega$:

$$\sum_{i,j=1}^n \frac{\partial a_i}{\partial \eta_j}(\eta) \xi_j \xi_j \geq c_0 |\eta|^{p-2} |\xi|^2, \quad (5.1)$$

$$\sum_{i,j=1}^n \left| \frac{\partial a_i}{\partial \eta_j}(\eta) \right| \leq c_1 |\eta|^{p-2}, \quad (5.2)$$

for some positive constants c_0 and c_1 .

We consider for a positive constant γ , the following obstacle problem for a homogeneous operator:

$$\begin{cases} \operatorname{div}(a(\nabla u)) = \gamma & \text{in } \{u > 0\}, \\ u \geq 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (5.3)$$

Then Proposition (3.1) and Theorem 4.1 apply, and we have:

Theorem 5.1. *Under the assumptions (5.1)-(5.2), the solution u to the obstacle problem (5.3), and its gradient have the following growth rates near a free boundary point $x_0 \in (\partial\{u > 0\}) \cap \Omega$*

$$0 \leq u(x) \leq \bar{C}_0 |x - x_0|^{\frac{p}{p-1}}, \quad \forall x \in B_R(x_0), \quad (5.4)$$

$$|\nabla u(x)| \leq \bar{C}_1 |x - x_0|^{\frac{1}{p-1}}, \quad \forall x \in B_R(x_0), \quad (5.5)$$

for all $R \in (0, 1/\Lambda)$ such that $\bar{B}_{4R}(x_0) \subset \Omega$ and for positive constants \bar{C}_0, \bar{C}_1 depending on n, p, c_0, c_1, γ , and $\|g\|_{L^\infty}$.

The main result of this section is the finiteness of the $(n - 1)$ -Hausdorff measure of the free boundary corresponding to the solution of (5.3). We follow the idea of the corresponding facts in [4] (section 6).

Theorem 5.2. *Under the assumptions (5.1)-(5.2), the free boundary of the solution of the obstacle problem (5.3) is locally of finite Hausdorff measure.*

Due to the local character of the result of Theorem 5.2, we will restrict ourselves to the unit ball and consider the solutions of the following class of problems

$$\mathcal{F}_A : \begin{cases} u \in W^{1,p}(B_1) \cap C_{loc}^{1,\alpha}(B_1) \\ \operatorname{div}(a(\nabla u)) = \gamma \quad \text{in } \{u > 0\} \cap B_1, \\ 0 \leq u \leq M_0 \quad \text{in } B_1, \\ 0 \in \partial\{u > 0\}, \end{cases}$$

where M_0 is a positive number. We may also assume that there exists a positive constant $M_1 = M_1(n, p, c_0, c_1, \gamma, M_0)$ such that

$$\|u\|_{C^{1,\alpha}(\overline{B}_{3/4})} \leq M_1 \quad \forall u \in \mathcal{F}_A. \quad (5.6)$$

We shall start by establishing local L^2 -estimate for the second derivatives of u . In order to do that, we define for each $r > 0$ and each function $u \in \mathcal{F}_A$, the quantity

$$\begin{aligned} E(r, u) &= \frac{1}{|B_r|} \int_{B_r \cap \{\nabla u(x) \neq 0\}} [|\nabla u|^{\frac{p-2}{2}} |D^2 u|]^2 dx \\ &= \frac{1}{|B_1|} \int_{B_1 \cap \{\nabla u(rx) \neq 0\}} [|\nabla u(rx)|^{\frac{p-2}{2}} |D^2 u(rx)|]^2 dx. \end{aligned}$$

For each $\epsilon \in (0, 1)$, we introduce the approximation a_ϵ of the vector function a defined by $a_\epsilon(\eta) = a(\eta) + \frac{\epsilon c_0}{n}(\epsilon^2 + |\eta|^2)^{\frac{p-2}{2}} \eta$. Then it is easy to verify that we have for the positive constants $c'_0 = c_0(1 + \epsilon \min(1, p - 1))$ and $c'_1 = c_1(1 + \epsilon \max(1, p - 1))$ for all $\xi \in \mathbb{R}^n$ and $\eta \in \Omega$:

$$\sum_{i,j=1}^n \frac{\partial a_{\epsilon i}}{\partial \eta_j}(\eta) \xi_j \xi_j \geq c'_0(\epsilon^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2, \quad (5.7)$$

$$\sum_{i,j=1}^n \left| \frac{\partial a_{\epsilon i}}{\partial \eta_j}(\eta) \right| \leq c'_1(\epsilon^2 + |\eta|^2)^{\frac{p-2}{2}}. \quad (5.8)$$

Then we consider, for $\epsilon > 0$, the unique solution of the problem

$$(P_\epsilon) \begin{cases} u_\epsilon - u \in W_0^{1,p}(B_1) \\ \operatorname{div}(a_\epsilon(\nabla u_\epsilon)) = \gamma H_\epsilon(u_\epsilon) \quad \text{in } B_1, \end{cases}$$

where $H_\epsilon(v) := \min(1, \frac{v^+}{\epsilon})$.

First, we have:

Lemma 5.1. *There exists a positive constant $C_2 = C_2(n, p, c_0, c_1, \gamma, M_0)$ such that we have for every $u \in \mathcal{F}_A$*

$$E(1/2, u) \leq C_2 \|\nabla u\|_{L^\infty(B_{3/4})}^{2(p-1)}. \quad (5.9)$$

To prove Lemma 5.1, we need the following lemma.

Lemma 5.2. *Let G be a smooth monotone function with $G(0) = 0$, and ζ a nonnegative smooth function with compact support in B_1 . Then we have for $C'_1 = \frac{2c'_1}{c'_0}$*

$$\begin{aligned} & \int_{B_1} \zeta^2 G'(u_{\epsilon x_i}) (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|^2 dx \\ & \leq C'_1 \int_{B_1} \zeta |G(u_{\epsilon x_i})| (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}| |\nabla \zeta| dx. \end{aligned} \quad (5.10)$$

Proof. Let G and ζ be as in the lemma. Note that $u_\epsilon \in W_{loc}^{2,2}(B_1)$ [17]. Moreover, from (5.7), we have

$$\begin{aligned} Da_\epsilon(\nabla u_\epsilon) \cdot \nabla u_{\epsilon x_i} \cdot \nabla u_{\epsilon x_i} &= \sum_{k,j} \frac{\partial a_{\epsilon k}}{\partial \eta_j} (\nabla u_\epsilon)_{\epsilon x_i x_k} u_{\epsilon x_i x_j} \\ &\geq c'_0 (\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|^2. \end{aligned} \quad (5.11)$$

Similarly, we get by using Cauchy-Schwarz inequality and (5.8)

$$\begin{aligned} |Da_\epsilon(\nabla u_\epsilon) \cdot \nabla u_{\epsilon x_i} \cdot \nabla \zeta| &= \left| \sum_{k,j} \frac{\partial a_{\epsilon k}}{\partial \eta_j} (\nabla u_\epsilon)_{\epsilon x_i x_k} \zeta_{x_j} \right| \\ &\leq \sum_{k,j} \left| \frac{\partial a_{\epsilon k}}{\partial \eta_j} (\nabla u_\epsilon) \right| |u_{\epsilon x_i x_k}| |\zeta_{x_j}| \\ &\leq \sum_j \left(\sum_k \left| \frac{\partial a_{\epsilon k}}{\partial \eta_j} (\nabla u_\epsilon) \right|^2 \right)^{1/2} \left(\sum_k |u_{\epsilon x_i x_k}|^2 \right)^{1/2} |\nabla \zeta| \\ &\leq |\nabla \zeta| \left(\sum_{k,j} \left| \frac{\partial a_{\epsilon k}}{\partial \eta_j} (\nabla u_\epsilon) \right|^2 \right)^{1/2} |\nabla u_{\epsilon x_i}| \\ &\leq c'_1 |\nabla \zeta| (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|. \end{aligned} \quad (5.12)$$

