On the Cauchy problem describing an electron-phonon interaction

João-Paulo Dias(1), Mário Figueira(1) and Filipe Oliveira(2)

(1) CMAF/UL and FCUL, Av. Prof. Gama Pinto, 2
1649-003 Lisboa-Portugal
(2) Centro de Matemática e Aplicações, FCT-UNL
Monte da Caparica-Portugal

Abstract

In this paper we derive a model to describe a quartic anharmonic interatomic interaction with an external potential involving a pair electron-phonon. We study the corresponding Cauchy Problem in the semilinear and quasilinear cases.

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1 Introduction

Electron-phonon interactions play a crucial role in the determination of the physical properties of many mixed cristals ([16]).

In the present paper, we study the well-posedness of a nonlinear dispersive system arizing in the frame of electron-phonon interaction in a one-dimensional lattice. In [10], V. Konotop treats the temporal dynamics of such a system in the presence of resonant interactions between the electron and phonon subsystems. The hamiltonian H for such a one-dimensional chain of particles is given by

$$H = H_{el} + H_{ph} + H_{el-ph},$$

where, denoting by a dot the time derivative, the hamiltonians for each subsystem and their interaction read in bra-ket notation

$$H_{el} = -J \sum_n (|n >< n+1| + n >< n+1|),$$

email adresses:
dias@ptmat.fc.ul.pt(J.P.Dias); figueira@ptmat.fc.ul.pt(M.Figueira); fso@fct.unl.pt(F.Oliveira)

$$H_{ph} = \frac{M}{2}\sum_n \dot{\rho_n}^2 + \frac{U}{2}\sum_n (\rho_{n+1} - \rho_n)^2,$$

and

$$H_{el-ph} = \chi \sum_n |n >< n|(\rho_{n+1} - \rho_{n-1}).$$

Here, ρ_n denotes the distance to the equilibrium position of the n^{th} atom of mass M, J is the energetical constant determined by the overlapping of the electronic orbitals, U is a force constant and χ represents the strenght of the electron-phonon interaction.

In the continuum limit, the above hamiltonians become

$$H_{el} = -J\int |u_x|^2, \qquad H_{ph} = \frac{M}{2}\int \rho_t^2 + \frac{U}{2}\int \rho_x^2 \qquad \text{and}$$

$$H_{el-ph} = \chi \int |u|^2 \rho_x dx,$$

where u is the electronic wave-function.

Putting $q = \rho$, $p = M\rho_t$, we obtain the Hamilton evolution set of equations

$$\begin{cases} \dot{q}_{ph} = \dfrac{\partial (H_{ph} + H_{el-ph})}{\partial p_{ph}} \\ \dot{p}_{ph} = -\dfrac{\partial (H_{ph} + H_{el-ph})}{\partial q_{ph}} \\ i\hbar u_t = \dfrac{\partial \left(H_{el} + H_{el-ph}\right)}{\partial u}. \end{cases} \tag{1}$$

In the present paper, we will treat the Cauchy problem associated with this evolution system. We will replace the Hamiltonian of the electronic and phonon subsystems respectively by

$$H_{el} = -J\int |u_x|^2 + \frac{\alpha}{4}\int |u|^4, \quad \alpha \in \mathbb{R} \tag{2}$$

and

$$H_{ph} = \frac{M}{2}\int \rho_t^2 + \frac{U}{2}\int \rho_x^2 - \frac{\beta}{4}\int \rho^4, \quad \beta \in \mathbb{R}, \tag{3}$$

allowing the possibility of nonlinear cubic potentials for the evolution of u and ρ. Also, we will incorporate in H_{el-ph} a term to account for the anharmonic interatomic interactions (see [1]):

$$H_{el-ph} = \chi \int |u|^2 \rho_x + \lambda \int (\rho_x)^4, \quad \lambda \geq 0. \tag{4}$$

By replacing (2), (3) and (4) in (1), we obtain the system

$$\begin{cases} i\hbar u_t + J u_{xx} = 2\chi u \rho_x + \alpha |u|^2 u, & x \in \mathbb{R}, \quad t \geq 0, \\ M\rho_{tt} - [U\rho_x + \lambda \rho_x^3]_x = \chi(|u|^2)_x + \beta\rho^3. \end{cases} \tag{5}$$

Finally, after putting all physical constants equal to the unity, and scaling out the remaining coefficient of the term $u\rho_x$ by the transformation $\tilde{\rho} = 2\rho$ and $\tilde{u} = \sqrt{2}u$, we obtain the Initial Value Problem

$$\begin{cases} iu_t + u_{xx} = u\rho_x + \alpha|u|^2u, & x \in \mathbb{R}, \quad t \geq 0, \\ \\ \rho_{tt} - [\rho_x + \lambda\rho_x^3]_x = (|u|^2)_x + \beta\rho^3, \\ \\ u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \quad \rho_t(0,x) = \rho_1(x). \end{cases} \tag{6}$$

For $\alpha = \beta = \lambda = 0$, by putting $n = \rho_x$, we obtain the classical Zakharov system

$$\begin{cases} iu_t + u_{xx} = un \\ \\ n_{tt} - n_{xx} = (|u|^2)_{xx}. \end{cases} \tag{7}$$

The Initial Value Problem for (7) is studied in [8],[12]. Also, in the case where $\beta = \lambda = 0$, $\alpha \neq 0$, (6) falls in the scope of the Zakharov-Rubenchik equation studied in [6],[11] for the global well-posedness and stability of solitary waves and in [7] for the adiabatic limit to the Cubic Nonlinear Schrödinger Equation.

The rest of this paper is organized as follows:

In Section 2 we treat the local well-posedness of (6). The main difficulty of this system is the presence of the strongly nonlinear term with derivative-loss $\rho_x^2\rho_{xx}$. In order to overcome this problem, we translate (6) in terms of its Riemann invariants. Next, we perform a change of functions technique developed in [15], [6] and [5] which takes care of the derivative-loss and use a variant of a result derived by Kato ([9]) to prove the existence and uniqueness of strong local solutions to (6) for initial data

$$(u_o, \rho_o, \rho_{to}) \in H^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^2(\mathbb{R}).$$

In Section 3, we derive some conservation laws for (6) and prove the existence of solutions which blow-up in L^2 in finite time (provided that $\beta > 0$) by adapting a result due to Reed and Simon ([13]). Also, for $\beta \leq 0$ and $\lambda = 0$, we prove that the solutions obtained in the previous section are in fact global in time.

