SHORT WAVE–LONG WAVE INTERACTIONS FOR COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We consider a model describing the short wave *vs.* long wave interactions where the short waves are governed by a nonlinear Scrhödinger equation, the long waves are governed by the Navier-Stokes equations for a compressible isentropic fluid flow, and these two modes are coupled nonlinearly by interaction terms. Our model is motivated by the general approach introduced by Benney in [4] and follows the reasoning adopted for the several models proposed and analyzed in [9]. For the model proposed here we address the problem of the global existence and uniqueness of the solution in the one-dimensional case. We also discuss the problem of the convergence of the sequence of solutions when the viscosity and the interaction coefficient vanish, applying in this part a recent compactness framework developed by Chen and Perepelitsa [8].

1. Introduction

In this paper we consider a model describing the short wave *vs.* long wave interactions where the short waves are governed by a nonlinear Scrhödinger equation, the long waves are governed by the Navier-Stokes equations for a compressible isentropic fluid flow, and these two modes are coupled nonlinearly by interaction terms. Our model is motivated by the general approach introduced by Benney in [4] and follows the reasoning adopted for the several models proposed and analyzed in [9]. For the model proposed here we address the problem of the global existence and uniqueness of the solution in the one-dimensional case. We also discuss the problem of the convergence of the sequence of solutions when the viscosity and the interaction coefficient vanish, applying in this part a recent compactness framework developed by Chen and Perepelitsa [8].

In order to justify our model, we begin with a general discussion in the multidimensional context. So, we consider the multidimensional Navier-Stokes equations for viscous compressible (isentropic polytropic) fluid flows (see, e.g., [19])

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\frac{\partial \rho u_i}{\partial t} + \operatorname{div}(\rho u u_i) - \mu \Delta u_i - \xi \partial_i \operatorname{div} u + \partial_i (a\rho^\gamma) = \rho F_i, \quad 1 \le i \le N, \tag{1.2}$$

where, as usual, ρ represents the gas density, $u = (u_1, \dots, u_N)$ represents the particle fluid velocity, $\mu > 0$, $\mu + \xi > 0$, $a > 0$, $\gamma \in (1, \infty)$ and $F_i = F_i(x,t)$, $i = 1, \dots, N$, are functions corresponding to external force terms on $\mathbb{R}^n \times (0,T)$ for some $T \in (0,\infty)$.

We recall the Lagrangian transformation $Y(x,t) = (y(x,t), t)$, where $y(x,t) = \Phi(t;x)$ and the latter is the solution of the (normalized) initial value problem

$$\frac{d\Phi}{dt} = u(\Phi(t;x), t), \tag{1.3}$$

$$\Phi(0;x) = (x_1, \dots, x_{N-1}, \int_0^{x_N} \rho_0(x_1, \dots, x_{N-1}, s)\, ds), \tag{1.4}$$

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where $\rho_0(x) = \rho(x,0)$. The (non-standard) choice of the initial data in (1.4) is motivated by the well known relation (for smooth solutions with $\rho(x,t) > 0$) for the Jacobian of the mapping $x \mapsto y(x,t)$, $J_x y(x,t)$, (see, e.g., [15, 23])

$$\frac{\partial}{\partial t}\left(\frac{J_x y(x,t)}{\rho(x,t)}\right) = 0,$$

which, with (1.4), implies $J_x y(x,t) = \rho(x,t)$.

We now consider short waves propagating along the flow path described by the nonlinear Schrödinger equation

$$i\frac{\partial w}{\partial t} + \Delta_y w = |w|^2 w + wG, \tag{1.5}$$

where w is the unknown complex valued wave-particle function, and G is a real valued function on $\mathbb{R}^n \times (0,T)$ corresponding to a potential due to external force terms.

We first observe that, since G is real, we have for granted the standard conservation of the L^2-norm of w, that is,

$$\frac{d}{dt}\int_{\mathbb{R}^N} |w|^2\,dy = 0, \tag{1.6}$$

which is easily obtained by multiplying (1.5) by $\bar{w}$, taking the imaginary part and integrating.

