MORSE-SMALE INDEX THEOREMS FOR ELLIPTIC BOUNDARY DEFORMATION PROBLEMS

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Abstract. Morse-type index theorems for self-adjoint elliptic second order boundary value problems arise as the second variation of an energy functional corresponding to some variational problem. The celebrated Morse index theorem establishes a precise relation between the Morse index of a geodesic (as critical point of the geodesic action functional) and the number of conjugate points along the curve. Generalization of this theorem to linear elliptic boundary value problems appeared since seventies. (See, for instance, [Sma65] [Uhl73] and [Sim68] among others). The aim of this paper is to prove a Morse-Smale index theorem for a second order self-adjoint elliptic boundary value problem in divergence form on a star-shaped domain of the N-dimensional Euclidean space with Dirichlet and Neumann boundary conditions. This result will be achieved by generalizing a recent new idea introduced by authors in [DJ10], based on the idea of shrinking the boundary.

1. Introduction

The celebrated Morse index theorem establishes a direct relationship between the Morse index of a geodesic in a Riemannian manifold (seen as a critical point of the geodesic action functional on a suitable Hilbert manifold of curves) and the total number of conjugate points along this geodesic. A lot of interesting and useful generalization were obtained in the last two decades starting with the paper of Duistermaat [Dui76]. In finite dimension, symplectic geometry, intersection theory and variational methods have evolved into a rich and active area. Since the literature on this subject is quite broad, we briefly review only some milestones on this subject. One of the most useful and popular symplectic invariant arising in this setting is the so-called *Maslov index*. We shall refer to the paper of Arnol'd [Arn67] in which the intersection theory was developed. Some years later, a beautiful paper dealing with these topics was written in 1976 by Duistermaat and it could be considered as the finite dimensional proof of the Morse index theorem for geodesics with general boundary conditions. In [Dui76] the author established a precise relation between the total number of conjugate points and the Maslov index for a suitable Lagrangian path induced by the Jacobi deviation equation along a geodesic.

In 1965 Smale [Sma65] proved a sort of generalized multidimensional Morse index theorem, namely an equivalent version of the classical Morse index theorem for partial differential operators. Later on, Simons in [Sim68] and Uhlenbeck [Uhl73] proved an analogous result in the context of minimal surfaces and for general Hilbert spaces respectively. Finally, some years later, Swanson in a series of papers [Swa78a] and [Swa78b] developed a suitable Lagrangian intersection theory in infinite dimension. In [Swa78a] Swanson developed a so-called *Fredholm*

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intersection theory and in [Swa78b] he gave an interesting application to general strongly elliptic boundary deformation problems. In particular, Swanson's construction affords new invariants for the study of infinite dimensional problems and it can be regarded as a sort of Arnol'd-Maslov intersection theory in infinite dimension. To the authors' knowledge this was the first paper in which such intersection theory in infinite dimension appears.

A lot of works concerning the study of the geometry and the topology of the set of all Lagrangian subspaces of an infinite Hilbert space have been realized by many authors, and the literature is very rich in contribution on these topics. A crucial difference between the infinite and finite dimension is that the Lagrangian Grassmannian manifold $\Lambda(H)$ (i.e. the set of all Lagrangian subspaces of an Hilbert space H) in infinite dimension is contractible (and hence no non-trivial topological invariant can arise). Thus, it is evident that in order to have a non-trivial topology able to produce a suitable intersection theory, we need to restrict ourselves to some subset of $\Lambda(H)$, and, more precisely, to the subset of the *Fredholm pairs of Lagrangian subspaces* of H. (See [Fur04] and Section 3 for more details). From the point of view of the elliptic boundary value problems this leads to some restriction on the elliptic differential operator as well as on the boundary operator. Very recently, in [DJ10] the authors introduced a new idea in order to study the Morse index of a second order semilinear elliptic boundary value problem under general Lagrangian boundary conditions. In particular, they gave a sort of dynamical system interpretation of Sturm-Liouville theory and Morse index theorem in terms of oscillations. The new key idea of the paper [DJ10] was to introduce a "sweeping" of the underlying spatial domain by shrinking the boundary. The authors of [DJ10] constructed in a very explicit way the *abstract trace map in the Cauchy data spaces*, which is the key in order to prove the spectral flow formulas. In spite of this simplicity, this theory requires a technical assumption on the domain: it only works for star-shaped domains.

Our goal consists in proving that the Maslov index and the generalized Morse index of a self-adjoint elliptic boundary value problem in divergence form coincide. To calculate the Maslov index we adopt and generalize the idea developed in [DJ10] of shrinking the star-shaped domain for second order elliptic operators in divergence form with Dirichlet and Neumann boundary conditions, by reformulating the trace map in a proper symplectic context (see Sections 3 and 6 for more details). Moreover, taking into account the domain shrinking methods, we define the generalized Morse index as the spectral flow of a family of self-adjoint operators related to the variational formulation of a one-parameter family of boundary deformation problems (see Sections 4 and 5 for more detalis). The last Section of this paper is devoted to demonstrate the equivalence between the two indices. The paper is organized as follows:

Contents

2. Description of the problem

For $N \geq 1$, let $\Omega \subset \mathbb{R}^N$ be a bounded star-shaped domain with (at least) $\mathscr{C}^3$ boundary $\Gamma := \partial\Omega$, and let us assume that $\mathscr{F} \in \mathscr{C}^3(\bar{\Omega} \times \mathbb{R}; \mathbb{R})$. We consider the second order elliptic equation

$$\mathscr{L} w + D_w \mathscr{F}(\boldsymbol{x}, w) = 0, \qquad \boldsymbol{x} \in \Omega \tag{1}$$

with $\mathscr{L} = \mathscr{L}(\boldsymbol{x}, D) := -\sum_{i,j=1}^N D_j\big(a_{ij}(\boldsymbol{x}) D_i\big)$, where $a_{ij} = a_{ji}$ are $\mathscr{C}^3(\bar{\Omega})$-functions satisfying

$$a_{ij}(\boldsymbol{x})\, \xi_i\, \xi_j > 0 \qquad \forall \boldsymbol{x} \in \bar{\Omega},\ \forall \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N \setminus \{\mathbf{0}\}, \tag{2}$$

for each $i, j = 1, \dots, N$ and where D_w denotes the derivative with respect to the second variable w. We consider the following boundary condition

$$\mathscr{B}(\boldsymbol{x}, D) w = 0, \qquad \boldsymbol{x} \in \Gamma, \tag{3}$$

where $\mathscr{B}$ is the operator given by

$$\mathscr{B}(\boldsymbol{x}, D) := \sum_{j=1}^N b_j(\boldsymbol{x})\, D_j + b_0,$$

with $b_j \in \mathscr{C}^2(\Gamma)$. From now on, we will restrict ourselves to the study of the *Neumann type boundary conditions* and of the *Dirichlet boundary conditions*.
The *Neumann type boundary conditions* correspond to the case

$$\boldsymbol{b}(\boldsymbol{x}) = A(\boldsymbol{x}) \boldsymbol{n}, \qquad b_0 = 0, \tag{4}$$

where $\boldsymbol{b}(\boldsymbol{x}) = (b_1(\boldsymbol{x}), \dots, b_N(\boldsymbol{x}))^T$, $A(\boldsymbol{x}) = (a_{i,j}(\boldsymbol{x}))_{i,j=1}^N$, and where $\boldsymbol{n} = (n_1, \dots, n_N)^T$ denotes the outer normal. Note that the condition

$$\boldsymbol{b}(\boldsymbol{x}) = A(\boldsymbol{x}) \boldsymbol{n}$$

has been chosen in order to guarantee the self-adjointness of the boundary value problem (1)-(3). For more details, we refer to [Ama83].
Obviously, the *Dirichlet boundary conditions* correspond to the case

$$\boldsymbol{b} \equiv 0, \qquad b_0 = 1. \tag{5}$$

We assume that there exists a $\mathscr{C}^2$-solution $\bar{w}$ of the boundary value problem

$$\begin{cases} \mathscr{L} w + D_w \mathscr{F}(\boldsymbol{x}, w) = 0 & \text{in } \Omega \\ \\ \mathscr{B}(\boldsymbol{x}, D) w = 0 & \text{on } \Gamma. \end{cases} \tag{6}$$

By linearizing the equation in (6) at $\bar{w}$ we get

$$\begin{cases} \mathscr{L} w + D_w^2 \mathscr{F}(\boldsymbol{x}, \bar{w}) w = 0 & \text{in } \Omega \\ \\ \mathscr{B}(\boldsymbol{x}, D) w = 0 & \text{on } \Gamma. \end{cases} \tag{7}$$