Let $\varphi = G(u_{\epsilon x_i}) \zeta^2$. By taking φ_{x_i} as a test function for (P_ϵ) and integrating by parts and using the monotonicity of H_ϵ and G , we get

$$\int_{B_1} (a_\epsilon(\nabla u_\epsilon))_{x_i} \cdot \nabla (G(u_{\epsilon x_i}) \zeta^2) dx = - \int_{B_1} \gamma H'_\epsilon(u_\epsilon) u_{\epsilon x_i} G(u_{\epsilon x_i}) \zeta^2 \leq 0,$$

which leads to

$$\begin{aligned} I_1^i &= \int_{B_1} \zeta^2 G'(u_{\epsilon x_i}) Da_\epsilon(\nabla u_\epsilon) \cdot \nabla u_{\epsilon x_i} \cdot \nabla u_{\epsilon x_i} dx \\ &\leq - \int_{B_1} 2\zeta G(u_{\epsilon x_i}) Da_\epsilon(\nabla u_\epsilon) \cdot \nabla u_{\epsilon x_i} \cdot \nabla \zeta dx = I_2^i. \end{aligned} \quad (5.13)$$

From (5.11) we have

$$I_1^i \geq c'_0 \int_{B_1} \zeta^2 G'(u_{\epsilon x_i}) (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|^2 dx. \quad (5.14)$$

Similarly, we get from (5.12)

$$\begin{aligned} |I_2^i| &\leq \int_{B_1} 2\zeta G(u_{\epsilon x_i}) |Da_\epsilon(\nabla u_\epsilon) \cdot \nabla u_{\epsilon x_i} \cdot \nabla \zeta| dx \\ &\leq 2c'_1 \int_{B_1} \zeta |G(u_{\epsilon x_i})| (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}| |\nabla \zeta| dx. \end{aligned} \quad (5.15)$$

Combining (5.13), (5.14) and (5.15), we get (5.10). \square

Proof of Lemma 5.1. We consider $\zeta \in \mathcal{D}(B_{3/4})$ such that

$$\begin{cases} 0 \leq \zeta \leq 1 & \text{in } B_{3/4} \\ \zeta = 1 & \text{in } B_{1/2} \\ |\nabla \zeta| \leq 4 & \text{in } B_{3/4}. \end{cases}$$

We shall consider the two possible cases.

1st Case: $p < 2$.

Let $G(t) = (\epsilon + t^2)^{\frac{p-2}{2}} t$. Then we have:

$$G'(t) = (\epsilon + t^2)^{\frac{p-2}{2}} \left[1 + \frac{(p-2)t^2}{\epsilon + t^2} \right] \geq (p-1)(\epsilon + t^2)^{\frac{p-2}{2}}.$$

Setting $t_\epsilon = (\epsilon + |\nabla u_\epsilon|^2)^{1/2}$ and $s_\epsilon = (\epsilon + |u_{\epsilon x_i}|^2)^{1/2}$ and using Young's inequality

and the fact that $|\nabla\zeta| \leq 4$, we get from (5.10) and the monotonicity of t^{p-2}

$$\begin{aligned}
& \int_{B_1} \zeta^2 s_\epsilon^{p-2} t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|^2 dx \\
& \leq \frac{4C_1'}{p-1} \int_{B_1} \zeta s_\epsilon^{p-2} t_\epsilon^{p-2} |u_{\epsilon x_i}| |\nabla u_{\epsilon x_i}| dx \\
& \leq \frac{1}{2} \int_{B_1} \zeta^2 s_\epsilon^{p-2} t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|^2 dx \\
& \quad + \frac{8C_1'^2}{(p-1)^2} \int_{B_{3/4}} s_\epsilon^{p-2} t_\epsilon^{p-2} |u_{\epsilon x_i}|^2 dx \\
& \leq \frac{1}{2} \int_{B_1} \zeta^2 s_\epsilon^{p-2} t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|^2 dx \\
& \quad + \frac{8C_1'^2}{(p-1)^2} \int_{B_{3/4}} t_\epsilon^{2(p-1)} dx.
\end{aligned}$$

Using again the monotonicity of t^{p-2} , the fact that $\zeta = 1$ in $B_{1/2}$, and summing up from $i = 1$ to $i = n$, we obtain

$$\int_{B_{1/2}} \left[t_\epsilon^{p-2} |D^2 u_\epsilon| \right]^2 dx \leq \frac{16nC_1'^2}{(p-1)^2} \int_{B_{3/4}} t_\epsilon^{2(p-1)} dx. \quad (5.16)$$

It follows from (5.16) (because ∇u_ϵ is uniformly bounded in $B_{3/4}$) that $t_\epsilon^{p-2} D^2 u_\epsilon$ is bounded in $L^2(B_{1/2})$. So there exists a subsequence and a function $W \in L^2(B_{1/2})$ such that

$$t_\epsilon^{p-2} D^2 u_\epsilon \rightharpoonup W \text{ weakly in } L^2(B_{1/2}).$$

Passing to the \liminf in (5.16), we obtain by taking into account the fact that ∇u_ϵ converges uniformly, up to a subsequence, to ∇u in $\overline{B}_{3/4}$

$$\begin{aligned}
\int_{B_{1/2}} |W|^2 dx & \leq \liminf_{\epsilon \rightarrow 0} \int_{B_{1/2}} \left[t_\epsilon^{p-2} |D^2 u_\epsilon| \right]^2 dx \\
& \leq \frac{16nC_1'^2}{(p-1)^2} \int_{B_{3/4}} |\nabla u|^{2(p-1)} dx.
\end{aligned}$$

Since ∇u_ϵ converges uniformly to ∇u in $\overline{B}_{3/4}$, we deduce that $D^2 u \in L_{loc}^2(B_{1/2} \cap \{\nabla u \neq 0\})$. Consequently we obtain $W = |\nabla u|^{p-2} D^2 u$ a.e. in $B_{1/2} \cap \{\nabla u \neq 0\}$, and therefore we get

$$E(1/2, u) \leq \frac{16nC_1'^2}{(p-1)^2} |B_{3/4}| \|\nabla u\|_{L^\infty(B_{3/4})}^{2(p-1)}.$$

2nd Case: $p \geq 2$.