We then propose to model the short wave *vs.* long wave interactions by taking F_i (in (1.2)) and G (in (1.5)) in the form

$$F_i = \frac{\alpha}{\rho}\partial_i(g'(\frac{1}{\rho})h(|w\circ Y|^2)), \quad G = \alpha g(v)h'(|w|^2), \tag{1.7}$$

where $\alpha > 0$ is a parameter which we call interaction coefficient,

$$v(y,t) = \frac{1}{\rho\circ X(y,t)}, \quad X(y,t) = Y^{-1}(y,t), \tag{1.8}$$

and $g, h : [0,\infty) \to [0,\infty)$ are smooth functions satisfying $\operatorname{supp} g'$ is compact in $(0,\infty)$, $\operatorname{supp} h'$ is compact in $[0,\infty)$, and $g(0) = h(0) = 0$. We then get the following system:

$$i\frac{\partial w}{\partial t} + \Delta_y w = |w|^2 w + \alpha g(v)h'(|w|^2)w, \tag{1.9}$$

$$\frac{\partial \rho u_i}{\partial t} + \operatorname{div}(\rho u u_i) - \mu\Delta u_i - \xi\partial_i \operatorname{div} u + \partial_i(a\rho^\gamma) = \alpha\partial_i(g'(\frac{1}{\rho})h(|w\circ Y|^2), \quad 1 \le i \le n, \tag{1.10}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \tag{1.11}$$

which is our model for short wave *vs.* long wave interactions involving compressible fluid flows.

The importance of the specific form of the nonlinear interaction terms F and G given in (1.7) comes from the main energy identity for the system (1.9), (1.10),(1.11), which we formally deduce as follows. Let us assume that $\lim_{|x|\to\infty}\rho_0(x) = \rho_*$, $\lim_{|x|\to 0} u_0(x) = 0$, and, as usual, we define the pressure

$$p(\rho) = a\rho^\gamma,$$

and the internal energy

$$e(\rho) = \frac{a}{\gamma - 1}\rho^\gamma.$$

We also denote

$$e^*(\rho) = e(\rho) - e(\rho_*) - e'(\rho_*)(\rho - \rho_*).$$

We recall that the energy identity for the Navier-Stokes equations reads (see, e.g., [19])

$$\frac{d}{dt}\int\left\{\rho\frac{|u|^2}{2}+e^*(\rho)\right\}dx+\int(\mu|Du|^2+\xi(\operatorname{div}u)^2)\,dx=\int\rho u\cdot F\,dx, \tag{1.12}$$

which is formally deduced in a standard way by multiplying (1.2) by u_i, summing up for $i=1,\dots,N$, using (1.1) and (1.2), and integrating. Now, using the specific definition of F in (1.7), (1.18) becomes, after integration by parts,

$$\frac{d}{dt}\int\left\{\rho\frac{|u|^2}{2}+e^*(\rho)\right\}dx+\int(\mu|Du|^2+\xi(\operatorname{div}u)^2)\,dx=\int(-\rho\operatorname{div}u)\big(\frac{\alpha}{\rho}g'(\frac{1}{\rho})h(|w\circ Y|^2)\big)\,dx, \tag{1.13}$$

which, by (1.1), is equal to

$$\frac{d}{dt}\int\left\{\rho\frac{|u|^2}{2}+e^*(\rho)\right\}dx+\int(\mu|Du|^2+\xi(\operatorname{div}u)^2)\,dx=\int(\rho_t+u\cdot\nabla\rho)\big(\frac{\alpha}{\rho}g'(\frac{1}{\rho})h(|w\circ Y|^2)\big)\,dx, \tag{1.14}$$

Then, using the Lagrangian transformation, for the right-hand side of (1.14), we get

$$\int(\rho_t+u\cdot\nabla\rho)\big(\frac{\alpha}{\rho^2}g'(\frac{1}{\rho})h(|w\circ Y|^2)\big)\rho\,dx=-\int\alpha g(v)_t h(|w|^2)\,dy. \tag{1.15}$$

On the other hand, multiplying (1.9) by $\bar{w}_t$ and taking the real part, we obtain

$$\frac{\partial}{\partial t}\left(|\nabla_y w|^2+\frac{1}{2}|w|^4+\alpha g(v)h(|w|^2)\right)=\alpha g(v)_t h(|w|^2), \tag{1.16}$$

and, so,

$$\int\alpha g(v)_t h(|w|^2)\,dy=\frac{\partial}{\partial t}\int\left(|\nabla_y w|^2+\frac{1}{2}|w|^4+\alpha g(v)h(|w|^2)\right)dy. \tag{1.17}$$