Now, since Ω is star-shaped, without loss of generalities, we can assume (up to translation) that it is star-shaped with respect to the origin. We define $\Omega^* := \Omega \setminus \{\mathbf{0}\}$; thus for each point

$\boldsymbol{x} \in \Omega^*$ there exists an instant $t \in (0,1)$ and $\boldsymbol{y} \in \Gamma$ such that $\boldsymbol{x} = t\,\boldsymbol{y}$. For each $t \in (0,1]$, we consider the domain

$$\Omega_t := \{\boldsymbol{x} \in \Omega \mid \boldsymbol{x} = s\,\boldsymbol{y},\ s \in [0,t),\ \boldsymbol{y} \in \Gamma\}\,.$$

The above deformation of domain induces a one-parameter family of linear second order elliptic boundary deformation problems parameterized by the unit interval

$$\begin{cases} \mathscr{L}w + D^2_w \mathscr{F}(\boldsymbol{x}, \bar{w})w = 0 & \text{in } \ \Omega_t \\ \\ \mathscr{B}(\boldsymbol{x}, D)w = 0 & \text{on } \ \Gamma_t, \end{cases} \tag{8}$$

where $\Gamma_t := \partial\Omega_t$. Note that $\Omega_1 = \Omega$.

DEFINITION 2.1. *The instant $t \in (0,1]$ is a* conjugate instant *if there exists a non-trivial solution of the boundary value problem given in* (8)*. If t is a conjugate instant, then the dimension of the space of solutions of the problem* (8) *is called the* multiplicity *of the conjugate instant t.*

DEFINITION 2.2. *A solution of the boundary value problem* (6) *is said to be* non-degenerate *if $t = 1$ is not a conjugate instant for the linearized boundary value problem* (8)

In order to state our main theorem, let us denote by $i_{\mathrm{spec}}(\bar{w})$ the (generalized) Morse index and by $i_{\mathrm{Mas}}(\bar{w})$ the Maslov index. We point out that the (generalized) Morse index $i_{\mathrm{spec}}(\bar{w})$ is defined by means of the spectral flow of a family of self-adjoint operators related to the variational formulation of a one-parameter family of boundary deformation problems associated with problem (8). For more details on the definition of $i_{\mathrm{spec}}(\bar{w})$, we refer to Sections 4 and 5. On the other hand, we also remark that the formulation of the Maslov index $i_{\mathrm{Mas}}(\bar{w})$ we will provide is based on the approach developed in [DJ10]. In particular, we will define the Maslov index $i_{\mathrm{Mas}}(\bar{w})$ associated with the evolution of the Lagrangian subspaces of the weak solutions of the equation in (7) as a tool allowing us to count the non-transverse intersections of the trace map of the solutions on a shrinking boundary with the Lagrangian subspace given by the initial data. We refer to the Sections 3 and 6 for the accurate definition of $i_{\mathrm{Mas}}(\bar{w})$. Our main result can be stated as follows.

THEOREM 1. **(Morse index theorem - Neumann case)** *Let $\bar{w}$ be a (at least) $\mathscr{C}^2$ non-degenerate solution of the boundary value problem*

$$\begin{cases} \mathscr{L}w + D_w \mathscr{F}(\boldsymbol{x}, w) = 0 & \textit{in } \ \Omega \\ \\ \langle A(\boldsymbol{x})\boldsymbol{n}, Dw\rangle = 0 & \textit{on } \ \Gamma, \end{cases} \tag{9}$$

where D denotes the gradient, $Du = (D_1, \dots, D_N)^T$. Then,

$$i_{\mathrm{spec}}(\bar{w}) = i_{\mathrm{Mas}}(\bar{w}).$$

THEOREM 2. **(Morse index theorem - Dirichlet case)** *Let $\bar{w}$ be a (at least) $\mathscr{C}^2$ non-degenerate solution of the boundary value problem*

$$\begin{cases} \mathscr{L}w + D_w \mathscr{F}(\boldsymbol{x}, w) = 0 & \textit{in } \ \Omega \\ \\ w = 0 & \textit{on } \ \Gamma. \end{cases} \tag{10}$$

Then,

$$i_{\mathrm{spec}}(\bar{w}) = i_{\mathrm{Mas}}(\bar{w}).$$

Moreover, $-i_{\mathrm{spec}}(\bar{w})$ *and* $-i_{\mathrm{Mas}}(\bar{w})$ *are equal to the sum of the multiplicity of all the conjugate instants* $t \in (0,1)$.

Remark 2.3. *We observe that our main results do not necessarily need a gradient-type non-linearity. Theorem 1 and Theorem 2 hold true by replacing the nonlinear term* $D_w\mathscr{F}(\boldsymbol{x}, w)$ *with a more general term of the form* $g(\boldsymbol{x}, w)$*, provided that* $g \in \mathscr{C}^2(\bar{\Omega} \times \mathbb{R})$.

3. Fredholm Lagrangian Grassmannian and Maslov index

In this section we briefly describe the Maslov index in the infinite dimensional setting in order to fix our notations. Our basic references for this section are provided by the two papers [Fur04] and [DJ10]. Let $(H, \langle\cdot,\cdot\rangle)$ be a real separable Hilbert space. A symplectic real Hilbert space is the pair (H, ω), where ω is a non-degenerate, skew-symmetric bounded bilinear form on H. We recall that the non-degeneracy of ω means that the map $u \longmapsto \omega(u, \cdot)$ is an isomorphism between the Hilbert space H and its dual space H^*.

Let (H, ω) be a *sh-space*, namely a symplectic Hilbert space. Given a subspace W of H we will denote by $W^\sharp$ the orthogonal of W with respect to the symplectic form ω. In other words,

$$W^\sharp = \{v \in H | \quad \omega(u, v) = 0 \quad \forall\, u \in W\}.$$

Let W be a closed subspace of a sh-space H. We say that W is *Lagrangian* if $W = W^\sharp$. Let $\Lambda(H)$ be the set of all Lagrangian subspaces in H called the *Lagrangian Grassmannian manifold* of the symplectic space H.

Example 3.1. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be any Hilbert space and let us consider the product* $\mathscr{H}(H) = H \times H$ *with the induced scalar product and with the canonical symplectic form coming from the identification of the space* H *with its dual* H^**; namely:*

$$\omega((u_1, v_1), (u_2, v_2)) = \langle u_1, v_2\rangle - \langle u_1, v_1\rangle.$$

In $\mathscr{H}(H)$ *the subspaces* $H_0 = H \times \{0\}$ *and* $H_1 = \{0\} \times H$ *are always Lagrangian subspaces.*

Definition 3.2. *Let* V, W *be two closed subspaces of* H*. We say that the pair* (V, W) *is a* Fredholm pair*, if the following conditions hold:*

(1) $V + W$ *is closed.*
(2) $\dim(V \cap W)$ *is finite*
(3) $\operatorname{codim}(V + W)$ *is finite.*

Moreover, denote by $\mathrm{Fp}(H)$ *the set of all Fredholm pairs in* H.

It is easy to show that if V and W are Lagrangian subspaces of a sh-space H, then the pair $(V, W) \in \mathrm{Fp}(H)$ if and only if it satisfies conditions (1)-(2) of the previous definition, being condition (3) equivalent to condition (2) in this setting.
In order to define the Maslov index, we need to introduce the following definition.

Definition 3.3. *Let* $\mu \in \Lambda(H)$ *be a (closed) Lagrangian subspace. The Fredholm Lagrangian Grassmannian with respect to* μ *is defined as*

$$\mathscr{F}\Lambda_\mu(H) = \{\eta \in \Lambda(H) | \ (\mu, \eta) \text{ is a Fredholm pair}\}.$$

Moreover, we call the subset

$$\mathscr{M}_\mu(H) = \{\eta \in \mathscr{F}\Lambda_\mu(H) | \ \eta \cap \mu \neq \{0\}\},$$

the Maslov cycle *with respect to* μ.