Finally, if $\lambda > 0$ and $\beta < 0$, we establish in Section 4 the global existence of weak solutions for (6) by applying a compensated-compactness method developed in [14] by Serre and Shearer (see also [2]). The adaptation of this method to a Schrödinger- Nonlinear Elasticity system was made in [4]. The technique of using this compensated-compactness result in order to prove the existence of global weak solutions was introduced in [3] in the frame of a Schrödinger - Conservation law system.

2 Local existence of strong solutions

In this section we adress the local-wellposdness of the I.V.P. (6).

Let $u_o \in H^3(\mathbb{R})$, $\rho_0 \in H^3(\mathbb{R}) and \rho_1 \in H^2(\mathbb{R})$.
By setting $v = \rho_x$, $w = \rho_t$ and $\sigma(v) = v + \lambda v^2$, the Cauchy problem (6) is equivalent to

$$\begin{cases} iu_t + u_{xx} = uv + \alpha|u|^2u \\ \\ \rho_t = w \\ \\ v_t - w_x = 0 \\ \\ w_t - (\sigma(v))_x = (|u|^2)_x + \beta\rho^3 \end{cases} \tag{8}$$

with initial data

$$u(.,0) = u_o \in H^3(\mathbb{R}), \quad \rho(.,0) = \rho_0 \in H^3(\mathbb{R}), \quad v(.,0) = v_o := \rho_{0x} \in H^2(\mathbb{R}) \tag{9}$$
$$\text{and } w(.,0) = w_0 := \rho_1 \in H^2(\mathbb{R}).$$

Let $\lambda \geq 0$. By introducing the Riemann invariants

$$l = w + \int_0^v \sqrt{1+3\lambda\xi^2}d\xi \quad \text{and} \quad r = w - \int_0^v \sqrt{1+3\lambda\xi^2}d\xi,$$

we derive

$$l - r = 2\int_0^v \sqrt{1+3\lambda\xi^2}d\xi = v\sqrt{1+3\lambda v^2} + \frac{1}{\sqrt{3\lambda}}\operatorname{arcsinh}(\sqrt{3\lambda}), \quad w = \frac{l+r}{2}.$$

Noticing that

$$f(v) = v\sqrt{1+3\lambda v^2} + \frac{1}{\sqrt{3\lambda}}\operatorname{arcsinh}(\sqrt{3\lambda})$$

is one-one and smooth, we put $v = f^{-1}(l-r) = v(l,r)$ and for classical solutions the Cauchy problem (8), (9) is equivalent to

$$\begin{cases} iu_t + u_{xx} = uv + \alpha|u|^2u \\ \\ \rho_t = \frac{1}{2}(l+r) \\ \\ l_t - \sqrt{1+3\lambda v^2}l_x = (|u|^2)_x + \beta\rho^3 \\ \\ r_t + \sqrt{1+3\lambda v^2}r_x = (|u|^2)_x + \beta\rho^3 \end{cases} \tag{10}$$

with initial data

$$u(.,0) = u_o \in H^3(\mathbb{R}), \quad \rho(.,0) = \rho_0 \in H^3(\mathbb{R}), \quad l(.,0) = l_0 \in H^2(\mathbb{R}), \tag{11}$$
$$\text{and } r(.,0) = r_0 \in H^2(\mathbb{R}),$$

where

$$l_0 = w_0 + \int_0^{v_0} \sqrt{1+3\lambda\xi^2}d\xi \quad \text{and } r_0 = w_0 - \int_0^{v_0} \sqrt{1+3\lambda\xi^2}d\xi. \tag{12}$$

In order to obtain a local classical solution for the Cauchy problem (10),(11) for a fixed $\lambda \geq 0$ we will follow the technique employed in [6] and in [5]:

We consider the auxiliary system with non-local source terms

$$\begin{cases} iF_t + F_{xx} = 2\alpha|u|^2F + \alpha u^2\overline{F} + Fv + \frac{1}{2}u(l_x + r_x) \\ \rho_t = \frac{1}{2}(l + r) \\ l_t - \sqrt{1 + 3\lambda v^2}l_x = (|\tilde{u}|^2)_x + \beta\rho^3 \\ r_t + \sqrt{1 + 3\lambda v^2}r_x = (|\tilde{u}|^2)_x + \beta\rho^3 \end{cases} \tag{13}$$

where $\overline{F}$ is the complex conjugate of F and

$$u(x,t) = u_0(x) + \int_0^t F(x,s)ds, \tag{14}$$

$$\tilde{u}(x,t) = (\Delta - 1)^{-1}(\alpha|u|^2u + u(v-1) - iF),$$

with initial data

$$F(.,0) = F_0 \in H^1(\mathbb{R}), \quad \rho(.,0) = \rho_0 \in H^3(\mathbb{R}), \quad l(.,0) = l_0 \in H^2(\mathbb{R}), \tag{15}$$

$$\text{and } r(.,0) = r_0 \in H^2(\mathbb{R}), \quad l_0 \text{ and } r_0 \text{ given by (12).}$$

We will prove the following result:

Theorem 2.1 *Let* $(F_0, \rho_0, l_0, r_0) \in H^1 \times H^3 \times H^2 \times H^2$.
There exists $T^* = T^*(F_0, \rho_0, l_0, r_0) > 0$ *such that for all* $T < T^*$ *there exists a unique solution* (F, ρ, l, r) *of the Cauchy problem (13), (15) with*

$$(F,\rho,l,r) \in C^j([0,T];H^{1-2j}) \times C^j([0,T];H^{3-j}) \times C^j([0,T];H^{2-j}) \times C^j([0,T];H^{2-j}), \quad j = 0,1.$$

From this result, we will prove the following Theorem

Theorem 2.2 *Let* $(u_0, \rho_0, \rho_1) \in H^3 \times H^3 \times H^2$.
There exists $T^* = T^*(F_0, \rho_0, \rho_1) > 0$ *such that for all* $T < T^*$ *there exists a unique solution* (F, ρ, l, r) *of the Cauchy problem (6), with*

$$(F,\rho) \in C^j([0,T];H^{3-2j}) \times \left(C^j([0,T];H^{3-2j}) \cap C^{j+1}([0,T];H^{2-2j})\right), \quad j = 0,1.$$

Proof of Theorem 2.1:

We want to apply a variant of Theorem 6 in [9], hence we need to put the Cauchy problem in the framework of real spaces. By introducing the new variables $F_1 = Re(F)$, $F_2 = Im(F)$, $u_1 = Re(u)$, $u_2 = Im(u)$.