Therefore, from (1.14), (1.15) and (1.17), we finally arrive at the energy identity

$$\begin{aligned}&\frac{d}{dt}\int\left(|\nabla_y w|^2+\frac{1}{2}|w|^4+\alpha g(v)h(|w|^2)\right)dy\\&+\frac{d}{dt}\int\left\{\rho\frac{|u|^2}{2}+\frac{a}{\gamma-1}\rho^\gamma\right\}dx+\int(\mu|Du|^2+\xi(\operatorname{div}u)^2)\,dx=0.\end{aligned} \tag{1.18}$$

In the one-dimensional context the Lagrangian transformation $Y(x,t)=(y(x,t),t)$ can be defined in a simpler way by

$$\partial_x y=\rho,\qquad \partial_t y=-\rho u,\qquad y(x,0)=\int_0^x\rho_0(z)\,dz, \tag{1.19}$$

and, so, it is more convenient to write the whole system (1.9),(1.10),(1.11) in Lagrangian coordinates, which gives,

$$iw_t+w_{yy}=|w|^2w+\alpha g(v)h'(|w|^2)w, \tag{1.20}$$

$$u_t+p(v)_y=\alpha(g'(v)h(|w|^2))_y+(\frac{\varepsilon}{v}u_y)_y, \tag{1.21}$$

$$v_t-u_y=0, \tag{1.22}$$

where $\varepsilon=\mu+\xi>0$, and, by abuse of notation, we denote $p(v)=\tilde{p}(v)$, with $\tilde{p}(v)=p(1/v)$, which through scaling gives

$$p(v)=\kappa v^{-\gamma},\quad \kappa=\frac{(\gamma-1)^2}{4\gamma},\quad \gamma>1.$$

The remaining parts of this paper are organized as follows. In Section 2, we prove the local existence and uniqueness of smooth solutions to the Cauchy problem for (1.20),(1.21),(1.22). In Section 3, we extend the local solution to the whole time-interval $[0,\infty)$. In Section 4, we analyze the vanishing viscosity limit when

$\alpha = O(\varepsilon^{1/2})$ by using the compactness framework developed in [8]. We also include an appendix at the end, where we explain the adaptations that need to be made in the calculations in [8] in order to obtain the same energy estimates as in [8], which allow to prove the compactness of the vanishing viscosity sequence.

2. Local existence and uniqueness

In this section we prove the local existence of a smooth solution to the Cauchy problem for (1.20),(1.21),(1.22) which is formed with prescribed initial data:

$$(w(y,0),u(y,0),v(y,0)) = (w_0(y),u_0(y),v_0(y)). \tag{2.1}$$

We assume that the initial data satisfy

$$w_0 \in W^{1,2}(\mathbb{R};\mathbb{C}), \quad u_0 \in W^{1,2}(\mathbb{R}), \tag{2.2}$$

and there exist $v_*, s_1, s_2 > 0$ such that $\lim_{|y|\to\infty} v_0(y) = v_*$, and

$$v_0 - v_* \in W^{1,2}(\mathbb{R}), \quad 0 < s_1 < v_0(y) < s_2,\ y \in \mathbb{R}, \tag{2.3}$$

where $W^{k,p}(\mathbb{R};\mathbb{C})$ and $W^{k,p}(\mathbb{R})$ denote, as usual, the complex-valued and real-valued Sobolev spaces of functions with distributional derivatives up to k-th order in $L^p(\mathbb{R})$. In what follows, we will frequently omit the target $\mathbb{C}$ for the spaces to which the complex-valued wave-function w belongs.

Theorem 2.1. *There exists $T > 0$ and a unique solution (w,u,v) of (1.20)-(2.1) satisfying*

$$\begin{gathered} w \in C([0,T];W^{1,2}(\mathbb{R})),\ v - v_* \in C([0,T];W^{1,2}(\mathbb{R})), \\ \frac{s_1}{2} \le v \le 2s_2, \quad u \in C([0,T];W^{1,2}(\mathbb{R})) \cap L^2([0,T];W^{2,2}(\mathbb{R})), \\ v_t, u_t \in L^2(\mathbb{R}\times[0,T]). \end{gathered} \tag{2.4}$$