It is possible to specify an intersection theory exactly as in the finite dimensional situation. More precisely, given $\mu \in \Lambda(H)$ and $l : [0,1] \to \mathscr{F}\Lambda_{\mu}(H)$ is a $\mathscr{C}^1$ path, we recall that the *Maslov index* $i_{\mathrm{Mas}}(l, \mu)$ is a semi-integer homotopy invariant of paths l which gives the algebraic counts of non-transverse intersections of the family $\{l(t)\}_{t\in[0,1]}$ with the Lagrangian subspace μ. For each C^1-curve $l\colon [0,1] \to \mathscr{F}\Lambda_{\mu}(H)$, we say that $t^* \in [0,1]$ is a *crossing instant* for the path l if $l(t^*) \in \mathscr{M}_\mu(H)$.

Let ν be a Lagrangian subspace transversal to $l(t^*)$, then $l(t)$ is transversal to ν for any t in a neighborhood $\mathscr{U}_{t^*}$ of t^*. Therefore, there exists a differentiable path of bounded operators $\phi_t : l(t^*) \to \nu$ so that $l(t) = graph \phi_t$ for every $t \in \mathscr{U}_{t^*}$. In other words, given $y \in l(t^*)$, then $\phi_t(y)$ is the unique vector such that

$$\phi_t(y) \in \nu, \qquad y + \phi_t(y) \in l(t).$$

At the crossing instant $t = t^*$, we define the bilinear form Q and the corresponding quadratic form Γ as follows:

$$Q(x, y) := \frac{d}{dt}\Big|_{t=t^*} \omega(x, \phi_t(y)), \qquad \forall x, y \in l(t^*) \cap \mu,$$

$$\Gamma(l, \mu; t^*) := Q(y, y), \qquad \forall y \in l(t^*) \cap \mu.$$

Note that both Q and Γ are independent on ν. The importance of the *crossing form* Γ is related to the fact that it gives the local contribution to the Maslov index and it will be useful in order to prove our result.
The crossing $t^* \in [0,1]$ will be called a *regular crossing* if Γ is non-degenerate. Any regular crossing instant t^* is isolated, and hence on $[0,1]$ the crossing instants are in a finite number. Assuming that l has only regular crossings, we can define the Maslov index as follows.

Definition 3.4. *The Maslov index of the Lagrangian path l relative to the Lagrangian subspace μ is the semi-integer defined by*

$$i_{\mathrm{Mas}}(l, \mu; [0,1]) := \frac{1}{2} \operatorname{sgn} \Gamma(l, \mu; 0) + \sum_{t \in (0,1)} \operatorname{sgn} \Gamma(l, \mu; t) + \frac{1}{2} \operatorname{sgn} \Gamma(l, \mu; 1), \tag{11}$$

where sgn *denotes the signature of a quadratic form and the summation runs over all crossings t.*

Moreover, the curve l will be termed *positive* (resp. *negative*) if each crossing is positive (resp. negative) as quadratic form. A positive (resp. negative) curve $l : [0,1] \to \mathscr{F}\Lambda_{\mu}(H)$ has only a finite number of crossings. The following result holds.

Proposition 3.5. *(well-known) Let $l : [0,1] \to \mathscr{F}\Lambda_{\mu}(H)$ be a negative $\mathscr{C}^1$ path. Then, we have*

$$i_{\mathrm{Mas}}(l, \mu; [0,1]) = -\frac{1}{2} \dim(l(0) \cap \mu) - \sum_{t \in (0,1)} \dim(l(t) \cap \mu) - \frac{1}{2} \dim(l(1) \cap \mu),$$

where the summation runs over all the crossing instants.

We conclude this section with some definitions and known lemmas useful for the proof of Lemma 6.2. Given the symplectic Hilbert space (H, ω), we denote by $GL_c(H)$ the Fredholm group of H consisting of linear invertible bounded operators of the form compact perturbation of the identity and we denote by $SP_c(H)$ the *Fredholm symplectic group* of the operators $\Psi \in GL_c(H)$ such that Ψ is symplectic, namely $\omega(x, y) = \omega(\Psi x, \Psi y)$. For a closed subspace

H_+ of H, $F_{res}(H_+)$ is the orbit of H_+ under the action of the Fredholm group $GL_c(H)$, called the *reduced Fredholm Grassmannian based on* H_+. Furthermore, if H_+ is a Lagrangian subspace, then the orbit of H_+ under the action of $Sp_c(H_+)$ is called the *reduced Fredholm Lagrangian Grassmannian based on* H_+, and will be denoted by $\mathscr{F}\Lambda_{res}(H_+)$.

Following [DN06] we give a useful criteria which we will need in the proof of Lemma 6.2. We consider a real separable Hilbert space H and we fix a polarization $H = H_+ \oplus H_-$, where H_+ and H_- are both infinite dimensional orthogonal closed subspaces. It is easy to observe that if $H_1 \in F_{res}(H_-)$ and $H_2 \in F_{res}(H_1)$, then $H_2 \in F_{res}(H_-)$.

LEMMA 3.6. *A closed subspace $H_1 \subset H$ lies in $F_{res}(H_-)$ if and only if there exists a linear operator $A = \mathrm{Id} + K$ where K is compact and $H_2 \in F_{res}(H_-)$, so that*

(1) $A(H) + H_2 = H$;
(2) $A^{-1}(H_2) = H_1$.

Proof. For the proof of this result we refer to [DN06, Lemma 6] and references therein. □

We close this section with the following lemmata proven in details in [DJ10, Section 3].

LEMMA 3.7. *For $\mu, \eta \in \Lambda(H)$, if $\dim(\mu \cap \eta)$ is finte dimensional, then (μ, η) is a Fredholm pair.*

Proof. See [DJ10, Lemma 3.2]. □

LEMMA 3.8. *Let $\xi \in \Lambda(H)$ and $\eta \in F_{res}(\xi)$. If ω vanishes on η (i.e. $\omega|_{\eta\times\eta} \equiv 0$), then $\eta \in \Lambda(H)$, namely η is a Lagrangian subspace.*

Proof. See [DJ10, Lemma 3.3]. □

LEMMA 3.9. *Let η, μ be two closed subspaces of H. If (η, μ) is a Fredholm pair and $\xi \in F_{res}(\eta)$, then (ξ, μ) is also a Fredholm pair.*

Proof. See [DJ10, Lemma 3.4]. □

4. Spectral flow for paths of Fredholm quadratic forms

We devote this section to introduce the concept of spectral flow of a family of Fredholm quadratic forms on a Hilbert bundle over the unit interval $I := [0, 1]$, which is the object that intrinsically arises in our framework, allowing us to define the generalized Mores index. To this aim, we strictly follows the description given in [MPP05] and in [FPR99]. Let S, T be two invertible self-adjoint operators on a Hilbert space H such that $S - T$ is compact. Denoting with $E_-(\cdot)$ and $E_+(\cdot)$ the negative and positive spectral subspaces of an operator, it follows that $E_-(S) \cap E_+(T)$ and $E_+(S) \cap E_-(T)$ have finite dimension and the *relative Morse index* of the pair (S, T) is defined by

$$i_{\mathrm{rel}}(S, T) = \dim\left(E_-(S) \cap E_+(T)\right) - \dim\left(E_+(S) \cap E_-(T)\right).$$

It is easy to see that when the negative spectral subspaces of both operators are finite dimensional $i_{\mathrm{rel}}(S, T)$ is given by the difference $i_{\mathrm{Mor}}(S) - i_{\mathrm{Mor}}(T)$ between the Morse indices.
A bounded self-adjoint operator A is Fredholm if $\ker A$ is finite dimensional. The topological group $GL(H)$ of all automorphisms of H acts naturally on the space of all self-adjoint Fredholm operators $\mathrm{Fred}_S(H)$ by *cogredience* sending $A \in \mathrm{Fred}_S(H)$ to S^*AS. This induces an action of paths in $GL(H)$ on paths in $\mathrm{Fred}_S(H)$. It is well-known fact that for any path $A\colon [a, b] \to \mathrm{Fred}_S(H)$ there exist a path $M\colon [a, b] \to GL(H)$, and a symmetry $\mathcal{J}(\mathcal{J}^2 = \mathrm{Id})$ such that $M^*(t)A(t)M(t) = \mathcal{J} + K(t)$ with $K(t)$ compact for each $t \in [a, b]$.

Let $A\colon [a, b] \to \mathrm{Fred}_S(H)$ be a path such that $A(a)$ and $A(b)$ are invertible operators.

Definition 4.1. *The spectral flow of the path A is the integer*

$$\mathrm{sf}\,(A,[a,b]) \equiv i_{\mathrm{rel}}(\mathcal{J}+K(a),\mathcal{J}+K(b)),$$

where $\mathcal{J}+K$ is any compact perturbation of a symmetry cogredient with A.