By setting $U = (F_1, F_2, \rho, l, r)$ and $F_{10} = Re(F_0)$, $F_{20} = Im(F_0)$, the I.V.P. (13), (15) can be written in the form

$$\begin{cases} \dfrac{\partial}{\partial t}U + A(U)U = g(t,U) \\ U(.,0) = U_0, \end{cases} \tag{16}$$

where

$$A(U) = \begin{bmatrix} 0 & \Delta & 0 & 0 & 0 \\ -\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1+3\lambda v^2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1+3\lambda v^2} \end{bmatrix},$$

$$g(t,U) = \begin{bmatrix} 2\alpha|u|^2F_2 - \alpha(u_1^2 - u_2^2)F_2 + 2\alpha u_1u_2F_1 + F_2v + \frac{1}{2}u_2(l_x + r_x) \\ 2\alpha|u|^2F_1 - \alpha(u_1^2 - u_2^2)F_1 - 2\alpha u_1u_2F_2 - F_1v - \frac{1}{2}u_2(l_x + r_x) \\ \frac{1}{2}(l+r) \\ (|\tilde{u}|^2)_x + \beta\rho^3 \\ (|\tilde{u}|^2)_x + \beta\rho^3 \end{bmatrix}$$

and

$$U_0 = (F_{10}, F_{20}, \rho_0, l_0, r_0) \in Y = (H^1(\mathbb{R}))^2 \times (H^2(\mathbb{R}))^3.$$

(The condition $\rho_0 \in H^3(\mathbb{R})$ will be used later).
Note that the source term $g(t,U)$ is non-local, due to the presence of $\tilde{u}$.

We now set $X = (H^{-1}(\mathbb{R})^2 \times (L^2(\mathbb{R}))^3$ and $S = (1-\Delta)I$, wich is an isomorphism $S : Y \to X$.

Furthermore, we denote by W_R the open ball in Y of radius R centered at the origin and by $G(X,1,\beta)$ the set of linear operators $\Lambda : D(\Lambda) \subset X \to X$ such that:

- $-\Lambda$ generates a C_o-semigroup $\{e^{-t\Lambda}\}_{t\in\mathbb{R}}$;
- for all $t \geq 0$, $\|e^{-t\Lambda}\| \leq e^{\beta t}$, where, for all $U \in W_R$,

$$\beta = \frac{1}{2}\sup_{x\in\mathbb{R}} \|\frac{\partial}{\partial x}a(\rho,l,r)\| \leq c(R), \quad c : [0;+\infty[\to [0;+\infty[\text{ continuous, and}$$

$$a(\rho,l,r) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{1+3\lambda v^2} & 0 \\ 0 & 0 & \sqrt{1+3\lambda v^2} \end{bmatrix}.$$

Following [9] (paragraph 12),

$$A : U = (F_1, F_2, \rho, l, r) \in W_R \to G(X,1,\beta).$$

It is easy to see that g verifies, for fixed $T > 0$, $\|g(y,U)\|_Y \leq \theta$, $t \in [0,T]$, $U \in W$.
For (ρ,l,r) in a ball $\tilde{W}$ in $(H^2(\mathbb{R}))^3$, we set (see [9]-12.6)

$$B_o(\rho,l,r) = [(1-\Delta), a(\rho,l,r)](1-\Delta)^{-1} \in \mathcal{L}((L^2(\mathbb{R}))^3).$$

We now introduce the operator $B(U) \in \mathcal{L}(X)$, $U = (F_1, F_2, \rho, l, r) \in\in W_R$ by

$$B(U) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & 0 & & B_0(\rho, l, r) & \\ 0 & 0 & & & \end{bmatrix}.$$

In [9] (paragraph 12), Kato proved that for $(\rho, l, r) \in \tilde{W}$ we have

$$(1-\Delta)a(\rho, l, r)(1-\Delta)^{-1} = a(\rho, l, r) + B_0(\rho, l, r).$$

Hence, we easily derive for $U \in W_R$, $SA(U)S^{-1} = A(U) + B(U)$.

Now, for each pair $U, U* \in W_R$, $U = (F_1, F_2, \rho, l, r)$, $U^* = (F_1^*, F_2^*, \rho^*, l^*, r^*)$, we claim that

$$\|g(t,U) - g(t,U^*)\|_{L^1(0,T';X)} \leq c(T') \sup_{0\leq t\leq T'} \|U(t) - U(t')\|_X, \tag{17}$$

where $0 \leq T' \leq T$ and and $c(T')$ is a non-decreasing function such that $c(0) = 0$. Indeed, let us point out that for $h \in L^2(\mathbb{R})$ and $w \in H^1(\mathbb{R})$,

$$\|hw\|_{H^{-1}} \leq \|h\|_{H^{-1}} \|w\|_{H^1}.$$

Hence, for example,

$$\|F_1 u_1(u_1^* - u_1)\|_{H^{-1}} \leq \|F_1\|_{H^1} \|u_1\|_{H^1} \|u_1^* - u_1\|_{H^{-1}},$$

and, for $t \leq T'$,

$$(l_x + r_x)\left(\int_0^t F_2 - \int_0^t F_2^*\right) \|_{H^{-1}} \leq \|l_x + r_x\|_{H^1} \int_0^t \|F - F*\|_{H^{-1}}$$
$$\leq c(T') \sup_{0\leq t\leq T'} \|U(t) - U^*(t)\|_X.$$