Let $G(t) = t$. Using Young's inequality and the fact that $|\nabla\zeta| \leq 4$, we get from (5.10)

$$\begin{aligned}
\int_{B_1} \zeta^2 t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|^2 dx &\leq 4C_1' \int_{B_1} \zeta t_\epsilon^{p-2} |u_{\epsilon x_i}| |\nabla u_{\epsilon x_i}| dx \\
&\leq \frac{1}{2} \int_{B_1} \zeta^2 t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|^2 dx \\
&\quad + 8C_1'^2 \int_{B_{3/4}} t_\epsilon^{p-2} |u_{\epsilon x_i}|^2 dx \\
&\leq \frac{1}{2} \int_{B_1} \zeta^2 t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|^2 dx \\
&\quad + 8C_1'^2 \int_{B_{3/4}} t_\epsilon^{2(p-1)} dx.
\end{aligned}$$

Using the the fact that $\zeta = 1$ in $B_{1/2}$ and summing up from $i = 1$ to $i = n$, we obtain

$$\int_{B_{1/2}} t_\epsilon^{p-2} |D^2 u_\epsilon|^2 dx \leq 16nC_1'^2 \int_{B_{3/4}} t_\epsilon^{2(p-1)} dx. \quad (5.17)$$

Using the monotonicity of t^{p-2} and (5.17), we get

$$\begin{aligned}
\int_{B_{1/2}} \left[t_\epsilon^{p-2} |D^2 u_\epsilon| \right]^2 dx &= \int_{B_{1/2}} t_\epsilon^{p-2} t_\epsilon^{p-2} |D^2 u_\epsilon|^2 dx \\
&\leq \|t_\epsilon\|_{L^\infty(B_{3/4})}^{p-2} \int_{B_{1/2}} t_\epsilon^{p-2} |D^2 u_\epsilon|^2 dx \\
&\leq 16nC_1'^2 \|t_\epsilon\|_{L^\infty(B_{3/4})}^{p-2} \int_{B_{3/4}} t_\epsilon^{2(p-1)} dx \\
&\leq 16nC_1'^2 |B_{3/4}| \cdot \|t_\epsilon\|_{L^\infty(B_{3/4})}^{p-2} \cdot \|t_\epsilon\|_{L^\infty(B_{3/4})}^{2(p-1)} \quad (5.18)
\end{aligned}$$

Passing to the limit, as in the first case, we obtain from (5.18)

$$\begin{aligned}
E(1/2, u) &\leq 16nC_1'^2 |B_{3/4}| \cdot \|\nabla u\|_{L^\infty(B_{3/4})}^{p-2} \cdot \|\nabla u\|_{L^\infty(B_{3/4})}^{2(p-1)} \\
&\leq 16nC_1'^2 |B_{3/4}| M_1^{p-2} \cdot \|\nabla u\|_{L^\infty(B_{3/4})}^{2(p-1)}.
\end{aligned}$$

Hence the lemma holds for $C_2 = \frac{16nC_1'^2 \max(1, M_1^{p-2})}{\min(1, (p-1)^2)}$. \square

Now we prove that $E(r, u)$ is uniformly bounded.

Lemma 5.3. *There exists a positive constant $C_3 = C_3(n, p, c_0, c_1, \gamma, M_0)$ such that we have*

$$E(r, u) \leq C_3, \quad \forall u \in \mathcal{F}_A, \quad \forall r \in (0, \frac{1}{2}).$$

Proof. Note that it is enough to prove the lemma for $r \in (0, \frac{1}{4})$. Indeed, for $r \in [\frac{1}{4}, \frac{1}{2})$, we have by Lemma 5.1

$$\begin{aligned}
E(r, u) &= \frac{1}{|B_r|} \int_{B_r \cap \{\nabla u \neq 0\}} [|\nabla u|^{p-2} |D^2 u|]^2 dx \\
&\leq \frac{1}{|B_{1/4}|} \int_{B_{1/2} \cap \{\nabla u \neq 0\}} [|\nabla u|^{p-2} |D^2 u|]^2 dx \\
&= \frac{|B_{1/2}|}{|B_{1/4}|} E(1/2, u) \\
&\leq C_2 \frac{|B_{1/2}|}{|B_{1/4}|} \|\nabla u\|_{L^\infty(B_{3/4})}^{2(p-1)} \\
&\leq C_3 = C_3(n, p, c_0, c_1, \gamma, M_0).
\end{aligned}$$

For $r \in (0, \frac{1}{4})$, we consider the function $v_r(x) = \frac{u(2rx)}{2r}$ defined in B_1 . We have by definition of v_r , and Theorem 5.1

$$0 \leq v_r \leq \bar{C}_0 \frac{(2r)^{\frac{p}{p-1}}}{2r} \leq \bar{C}_0 2^{\frac{1}{p-1}} r^{\frac{1}{p-1}} \quad \text{in } B_1, \quad (5.19)$$

$$|\nabla v_r(x)| = |\nabla u(2rx)| \leq \bar{C}_1 (2r)^{\frac{1}{p-1}} \leq \bar{C}_1 2^{\frac{1}{p-1}} r^{\frac{1}{p-1}} \quad \text{in } B_1, \quad (5.20)$$

$$D^2 v_r(x) = 2r(D^2 u)(2rx). \quad (5.21)$$

Using (5.20)-(5.21), we compute

$$\begin{aligned}
E(1/2, v_r) &= \frac{1}{|B_1|} \int_{B_1 \cap \{\nabla v_r(\frac{1}{2}x) \neq 0\}} [|\nabla v_r(\frac{1}{2}x)|^{p-2} |D^2 v_r(\frac{1}{2}x)|]^2 dx \\
&= \frac{1}{|B_1|} \int_{B_1 \cap \{\nabla u(rx) \neq 0\}} [|\nabla u(rx)|^{p-2} 2r |D^2 u(rx)|]^2 dx \\
&= 4r^2 E(r, u).
\end{aligned} \quad (5.22)$$

Moreover, we have

$$\begin{aligned}
\operatorname{div}(a(\nabla v_r(x))) &= \operatorname{div}(a(\nabla u(2rx))) \\
&= 2r \operatorname{div}(a(\nabla u(\cdot)))(2rx) \\
&= 2r\gamma \quad \text{in } \{v_r > 0\}.
\end{aligned}$$

Using (5.5) and Lemma 5.1, we obtain

$$\begin{aligned}
E(1/2, v_r) &\leq C_2 \|\nabla v_r\|_{L^\infty(B_{3/4})}^{2(p-1)} \\
&\leq C_2 (\bar{C}_1 2^{\frac{1}{p-1}} r^{\frac{1}{p-1}})^{2(p-1)} \\
&\leq 4C_2 \bar{C}_1^{2(p-1)} r^2.
\end{aligned} \quad (5.23)$$

Taking into account (5.22) and (5.23), we get

$$E(r, u) \leq C_2 C_1^{2(p-1)} = C_3(n, p, c_0, c_1, \gamma, M_0).$$

□

Lemma 5.4. *We have*

$$\gamma^2 \frac{(H_\epsilon(u_\epsilon))^2}{4c_1^2} \leq \left[(\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |D^2 u_\epsilon| \right]^2 \quad \text{a.e. in } B_1. \quad (5.24)$$

Proof. Since $u_\epsilon \in W_{loc}^{2,2}(B_1)$, we obtain from (P_ϵ) by using (5.8) and Schwarz inequality

$$\begin{aligned} (\gamma H_\epsilon(u_\epsilon))^2 &= (\operatorname{div}(a_\epsilon(\nabla u_\epsilon)))^2 \\ &= \left(\sum_i \frac{\partial}{\partial x_i} a_{\epsilon i}(\nabla u_\epsilon) \right)^2 \\ &= \left(\sum_{i,j} \frac{\partial a_{\epsilon i}}{\partial \eta_j}(\nabla u_\epsilon) u_{\epsilon x_i x_j} \right)^2 \\ &\leq \left(\sum_{i,j} \left| \frac{\partial a_{\epsilon i}}{\partial \eta_j}(\nabla u_\epsilon) \right|^2 \right) \left(\sum_{i,j} |u_{\epsilon x_i x_j}|^2 \right) \\ &\leq \left(\sum_{i,j} \left| \frac{\partial a_{\epsilon i}}{\partial \eta_j}(\nabla u_\epsilon) \right|^2 |D^2 u_\epsilon|^2 \right) \\ &\leq c_1'^2 (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |D^2 u_\epsilon|^2. \end{aligned}$$

Hence (5.24) holds. □

Lemma 5.5. *There exists a positive constant $C_4 = C_4(n, p, c_0, c_1, \gamma, M_0)$ such that for any $\delta > 0$ and $r < 1/4$ with $B_{2r}(x_0) \subset B_1$ and $x_0 \in \partial\{u > 0\} \cap B_{1/2}$, we have*

$$\mathcal{L}^n(O_\delta \cap B_r(x_0) \cap \{u > 0\}) \leq C_4 \delta r^{n-1}, \quad (5.25)$$

where $O_\delta = \{|\nabla u| < \delta^{\frac{1}{p-1}}\} \cap B_{1/2}$.