A *Fredholm quadratic form* is a function $q\colon H \to \mathbb{R}$ such that there exists a bounded symmetric bilinear form $b = b_q\colon H\times H \to \mathbb{R}$ with $q(u) = b(u,u)$ and with $\ker b$ of finite dimension. Here $\ker b = \{u : b(u,v) = 0 \text{ for all } v\}$. The set $Q_F(H)$ of all Fredholm quadratic forms is an open subset of the set of bounded quadratic forms $Q(H)$ that is stable under perturbations by weakly continuous quadratic forms.
A quadratic form is called *non-degenerate* if the map $u \to b_q(u,\cdot)$ is an isomorphism between H and H^*. By Riesz representation theorem, $Q_F(H)$ is isometrically isomorphic to $\mathrm{Fred}_S(H)$. Clearly this isometry sends the set of all non-degenerate quadratic forms onto $GL(H)$. Let us observe that a Fredholm quadratic form q is non-degenerate if and only if $\ker b_q = 0$.

A path of quadratic forms $q\colon [a,b] \to Q_F(H)$ with non-degenerate end points $q(a)$ and $q(b)$ will be called *admissible*.

Definition 4.2. *The spectral flow of an admissible path $q\colon [a,b]\to Q_F(H)$ is given by*

$$\mathrm{sf}\,(q,[a,b]) = \mathrm{sf}\,(A_q,[a,b])$$

where $A_{q(t)}$ is the unique self-adjoint Fredholm operator such that $\langle A_{q(t)}u,u\rangle = q(t)(u)$ for all $u\in H$.

We refer to [MPP05] and references therein for a detailed description of the properties of the spectral flow of an admissible path of quadratic forms.

We need a formula that leads to the calculation of the spectral flow for paths with only regular crossing instants. If a path $q\colon [a,b]\to Q_F(H)$ is differentiable at t, then the derivative $\dot q(t)$ (with respect to t) is also a quadratic form. We will say that a point t is a crossing instant if $\ker b_{q(t)} \neq \{0\}$, and we will say that the crossing instant t is regular if the *crossing form* $\Gamma(q,t)$ defined by

$$\Gamma(q,t) := \dot q(t)|_{\ker b_{q(t)}}$$

is non-degenerate. It is easy to see that regular crossing instants are isolated. From [MPP05] and references therein, we obtain a formula which allows us to calculate the spectral flow.

Proposition 4.3. *If all crossing instants t_i of the path are regular, then they are finite in number and*

$$\mathrm{sf}\,(q,[a,b]) = \sum_i \mathrm{sgn}\,\Gamma(q,t_i). \tag{12}$$

5. A variational framework

The aim of this section is to define the generalized Morse index $i_{\mathrm{spec}}(\bar w)$ of a non-degenerate solution $\bar w$ of (6). To this purpose, we introduce the variational set-up associated with the rescaled parameter-dependent boundary value problem (8).

Let us set $H^1(\Omega) := H^1(\Omega;\mathbb{R})$. If clear from the context, we shall drop out Ω.
Let w_t be the rescaled function defined below

$$w_t(\boldsymbol{x}) := w(t\,\boldsymbol{x}). \tag{13}$$

By an elementary calculation, it follows that w is a solution of (8) if and only w_t solves

$$\begin{cases} \mathscr{A}_t(\boldsymbol{x}, D) := \mathscr{L}_t w + t^2 D_w \mathscr{F}(t\boldsymbol{x}, \bar{w}) w = 0 & \text{in } \Omega \\ \mathscr{B}_t(\boldsymbol{x}, D) w = 0 & \text{on } \Gamma, \end{cases} \tag{14}$$

where $\mathscr{L}_t := -\sum_{i,j=1}^N D_j\big(a_{ij}(t\,\boldsymbol{x}) D_i\big)$, $\mathscr{B}_t(\boldsymbol{x}, D)w = \langle A(t\boldsymbol{x})\boldsymbol{n}, Dw\rangle$ in the case of Neumann boundary conditions, and $\mathscr{B}_t(\boldsymbol{x}, D)w = w$ in the case of Dirichlet boundary conditions. First of all, we observe that the boundary value problem (14) has a variational structure in the Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$ in the case of Neumann and Dirichlet boundary conditions respectively, and, hence, its weak solutions can be characterized as critical points of a functional. More precisely, the following results hold in the Neumann and Dirichlet setting, respectively.

Lemma 5.1. *For each $t \in [0, 1]$, let $H_t : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the symmetric bilinear form defined by*

$$H_t[u, v] = \int_\Omega \langle A(t\boldsymbol{x}) Du, Dv\rangle \, d\boldsymbol{x} + t^2 \int_\Omega D^2_w \mathscr{F}(t\boldsymbol{x}, \bar{w})\, u\, v\, d\boldsymbol{x}. \tag{15}$$

Then, the subspace of all the weaks solutions of (14) with $\mathscr{B}_t(\boldsymbol{x}, D)w = \langle A(t\boldsymbol{x})\boldsymbol{n}, Dw\rangle$ coincides with $\ker h_t$*, where h_t is the quadratic form associated with H_t.*

Proof. The general Green's formula ensures that

$$\int_\Omega \sum_{i,j=1}^N a_{ij}(t\boldsymbol{x})\, D_j u\, D_i v\, d\boldsymbol{x} + t^2 \int_\Omega D^2_w \mathscr{F}(t\boldsymbol{x}, \bar{w})\, u\, v\, d\boldsymbol{x} = \tag{16}$$

$$= \int_\Omega -v \sum_{i,j=1}^N D_i\left(a_{ij}(t\boldsymbol{x})\, D_j u\right) \, d\boldsymbol{x} + t^2 \int_\Omega D^2_w \mathscr{F}(t\boldsymbol{x}, \bar{w})\, u\, v\, d\boldsymbol{x} + \int_\Gamma v_\Gamma \sum_{i,j=1}^N n_i a_{ij}(t\boldsymbol{x})\, D_j u_\Gamma\, d\Gamma,$$

where the restriction of a given function to the boundary Γ is denoted with the subscript Γ, and n_j are the components of the normal direction $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$, $\boldsymbol{x} \in \Gamma$. The thesis easily follows by taking into account the symmetry of the matrix $A = (a_{i,j})_{i,j=1}^N$. □

Lemma 5.2. *For each $t \in [0, 1]$, let $H_t : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ be the symmetric bilinear form defined as before by*

$$H_t[u, v] = \int_\Omega \langle A(t\boldsymbol{x}) Du, Dv\rangle \, d\boldsymbol{x} + t^2 \int_\Omega D^2_w \mathscr{F}(t\boldsymbol{x}, \bar{w})\, u\, v\, d\boldsymbol{x}. \tag{17}$$

Then, the subspace of all the weaks solutions of (14) with $\mathscr{B}_t(\boldsymbol{x}, D)w = w$ coincides with $\ker h_t$*, where h_t is the quadratic form associated with H_t.*

Proof. The proof easily follows by combining the Green's formula (16) with the fact the bilinear form H_t is defined on the space $H^1_0(\Omega) \times H^1_0(\Omega)$. □

For each $t \in [0, 1]$, let g_t be the Riemannian metric given by

$$g_t : \Omega \to \operatorname{Sym}^+(\mathbb{R}^N \oplus \mathbb{R}^N) : \boldsymbol{x} \mapsto g_t(\boldsymbol{x}) := \langle A(t\boldsymbol{x})\cdot, \cdot\rangle \tag{18}$$

where Sym^+ denotes the cone of all (non-degenerate) positive definite symmetric matrices. Moreover, let us set $g := g_1$.