Finally, applying Theorem 6 in [9], replacing the local condition (7.7) by (17), we obtain the result described in Theorem 2.1, but with $\rho \in C^j([0;T], H^{2-j}), j = 0, 1$. To obtain $\rho \in C^j([0,T], H^{3-j})$, it is enough to remark that, since $\rho_t = w$, $\rho_0 \in H^3$, $v_0 = \rho_{0x} \in H^2$, $w_0 \in \rho_1 \in H^2$, we derive $\rho_x = v \in C^j([0,T], H^{2-j})$. ■

Proof of Theorem 2.2:

We will follow here the ideas in [5]:

If (F, ρ, l, r) is a solution of (13), (15), by differenciating (14) with respect to t we obtain $u = F$. Replacing in the first equation of (13), we obtain

$$(iu_t + u_{xx})_t = 2\alpha|u|^2 F + \alpha u^2 \overline{F} + Fv + \frac{1}{2}u(l_x + r_x) = 2\alpha|u|^2 u_t + \alpha u^2 \overline{u}_t + u_t v + uv_t.$$

Hence, $(iu_t + u_x x - \alpha|u|^2 u - uv)_t = 0$, and we get $iu_t + u_x x - \alpha|u|^2 u - uv = \phi_0(x)$, where

$$\phi_0(x) = iF_0 + u_0'' - \alpha|u_0|^2 u_0 - u_0 v_0.$$

By choosing $F_0 = i(u_0'' - \alpha|u_0|^2 u_0 - u_0 v_0)$, we obtain $\phi_0 = 0$ and (u, v) satisfy the first equation in (10).

Furthermore, from this equation we derive

$$u = (\Delta - 1)^{-1}(\alpha|u|^2 u + u(v-1) - iu_t). \tag{18}$$

Therefore $u = \tilde{u}$ and (u, ρ, l, r) satisfy (10), (11). Note that $u_t = F \in C([0,T]; H^1)$. Moreover

$$u(x,t) = u_0(x) + \int_0^t F(x,s)ds \in C([0,T]; H^1),$$

but from (18) we have in fact $u \in C([0,T]; H^3)$. ■

3 Global well-posedness for $\lambda = 0$ and blow-up results

In this section, we prove that the local solutions obtained in Theorem 2.2 are in fact global in time in the case where $\beta \leq 0$ and $\lambda = 0$. Conversely, if $\beta > 0$, we show the blow-up of the local solutions in finite time under some conditions on the initial data.

We consider initial data $(u_0, \rho_0, \rho_1) \in H^3 \times H^3 \times H^2$. Let

$$(u, \rho) \in C^j([0,T], H^{3-2j}), \quad j = 0, 1$$

the unique corresponding maximal solution of the Cauchy problem (6). We begin by deriving the following conservation laws:

$$\frac{\partial}{\partial t}\int |u|^2 dx = 0, \quad t \in [0, T[, \tag{19}$$

$$\frac{\partial}{\partial t}E(t) = 0, \quad t \in [0; T[, \tag{20}$$

where the energy $E(t)$ is given by

$$E(t) = \frac{1}{2}\int (\rho_t)^2 dx + \frac{1}{2}\int (\rho_x)^2 dx + \frac{\lambda}{4}\int (\rho_x)^4 dx - \frac{\beta}{4}\int \rho^4 dx + \int \rho_x |u|^2 dx$$
$$+ \int |u_x|^2 dx + \frac{\alpha}{2}\int |u|^4 dx.$$

For the first one we multiply the first equation in (6) by $\overline{u}$ and we integrate the imaginary part. To obtain the conservation of energy, we derive from (6)

$$Re\int iu_t \overline{u}_t dx + Re\int u_{xx}\overline{u}_t dx = Re\int \rho_x u \overline{u}_t dx + \alpha Re\int |u|^2 u \overline{u}_t dx$$

and

$$-\frac{1}{2}\frac{\partial}{\partial t}\int |u_x|^2 dx = \frac{1}{2}\int \rho_x \frac{\partial}{\partial t}|u|^2 dx + \frac{\alpha}{4}\frac{\partial}{\partial t}\int |u|^4 dx$$
$$= \frac{1}{2}\frac{\partial}{\partial t}\int \rho_x |u|^2 dx - \frac{1}{2}\int \frac{\partial}{\partial t}\rho_x |u|^2 dx + \frac{\alpha}{4}\frac{\partial}{\partial t}\int |u|^4 dx.$$

Finally,

$$-\frac{1}{2}\int \frac{\partial^2 \rho}{\partial x \partial t}|u|^2 dx - \frac{1}{2}\int \frac{\partial \rho}{\partial t}(|u|^2)_x dx = \frac{1}{2}\int \frac{\partial \rho}{\partial t}\left\{\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial x}\left[\rho_x + \lambda(\rho_x)^3\right] - \beta\rho^3\right\} dx$$
$$= \frac{1}{4}\frac{\partial}{\partial t}\int (\rho_t)^2 dx + \frac{1}{4}\frac{\partial}{\partial t}\int (\rho_x)^2 dx + \frac{\lambda}{8}\frac{\partial}{\partial t}\int (\rho_x)^4 - \frac{\beta}{8}\frac{\partial}{\partial t}\int \rho^4 dx,$$

and (20) is proved.

Next, we will prove the following result:

Theorem 3.1 *Let $\beta \leq 0$ and $\lambda = 0$.*
Then Theorem 2.2 holds for $T^* = +\infty$.

Proof of Theorem 3.1:

In order to prove this result, we is sufficient to deriva *a priori* bounds for the norms $\|u\|_{H^3}$, $\|\rho\|_{H^3}$ and $\|\rho_t\|_{H^2}$.
Let us begin by noticing that $|\int \rho_x |u|^2 dx| \leq \frac{1}{4}\int (\rho_x)^2 dx + \int |u|^4 dx$.
By the Gagliardo-Nirenberg inequality and (19),

$$\|u\|_{L^4}^4 \leq c_0 \|u\|_{L^2}^3 \|u_x\|_{L^2} \leq c_0 \|u_0\|_{L^2}^3 \|u_x\|_{L^2} \leq c\|u_0\|_{L^2}^6 + \frac{1}{2}\|u_x\|_{L^2}^2.$$

Since $\beta \leq 0$, we obtain from (20)

$$\int (\rho_t)^2 dx + \int [(\rho_x)^2 + \lambda(\rho_x)^4] dx + \int |u_x|^2 dx \leq c, \tag{21}$$

with c depending only on $(\|u_0\|_{H^1}, \|\rho_0\|_{H^2}, \|\rho_1\|_{H^1})$.