Proof. We first consider the Cauchy problem for (1.20),(1.21),(1.22) with periodic initial data of period $2k$, $k \in \mathbb{N}$, defined as

$$\begin{aligned} &(w(y,0),u(y,0),v(y,0)) = (w_{0,k}(y),u_{0,k}(y),v_{0,k}(y)), \\ &w_{0,k}(y) = w_0(y) - \varphi_k(y)\frac{w_0(k)-w_0(-k)}{2k}(y+k), \quad y \in [-k,k], \\ &u_{0,k}(y) = u_0(y) - \varphi_k(y)\frac{u_0(k)-u_0(-k)}{2k}(y+k), \quad y \in [-k,k], \\ &v_{0,k}(y) = \varphi_k(y)v_* + (1-\varphi_k(y))v_0(y), \quad y \in [-k,k], \\ &(w_{0,k}(y+2k),u_{0,k}(y+2k),v_{0,k}(y+2k)) = (w_{0,k}(y),u_{0,k}(y),v_{0,k}(y)), \end{aligned} \tag{2.5}$$

where φ_k is a smooth periodic function of period $2k$ with all its derivatives bounded, uniformly in k, satisfying $0 \le \varphi_k \le 1$, $\varphi_k(-k) = \varphi_k(k) = 1$, $\varphi_k(y) = 0$, for $-k+1 \le y \le k-1$.

We will solve locally the Cauchy problem (1.20),(1.21),(1.22),(2.5) by using Faedo-Galerkin method following Kazhikhov's ideas (see [2], chapter 2). For simplicity of notation, we drop the subscripts k which refer to the period $2k$ and denote $\Omega = (-k,k)$. Let

$$X_n = \text{span}_{\mathbb{C}}\{\cos\frac{j\pi y}{k},\ \sin\frac{j\pi y}{k}\ :\ j = 0,1,\dots,n\},$$

with corresponding projection $P_n : L^2((-k,k);\mathbb{C}) \to X_n$. We seek for a local solution $(w(y,t),u(y,t),v(y,t))$ of problem (1.20),(1.21),(1.22),(2.5), so we consider the projections $w_n = P_n w$ and $u_n = P_n u$, which may be written as

$$w_n = \sum_{j=0}^n \left(a_j(t)\cos\frac{j\pi y}{k} + b_j(t)\sin\frac{j\pi y}{k}\right), \quad u_n = \sum_{j=0}^n \left(c_j(t)\cos\frac{j\pi y}{k} + d_j(t)\sin\frac{j\pi y}{k}\right).$$

As usual, we identify w_n, u_n with the vector functions

$w_n(t) \equiv (a_0(t), \dots, a_n(t), b_1(t), \dots, b_n(t)) \in \mathbb{C}^{2n+1}$ and $u_n(t) \equiv (c_0(t), \dots, c_n(t), d_1(t), \dots, d_n(t)) \in \mathbb{R}^{2n+1}$,

respectively. We then consider the system

$$P_n \left(iw_{nt} + w_{nyy} - |w_n|^2 w_n - \alpha g(v_n) h'(|w_n|^2) w_n \right) = 0, \tag{2.6}$$

$$P_n \left(u_{nt} + \left(p(v_n) - \alpha g'(v_n) h(|w_n|^2) - \frac{\varepsilon}{v_n} u_{ny} \right)_y \right) = 0, \tag{2.7}$$

$$v_{nt} - u_{ny} = 0, \tag{2.8}$$

with initial data

$$(w_n(y,0), u_n(y,0)) = (P_n w_{0,k}(y), P_n u_{0,k}(y)), \qquad v_n(y,0) = v_{0,k}(y). \tag{2.9}$$

Observe that (2.8) and the second equation in (2.9) allow us to obtain v_n as a functional of u_n. Setting

$$z_n(t) \equiv \int_0^t u_n(s)\, ds,$$

we obtain from system (2.6)–(2.8) a system of ordinary differential equations

$$\frac{dw_n}{dt} = \Phi_n(w_n, u_n, z_n), \tag{2.10}$$

$$\frac{du_n}{dt} = \Psi_n(w_n, u_n, z_n), \tag{2.11}$$

$$\frac{dz_n}{dt} = u_n, \tag{2.12}$$

where Φ_n, Ψ_n are determined from

$$\Phi_n \equiv iP_n \left(w_{nyy} - |w_n|^2 w_n - \alpha g(v_n) h'(|w_n|^2) w_n \right),$$
$$\Psi_n \equiv P_n \left(-p(v_n)_y + \alpha (g'(v_n) h(|w_n|^2))_y + (\frac{\varepsilon}{v_n} u_{ny})_y \right).$$

Initial values are given by

$$(w_n(0), u_n(0), z_n(0)) = ((a_0, \dots, a_n, b_1, \dots, b_n), (c_0, \dots, c_n, d_1, \dots, d_n), (0, \dots, 0, 0, \dots, 0)), \tag{2.13}$$

where $a_0, \dots, a_n, b_1, \dots, b_n, c_0, \dots, c_n, d_1, \dots, d_n$ are determined by

$$P_n w_{0,k} = \sum_{j=0}^n \left(a_j \cos \frac{j\pi y}{k} + b_j \sin \frac{j\pi y}{k} \right), \quad P_n u_{0,k} = \sum_{j=0}^n \left(c_j \cos \frac{j\pi y}{k} + d_j \sin \frac{j\pi y}{k} \right).$$

The existence and uniqueness of a solution of (2.10)–(2.13) over an interval $[0, \tau_n]$ is guaranteed by well known classical results on the theory of ordinary differential equations.