Proof. Let $r \in (0, 1/4)$, $\delta > 0$ and $x_0 \in \partial\{u > 0\} \cap B_{1/2}$ such that $B_{2r}(x_0) \subset B_1$. Let G be a smooth monotone function with $G(0) = 0$, and $\zeta \in \mathcal{D}(B_1)$ such that

$$\begin{cases} 0 \leq \zeta \leq 1 & \text{in } B_{2r}(x_0) \\ \zeta = 1 & \text{in } B_r(x_0) \\ |\nabla \zeta| \leq \frac{2}{r} & \text{in } B_{2r}(x_0). \end{cases}$$

We denote by u_ϵ the unique solution of the problem (P_ϵ) . First, we have from (5.10)

$$\begin{aligned}
& \int_{B_1} \zeta^2 G'(u_{\epsilon x_i}) (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|^2 dx \\
& \leq C'_1 \int_{B_1} \zeta G(u_{\epsilon x_i}) (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}| |\nabla \zeta| dx \\
& \leq \frac{2C'_1}{r} \int_{B_{2r}(x_0)} |G(u_{\epsilon x_i})| (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}| dx \\
& \leq \frac{2C'_1}{r} \int_{B_{2r}(x_0)} |G(u_{\epsilon x_i}) - G(u_{x_i})| (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}| dx \\
& \quad + \frac{2C'_1}{r} \int_{B_{2r}(x_0)} |G(u_{x_i})| (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}| dx. \tag{5.26}
\end{aligned}$$

We shall discuss two cases:

1st Case: $p < 2$.

For each $\epsilon > 0$ and $\eta = 2^{p-1}\delta$, we consider the function

$$G(t) = \begin{cases} (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-2}{2}} \eta^{\frac{1}{p-1}} & \text{if } t > \eta^{\frac{1}{p-1}} \\ (\epsilon + t^2)^{\frac{p-2}{2}} t & \text{if } |t| \leq \eta^{\frac{1}{p-1}} \\ -(\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-2}{2}} \eta^{\frac{1}{p-1}} & \text{if } t < -\eta^{\frac{1}{p-1}}. \end{cases}$$

G is Lipschitz continuous with $G'(t) = (\epsilon + t^2)^{\frac{p-2}{2}} \left[1 + \frac{(p-2)t^2}{\epsilon + t^2}\right] \chi_{\{|t| < \eta^{\frac{1}{p-1}}\}}$ and

$$(p-1)(\epsilon + t^2)^{\frac{p-2}{2}} \chi_{\{|t| < \eta^{\frac{1}{p-1}}\}} \leq G'(t) \leq (\epsilon + t^2)^{\frac{p-2}{2}} \chi_{\{|t| < \eta^{\frac{1}{p-1}}\}}.$$

Let $t_\epsilon = (\epsilon + |\nabla u_\epsilon|^2)^{1/2}$. Since $\zeta = 1$ in $B_r(x_0)$, and $\{|\nabla u_\epsilon| < \eta^{\frac{1}{p-1}}\} \subset \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}$, we obtain from (5.26) and the monotonicity of $(\epsilon + t^2)^{\frac{p-2}{2}}$

$$\begin{aligned}
& \int_{B_r \cap \{|\nabla u_\epsilon| < \eta^{\frac{1}{p-1}}\}} t_\epsilon^{2(p-2)} |\nabla u_{\epsilon x_i}|^2 dx \\
& \leq \int_{B_r \cap \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}} t_\epsilon^{2(p-2)} |\nabla u_{\epsilon x_i}|^2 dx \\
& \leq \frac{2C'_1}{(p-1)r} \int_{B_r \cap \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}} |G(u_{\epsilon x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| dx \\
& = \frac{2C'_1}{(p-1)r} \int_{B_r \cap \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}} |G(u_{\epsilon x_i}) - G(u_{x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| dx \\
& \quad + \frac{2C'_1}{(p-1)r} \int_{B_r \cap \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}} |G(u_{x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| dx \\
& = \frac{2C'_1}{(p-1)r} (J_1^i + J_2^i). \tag{5.27}
\end{aligned}$$

Since $|G(u_{\epsilon x_i}) - G(u_{x_i})| \rightarrow 0$ in $L^2(B_{2r}(x_0))$, as $\epsilon \rightarrow 0$ (see [4], proof of Lemma 6.4) and $t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|$ is bounded in $L^2(B_{2r}(x_0))$ independently of ϵ (see proof of Lemma 5.1), we obtain

$$J_1^i = \int_{B_{2r}(x_0)} |G(u_{\epsilon x_i}) - G(u_{x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| dx \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (5.28)$$

For J_2^i , we have since $|G(t)| \leq (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-1}{2}}$

$$\begin{aligned} J_2^i &= \int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} |G(u_{x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| \\ &\leq (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-1}{2}} \int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| \\ &\leq (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-1}{2}} |B_{2r}(x_0)|^{1/2} \cdot \left(\int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} [t_\epsilon^{p-2} |D^2 u_\epsilon|]^2 dx \right)^{1/2} \end{aligned} \quad (5.29)$$

We claim that $O_\delta \subset \{|\nabla u_\epsilon| < \eta^{\frac{1}{p-1}}\}$. Indeed since ∇u_ϵ converges uniformly to ∇u in $\overline{B}_{1/2}$, there exists $\epsilon_0 > 0$ such that

$$\forall \epsilon \in (0, \epsilon_0), \quad \|\nabla u_\epsilon - \nabla u\|_{L^\infty(\overline{B}_{1/2})} < \delta^{\frac{1}{p-1}}/2.$$

We deduce that for $x \in B_r(x_0) \cap O_\delta$,

$$\begin{aligned} \forall \epsilon \in (0, \epsilon_0), \quad |\nabla u_\epsilon(x)| &\leq |\nabla u_\epsilon(x) - \nabla u(x)| + |\nabla u(x)| \\ &< \delta^{\frac{1}{p-1}}/2 + \delta^{\frac{1}{p-1}} \\ &< 2\delta^{\frac{1}{p-1}} = \eta^{\frac{1}{p-1}}. \end{aligned}$$

We obtain from (5.27)

$$\int_{B_r(x_0) \cap O_\delta} [t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|]^2 dx \leq \frac{2C'_1}{(p-1)r} (J_1^i + J_2^i). \quad (5.30)$$

Summing up from $i = 1$ to n in (5.30) and using (5.24) and (5.29), we get

$$\begin{aligned} \frac{1}{4c_1^2} \int_{B_r(x_0) \cap O_\delta} (\gamma H_\epsilon(u_\epsilon))^2 dx &\leq \frac{2C'_1}{(p-1)r} \left\{ \sum_{i=1}^n J_1^i \right. \\ &\left. + n(\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-1}{2}} |B_{2r}(x_0)|^{1/2} \left(\int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} [t_\epsilon^{p-2} |D^2 u_\epsilon|]^2 dx \right)^{1/2} \right\}. \end{aligned} \quad (5.31)$$

Letting $\epsilon \rightarrow 0$ in (5.31) and using (5.28) and Lemma 5.3, we obtain

$$\begin{aligned} & \int_{B_r(x_0) \cap O_\delta \cap \{u > 0\}} dx \\ & \leq \frac{8nc_1^2 C_1'}{\gamma^2(p-1)r} \eta |B_{2r}(x_0)|^{1/2} \left(\int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} \left[|\nabla u|^{p-2} |D^2 u|^2 \right] dx \right)^{1/2} \\ & \leq \frac{\eta}{r} |B_{2r}(x_0)|^{1/2} (C(n, p, c_0, c_1, \gamma, M_0) |B_{2r}(x_0)|)^{1/2}, \end{aligned}$$

which can be written

$$\mathcal{L}^n(O_\delta \cap B_r(x_0) \cap \{u > 0\}) \leq C(n, p, c_0, c_1, \gamma, M_0) \delta r^{n-1}.$$

2nd Case: $p \geq 2$.