Moreover, since $\rho(t) = \rho_0(x) + \int_0^t \rho_t(x, \tau, d\tau)$,

$$\|\rho(t)\|_{L^2} \leq \|\rho_0\|_{L^2} + \int_0^t \|\rho_t(\tau)\|_{L^2} d\tau,$$

hence, since $\beta \leq 0$,

$$\int (\rho_t)^2 dx + \int (\rho)^2 dx + \int (\rho_x)^2 dx + \int |u|^2 dx + \int |u_x|^2 dx \leq C(1+t), \tag{22}$$

with C depending exclusively on the initial data.

Next, we estimate $\|u_{xx}\|_{L^2}$, $\|\rho_{xt}\|_{L^2}$ and $\|\rho_{xx}\|_{L^2}$. For $\lambda = 0$, the system (10) reads

$$\begin{cases} iu_t + u_{xx} = uv + \alpha|u|^2u \\ \rho_t = \frac{1}{2}(l+r) \\ l_t - l_x = (|u|^2)_x + \beta\rho^3 \\ r_t + r_x = (|u|^2)_x + \beta\rho^3 \end{cases} \tag{23}$$

We put

$$\alpha(t) = \int (r_x)^2 dx + \int (l_x)^2 + \int |u_t|^2 dx.$$

In what follows we will denote by $A(t)$ a generic positive continuous function

$$A : \mathbb{R}_+ \to \mathbb{R}_+,$$

which can change from line to line.
By deriving with respect to x the last equation in (23), multiplying by r_x and integrating, we get

$$\frac{1}{2}\frac{d}{dt}\int (r_x)^2 dx \le 2\int |uu_x r_x| dx + 2\int |u_x^2 r_x| dx + 3|\beta| \int \rho^2 |\rho_x r_x| dx$$

$$\le A(t)\left[\left(\int r_x^2 dx\right)^{\frac{1}{2}}\left(\int |u_{xx}|^2 dx\right)^{\frac{1}{2}} + \|u\|_\infty \left(\int |u_x|^2 dx\right)^{\frac{1}{2}}\left(\int r_x^2 dx\right)^{\frac{1}{2}} + \left(\int r_x^2 dx\right)^{\frac{1}{2}}\right]$$

$$\le A(t)\left[\left(\int r_x^2 dx\right)^{\frac{1}{2}}\left(\int |u_{xx}|^2 dx\right)^{\frac{1}{2}} + \left(\int r_x^2 dx\right)^{\frac{1}{2}}\right],$$

where we have used the Sobolev injection $\|u_x\|_\infty \le c\|u_x\|_{H^1}$ and (22).
By writing a similar estimate for l_x, we obtain

$$\frac{1}{2}\frac{d}{dt}\int ((r_x)^2 + (l_x)^2) dx \le A(t)\left[\alpha^{\frac{1}{2}}(t) + \alpha^{\frac{1}{2}}(t)\left(\int |u_{xx}|^2 dx\right)^{\frac{1}{2}}\right]. \tag{24}$$

From the first equation in (23),

$$\|u_{xx}\|_{L^2} \le \|u_t\|_{L^2} + A(t) \le \alpha^{\frac{1}{2}}(t) + A(t). \tag{25}$$

By replacing in (24),

$$\frac{1}{2}\frac{d}{dt}\int ((r_x)^2 + (l_x)^2) dx \le A(t)\left[\alpha^{\frac{1}{2}}(t) + \alpha(t)\right]. \tag{26}$$

Moreover, since $\rho_t = \frac{1}{2}(l+r)$,

$$\|\rho_{xt}\|_{L^2} \le c\alpha^{\frac{1}{2}}(t). \tag{27}$$

Now, by multiplying the first equation in (23) by $\overline{u}_t$, integrating the imaginary part and using the Cauchy-Schwarz inequality,

$$\frac{1}{2}\frac{d}{dt}\int |u_t|^2 dx = \int \rho_{xt} Im(u\overline{u}_t) + \alpha \int (|u|^2)_t Im(u\overline{u}_t)$$

$$\leq \|u\|_\infty \|\rho_{xt}\|_{L^2} \left(\int |u_t|^2 dx\right)^{\frac{1}{2}} \leq c\alpha(t).$$

Finally, using (26), we get

$$\frac{d}{dt}\alpha(t) \leq A(t)\left[\alpha^{\frac{1}{2}}(t) + \alpha(t)\right] \leq A(t)\left[1 + \alpha(t)\right],$$

and

$$\alpha(t) \leq (1 + \alpha(0)) e^{\int_0^t A(\tau) d\tau} - 1.$$

Hence, by (25) and (27),

$$\|u_{xx}\|_{L^2} + \|\rho_{xt}\|_{L^2} \leq A(t).$$

By, the second and third equation in (23), $\|l_t\|_{L^2} + \|r_t\|_{L^2} \leq A(t)$, therefore

$$\|\rho_{tt}\|_{L^2} = \frac{1}{2}\|l_t + r_t\|_{L^2} \leq A(t)$$

and

$$\|\rho_{xx}\|_{L^2} = \|\rho_{tt} - (|u|^2)_x - \beta\rho^3\|_{L^2} \leq A(t).$$

To obtain a continuous bound on $\|\rho_{xxx}\|_{L^2}$, $\|u_{xxx}\|_{L^2}$ and $\|\rho_{txx}\|_{L^2}$, the exact same method can be used by setting

$$\alpha(t) = \int (r_{xx})^2 dx + \int (l_{xx})^2 + \int |u_{xt}|^2 dx$$

and deriving system (23) with respect to x. ■

We now assume $\beta > 0$. In what follows, we will consider the following conditions on the initial data:

$$\int \rho_0 \rho_1 dx > 0 \tag{28}$$

and

$$E(0) < -\frac{1}{64}(\frac{9}{4} + 2\alpha)^2 c_o^2 \|u_o\|_{L^2}^6. \tag{29}$$

We will prove the following blow-up result:

Theorem 3.2 *Let* $\beta > 0$.
Under the conditions of Theorem 2.2 and assuming that the initial data (u_o, ρ_0, ρ_1) *satisfies conditions (28) and (29), there exists a time* $0 < T^* \leq T_0 := (\int \rho_0^2)(\int \rho_0\rho_1)^{-1}$ *such that, if the solution exists in* $[0, T^*]$, *then*

$$\lim_{t \to T^{*-}} \int \rho^2 dx = +\infty.$$

Proof of Theorem 3.2:

Following [13] (chapter 10, paragraph 13), let us put

$$G(t) = \int \rho^2 dx \quad \text{and} \quad F(t) = (G(t))^{-\frac{1}{2}}. \tag{30}$$

We have $F'(t) = -\dfrac{1}{2}G(t)^{-\frac{3}{2}}G'(t) = -G(t)^{-\frac{3}{2}}\displaystyle\int \rho\rho_t dx$, and, from (28), $F'(0) < 0$.
Furthermore, we set $Q(t) = -2G(t)^{\frac{5}{2}}F''(t) = G''(t)G(t) - \frac{3}{2}G'(t)^2$, with

$$G''(t) = 6\int (\rho_t)^2 dx + 2H(t) \quad \text{and} \quad H(t) = \int [\rho\rho_{tt} - \frac{5}{2}(\rho_t)^2]dx.$$

We have

$$Q(t) = 6[\Big(\int \rho^2 dx\Big)\Big(\int (\rho_t)^2 dx - \int (\rho\rho_t)^2 dx\Big)] + 2G(t)H(t),$$

and by the Cauchy-Schwarz inequality we obtain $Q(t) \geq 0$, and consequently $F''(t) \leq 0$, provided $H(t) \geq 0$.
This last fact is easy to check. From (6) and (20),

$$\begin{aligned} H(t) &= -4E(t) + 4[\int |u_x|^2 dx + \frac{\alpha}{2}\int |u|^4 dx + \frac{3}{4}\int \rho_x|u|^2]dx + \int (\rho_x)^2 dx \\ &= -4E(0) + 4[\int |u_x|^2 dx + \frac{\alpha}{2}\int |u|^4 dx + \frac{3}{4}\int \rho_x|u|^2]dx + \int (\rho_x)^2 dx. \end{aligned}$$

We have

$$3\int \rho_x|u|^2 dx \leq \int (\rho_x)^2 dx + \frac{9}{4}\int |u|^4 dx$$

and, by the Gagliardo-Nirenberg inequality and (19),

$$(\frac{9}{4}+2|\alpha|)\int |u|^4 dx \leq (\frac{9}{4}+2|\alpha|)c_0\|u_0\|^3_{L^2}\|u_x\|_{L^2} \leq 4\int |u_x|^2 dx + \frac{1}{16}(\frac{9}{4}+2|\alpha|)^2 c_0^2\|u_0\|^6_{L^2}.$$

From condition (29), $H(t) \geq -4E(0) - \frac{1}{16}(\frac{9}{4} + 2|\alpha|)^2 c_0^2\|u_0\|^6_{L^2} \geq 0$.
Hence, we have shown that for all $t \in [0, T[$, $F''(t) \leq 0$, which implies Theorem 3.2.
■

4 Global existence of weak solutions for the quasilinear system

For the study of the existence of a global weak solution to the Cauchy problem (6), we will consider, for $\epsilon > 0$, the regularized problem (cf. [4] for the case $\beta = 0$)

$$\begin{cases} iu_t + u_{xx} = u\rho_x + \alpha|u|^2 u \\ \rho_t = w \\ w_t - \epsilon w_{xx} = \beta\rho^3 + (\sigma(\rho_x))_x + (|u|^2)_x \end{cases} \tag{31}$$

with initial data (we have dropped the ϵ parameter on u, w and ρ)

$$u(0,x)=u_0(x)\in H^1(\mathbb{R}),\ \rho(0,x)=\rho_0(x)\in H^2(\mathbb{R}),\ w(x,0)=\rho_t(0,x)=\rho_1(x)\in H^1(\mathbb{R}). \tag{32}$$

Here, $\sigma(v)=v+\lambda v^3$ and $\lambda>0$ (hence $\sigma'(v)=1+3\lambda v^2>0$).
For a smooth solution of (31)-(32), the energy identity (20) takes the form

$$\frac{d}{dt}\left\{\frac{1}{2}\int w^2dx+\frac{1}{2}\int v^2dx+\frac{\lambda}{4}\int v^4dx-\frac{\beta}{4}\int\rho^4dx+\right.$$
$$\left.\int v|u|^2dx+\int|u_x|^2dx+\frac{\alpha}{2}\int|u|^4dx\right\}=-\epsilon\int(w_x)^2dx, \tag{33}$$

where we have put $v=\rho_x$. On the other hand, the conservation law

$$\frac{d}{dt}\left(\int|u|^2dx\right)=0 \tag{34}$$

still holds. Also, we deduce (cf. [4] and following [14])

$$\int[w_tv_x-\sigma'(v)(v_x)^2]dx=\int(|u|^2)_xv_xdx+\beta\int\rho^3v_xdx+\epsilon\int w_{xx}v_xdx$$

and

$$-\frac{d}{dt}\int w_xvdx+\int(w_x)^2dx-\int\sigma'(v)(v_x)^2dx=\int(|u|^2)_xv_xdx+\beta\int\rho^3\rho_{xx}dx+\frac{\epsilon}{2}\frac{d}{dt}\int(v_x)^2dx$$

since

$$-\frac{d}{dt}\int w_xvdx=-\int w_{xt}vdx-\int w_xv_tdx=\int w_tv_xdx-\int w_xv_tdx$$

and

$$v_t=\rho_{xt}=w_x.$$

Integrating this identity over the time interval $[0,t]$ we obtain, with $v_0(x)=v(x,0)$,

$$-\int w_xvdx+\int\rho_{1x}v_0dx+\int_0^t\int(w_x)^2dxd\tau-\int_0^t\int\sigma'(v)(v_x)^2dxd\tau$$
$$=\int_0^t\int(|u|^2)_xv_xdxd\tau-3\beta\int_0^t\int\rho^2(\rho_x)^2dxd\tau+\frac{\epsilon}{2}\int(v_x)^2dx-\frac{\epsilon}{2}\int(v_{0x})^2dx.$$