We are going to prove that there exists an interval $[0, \tau_*]$, independent of n, over which all the solutions $(w_n(t), u_n(t), v_n(t))$ of the problems (2.10)–(2.13) may be extended, uniformly with respect to $n \in \mathbb{N}$. Moreover, we have

$$\|(w_n(t), u_n(t))\|^2_{L^2(\Omega)} + \|(w_{ny}(t), u_{ny}(t))\|^2_{L^2(\Omega)} + \int_0^t \left(\|u_{nt}(s)\|^2_{L^2(\Omega)} + \|u_{nyy}(s)\|^2_{L^2(\Omega)} \right) ds \le c, \tag{2.14}$$

$$\frac{1}{2} s_1 \le v_n(t) \le 2 s_2, \tag{2.15}$$

for $t \in [0, \tau_*]$, for some constant $c > 0$ independent of $n \in \mathbb{N}$.

Indeed, multiplying (2.6) by $\bar w_n$, taking the imaginary part, and integrating over Ω, we obtain

$$\frac{d}{dt}\|w_n(t)\|^2_{L^2(\Omega)} = 0. \tag{2.16}$$

Further, multiplying (2.6) by $\bar w_{nt}$, taking the real part, and integrating over Ω, we obtain

$$\frac{d}{dt}\left(\|w_{ny}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\||w_n(t)|^4\|_{L^1(\Omega)} + \alpha\|g(v_n)h(|w_n|^2)\|_{L^1(\Omega)}\right) = \int_\Omega \alpha g(v_n)_t h(|w_n|^2)\,dy. \tag{2.17}$$

Concerning the right hand side of (2.17), multiplying (2.8) by $\alpha g'(v_n)h(|w_n|^2)$, integrating over Ω and using integration by parts, we obtain

$$\int_\Omega \alpha g(v_n)_t h(|w_n|^2)\,dy = -\alpha\int_\Omega (g'(v_n)h(|w_n|^2))_y u_n\,dy.$$

Now, the right-hand side of the above equation, can be evaluated by multiplying (2.7) by u_n, (2.8) by $p(v_*)-p(v_n)$, adding up, integrating over Ω and using integration by parts, which gives

$$-\alpha\int_\Omega (g'(v_n)h(|w_n|^2))_y u_n\,dy = -\frac{d}{dt}\int_\Omega (\frac{1}{2}|u_n|^2 + P(v_n))\,dy - \varepsilon\|\frac{u_{ny}}{v_n^{1/2}}\|^2_{L^2(\Omega)},$$

where

$$P(v) = \int_{v_*}^{v} (p(v_*)-p(s))\,ds.$$

Substituting in (2.17), we finally obtain the following identity

$$\begin{aligned}\frac{d}{dt}&\left(\|w_{ny}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} + \|P(v_n)\|_{L^1(\Omega)} + \frac{1}{2}\||w_n(t)|^4\|_{L^1(\Omega)} + \alpha\|g(v_n)h(|w_n|^2)\|_{L^1(\Omega)}\right)\\ &+\varepsilon\|\frac{u_{ny}}{v_n^{1/2}}\|^2_{L^2(\Omega)} = 0.\end{aligned} \tag{2.18}$$

In particular, we get

$$\|(w_n(t),u_n(t))\|^2_{L^2(\Omega)} + \|w_{ny}(t)\|^2_{L^2(\Omega)} + \varepsilon\int_0^t \|\frac{u_{ny}}{v_n^{1/2}}(s)\|^2_{L^2(\Omega)}\,ds \le c, \tag{2.19}$$

for some constant $c>0$ independent of n. Next, from (2.8), we obtain

$$\begin{aligned}s_1 &- (2t)^{1/2}\Big(\int_0^t \|u_{ny}(s)\|^2_{L^2(\Omega)}\,ds\Big)^{1/4}\Big(\int_0^t \|u_{nyy}(s)\|^2_{L^2(\Omega)}\,ds\Big)^{1/4}\\ &\le v_n(t) \le s_2 + (2t)^{1/2}\Big(\int_0^t \|u_{ny}(s)\|^2_{L^2(\Omega)}\,ds\Big)^{1/4}\Big(\int_0^t \|u_{nyy}(s)\|^2_{L^2(\Omega)}\,ds\Big)^{1/4},\end{aligned} \tag{2.20}$$