Lemma 5.3. *The form h_t is a Fredholm quadratic form. Moreover, h_t is non-degenerate if and only if t is not a conjugate instant.*

Proof. The Hessian quadratic form h_t is given by the sum of two terms, q_t and r_t, defined respectively by

$$q_t(u) := \langle Du, Du\rangle_{L^2(\Omega,g_t)} = \int_\Omega \langle A(t\boldsymbol{x})Du, Du\rangle\, d\boldsymbol{x}, \qquad r_t(u) := \langle u, a_t(\boldsymbol{x})u\rangle_{L^2(\Omega)},$$

where $a_t(\boldsymbol{x}) := t^2 D_w^2 \mathscr{F}(t\boldsymbol{x}, \bar{w})$.
By definition of q_t, it follows that ker $q_t = \mathbb{R}$. This implies that ker q_t has a finite dimension. Moreover, the form r_t is the restriction to H^1 in case of Neumann boundary conditions (or to H_0^1 in case of Dirichlet boundary conditions) of a quadratic form defined on L^2 and, since the inclusions $H^1 \hookrightarrow L^2$ and $H_0^1 \hookrightarrow L^2$ are compact, it follows that r_t is weakly continuous on L^2. Thus, we infer that h_t is Fredholm quadratic form.
The proof of the second assertion follows by taking into account that the transformation $w \mapsto w_t$ is a bijection between the set of solutions of (8) and the solutions of (14), Definition 2.1 and Lemma 5.1 (or Lemma 5.2, respectively). □

For any $t \in [0,1]$, we denote by $\mathscr{S}_t \subset H^1(\Omega_t)$ the space of all the weak solutions of the boundary value problem (8). The second assertion of Lemma 5.3 guarantees that $\mathscr{S}_t =$ ker h_t. It is easy to verify that the family of Hessians $\{h_t \mid t \in [0,1]\}$ defines a $\mathscr{C}^1$ function $\boldsymbol{h}$ of quadratic forms.
Observe that h_0 is non-degenerate as direct consequence of the fact that A is non-degenerate in the whole set Ω. We are now in position to define the generalized Morse index $i_{\text{spec}}(\bar{w})$.

Definition 5.4. *We define the* generalized Morse index $i_{\text{spec}}(\bar{w})$ *of a (at least)* $\mathscr{C}^2$ *non-degenerate solution of the boundary value problem* (6) *the integer*

$$i_{\text{spec}}(\bar{w}) := \operatorname{sf}(\boldsymbol{h}, [0,1])$$

6. A symplectic formulation

This section is devoted to introduce the definition of the Maslov index $i_{\text{Mas}}(\bar{w})$, and follows the same approach of [DJ10]. By taking into account the Green's formula and the symmetry of $A(\boldsymbol{x})$, we have

$$\int_\Omega \big(\mathscr{A}w\cdot z - \mathscr{A}w\cdot u\big)\, dx = \int_\Omega \big(\mathscr{L}w\cdot z - \mathscr{L}w\cdot u\big)\, dx = \int_\Gamma \big(z_\Gamma\cdot D_{\boldsymbol{n}_g} w_\Gamma - w_\Gamma\cdot D_{\boldsymbol{n}_g} z_\Gamma\big) d\Gamma, \tag{19}$$

where $\mathscr{A} := \mathscr{L} + D_w^2\mathscr{F}(\boldsymbol{x}, \bar{w})$ and $D_{\boldsymbol{n}_g} w_\Gamma = \langle A(\boldsymbol{x})\boldsymbol{n}, Dw_\Gamma\rangle$.

Remark 6.1. *This equation offers a connection between the set of solutions w of* (7) *and their boundary information* $(w_\Gamma, D_{\boldsymbol{n}_g} w_\Gamma)$.

According to the trace theorem, the trace operator $w \mapsto w|_\Gamma$ is surjective from $W^{1,2}(\Omega)$ to $E := H^{1/2}(\Gamma)$, and its kernel is $W_0^{1,2}(\Omega)$. Denoting by $E^* := H^{-1/2}(\Gamma)$ the dual space of E, we define the direct sum $\mathscr{H}_E := E \oplus E^*$. The space $\mathscr{H}_E$ has a natural symplectic structure $\omega : \mathscr{H}_E \times \mathscr{H}_E \to \mathbb{R}$ given by:

$$\omega\big((w_1, z_1), (w_2, z_2)\big) = \langle w_1, z_2\rangle - \langle w_2, z_1\rangle, \tag{20}$$

where $(w_i, z_i) \in \mathscr{H}_E$, for $i = 1, 2$, and $\langle\cdot,\cdot\rangle$ is the duality pairing between E and E^*. In order to define a suitable intersection theory, we define the following *trace map*

$$\mathscr{T} : W^{1,2}(\Omega) \longrightarrow \mathscr{H}_E : w \longmapsto \big(w|_\Gamma, D_{\boldsymbol{n}_g} w|_\Gamma\big).$$

Notice that $\omega(\mathscr{T}z, \mathscr{T}w) = \int_\Gamma \big(z_\Gamma \cdot D_{\boldsymbol{n}_g} w_\Gamma - w_\Gamma \cdot D_{\boldsymbol{n}_g} z_\Gamma\big) d\Gamma$. Denote by $\mathscr{I}$ the set of all the weak solutions $w \in W^{1,2}(\Omega)$ satisfying the differential equation given in the boundary value problem (7) without any boundary condition. From equation (19), we immediately infer that

$$\omega|_{\mathscr{T}(\mathscr{I})\times\mathscr{T}(\mathscr{I})} \equiv 0,$$

from which we can easily prove that the subspace $\mathscr{T}(\mathscr{I})$ is a Lagrangian subspace of $\mathscr{H}_E$. Moreover, let us define the Lagrangian subspace μ of $\mathscr{H}_E$, by setting

$$\mu = H^{1/2}(\Gamma) \times \{0\} \qquad \text{under Neumann boundary conditions,} \tag{21}$$

$$\mu = \{0\} \times H^{-1/2}(\Gamma) \qquad \text{under Dirichlet boundary conditions.} \tag{22}$$

As observed in [DJ10], the existence problem (7) with $\mathscr{B}(\boldsymbol{x}, D)w = \langle A(\boldsymbol{x})\boldsymbol{n}, Dw\rangle$ is so reduced to the existence of a non-trivial Lagrangian intersection between $\mathscr{T}(\mathscr{I})$ and $\mu = H^{1/2}(\Gamma) \times \{0\}$. Analogously, the existence problem (7) with $\mathscr{B}(\boldsymbol{x}, D)w = w$ is reduced to the existence of a non-trivial Lagrangian intersection between $\mathscr{T}(\mathscr{I})$ and $\mu = \{0\} \times H^{-1/2}(\Gamma)$. We now focus our attention on the one-parameter problem (14) in order to develope an intersection theory. In particular, for each $t \in (0, 1]$, we denote by $\tilde{\mathscr{I}}_t$ the set of all the $W^{1,2}$ weak solutions of the differential equation in (14), and, as before, we define the *trace map* as follows:

$$\tilde{\mathscr{T}}_t : W^{1,2} \longrightarrow \mathscr{H}_E : w \longmapsto \big(w_\Gamma, D_{\boldsymbol{n}_{g_t}} w_\Gamma\big), \tag{23}$$

where, according to (18), $D_{\boldsymbol{n}_{g_t}} w_\Gamma = \langle A(t\boldsymbol{x})\boldsymbol{n}, Dw_\Gamma\rangle$. The next lemma is crucial in order to associate the elliptic boundary deformation problem with a differentiable curve in the Fredholm Lagrangian Grassmannian with respect to μ. We remark that in order to prove this result the unique continuation property is crucial.

Lemma 6.2. *For each $t \in (0, 1]$, we have $\tilde{\mathscr{T}}_t(\tilde{\mathscr{I}}_t) \in \mathscr{F}\Lambda_\mu(\mathscr{H}_E)$.*

Proof. We argue as in the proof of [DJ10, Proposition 4.1], combined with the extension for more general operators given in [DN08]. Let us set $\tilde{\mathscr{H}} := W^{1,2}(\Omega)$ and for each $t \in [0, 1]$ let us consider the polarization of $\tilde{\mathscr{H}}$ given by $W_0^{1,2}(\Omega) \oplus \tilde{L}_t$ where

$$\tilde{L}_t = \{w \in \tilde{\mathscr{H}} |\ \mathscr{L}_t(w) = 0\},$$

i.e. the set of *generalized harmonic functions*. For each t, this decomposition induces two projections

$$\pi_1^t, \pi_2^t : \tilde{\mathscr{H}} \to \tilde{\mathscr{H}}, \qquad \pi_1^t(w) = w_1^t, \quad \pi_2^t(w) = w_2^t,$$

such that each $w \in \tilde{\mathscr{H}}$ can be written as $w = w_1^t + w_2^t$, where $w_1^t = \pi_1^t(w) \in W_0^{1,2}(\Omega)$ and $w_2^t = \pi_2^t(w) \in \tilde{L}_t$. The equation in (14) can be written as

$$w - F(t)w = \pi_2^t\, w$$

where

$$F(t) : \tilde{\mathscr{H}} \to W_0^{1,2}(\Omega), \qquad F(t)w := \mathscr{L}_t^{-1}[-t^2\, D_w\mathscr{F}(t\boldsymbol{x}, \bar{w})w]$$

is compact. We define the function $G(t) : \tilde{\mathscr{H}} \to \tilde{\mathscr{H}}$, by setting

$$G(t)\, w := w - F(t)\, w.$$

According to [DJ10] and [DN08], and taking into account the unique continuation property of second order elliptic operators with regular coefficients, it is possible to deduce that

(1) $\tilde{\mathscr{I}}_t = G^{-1}(t)(\tilde{L}_t)$;
(2) $G(t)(\tilde{\mathscr{H}}) + \pi_2^t(\tilde{\mathscr{H}}) = \tilde{\mathscr{H}}$;

(3) G is a compact perturbation of the identity.