Since $-\int w_xvdx=\int wv_xdx$, we get

$$\int_0^t\int\sigma'(v)(v_x)^2dxd\tau+\frac{\epsilon}{2}\int(v_x)^2dx\le\frac{\epsilon}{4}\int(v_x)^2dx+\frac{1}{\epsilon}\int w^2dx+\int|v_0\rho_{1x}|dx+$$
$$\frac{\epsilon}{2}\int(v_{0x})^2dx+3\beta\int_0^t\int\rho^2v^2dxd\tau+\epsilon\int_0^t\int(w_x)^2dxd\tau+2\int_0^t\int|uu_xv_x|dxd\tau \tag{35}$$

and

$$2\int_0^t\int|uu_xv_x|dxd\tau \le 2\int_0^t\int|uu_x|^2dxd\tau + \frac{1}{2}\int_0^t\int(v_x)^2dxd\tau. \qquad (36)$$

Now, let us assume $\beta \le 0$. Since $\epsilon > 0$, we can derive from (33), as in (21),

$$\int w^2dx + \int(v^2+\lambda v^4)dx + \int|u_x|^2 + \epsilon\int_0^t(w_x)^2dxd\tau \le C, \qquad (37)$$

where C only depends on $(\|u_0\|_{H^1}, \|\rho_0\|_{H^2}, \|\rho_1\|_{H^1})$.
Hence, from (34), (36) and (37),

$$2\int_0^t\int|uu_xv_x|dxd\tau \le Ct + \frac{1}{2}\int_0^t\int\sigma'(v)(v_x)^2dxd\tau. \qquad (38)$$

Taking $\epsilon \le 1$, we deduce from (35), (36), (37) and (38)

$$\epsilon\int_0^t\int\sigma'(v)(v_x)^2dxd\tau + \epsilon^2\int(v_x)^2dx + \epsilon|\beta|\int_0^t\int\rho^2v^2dxd\tau \le C(1+t). \qquad (39)$$

Let us now analyse the problem of the existence and uniqueness of a solution

$$(u,\rho,w) \in C([0,+\infty[;H^1) \times C([0,+\infty[;H^2) \times C([0,+\infty[;H^1)$$

to the Cauchy problem (31)-(32). Without loss of generality, we may assume $\epsilon = 1$. We start with the existence and uniqueness of a local (in time) solution. We fix $0 < T < +\infty$ and introduce the Banach spaces $X_T = C([0,T];H^1)$ (complex) and $Y_T = C([0,T];H^2)$ (real) endowed with the usual norms. Furthermore, we consider the product space $\tilde{B}_R^T \times B_R^T$ where

$$\tilde{B}_R^T = \{u \in X_T \,:\, \|u\|_{X_T} \le R\} \qquad \text{and} \qquad B_R^T = \{u \in Y_T \,:\, \|u\|_{Y_T} \le R\}.$$

Finally, we consider the application

$$\Phi \,:\, (\tilde{u},\tilde{\rho}) \in \tilde{B}_R^T \times B_R^T \to (u,\rho) \in X_T \times Y_T.$$

Here, u denotes the solution of the linear problem

$$\begin{cases} iu_t + u_{xx} = \tilde{\rho}_x\tilde{u} + \alpha|\tilde{u}|^2\tilde{u}, \\ u(.,0) = u_0 \in H^1, \end{cases} \qquad (40)$$

and

$$\rho(t) = \rho_0 + \int_0^t wd\tau, \quad \rho(.,0) = \rho_0 \in H^2, \qquad (41)$$

where w is the unique solution of

$$\begin{cases} w_t - w_{xx} = \beta\tilde{\rho}^3 + (\sigma(\tilde{\rho}_x))_x + (|\tilde{u}|^2)_x, \\ w(.,0) = w_0(x) \in H^1, \end{cases} \qquad (42)$$

verifying $w \in L^2(0,T;H^2)$, $w_t \in L^2(0,T;L^2)$. We have

$$u(t) = e^{it\partial_{xx}}u_0 - i\int_0^t e^{i(t-s)\partial_{xx}}(\tilde{\rho}_x\tilde{u} + \alpha|\tilde{u}|^2\tilde{u})(s)ds$$

and $\beta\tilde{\rho}^3 + (\sigma(\tilde{\rho}_x))_x + (|\tilde{u}|^2)_x \in C([0,T];L^2)$.

The existence and uniqueness of a local solution is a consequence of the Banach fixed-point theorem for a convenient choice of R and T, $R > max(\|u_0\|_{H^1}, \|\rho_0\|_{H^2})$. We have

$$w_t - w_{xx} = \beta\rho^3 + (\sigma(\rho_x))_x + (|u|^2)_x.$$

From (33), (34), (37), (39), (40) and (41), we derive the *a priori* estimate

$$|w_t - w_{xx}|_{L^2(0,T;L^2)} \leq C(T), \qquad C \in C([0,+\infty[;\mathbb{R}_+),$$

which implies $w \in L^2(0,T;H^2)$ and a similar *a priori* estimate for $\|w\|_{L^2(0,T;H^2)}$ and so for $\|w_t\|_{L^2(0,T;L^2)}$ and $\|w\|_{C([0,T];H^1)}$.

We conclude that $\rho \in Y_T$ and $u \in X_T$, with similar estimates for $\|\rho\|_{Y_T}$ and $\|u\|_{X_T}$, hence we can extend the solution to $[0,+\infty[$.
Hence, if we write

$$\rho_\epsilon(t) = \rho_0 + \int_0^t w_\epsilon d\tau, \quad \rho_0 \in H^2(\mathbb{R}), \quad 0 < \epsilon \leq 1, \tag{43}$$

we get, with

$$u_\epsilon(0,x) = u_0(x) \in H^1, \quad v_\epsilon(0,x) = v_0(x) \in H^1, \quad w_\epsilon(0,x) = \rho_t(0,x) = \rho_1(x) \in H^1, \tag{44}$$

a unique solution

$$(u_\epsilon, v_\epsilon, w_\epsilon) \in (C([0,+\infty[;H^1))^3 \tag{45}$$

of the Cauchy problem

$$\begin{cases} iu_{\epsilon t} + u_{\epsilon xx} = u_\epsilon v_\epsilon + \alpha|u_\epsilon|^2 u_\epsilon \\ v_{\epsilon t} = w_{\epsilon x} \\ w_{\epsilon t} = (\sigma(v))_x + (|u_\epsilon|^2)_x + \beta{\rho_\epsilon}^3 + \epsilon w_{\epsilon xx} \end{cases} \tag{46}$$

with the initial data (44).