and, for $t\le 1$,

$$\|v_{ny}\|^2_{L^2(\Omega)} \le 3\left(\|v_{0y}\|^2_{L^2(\Omega)} + \int_0^t \|u_{nyy}(s)\|^2_{L^2(\Omega)}\,ds\right). \tag{2.21}$$

Let us define

$$A(t) := \|u_{ny}(t)\|^2_{L^2(\Omega)} + \int_0^t \frac{\varepsilon}{2s_2}\|u_{nyy}(s)\|^2_{L^2(\Omega)}\,ds.$$

We will seek for $\tau_*\in(0,1)$, independent of n, such that

$$(2\tau_*)^{1/2}\frac{(cs_2)^{1/4}}{\varepsilon^{1/2}}\sup_{t\in[0,\tau_*]} A^{1/4}(t) \le \frac{s_1}{2}, \tag{2.22}$$

so that, in particular, from (2.20) and (2.18), (2.15) will be satisfied. Now, multiplying (2.7) by $-u_{nyy}$, integrating over Ω, using integration by parts, Cauchy and Young inequalities, (2.15), (2.18), after some standard manipulation we finally arrive at

$$A'(t) \le c + cA^4(t), \tag{2.23}$$

for some $c > 0$ independent of n. More specifically, we use estimates such as

$$-\int_\Omega u_{nyy}(u_{nt} - \frac{\varepsilon}{v_n} u_{nyy})\,dy = \int_\Omega (u_{ny}u_{nyt} + \frac{\varepsilon}{v_n} u_{nyy}^2)\,dy \ge \frac{1}{2}A' + \frac{\varepsilon}{4s_2}\|u_{nyy}\|^2_{L^2(\Omega)},$$

$$\begin{aligned}\left|\int_\Omega \frac{\varepsilon}{v_n^2} v_{ny}u_{ny}u_{nyy}\,dy\right| &\le \frac{4\varepsilon}{s_1^2}\|u_{ny}\|_\infty \|u_{nyy}\|_{L^2(\Omega)}\|v_{ny}\|_{L^2(\Omega)} \\ &\le c\|u_{ny}\|^{1/2}_{L^2(\Omega)}\|u_{nyy}\|^{3/2}_{L^2(\Omega)}\|v_{ny}\|_{L^2(\Omega)} \\ &\le \frac{\varepsilon}{8s_2}\|u_{nyy}\|^2_{L^2(\Omega)} + c(\|u_{ny}\|^4_{L^2(\Omega)} + \|v_{ny}\|^8_{L^2(\Omega)}) \\ &\le \frac{\varepsilon}{8s_2}\|u_{nyy}\|^2_{L^2(\Omega)} + c(1 + A^4),\end{aligned}$$

and so on. Hence, from (2.23) and (2.22) we obtain τ_* independent of n such that (2.14) and (2.15) hold for $t \in [0, \tau_*]$, uniformly in $n \in \mathbb{N}$.

Next, since (2.14) and (2.15) are uniform in $n \in \mathbb{N}$, we may extract a subsequence, which we keep denoting by (w_n, u_n, v_n), such that $w_n \to w^{(k)}$, $u_n \to u^{(k)}$, $v_n \to v^{(k)}$ and $u_{ny} \to u_y^{(k)}$, in $L^2(\Omega \times [0, \tau_*])$ and a.e., where $(w^{(k)}, u^{(k)}, v^{(k)})$ is a generalized solution of the Cauchy problem (1.20),(1.21),(1.22),(2.5).

Now, we observe that estimates (2.14) and (2.15) hold for $(w^{(k)}, u^{(k)}, v^{(k)})$ uniformly with respect to $k \in \mathbb{N}$. Therefore, we can extract a subsequence which we keep denoting $(w^{(k)}, u^{(k)}, v^{(k)})$ such that $w^{(k)} \to w$, $u^{(k)} \to u$, $v^{(k)} \to v$ and $u_y^{(k)} \to u_y$, in $L^2_{\text{loc}}(\mathbb{R} \times [0, \tau_*])$ and a.e., where (w, u, v) is a generalized solution of (1.20)–(2.1) satisfying (2.4).