By using Lemma 3.6, we can conclude that $\tilde{\mathscr{I}_t} = G^{-1}(t)(\tilde{L}_t) \in F_{res}(\tilde{L}_t)$. Moreover, for each t, the subspace $L_t = \tilde{\mathscr{T}_t}(\tilde{L}_t)$ is Lagrangian and, by the previous discussion, it follows that $\tilde{\mathscr{T}_t}(\tilde{\mathscr{I}_t}) \in F_{res}(L_t)$, since, according to [DN08], we can show that the map

$$\tilde{\mathscr{T}_t} : \tilde{\mathscr{I}_t} \to \tilde{\mathscr{T}_t}(\tilde{\mathscr{I}_t})$$

is an isomorphism. By Lemma 3.7 combined with the definition of the Lagrangian space μ given in (21)-(22), we infer that (L_t, μ) is a Fredholm pair; thus $(\tilde{\mathscr{T}_t}(\tilde{\mathscr{I}_t}), \mu)$ is also a Fredholm pair, as a direct consequence of Lemma 3.9. In order to complete the proof, we need to show that $\tilde{\mathscr{T}_t}(\tilde{\mathscr{I}_t})$ is a Lagrangian subspace in $\mathscr{H}_E$. The thesis readily follows from Lemma 3.8. □

Denoting by $l : (0, 1] \to \mathscr{F}\Lambda_\mu(\mathscr{H}_E)$ the map defined by

$$l(t) := \tilde{\mathscr{T}_t}(\tilde{\mathscr{I}_t}), \tag{24}$$

we can state the following proposition.

Proposition 6.3. *The map $l : (0, 1] \to \mathscr{F}\Lambda_\mu(\mathscr{H}_E)$ is of class $\mathscr{C}^1$.*

Proof. For the proof of this result we refer to [DJ10, Proposition 4.3]. □

For each $t \in (0, 1]$, we denote by E_t the Hilbert space $H^{\frac{1}{2}}(\Gamma_t)$ and by E_t^* its dual space $H^{-\frac{1}{2}}(\Gamma_t)$. Consider $\mathscr{H}_t := E_t \oplus E_t^*$ equipped with the symplectic structure ω defined in (20). For each $t \in (0, 1]$, let us consider the Hilbert spaces represented respectively by $(\mathscr{H}_t, \omega)$ and $(\mathscr{H}, \omega_t)$ where $\omega_t := t^{N-1}\omega$,[1] and define the symplectic map

$$\Psi_t : (\mathscr{H}_t, \omega) \longrightarrow (\mathscr{H}, \omega_t) : \quad z(\boldsymbol{x}) \longmapsto z_t(\boldsymbol{y}), \text{ where } \boldsymbol{x} = t\boldsymbol{y},$$

induced by the scaling. As observed in [DJ10, Lemma 4.4 and Remark 4.5] the following result holds.

Lemma 6.4. *For each $t \in (0, 1]$, the map Ψ_t is symplectic between the Hilbert spaces $(\mathscr{H}_t, \omega)$ and $(\mathscr{H}, \omega_t)$. Moreover, it maps Lagrangian subspaces of $(\mathscr{H}_t, \omega)$ in Lagrangian subspaces of $(\mathscr{H}, \omega)$ and viceversa.*

We observe that the map Ψ_t is not a symplectic map between $(\mathscr{H}_t, \omega)$ and $(\mathscr{H}, \omega)$. Denoted by $\mathscr{I}_t$ the space of the weak solutions of the differential equation in (8) in $W^{1,2}(\Omega_t)$ for each $t \in (0, 1]$, we define the following *trace map*

$$\mathscr{T}_t : \mathscr{I}_t \longrightarrow (\mathscr{H}_t, \omega) : w \longmapsto \big(w_{\Gamma_t}, t\ D_{\boldsymbol{n}_g} w_{\Gamma_t}\big).$$

Lemma 6.5. *For each $t \in (0, 1]$, we have*

$$\Psi_t(\mathscr{T}_t(\mathscr{I}_t)) = l(t).$$

Summing up the previous results, we can conclude the following.

Proposition 6.6. *The curve l is $\mathscr{C}^1$ in the Fredholm Lagrangian Grassmannian $\mathscr{F}\Lambda_\mu(\mathscr{H}_E)$.*

This curve describes the evolution of Lagrangian subspaces of weak solutions of the equation in (8) as t increases. Since the conjugate instants cannot accumulate at 0, we can find $\varepsilon > 0$ such that there are no conjugate instants in $[0, \varepsilon]$. Thus, the Maslov index of the path $l : [\varepsilon, 1] \to \mathscr{F}\Lambda_\mu(\mathscr{H}_E)$ is well-defined and independent on the choice of ε.

[1] We observe that in dimension greater than one we have to change the symplectic structure by the conformal factor t^{N-1}. In dimension 1, which naturally arises for example in the geodesic case, this conformal factor is 1. (See, for instance, [MPP05]).

Definition 6.7. *We define the Maslov index of $\bar w$ as:*

$$i_{\mathrm{Mas}}(\bar w) := i_{\mathrm{Mas}}(l, \mu; [\varepsilon, 1]).$$

7. Main results

In this section we provide the proofs of Theorem 1 and Theorem 2. The main idea consists in showing that the Maslov index and the spectral flow coincide, by using the formula (11) for the computation of the Maslov index for $\mathscr{C}^1$ paths in the Fredholm Lagrangian Grassmannian as well as the formula (12) for the computation of the spectral flow for $\mathscr{C}^1$ paths of Fredholm quadratic forms. More precisely, since in the case we are dealing with both the Maslov index and the spectral flow can be computed by calculating the contribution at each crossing point, the key idea to prove the equality between these two numbers is to construct explicitly an isomorphism transforming the crossing forms needed to compute the spectral flow into the ones needed to compute the Maslov index. Moreover, we will show that these crossing forms have the same signature in the corresponding crossing points. (Compare with [DJ10, Theorem 2.4, Theorem 2.5 and Corollary 2.6], by taking into account that the spectral index agrees with minus the Morse index and that we assume that $\bar w$ is a non-degenerate solution). For the sake of clarity, we shall prove these results independently.

Proof of Theorem 1. We first concentrate on the computation of the generalized Morse index under Neumann boundary conditions. According to the definition of generalized Morse index $i_{\mathrm{spec}}(\bar w)$ and to Lemma 5.1, we know that

$$i_{\mathrm{spec}}(\bar w) = \mathrm{sf}\,(\boldsymbol{h}, [0, 1]),$$

where

$$h_t[z] = \int_\Omega \langle A(t\boldsymbol{x})Dz, Dz\rangle \, d\boldsymbol{x} + t^2 \int_\Omega D^2_w \mathscr{F}(t\boldsymbol{x}, \bar w)\, z^2 \, d\boldsymbol{x} \qquad \forall\, t \in [0, 1],\ z \in H^1(\Omega).$$

More precisely, our aim consists in writing the explicit expression of the crossing form $\Gamma(\boldsymbol{h}, t)$ at each crossing point t. By using the perturbation theorem given in [RS95] we can always assume (up to perturbation) that the crossing points are regular. Thus, they are isolated and they are in a finite number in $[0, 1]$. If t is a regular crossing point, the crossing form $\Gamma(\boldsymbol{h}, t)$ defined as the restriction of the derivative of h_t with respect to t to the subspace $\ker h_t$. In particular, for each $z \in \ker h_t$ it is easy to show that the crossing form is given by

$$\Gamma(\boldsymbol{h}, t)(z) = \int_\Omega \sum_{i,j=1}^N \langle \nabla a_{i,j}(t\boldsymbol{x}), \boldsymbol{x}\rangle D_j z\, D_i z \, d\boldsymbol{x} \ + \int_\Omega \partial_t a_t(\boldsymbol{x}) z^2 \, d\boldsymbol{x}, \tag{25}$$

where $a_t(\boldsymbol{x}) := t^2 D^2_w F(t\,\boldsymbol{x}, \bar w)$.