Moreover, we have for each $T > 0$, by (34), (37) and the first equation in (31),

$$\{u_\epsilon\}_\epsilon \text{ bounded in } L^\infty(0,+\infty;H^1),$$

$$\{u_{\epsilon t}\}_\epsilon \text{ bounded in } L^\infty(0,+\infty;H^{-1}),$$

hence $\{u_\epsilon\}_\epsilon$ belongs to a compact set of $L^2(0,T;L^2(I_R))$ for each interval $I_R = [-R,R]$, $R \geq 0$. By applying a standard diagonalization method we conclude that there exists $u \in L^\infty(0,+\infty;H^1)$ and a subsequence of $\{u_\epsilon\}_\epsilon$, still denoted $\{u_\epsilon\}_\epsilon$, such that

$$u_\epsilon \to u \text{ in } L^\infty(0,+\infty;H^1) \text{ weak* and in } L^1_{loc}(\mathbb{R}\times[0,\infty[).$$

We also have, by (37) and (39), $\{w_\epsilon\}_\epsilon$ bounded in $L^2_{loc}(\mathbb{R}\times[0,\infty[)$ and, with $\sum(v) = \frac{1}{2}v^2 + \frac{\lambda}{4}v^4$, $\{v_\epsilon\}_\epsilon$ bounded in $L^{\sum}_{loc}(\mathbb{R}\times[0,\infty[)$, where $v \in L^{\sum}_{loc}(\mathbb{R}\times[0,\infty[)$ means

$\int_K \sum(v)dxdt < +\infty$ for each compact $K \subset \mathbb{R} \times [0, +\infty[$. Finally we have, by (43), $\{\rho_\epsilon\}_\epsilon$ bounded in $L^2_{loc}(\mathbb{R} \times [0, +\infty[)$.
By (37) and (39) we derive, for $\epsilon \leq 1$,

$$\epsilon \int_0^t \int [(u_{\epsilon x})^2 + \sigma'(v_\epsilon)(u_{\epsilon x})^2] dx d\tau \leq C(1+t), \tag{47}$$

where C only depends on $(\|u_0\|_{H^1}, \|\rho_0\|_{H^2}, \|\rho_1\|_{H^1})$.

Now we consider the quasilinear hyperbolic system

$$\begin{cases} v_t = w_x \\ w_t = (\sigma(v))_x \end{cases} \tag{48}$$

and let $(\eta(v, w), q(v, w))$, $(v, w) \in \mathbb{R}^2$, be a pair of smooth convex entropy-entropy flux for (48) such that η_w, η_{ww} and $\eta_{vw}/\sqrt{\sigma'}$ are bounded in $\mathbb{R}^2$.
From (34) and the estimates (37) and (47), we can deduce that (cf. [14], [2] and [4])

$$\frac{\partial}{\partial t}\eta(w_\epsilon, v_\epsilon) + \frac{\partial}{\partial x} q(w_\epsilon, v_\epsilon)$$

belongs to a compact subset of $W^{-1,2}_{loc}(\mathbb{R} \times [0, +\infty[)$.
Hence, we can apply a result on compensated compactness of Serre and Shearer ([14]) to conclude that $\{(w_\epsilon, v_\epsilon)\}_\epsilon$ is pre-compact in $(L^1_{loc}(\mathbb{R} \times [0, +\infty[)^2$. Hence, there exists a subsequence $\{(u_\epsilon, v_\epsilon, w_\epsilon)\}_\epsilon$ and

$$(u, v, w) \in L^\infty(]0, +\infty[; H^1) \times L^{\sum}_{loc}(\mathbb{R} \times [0, +\infty[) \times L^2_{loc}(\mathbb{R} \times [0, +\infty[)$$

such that

$$(u_\epsilon, v_\epsilon, w_\epsilon) \to (u, v, w) \text{ in } (L^1_{loc}(\mathbb{R} \times [0, +\infty[)^3$$

and

$$\rho_\epsilon = \rho_0 + \int_0^t w_\epsilon d\tau \to \rho = \rho_0 + \int_0^t w d\tau \text{ in } L^1_{loc}(\mathbb{R} \times [0, +\infty[).$$

Hence, we obtain from (46) the following result:

Theorem 4.1 *Assume* $(u_o, \rho_0, \rho_1) \in H^1 \times H^2 \times H^1$, $\lambda > 0$ *and* $\beta \leq 0$. *Then, there exists*

$$(u, v, w) \in L^\infty(0, +\infty; H^1) \times L^{\sum}_{loc}(\mathbb{R} \times [0, +\infty[\times L^2_{loc}(\mathbb{R} \times [0, +\infty[)$$

such that, with $\rho(x,t) = \rho_0(x) + \int_0^t w(x, \tau) d\tau$, *we have*

$$-i \int_0^{+\infty} \int u \theta_t dx dt - \int_0^{+\infty} \int u_x \theta_x dx dt + \int u_0(x) \theta(x, 0) dx =$$

$$\int_0^{+\infty} \int v u \theta dx dt + \alpha \int_0^{+\infty} \int |u|^2 u \theta dx dt$$

for all $\theta \in C_0^1(\mathbb{R} \times [0, +\infty[)$ *(complex-valued), and*

$$\int_0^{+\infty} \int (v\phi_t - w\phi_x) dxdt + \int \rho_{0x} \phi(x,0) dx + \int_0^{+\infty} \int (w\psi_t - \sigma(v)\psi_x + \beta\rho^3\psi) dxdt$$

$$+ \int \rho_1 \psi(x,0) dx - \int_0^{+\infty} \int (|u|^2)_x \psi dxdt = 0$$

for all $\phi, \psi \in C_0^1(\mathbb{R} \times [0, +\infty[$ *(real-valued).*

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