For each $\epsilon > 0$ and $\eta = 2^{p-1} \delta$, we consider the function

$$G(t) = \begin{cases} (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-2}{2}} \eta^{\frac{1}{p-1}} & \text{if } t > \eta^{\frac{1}{p-1}} \\ (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-2}{2}} t & \text{if } |t| \leq \eta^{\frac{1}{p-1}} \\ -(\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-2}{2}} \eta^{\frac{1}{p-1}} & \text{if } t < -\eta^{\frac{1}{p-1}}. \end{cases}$$

G is Lipschitz continuous and we have

$$0 \leq G'(t) = (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-2}{2}} \chi_{\{|t| < \eta^{\frac{1}{p-1}}\}}.$$

Using the fact that $\zeta = 1$ in B_r , and since $|G(t)| \leq (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-1}{2}}$, we obtain from (5.26)

$$\begin{aligned} & \int_{B_r \cap \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}} (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-2}{2}} t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|^2 dx \\ & \leq \frac{2C_1'}{r} \int_{B_{2r}(x_0)} |G(u_{\epsilon x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| dx. \end{aligned}$$

Since $\{|\nabla u_\epsilon| < \eta^{\frac{1}{p-1}}\} \subset \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}$, by using the monotonicity of t^{p-2} , we obtain

$$\begin{aligned} & \int_{B_r \cap \{|\nabla u_\epsilon| < \eta^{\frac{1}{p-1}}\}} \left[t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| \right]^2 dx \\ & \leq \int_{B_r \cap \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}} (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-2}{2}} t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|^2 dx \\ & \leq \frac{2C_1'}{r} \int_{B_r \cap \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}} |G(u_{\epsilon x_i}) - G(u_{x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| dx \\ & \quad + \frac{2C_1'}{r} \int_{B_r \cap \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}} |G(u_{x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| dx \\ & = \frac{2C_1'}{r} (J_1^i + J_2^i). \end{aligned} \tag{5.32}$$

Since

$$|G(u_{\epsilon x_i}) - G(u_{x_i})| \leq (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-2}{2}} |u_{\epsilon x_i} - u_{x_i}|, \quad \forall x \in B_{2r}(x_0),$$

we deduce that $|G(u_{\epsilon x_i}) - G(u_{x_i})| \rightarrow 0$ in $L^2(B_{2r}(x_0))$, as $\epsilon \rightarrow 0$.

As in the first case, $t_\epsilon^{p-2} \nabla u_{\epsilon x_i}$ is bounded in $L^2(B_{2r}(x_0))$ independently of ϵ . Therefore

$$J_1^i = \int_{B_{2r}(x_0)} |G(u_{\epsilon x_i}) - G(u_{x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| dx \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (5.33)$$

For J_2^i , we have since $|G(t)| \leq (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-1}{2}}$

$$\begin{aligned} J_2^i &= \int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} |G(u_{x_i})| t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| \\ &\leq (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-1}{2}} \int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}| \\ &\leq (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-1}{2}} |B_{2r}(x_0)|^{1/2} \left(\int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} [t_\epsilon^{p-2} |D^2 u|]^2 \right)^{1/2} \end{aligned} \quad (5.34)$$

As in the first case, we have $O_\delta \subset \bar{B}_{1/2} \cap \{|\nabla u_\epsilon| < \eta^{\frac{1}{p-1}}\}$. Since we have also $\{|\nabla u_\epsilon| < \eta^{\frac{1}{p-1}}\} \subset \{|u_{\epsilon x_i}| < \eta^{\frac{1}{p-1}}\}$, we obtain from (5.32)

$$\int_{B_r(x_0) \cap O_\delta \cap \{u > 0\}} [t_\epsilon^{p-2} |\nabla u_{\epsilon x_i}|]^2 dx \leq \frac{2C'_1}{r} (J_1^i + J_2^i). \quad (5.35)$$

Using (5.24), (5.34) and (5.35), we get by summing up from $i = 1$ to $i = n$

$$\begin{aligned} \frac{1}{4c_1^2} \int_{B_r(x_0) \cap O_\delta} (\gamma H_\epsilon(u_\epsilon))^2 dx &\leq \frac{2C'_1}{r} \left\{ \sum_{i=1}^n J_1^i \right. \\ &\quad \left. + \frac{2nC'_1}{r} (\epsilon + \eta^{\frac{2}{p-1}})^{\frac{p-1}{2}} |B_{2r}(x_0)|^{1/2} \cdot \left(\int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} [t_\epsilon^{p-2} |D^2 u_\epsilon|]^2 dx \right)^{1/2} \right\} dx. \end{aligned} \quad (5.36)$$

Letting $\epsilon \rightarrow 0$ in (5.36) and using (5.33) together with Lemma 5.3, we obtain

$$\begin{aligned} &\int_{B_r(x_0) \cap O_\delta \cap \{u > 0\}} dx \\ &\leq \frac{8nc_1^2 C'_1}{\gamma^2 r} \eta |B_{2r}(x_0)|^{1/2} \left(\int_{B_{2r}(x_0) \cap \{\nabla u \neq 0\}} [|\nabla u|^{p-2} |D^2 u|]^2 dx \right)^{1/2} \\ &\leq \frac{\eta}{r} |B_{2r}(x_0)|^{1/2} C_3(n, p, c_0, c_1, \gamma, M_0) |B_{2r}(x_0)|^{1/2}, \end{aligned}$$

which can be written

$$\mathcal{L}^n(O_\delta \cap B_r(x_0) \cap \{u > 0\}) \leq C_4 = C_4(n, p, c_0, c_1, \gamma, M_0) \delta r^{n-1}.$$

□

Proof of Theorem 5.2. Using Lemma 5.5, the proof follows step by step the one of Theorem 1.5 of [4]. \square

6 Hausdorff measure of the free boundary

In this section, we prove that the free boundary has finite $(n - 1)$ -Hausdorff measure. The proof is done by contradiction, using the fact that the result is true when $p(x)$ is constant, which has been established in [1] for the Laplacian and in [11] for the p -Laplacian, and more generally in [4], for the A -obstacle problem. The homogeneous case of p -Laplacian type has been investigated in the previous section.