In what follows, we fix a regular crossing point t, and we assume that $z \in \ker h_t$. In particular, z solves (14) under Neumann boundary conditions, namely

$$\begin{cases} \mathscr{L}_t w + t^2 D_w \mathscr{F}(t\boldsymbol{x}, \bar w) w = 0 & \text{in } \ \Omega \\[2ex] \langle A(t\boldsymbol{x})\boldsymbol{n}, Dw\rangle = 0 & \text{on } \ \Gamma. \end{cases}$$

For every $s \in (0, 1]$ we set

$$z^t_s(\boldsymbol{x}) := z(s\boldsymbol{x}/t).$$

Clearly $z_t^t = z$ and, for every s, we get

$$(\mathscr{L}_s z_s^t)(\boldsymbol{x}) + a_s(\boldsymbol{x}) z_s^t(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega \tag{26}$$

which follows from the fact that $\mathscr{L}_s\, z_s^t(\boldsymbol{x}) = (s^2/t^2)\mathscr{L}_t\, z(s\boldsymbol{x}/t)$ combined with the definition of a_s given by $a_s(\boldsymbol{x}) := s^2 D_w^2 F(t\,\boldsymbol{x}, \bar{w})$.

Now if we differenciate the equation (26) with respect to s and evaluate in $s = t$, we obtain

$$\frac{d}{ds}\Big|_{s=t} \mathscr{L}_s\, z(\boldsymbol{x}) \,+\, \frac{d}{ds}\Big|_{s=t} \mathscr{L}_t\, z_s^t(\boldsymbol{x}) \,+\, \left(\frac{d}{ds}\Big|_{s=t} a_s(\boldsymbol{x})\right) z(\boldsymbol{x}) \,+\, a_t(\boldsymbol{x})\, \frac{d}{ds}\Big|_{s=t} z_s^t(\boldsymbol{x}) \,=\, 0. \tag{27}$$

Denoting by $\dot{z}_s^t(\boldsymbol{x})$ the derivative of z_s^t with respect to s, we can rewrite (27) as follows

$$-\sum_{i,j=1}^N D_j\left(\langle \nabla a_{i,j}(t\boldsymbol{x}), \boldsymbol{x}\rangle\, D_i z(\boldsymbol{x})\right) \,+\, \mathscr{L}_t\, \dot{z}(\boldsymbol{x}) \,+\, \partial_t a_t(\boldsymbol{x}) z(\boldsymbol{x}) \,+\, a_t(\boldsymbol{x})\, \dot{z}(\boldsymbol{x}) \,=\, 0. \tag{28}$$

Multipling (27) by z and integrating over Ω, we infer

$$\begin{aligned}\Gamma(\boldsymbol{h}, t)(z) = -\int_\Omega \left(\mathscr{L}_t\, \dot{z}(\boldsymbol{x})\, z(\boldsymbol{x}) \,+\, a_t(\boldsymbol{x})\, \dot{z}(\boldsymbol{x})\, z(\boldsymbol{x})\right) d\boldsymbol{x} \,+ \\ + \int_\Omega \left(\sum_{i,j=1}^N \langle \nabla a_{i,j}(t\boldsymbol{x}), \boldsymbol{x}\rangle D_j z(\boldsymbol{x})\, D_i z(\boldsymbol{x}) \,+\, \sum_{i,j=1}^N D_j\left(\langle \nabla a_{i,j}(t\boldsymbol{x}), \boldsymbol{x}\rangle\, D_i z(\boldsymbol{x})\right)\; z(\boldsymbol{x})\right) d\boldsymbol{x}.\end{aligned}$$

By applying the Green's formula, we get

$$\begin{aligned}\Gamma(\boldsymbol{h}, t)(z) = -\int_\Omega \left(\mathscr{L}_t\, z(\boldsymbol{x}) \,+\, a_t(\boldsymbol{x})\, z(\boldsymbol{x})\right) \dot{z}(\boldsymbol{x})\, d\boldsymbol{x} \,-\, \int_\Gamma \sum_{i,j=1}^N a_{i,j}(t\boldsymbol{x})\, D_i z_\Gamma(\boldsymbol{x})\, n_j(\boldsymbol{x})\, \dot{z}_\Gamma(\boldsymbol{x})\, d\Gamma \\ + \int_\Gamma \sum_{i,j=1}^N \left(a_{i,j}(t\boldsymbol{x})\, D_i \dot{z}_\Gamma(\boldsymbol{x}) \,+\, \langle \nabla a_{i,j}(t\boldsymbol{x}), \boldsymbol{x}\rangle D_i z_\Gamma(\boldsymbol{x})\right)\; n_j(\boldsymbol{x})\, z_\Gamma(\boldsymbol{x})\, d\Gamma.\end{aligned}$$

Since $z \in \ker h_t$, according to Lemma 5.1, we conclude that

$$\Gamma(\boldsymbol{h}, t)(z) = \int_\Gamma \sum_{i,j=1}^N \left(a_{i,j}(t\boldsymbol{x})\, D_i \dot{z}_\Gamma(\boldsymbol{x}) \,+\, \langle \nabla a_{i,j}(t\boldsymbol{x}), \boldsymbol{x}\rangle D_i z_\Gamma(\boldsymbol{x})\right)\; n_j(\boldsymbol{x})\, z_\Gamma(\boldsymbol{x}) d\Gamma. \tag{29}$$

Note that (29) can be equivalently written

$$\Gamma(\boldsymbol{h}, t)(z) = \int_\Gamma z_\Gamma(\boldsymbol{x})\; \frac{d}{ds}\Big|_{s=t} D_{\boldsymbol{n}_{g_s}} z_s^t|_\Gamma(\boldsymbol{x})\; d\Gamma, \tag{30}$$

where, according to (18), $D_{\boldsymbol{n}_{g_t}} z_\Gamma = \langle A(t\boldsymbol{x})\boldsymbol{n}, Dz_\Gamma\rangle$. By combining Proposition 4.3 with Definition 5.4, we point out that

$$i_{\text{spec}}(\bar{w}) := \sum_{t\in(0,1)} \operatorname{sgn}\, \Gamma(\boldsymbol{h}, t), \tag{31}$$

where the summation runs over all regular crossing instants t.

We are now interested in calculating the Maslov index $i_{\text{Mas}}(\bar{w})$, in order to prove that it coincides with to the generalized Morse index $i_{\text{spec}}(\bar{w})$, given by the above formula. To this aim, we study the associated crossing form $\Gamma(l, \mu; t)$ at each regular crossing point t. We refer to (21) and (24) for the definitions of the Lagrangian spaces μ and l, respectively. Following the approach of [DJ10, Claim 5.8], we will show that the quadratic form $\Gamma(l, \mu; t)$ coincides exactly with $\Gamma(\boldsymbol{h}, t)$.

Let us fix a crossing point t of $\Gamma(l,\mu)$. By definition, $l(t)\cap\mu \neq \{0\}$. In order to write the explicit expression of $\Gamma(l,\mu;t)$, we consider (as descrived in Section 3) a Lagrangian subspace ν transversal to $l(t)$. Therefore, there exists a differentiable path of bounded operators $\phi_s : l(t)\to\nu$ so that $l(s) = graph\phi_s$ for every s in a suitable small neighborhood of t. In other words, given $y\in l(t)$, then $\phi_s(y)$ is the unique vector such that

$$\phi_s(y)\in\nu, \qquad y+\phi_s(y)\in l(s).$$

Let us recall that $\Gamma(l,\mu;t)$ is the quadratic form associated with

$$Q(x,y) := \frac{d}{ds}\Big|_{s=t}\omega(x,\phi_s(y)), \qquad \forall x,y\in l(t)\cap\mu.$$

Fix $y\in l(t)\cap\mu$, then $y=\tilde{\mathscr{T}}_t(z)=\big(z_\Gamma, D_{\boldsymbol{n}_{g_t}}z_\Gamma\big)$, where z solves the equation in (14). The fact that $z\in\mu := H^{1/2}(\Gamma)\times\{0\}$ implies also that $D_{\boldsymbol{n}_{g_t}}z_\Gamma = 0$. As before, we can immediately prove that z_s^t solves equation (26). If we define $X(s) := \tilde{\mathscr{T}}_s(z_s^t) = \big(z_s^t|_\Gamma, D_{\boldsymbol{n}_{g_s}}z_s^t|_\Gamma\big)$, we note that $X(s)\in l(s)$. Hence, $X(s)=c(s)+\phi_s(c(s))$, with $c(s)\in l(t)$. Observe that $X(t)=y=c(t)$. Taking into account that $\dot c(t)+\phi_t(\dot c(t))\in l(t)$, we get

$$\omega(X(t),\frac{dX}{ds}(t)) = \omega(y,\dot c(t)+\phi_t(\dot c(t))+\dot\phi_t(c(t))) = \omega(y,\dot\phi_t(y)).$$