Uniqueness of the local solution of (1.20)–(2.1) satisfying (2.4), whose existence has just been proved, follows in a standard way. We consider two such solutions (w_1, u_1, v_1), (w_2, u_2, v_2), set $w = w_1 - w_2$, $u = u_1 - u_2$, $v = v_1 - v_2$, take the difference between the equations satisfied by (w_1, u_1, v_1) and (w_2, u_2, v_2), obtaining a system of three equations satisfied by (w, u, v) of the form

$$\begin{aligned} iw_t + w_{yy} &= A_1(w_1, w_2)w + A_2(w_1, w_2)\bar{w} + g(v_1)(A_3(w_1, w_2)w + A_4(w_1, w_2)\bar{w}) \\ &\quad + h'(|w_2|^2)w_2 A_5(v_1, v_2)v, \end{aligned} \tag{2.24}$$

$$\begin{aligned} u_t + (A_6(v_1, v_2)v)_y &= (A_7(v_1, v_2)h(|w_1|^2)v)_y + \alpha(g'(v_1)A_8(|w_1|, |w_2|)(|w_1| - |w_2|))_y \\ &\quad + (\frac{\varepsilon}{v_1}u_y)_y + (A_9(v_1, v_2)vu_{1y})_y, \end{aligned} \tag{2.25}$$

$$v_t - u_y = 0, \tag{2.26}$$

where $A_1, \dots, A_9$ are smooth functions of their arguments. We multiply (2.24) by $\bar{w}$ and take the imaginary part, multiply (2.25) by u and (2.26) by v, and integrate the resulting equations in $\mathbb{R} \times [0, t]$. From the last equation we obtain

$$\|v(t)\|^2 \le C\int_0^t \|u_y(s)\|^2\,ds + C\int_0^t \|v(s)\|^2\,ds, \tag{2.27}$$

where $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R})$. From the first and second equations, using integration by parts and Cauchy inequality, we obtain

$$\|w(t)\|^2 \le C\int_0^t (\|w(s)\|^2 + \|v(s)\|^2)\,ds, \tag{2.28}$$

$$\|u(t)\|^2 + \int_0^t \|u_y(s)\|^2\,ds \le C\int_0^t (H(s)\|v(s)\|^2 + \|w(s)\|^2)\,ds, \tag{2.29}$$

where we have used (2.4), $H \in L^1([0,\tau_*])$ and $t\in[0,\tau_*]$. Combining (2.27), (2.28) and (2.29), we obtain

$$\|w(t)\|^2 + \|u(t)\|^2 + \|v(t)\|^2 \le \int_0^t I(s)(\|v(s)\|^2 + \|w(s)\|^2)\,ds, \tag{2.30}$$

with $I \in L^1([0,\tau_*])$, from which it follows $\|w(t)\| = \|u(t)\| = \|v(t)\| = 0$, for $t\in[0,\tau_*]$.

□

3. Global existence

We are now going to extend to $[0,\infty)$ the time interval of existence of the local solution of (1.20)–(2.1) obtained through Theorem 2.1. For that, we need a priori estimates that we establish in the following lemmas. The first of them states the trivial extension of identities (2.16) and (2.18).

Lemma 3.1. *The local solution (w,u,v) of (1.20)–(2.1) obtained by Theorem 2.1 satisfies the identities*

$$\frac{d}{dt}\|w\|^2 = 0, \tag{3.1}$$

$$\frac{d}{dt}\left(\|w_y(t)\|^2 + \frac12\|u\|^2 + \|P(v)\|_{L^1(\mathbb{R})} + \frac12\||w(t)|^4\|_{L^1(\mathbb{R})} + \alpha\|g(v)h(|w|^2)\|_{L^1(\mathbb{R})}\right) + \varepsilon\|\frac{u_y}{v^{1/2}}\|^2 = 0. \tag{3.2}$$

In particular, from (3.1), (3.2) and the inequality $\|w(t)\|_\infty \le 2^{1/2}\|w\|^{1/2}\|w_y\|^{1/2}$, we deduce that

$$\|w(t)\|_\infty \le c, \tag{3.3}$$

for some constant $c>0$ depending only on the data.

Proof. Both identities are obtained in exactly the same way as (2.16) and (2.18). □

We now obtain an a priori pointwise bound for the specific volume v from above and away from zero.

