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A generalized notion of compliance

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Abstract

It is a known fact in structural optimization that for structures subject to prescribed non-zero displacements the work done by the loads is not a good measure of compliance, neither is the stored elastic energy. We briefly discuss a possible alternative measure of compliance in such cases.

1 Introduction

It is common in structural optimization to look for stiff structures (that is, structures which do not deform much) : beams that do not bend much, bridges that can stand heavy loads, and so on (see, for instance, [3]). This is usually achieved by minimizing the work done by the applied loads (this quantity is sometimes called "compliance", as opposed to "stiffness"). If all Dirichlet boundary conditions are homogeneous, then the work of the applied loads is equal to the elastic energy stored into the body. However, it has been observed that, if non-zero Dirichlet boundary conditions are considered, the work of the applied loads is no longer a good measure of the compliance of the structure, neither is the stored elastic energy (see [7] and [6, Section 3.1]. The goal of the present note is to propose a quantity called "generalized compliance" which describes correctly the behaviour of a structure under any boundary conditions.

The *design domain* (which may be under body loads) is an open set $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$; E will stand for the *elasticity tensor*, u designates the state variable, that is, the *displacement*, and $e(u)$ is the symmetric part of the jacobian matrix of u. The boundary $\partial\Omega$ of the design domain is partitioned into disjoint parts Γ_N (where some surface loads may be applied) and Γ_D (where some displacements are imposed); ν denotes the outward unit normal to $\partial\Omega$.

2 Zero Dirichlet boundary conditions

We start by considering the following elliptic problem describing the equilibrium of a linearly elastic structure :

$$\begin{cases} -\operatorname{div} Ee(u) = f, & \textit{in } \Omega, \\ u = 0, & \textit{in } \Gamma_D, \\ Ee(u)\nu = g, & \textit{in } \Gamma_N; \end{cases} \tag{1}$$

In the above, only zero Dirichlet boundary conditions are allowed (or, no Dirichlet boundary conditions appear at all, when $\Gamma_D = \emptyset$).

We are interested in optimizing the structure (defined by the domain Ω and by the elasticity tensor $E \in L^\infty(\Omega; \mathbb{R}^{d^4})$) in order to obtain a stiff structure. Note that high stiffness is equivalent to low compliance. It does not matter whether we are talking about shape optimization (when the shape of Ω varies), free material optimization (when the elastic tensor E is the optimization parameter), or any other kind of optimization.

The most common way to evaluate the compliance of an elastic structure is through the work done by the applied loads

$$\int_\Omega f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, ds \, . \tag{2}$$

It is not difficult to see that the smaller the work done by the applied loads, the stiffer the structure. This is so because f and g are fixed (they are data of the problem); thus, the work done by the loads f and g is small if the displacement u is small, that is, if the structure does not deform much.

It is a simple exercise to check that the quantity (2) is equal to the elastic energy stored in the body

$$\int_\Omega Ee(u) : e(u) \, dx \, . \tag{3}$$

It suffices to take u as test function in the variational formulation of (1) :

$$\begin{cases} \text{find } u \in V \text{ such that} \\ \displaystyle\int_\Omega Ee(u) : e(v) \, dx = \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds \, , \ \forall \, v \in V, \end{cases}$$

where V is the space of functions in $H^1(\Omega)$ whose trace on Γ_D is zero.

As a side note, we observe that, when computing the derivative of (2), with respect to some structural parameter, an adjoint state emerges $p = u$ (that is, the problem is self-adjoint), whereas the derivative of (3) involves the adjoint state $p = 2u$. It may seem strange that two quantities which are equal give rise to different adjoint states; recall, however, that (2) and (3) coincide for a displacement u satisfying problem (1) — if u is some arbitrary function in V, the quantities (2) and (3) have no reason to be equal. See [4, Chapter 5], or [5, Theorem 4.1], or [1, Section 4.3], for the definition and properties of the adjoint state.

3 Structures subject to prescribed displacement

The extreme opposite case is when no loads are applied at all, the deformation being caused by non-zero Dirichlet conditions :

$$\begin{cases} -\operatorname{div} Ee(u) = 0, & \textit{in } \Omega, \\ u = \bar{u}, & \textit{in } \Gamma_D, \\ Ee(u)\nu = 0, & \textit{in } \Gamma_N. \end{cases} \tag{4}$$

The description of a stiff structure as one which "does not deform much" is no longer valid; the displacement is prescribed on Γ_D. Such a structure should be called stiff if the effort it takes to impose the displacement $\bar{u}$ on Γ_D is large. That is, the work done by the force $g = Ee(u)$ should be large. Note that g cannot be called "applied load" now, since it is not a datum of problem (4).

So, a structure described by problem (4) should be called "stiff" if the work

$$\int_{\Gamma_D} Ee(u)\nu \cdot u \, ds \tag{5}$$

is *large*, which is very different from the situation described in Section 2, where for a structure to be stiff the work done by the applied loads had to be *small*.

Again, it is easy to check that the quantity (5) is equal to the elastic energy stored in the body (3). It suffices to integrate by parts in formula (3) and use the boundary conditions in (4).

So, in this case one should *maximize* the stored elastic energy in order to obtain a stiff structure, which is the opposite of the situation described in Section 2.

This phenomenon has already been observed in the literature; see [7], [2, Remark 11] and [8, Remark 9].

4 Loads and prescribed displacement

Consider now the general case :

$$\begin{cases} -\operatorname{div} Ee(u) = f, & \textit{in } \Omega, \\ u = \bar{u}, & \textit{in } \Gamma_D, \\ Ee(u)\nu = g, & \textit{in } \Gamma_N; \end{cases} \tag{6}$$

How can one define the notion of stiff structure (or its reverse, the compliance) ? Should one minimize or maximize the stored elastic energy (3) ?

We propose the difference

$$J(u) = 2\left(\int_\Omega f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, ds\right) - \int_\Omega Ee(u) : e(u) \, dx \tag{7}$$

as a measure of compliance. It is clear that, for structures governed by (1), the quantities (7), (2) and (3) are equal. On the other hand, for structures governed by (4), the quantities (7) and (3) are equal in modulus and have opposite signs.

All of the previous considerations are in accordance with the findings in [7].

It is interesting to note that the adjoint state emerging from the computation of the derivative of (7), with respect to some structural parameter, is equal to zero (see the remarks at the end of Section 2).

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