We consider, for $M > 0$, the family $\mathcal{G}_{p,f,M}$ of problems defined on the unit ball B_1 defined by $u \in \mathcal{G}_{A,f,M}$ if it satisfies :

$$\begin{cases} u \in W^{1,p(\cdot)}(B_1), & 0 \leq u \leq M \quad \text{in } B_1, \\ Au = f\chi_{\{u>0\}} & \text{in } B_1. \end{cases}$$

We know [6] that $u \in C_{loc}^{1,\alpha}(B_1)$ for some $\alpha \in (0, 1)$. We shall denote by $\mathcal{G}_{A,f,M}^0$ the subclass of $\mathcal{G}_{p,f,M}$ of those functions satisfying $u(0) = 0$.

Theorem 6.1. *Let $u \in \mathcal{G}_{p,f,M}$ and assume that $f \in C^0(B_1)$. Then there exists a positive constant C depending only on $n, p_-, p_+, L, c_0, c_1, c_2, M, \lambda, \Lambda$ such that for each $x_0 \in \partial\{u > 0\} \cap B_{1/2}$ and $r \in (0, 1/4)$, we have*

$$\mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r(x_0)) \leq Cr^{n-1}.$$

Proof. The proof is given in several steps.

Step 1. We observe that it is enough to prove the following: there exists a constant $C = C(n, p_-, p_+, L, c_0, c_1, c_2, M, \lambda, \Lambda)$ such that

$$\forall u \in \mathcal{G}_{A,f,M}^0, \quad \forall r \in (0, 1/4), \quad \mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r) \leq Cr^{n-1}. \quad (6.1)$$

Indeed, assume that (6.1) holds and let $x_0 \in (\partial\{u > 0\} \cap B_{1/2})$ and $r \in (0, 1/4)$. We consider the functions of the variable y defined in B_1 by

$$\tilde{u}(y) = 2u(x_0 + \frac{1}{2}y), \quad \tilde{a}(y, \xi) = a(x_0 + \frac{1}{2}y, \xi), \quad \tilde{f}(y) = \frac{1}{2}f(x_0 + \frac{1}{2}y).$$

Then we have $\tilde{u}(0) = 0$, $0 \leq \tilde{u} \leq 2M$ in B_1 , and

$$\begin{aligned} (\tilde{A}\tilde{u})(y) &= \operatorname{div}(\tilde{a}(y, \nabla\tilde{u})) = \operatorname{div}(a(x_0 + \frac{1}{2}y, \nabla u(x_0 + \frac{1}{2}y))) \\ &= \frac{1}{2} \operatorname{div}(a(\cdot, \nabla u(\cdot))(x_0 + \frac{1}{2}y)) = \frac{1}{2} f(x_0 + \frac{1}{2}y) \chi_{\{u>0\}}(x_0 + \frac{1}{2}y) \\ &= \tilde{f}(y) \chi_{\{u>0\}}(y). \end{aligned}$$

So $\tilde{u} \in \mathcal{G}_{\tilde{A}, \tilde{f}, 2M}^0$, and then we deduce that

$$\mathcal{H}^{n-1}(\partial\{\tilde{u} > 0\} \cap B_r) \leq Cr^{n-1}, \quad \forall r \in (0, 1/4).$$

Now, let $r \in (0, 1/4)$ and observe that

$$\partial\{u > 0\} \cap B_r(x_0) = x_0 + \frac{1}{2}(\partial\{\tilde{u} > 0\} \cap B_{2r}).$$

Then we have

$$\begin{aligned} \mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r(x_0)) &= \mathcal{H}^{n-1}\left(\frac{1}{2}\partial\{\tilde{u} > 0\} \cap B_{2r}\right) \\ &= \left(\frac{1}{2}\right)^{n-1} \mathcal{H}^{n-1}(\partial\{\tilde{u} > 0\} \cap B_{2r}) \leq \left(\frac{1}{2}\right)^{n-1} C(2r)^{n-1} = Cr^{n-1}. \end{aligned}$$

Step 2. To prove (6.1), it suffices to establish that there exists a constant $C = C(n, p_-, p_+, L, c_0, c_1, c_2, M, \lambda, \Lambda)$ such that

$$\forall u \in \mathcal{G}_{A, f, M}^0 \quad \forall k \in \mathbb{N} \setminus \{1\}, \quad \mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_{2^{-k}}) \leq C(2^{-k})^{n-1}. \quad (6.2)$$

Indeed, if $r \in (0, 1/4)$, there exists $k \in \mathbb{N} \setminus \{1\}$ such that $2^{-k-1} \leq r \leq 2^{-k}$. Then $B_r \subset B_{2^{-k}}$ and

$$\begin{aligned} \mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r) &\leq \mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_{2^{-k}}) \\ &\leq C(2^{-k})^{n-1} \leq C(2r)^{n-1} = 2^{n-1}Cr^{n-1}. \end{aligned}$$

Step 3. Proof of (6.2).

We argue by contradiction and assume that there exist a sequence $(j_k)_k$ of natural numbers and a sequence $(u_k)_k$ in $\mathcal{G}_{A, f, M}^0$ such that

$$\forall k \in \mathbb{N}, \quad j_k < j_{k+1}, \quad \mathcal{H}^{n-1}(\partial\{u_k > 0\} \cap B_{2^{-j_k}}) > k(2^{-j_k})^{n-1}. \quad (6.3)$$

We introduce v_k as in the proof of Lemma 4.1, defined in B_1 by

$$v_k(x) = \frac{u_k(2^{-j_k}x)}{S(2^{-j_k-1}, u_k)}.$$

We have, by Proposition 3.1 and Lemma 3.1, for all $k \geq k_0$, for some k_0 large enough

$$\begin{aligned} 0 \leq u_k(2^{-j_k}x) &\leq S(2^{-j_k}, u_k) \leq C_1(2^{-j_k})^{\frac{p_0}{p_0-1}} \quad \text{in } B_1, \\ S(2^{-j_k-1}, u_k) &\geq C(0)(2^{-j_k-1})^{\frac{p_0}{p_0-1}}. \end{aligned}$$

Consequently we have for $k \geq k_0$

$$0 \leq v_k \leq \frac{S(2^{-j_k}, u_k)}{S(2^{-j_k-1}, u_k)} \leq \frac{2^{q_0}C_1}{C(0)} \quad \text{in } B_1, \quad (6.4)$$

$$\sup_{x \in B_{1/2}} v_k(x) = 1, \quad v_k(0) = 0. \quad (6.5)$$

Now let $p_k(x) = p(2^{-j_k}x)$ and $a^k(x, \xi) := s_k^{p_k(x)-1} a(2^{-j_k}x, \frac{1}{s_k}\xi)$, where $s_k = \frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)}$, $\xi \in \mathbb{R}^n$. Then we have

$$\begin{aligned} (A_k v_k)(x) &= \operatorname{div}(a^k(x, \nabla v_k(x))) \\ &= 2^{-j_k} s_k^{p_k(x)-1} f(2^{-j_k}x) \chi_{\{u_k > 0\}} \\ &\quad + 2^{-j_k} (\ln(s_k)) s_k^{p_k(x)-1} a(2^{-j_k}x, \nabla u(2^{-j_k}x)) \nabla p(2^{-j_k}x). \end{aligned} \quad (6.6)$$

We claim that there exists $k_1 \in \mathbb{N}$ greater than k_0 , and a positive constant C independent of k such that we have, for $k \geq k_1$,

$$|A_k v_k| \leq C(1 + j_k 2^{-j_k}). \quad (6.7)$$

Indeed, first note that we have by Proposition 3.1 and Lemma 3.1, for some $k_1 > k_0$, and two positive constants \tilde{C}_1, \tilde{C}_2 , with $\tilde{C}_1 > 1$

$$\tilde{C}_2^{-1} (2^{j_k})^{\frac{1}{p_0-1}} \leq s_k = \frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)} \leq \tilde{C}_1^{-1} (2^{j_k})^{\frac{1}{p_0-1}} \quad \forall k \geq k_1, \quad (6.8)$$