Hence,

$$\Gamma(l,\mu;t) := \frac{d}{ds}\Big|_{s=t}\omega(y,\phi_s(y)) = \omega(X(t),\frac{dX}{ds}(t)) = \tag{32}$$

$$= \omega\left((z_\Gamma(\boldsymbol{x}), D_{\boldsymbol{n}_{g_t}}z_\Gamma(\boldsymbol{x})), \left(\dot z_\Gamma(\boldsymbol{x}), \frac{d}{ds}\Big|_{s=t}D_{\boldsymbol{n}_{g_s}}z_s^t|_\Gamma(\boldsymbol{x})\right)\right).$$

Combining the definition of the symplectic structure ω given in (20) with the fact that $y\in\mu$, we conclude that

$$\Gamma(l,\mu;t) := \int_\Gamma z_\Gamma(\boldsymbol{x})\ \frac{d}{ds}\Big|_{s=t}D_{\boldsymbol{n}_{g_s}}z_s^t|_\Gamma(\boldsymbol{x})\ d\Gamma,$$

which coincides exactly with the formula obtined for $\Gamma(\boldsymbol{h},t)(z)$ in (30). Recalling Definitions 3.4 and 6.7, we know that

$$i_{\mathrm{Mas}}(\bar w) := \sum_{t\in(0,1)} \mathrm{sgn}\ \Gamma(l,\mu;t), \tag{33}$$

where the summation runs over all crossings t. Observing that, by definition, the crossing forms $\Gamma(\boldsymbol{h},t)$ and $\Gamma(l,\mu;t)$ have the same crossing points t, from (31) and (33) we immediately achieve the thesis. □

Proof of Theorem 2. We first focus on the computation of the Maslov index $i_{\mathrm{Mas}}(\bar w)$. Let us fix a crossing point t of $\Gamma(l,\mu)$, where $\mu := \{0\}\times H^{-1/2}(\Gamma)$ as defined in (22). Consider $y\in l(t)\cap\mu$, then $y=\tilde{\mathscr{T}}_t(z)=\big(z_\Gamma, D_{\boldsymbol{n}_{g_t}}z_\Gamma\big)$, where z solves the equation in (14), and $z_\Gamma=0$. Arguing exactly as in the proof of previous Theorem 1, we notice that (32) holds true. Then, by combining the definition of the symplectic structure ω given in (20) with the fact that $y\in\mu$, we conclude that

$$\Gamma(l,\mu;t) := \int_\Gamma \dot z_\Gamma(\boldsymbol{x})\ \ D_{\boldsymbol{n}_{g_t}}z_\Gamma(\boldsymbol{x})\ d\Gamma. \tag{34}$$

Furthermore, since $y \in \mu$, it follows that

$$\Gamma(l,\mu;t) = -\int_\Gamma \frac{1}{t}\,\langle \boldsymbol{x}, \nabla z_\Gamma(\boldsymbol{x})\rangle\,\langle A(t\boldsymbol{x})\boldsymbol{n}, Dz_\Gamma\rangle\; d\Gamma = -\int_\Gamma \frac{1}{t}\,|\nabla z_\Gamma(\boldsymbol{x})|^2 \langle \boldsymbol{x}, \boldsymbol{n}^T\rangle\,\langle A(t\boldsymbol{x})\boldsymbol{n}, \boldsymbol{n}\rangle\; d\Gamma,$$

which is negative due to the uniform ellipticity assumption (2) combined with the fact that $\langle \boldsymbol{x}, \boldsymbol{n}^T\rangle > 0$ for every $x \in \Gamma$. We have so shown that l is a negative $\mathscr{C}^1$ path, and all the crossing points are regular. Then, Proposition 3.5 enables us to calculate explicitly the Maslov index, by means of the following formula

$$i_{\mathrm{Mas}}(\bar w) := \sum_{t\in(0,1)} \mathrm{sgn}\;\Gamma(l,\mu;t) = -\sum_{t\in(0,1)} \dim(l(t)\cap\mu), \tag{35}$$

where the summation runs over all crossings t.

Our aim consists now in showing that the generalized Morse index $i_{\mathrm{spec}}(\bar w)$ coincides with $i_{\mathrm{Mas}}(\bar w)$. According to Proposition 4.3, Definition 5.4 and Lemma 5.2, we know that

$$i_{\mathrm{spec}}(\bar w) = \mathrm{sf}\,(\boldsymbol{h}, J) = \sum_{t\in(0,1)} \mathrm{sgn}\;\Gamma(\boldsymbol{h}, t), \tag{36}$$

where the summation runs over all regular crossing instants t and

$$h_t[z] = \int_\Omega \langle A(t\boldsymbol{x})Dz, Dz\rangle\, d\boldsymbol{x} + t^2 \int_\Omega D_w^2\mathscr{F}(t\boldsymbol{x}, \bar w)\, z^2\, d\boldsymbol{x} \qquad \forall\, t\in[0,1],\; z\in H_0^1(\Omega).$$

The crossing form $\Gamma(\boldsymbol{h}, t)$ is defined as the restriction of the derivative of h_t with respect to t to the subspace $\ker h_t$. In particular, for each $z \in \ker h_t$ we can show, as before, that the crossing form is given again by (25).

Let us fix a regular crossing point t, and let us assume that $z \in \ker h_t$. In particular, z solves (14) under Dirichlet boundary conditions, namely

$$\begin{cases} \mathscr{L}_t w + t^2 D_w\mathscr{F}(t\boldsymbol{x}, \bar w) w = 0 & \text{in } \;\Omega \\[2ex] w = 0 & \text{on } \;\Gamma. \end{cases}$$

Exactly as in the proof of Theorem 1, we easily get

$$\begin{aligned}\Gamma(\boldsymbol{h}, t)(z) = &-\int_\Omega \left(\mathscr{L}_t\, z(\boldsymbol{x}) \;+\; a_t(\boldsymbol{x})\, z(\boldsymbol{x})\right) \dot z(\boldsymbol{x})\, d\boldsymbol{x} \;-\int_\Gamma \sum_{i,j=1}^N a_{i,j}(t\boldsymbol{x})\, D_i z_\Gamma(\boldsymbol{x})\, n_j(\boldsymbol{x})\, \dot z_\Gamma(\boldsymbol{x})\, d\Gamma \\ &+\int_\Gamma \sum_{i,j=1}^N \left(a_{i,j}(t\boldsymbol{x})\, D_i \dot z_\Gamma(\boldsymbol{x}) \;+\; \langle \nabla a_{i,j}(t\boldsymbol{x}), \boldsymbol{x}\rangle D_i z_\Gamma(\boldsymbol{x})\right)\; n_j(\boldsymbol{x})\, z_\Gamma(\boldsymbol{x}) d\Gamma.\end{aligned}$$

Since $z \in H_0^1(\Omega)$, by Lemma 5.2 we conclude that

$$\Gamma(\boldsymbol{h}, t)(z) = -\int_\Gamma \sum_{i,j=1}^N a_{i,j}(t\boldsymbol{x})\, D_i z_\Gamma(\boldsymbol{x})\, n_j(\boldsymbol{x})\, \dot z_\Gamma(\boldsymbol{x})\, d\Gamma - \int_\Gamma \dot z_\Gamma(\boldsymbol{x})\;\; D_{\boldsymbol{n}_{g_t}} z_\Gamma(\boldsymbol{x})\; d\Gamma, \tag{37}$$

where, as usual, $D_{\boldsymbol{n}_{g_t}} z_\Gamma = \langle A(t\boldsymbol{x})\boldsymbol{n}, Dz_\Gamma\rangle$.

By comparing (34) with (37), it is immediate to note that the two crossing forms $\Gamma(l,\mu;t)$ and $\Gamma(\boldsymbol{h}, t)(z)$ have the same regular crossing points t, and, moreover, they coincide at each crossing. Thus, from (35) and (36), the thesis follows. □

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