We deduce from the left hand side of (6.8) that

$$\begin{aligned} 2^{-j_k} s_k^{p_k(x)-1} &= 2^{-j_k} \left(\frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)} \right)^{p_k(x)-1} \\ &\leq 2^{-j_k} \tilde{C}_1^{p_+-1} (2^{j_k})^{\frac{p_k(x)-1}{p_0-1}} \\ &= \tilde{C}_1^{p_+-1} (2^{j_k})^{\frac{p_k(x)-p_0}{p_0-1}} \\ &= \tilde{C}_1^{p_+-1} e^{\frac{p_k(x)-p_0}{p_0-1} \ln(2^{j_k})} \\ &= \tilde{C}_1^{p_+-1} e^{\frac{L}{p_0-1} 2^{-j_k} |\ln(2^{j_k})|} \\ &\leq C(p_-, p_+, L). \end{aligned} \quad (6.9)$$

To estimate the second term in the righthand side of (6.6), we have by (2.1), the structural assumptions (second inequality in Remark 1.1), and Theorem 4.1

$$\begin{aligned} |a(2^{-j_k}x, \nabla u(2^{-j_k}x)) \nabla p(2^{-j_k}x)| &\leq c_4 |\nabla u_k(2^{-j_k}x)|^{p_k(x)-1} |\nabla p(2^{-j_k}x)| \\ &\leq c_4 L C_1 (2^{-j_k})^{\frac{p_k(x)-1}{p_0-1}}. \end{aligned} \quad (6.10)$$

Moreover, we obtain from (6.8)

$$|\ln(s_k)| \leq \tilde{C}_3 j_k \quad \forall k \geq k_1. \quad (6.11)$$

We deduce from (6.9)-(6.11) for $k \geq k_1$

$$\begin{aligned} 2^{-j_k} |\ln(s_k)| s_k^{p_k(x)-1} |a(2^{-j_k}x, \nabla u(2^{-j_k}x)) \nabla p(2^{-j_k}x)| \\ \leq c_4 L C_1 \tilde{C}_3 \tilde{C}_1^{p_+-1} 2^{-j_k} j_k (2^{-j_k})^{\frac{p_k(x)-1}{p_0-1}} (2^{j_k})^{\frac{p_k(x)-1}{p_0-1}} \\ = \tilde{C}_4 j_k 2^{-j_k}. \end{aligned} \quad (6.12)$$

Hence, we conclude from (6.6), (6.9), (6.12) and (3.1) that (6.7) holds.

We claim that there exist a subsequence still denoted by $(j_k)_k$ and a positive constant s_* such that

$$\lim_{k \rightarrow \infty} 2^{-j_k} s_k^{p_k(x)-1} = s_* \quad \forall x \in B_1. \quad (6.13)$$

Indeed, we first observe that

$$2^{-j_k} s_k^{p_k(x)-1} = 2^{-j_k} s_k^{p_0-1} s_k^{p_k(x)-p_0}. \quad (6.14)$$

From (6.8), we have

$$\tilde{C}_2^{1-p_0} \leq 2^{-j_k} s_k^{p_0-1} \leq \tilde{C}_1^{1-p_0} \quad \forall k \geq k_1,$$

which leads, up to a subsequence, still denoted by $(j_k)_k$ and a positive constant s_* to

$$\lim_{k \rightarrow \infty} 2^{-j_k} s_k^{p_0-1} = s_*. \quad (6.15)$$

Moreover, we have $s_k^{p_k(x)-p_0} = e^{(p(2^{-j_k}x)-p(0)) \ln(s_k)}$ and by (2.1) and (6.11)

$$|(p(2^{-j_k}x) - p(0)) \ln(s_k)| \leq L 2^{-j_k} |\ln(s_k)| \leq L \tilde{C}_3 j_k 2^{-j_k} \quad \forall x \in B_1. \quad (6.16)$$

We deduce from (6.14) – (6.16) that (6.13) holds.

Taking into account the uniform bound (6.4) of v_k , (6.7) and the fact that p_k satisfies (1.1) and (2.1) with the same constants, we deduce [6] that there exist two positive constants δ and C independent of k such that for all $k \geq k_1$, $v_k \in C^{1,\delta}(\overline{B}_{9/10})$ and $\|v_k\|_{C^{1,\delta}(\overline{B}_{9/10})} \leq C$. It follows then from Ascoli-Arzelà's theorem that there exists a subsequence, still denoted by v_k and a function $v \in C^{1,\delta'}(\overline{B}_{9/10})$ such that $v_k \rightarrow v$ in $C^{1,\delta'}(\overline{B}_{9/10})$, $\forall \delta' \in (0, \delta)$. Moreover, since $0 \leq \chi_{\{v_k > 0\}} \leq 1$ and f is continuous at 0, we have

$$\chi_{\{v_k > 0\}} \rightarrow 1 \text{ a.e. in } \{v > 0\} \cap B_{9/10} \quad (6.17)$$

$$f(2^{-j_k}x) \rightarrow f(0). \quad (6.18)$$

Using (6.4)-(6.7), (6.13), and (6.17)-(6.18), we see that v satisfies for $\gamma = s_* f(0) \in [s_* \lambda, s_* \Lambda]$

$$\begin{cases} A_0 v := \operatorname{div}(\hat{a}(\nabla v)) = \gamma & \text{in } B_{9/10} \cap \{v > 0\}, \quad v \geq 0 & \text{in } B_{9/10}, \\ \sup_{x \in B_{1/2}} v(x) = 1, \quad v(0) = 0, \end{cases}$$

with \hat{a} satisfying the conditions (5.1)-(5.2) with $p = p_0 = p(0)$.

We conclude (see Theorem 5.2) that $\partial\{v > 0\}$ has locally finite $(n-1)$ -Hausdorff measure, and for each $r \in (0, 9/40)$ we have

$$\mathcal{H}^{n-1}(\partial\{v > 0\} \cap B_r) \leq C r^{n-1}$$

where C is a positive constant depending only on n, p_0, L, c_0, c_1, c_2 and $s_* f(0)$. In particular we have

$$\mathcal{H}^{n-1}(\partial\{v > 0\} \cap B_{1/5}) \leq C2^{n-1}. \quad (6.19)$$

Now we claim that $\forall \epsilon \in (0, 1/10)$ there exists k_0 such that for all $k \geq k_0$,

$$\partial\{v_k > 0\} \cap B_{1/10} \subset N_\epsilon(\partial\{v > 0\} \cap B_{1/5}), \quad (6.20)$$

where $N_\epsilon(E) = \{x \in B_{1/5} : d(x, E) < \epsilon\}$ for $E \subset B_{1/5}$.

We argue by contradiction and assume that there exists $\epsilon_0 \in (0, 1/10)$ such that

$$\forall j \in \mathbb{N}, \quad \exists k_j \geq j : \quad \partial\{v_{k_j} > 0\} \cap B_{1/10} \not\subset N_{\epsilon_0}(\partial\{v > 0\} \cap B_{1/5}).$$

Therefore, there exists a subsequence which we denote also by v_k such that

$$\forall k \geq 1 \quad \partial\{v_k > 0\} \cap B_{1/10} \not\subset N_{\epsilon_0}(\partial\{v > 0\} \cap B_{1/5}).$$

It follows that

$$\forall k \geq 1, \quad \exists x_k \in \partial\{v_k > 0\} \cap B_{1/10} \quad \text{such that} \quad x_k \notin N_{\epsilon_0}(\partial\{v > 0\} \cap B_{1/5}),$$

that is, $d(x_k, \partial\{v > 0\} \cap B_{1/5}) \geq \epsilon_0$.

Up to a subsequence, we have $x_k \rightarrow x_* \in \overline{B}_{1/10}$, and we deduce by continuity that $d(x_*, \partial\{v > 0\} \cap B_{1/5}) \geq \epsilon_0$. This means that $B_{\epsilon_0}(x_*) \cap (\partial\{v > 0\} \cap B_{1/5}) = \emptyset$. In fact, $B_{\epsilon_0}(x_*) \cap \partial\{v > 0\} = \emptyset$, since $x_* \in \overline{B}_{1/10}$ and $\epsilon_0 < 1/10$ implies $B_{\epsilon_0}(x_*) \subset B_{1/5}$.

Now we can write $B_{\epsilon_0}(x_*) = (B_{\epsilon_0}(x_*) \cap \{v > 0\}) \cup (B_{\epsilon_0}(x_*) \cap \text{Int}(\{v = 0\}))$, where $\text{Int}(E)$ denotes the interior of the set E . Since $B_{\epsilon_0}(x_*)$ is connected, we deduce that either $B_{\epsilon_0}(x_*) = B_{\epsilon_0}(x_*) \cap \{v > 0\}$ or $B_{\epsilon_0}(x_*) = B_{\epsilon_0}(x_*) \cap \text{Int}(\{v = 0\})$. But since $v(x_*) = \lim_{k \rightarrow \infty} v_k(x_k) = 0$, the second alternative holds and then we have $v \equiv 0$ in $B_{\epsilon_0}(x_*)$.

Finally, we have by the uniform convergence of v_k to v in $B_{\epsilon_0}(x_*)$

$$\sup_{B_{\epsilon_0}(x_*)} v_k = \sup_{B_{\epsilon_0}(x_*)} |v_k - v| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (6.21)$$

On the other hand, for $\delta_0 \in (0, \epsilon_0)$, there exists k_0 such that $|x_k - x_*| < \delta_0/2$, $x_k \in B_{\delta_0}(x_*) \cap (\partial\{v_k > 0\})$, and $B_{\delta_0/2}(x_k) \subset B_{\delta_0}(x_*)$. Then, by applying Lemma 3.1, we have

$$\sup_{\overline{B}_{\delta_0}(x_*)} v_k \geq \sup_{\overline{B}_{\delta_0/2}(x_k)} v_k \geq C \left(\frac{\delta_0}{2} \right)^{\frac{p_k(x_k)}{p_k(x_k)-1}} + v_k(x_k) = C \left(\frac{\delta_0}{2} \right)^{\frac{p_k(x_k)}{p_k(x_k)-1}}.$$

Letting $k \rightarrow \infty$, we get a contradiction with (6.21). Hence (6.20) holds.

Conclusion: Note that since

$$\partial\{u_k > 0\} \cap B_{2^{-j_k}} = (10)(2^{-j_k})(\partial\{v_k > 0\} \cap B_{1/10}),$$

we have by (6.3)

$$\begin{aligned} ((10)(2^{-j_k}))^{n-1} \mathcal{H}^{n-1}(\partial\{v_k > 0\} \cap B_{1/10}) &= \mathcal{H}^{n-1}(\partial\{u_k > 0\} \cap B_{2^{-j_k}}) \\ &> k(2^{-j_k})^{n-1} \end{aligned}$$

which leads to

$$\mathcal{H}^{n-1}(\partial\{v_k > 0\} \cap B_{1/10}) > 10^{1-n}k. \quad (6.22)$$

We deduce from (6.20) and (6.22) that, $\forall \epsilon \in (0, 1/10)$, $\exists k_0$ such that $\forall k \geq k_0$

$$\mathcal{H}^{n-1}(N_\epsilon(\partial\{v > 0\} \cap B_{1/5})) \geq \mathcal{H}^{n-1}(\partial\{v_k > 0\} \cap B_{1/10}) > 10^{1-n}k.$$

In particular, we have $\forall j \geq 4$, $\exists k_j \geq k_0$, $k_{j+1} > k_j$ such that

$$\mathcal{H}^{n-1}(N_{2^{-j}}(\partial\{v > 0\} \cap B_{1/5})) \geq 10^{1-n}k_j.$$

Then

$$\begin{aligned} \infty &= \lim_{j \rightarrow +\infty} \mathcal{H}^{n-1}(N_{2^{-j}}(\partial\{v > 0\} \cap B_{1/5})) \\ &= \mathcal{H}^{n-1}\left(\bigcap_{j \geq 4} N_{2^{-j}}(\partial\{v > 0\} \cap B_{1/5})\right) = \mathcal{H}^{n-1}(\partial\{v > 0\} \cap B_{1/5}). \end{aligned}$$

This contradicts (6.19). \square

Theorem 6.2. *Let u be the solution of (P). For each $R > 0$ and $x_0 \in \Omega$ with $\overline{B_{2R}(x_0)} \subset \Omega$, there exists a positive constant C depending only on $n, p_-, p_+, c_0, c_1, c_2, LR, \frac{M}{R}, R\lambda$ and $R\Lambda$ such that we have for each $y_0 \in (\partial\{u > 0\}) \cap B_R(x_0)$ and $r \in (0, R/4)$*

$$\mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r(y_0)) \leq Cr^{n-1}.$$

Proof. Let $R > 0$ and $x_0 \in \Omega$ such that $\overline{B_{2R}(x_0)} \subset \Omega$. Let $y_0 \in (\partial\{u > 0\}) \cap B_R(x_0)$ and $r \in (0, R/4)$. We remark that $\overline{B_R(y_0)} \subset \overline{B_{2R}(x_0)} \subset \Omega$, and we introduce the functions defined in $\overline{B_1}$ by

$$\tilde{a}(z, \xi) = a(y_0 + Rz, \xi), \quad \tilde{u}(z) = \frac{u(y_0 + Rz)}{R}.$$

Then, we have

$$(\tilde{A}\tilde{u})(y) = \operatorname{div}(\tilde{a}(y, \nabla u(y))) = Rf\chi_{\{\tilde{u} > 0\}}, \quad 0 \leq \tilde{u} \leq \frac{M}{R}.$$

It follows that $\tilde{u} \in \mathcal{G}_{A, Rf, M/R}^0$. Applying Theorem 6.1, we obtain the existence of a positive constant C , depending only on $n, p_-, p_+, LR, \frac{M}{R}, c_0, c_1, c_2, R\lambda$ and $R\Lambda$, such that for $\rho = r/R \in (0, 1/4)$

$$\mathcal{H}^{n-1}(\partial\{\tilde{u} > 0\} \cap B_\rho) \leq C\rho^{n-1}. \quad (6.23)$$

Using the fact that

$$y_0 + R(\partial\{\tilde{u} > 0\} \cap B_\rho) = \partial\{u > 0\} \cap B_{\rho R}(y_0),$$

we deduce from (6.23) that

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r(y_0)) &= \mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_{\rho R}(y_0)) \\
&= \mathcal{H}^{n-1}(R(\partial\{\tilde{u} > 0\} \cap B_\rho)) \\
&= R^{n-1} \mathcal{H}^{n-1}(\partial\{\tilde{u} > 0\} \cap B_\rho) \\
&\leq CR^{n-1} \rho^{n-1} = Cr^{n-1}.
\end{aligned}$$

□

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