Lemma 3.2. *There exist constants $\alpha_0, m, M > 0$, depending only on the data, such that, for $0<\alpha<\alpha_0$, we have*

$$m < v(y,t) < M, \tag{3.4}$$

for all $(y,t)\in\mathbb{R}\times[0,\tau_*]$.

Proof. We first observe that (3.3) and the fact that g'' has compact support in $(0,\infty)$ imply that there exists $\alpha_0>0$ such that for $0<\alpha<\alpha_0$ we have

$$p'(v) - \alpha g''(v)h(|w|^2) < 0.$$

To prove (3.4) we follow an idea in [12]. If we set $E_1(t) := \{y\in\mathbb{R} : v(y,t) < \frac{v_*}{2}\}$ and $E_2(t) := \{y\in\mathbb{R} : v(y,t) > 2v_*\}$, from (3.2), we obtain

$$|E_1(t)|P(\frac{v_*}{2}) \le c, \qquad |E_2(t)|P(2v_*) \le c, \tag{3.5}$$

where by $|E|$ we denote the one-dimensional Lebesgue measure of the set E, and $c > 0$ depends only on the data. Hence, given $(y_0, t_0) \in \mathbb{R} \times [0, \tau_*]$, we may find a point (y_1, t_0) with

$$|y_1 - y_0| \le c\Big(\frac{1}{P(\frac{v_*}{2})} + \frac{1}{P(2v_*)}\Big) \tag{3.6}$$

and

$$\frac{v_*}{2} \le v(y_1, t_0) \le 2v_*. \tag{3.7}$$

We now analyze the difference $\Delta L(t) = \log v(y_1, t) - \log v(y_0, t)$. We have

$$\begin{aligned} \frac{d}{dt}\Delta L &= \frac{u_y}{v}\Big|_{y_0}^{y_1} = \frac{1}{\varepsilon}\int_{y_0}^{y_1} \big(u_t + (p(v) - \alpha g'(v)h(|w|^2))_y\big)\, dy \\ &= \frac{1}{\varepsilon}\frac{d}{dt}I + \frac{1}{\varepsilon}\Delta Q, \end{aligned} \tag{3.8}$$

where $I(t) = \int_{y_0}^{y_1} u(y,t)\, dy$ and $\Delta Q = (p(v) - \alpha g'(v)h(|w|^2))\big|_{y_0}^{y_1}$. We define $\lambda(t) := \Delta Q(t)/\Delta L(t)$ and observe that λ is negative. We can write (3.8) in the form

$$\frac{d}{dt}\Delta L - \frac{\lambda}{\varepsilon}\Delta L = \frac{1}{\varepsilon}\frac{d}{dt}I. \tag{3.9}$$

We then obtain

$$L(t) - e^{\int_0^t \frac{\lambda(s)}{\varepsilon}\, ds} L(0) = \frac{1}{\varepsilon}I(t) - e^{\int_0^t \frac{\lambda(s)}{\varepsilon}\, ds}\frac{1}{\varepsilon}I(0) + \int_0^t \frac{\lambda(s)}{\varepsilon^2} e^{\int_s^t \frac{\lambda(\tau)}{\varepsilon}\, d\tau} I(s)\, ds,$$

from which it follows

$$|L(t)| \le |L(0)| + \frac{1}{\varepsilon}(|I(t)| + |I(0)|) + \int_0^t \frac{-\lambda(s)}{\varepsilon^2} e^{\int_s^t \frac{\lambda(\tau)}{\varepsilon}\, d\tau} |I(s)|\, ds,$$

which, by Gronwall's inequality gives

$$|L(t)| \le (|L(0)| + \frac{1}{\varepsilon}(|I(t)| + |I(0)|)) e^{\frac{1}{\varepsilon} e^{\int_s^t \frac{\lambda(\tau)}{\varepsilon}\, d\tau}},$$

and, since λ is negative, we get

$$|L(t)| \le (|L(0)| + \frac{1}{\varepsilon}(|I(t)| + |I(0)|)) e^{\frac{1}{\varepsilon}}.$$

Now, we have

$$|I(t)| \le |y_1 - y_0|^{1/2} \|u(t)\|_{L^2(\mathbb{R})}$$

so that (3.3) and (3.6) imply $|I(t)| \le c$, where $c > 0$ depends only on the data. Therefore, we get

$$|L(t)| \le c,$$

for some $c > 0$ depending only on the data, which, together with (3.7), implies (3.4).

□

Next, we establish an a priori estimate for $\|v_y(t)\|$.

Lemma 3.3. *Let (w, u, v) be the solution of (1.20)–(2.1) obtained by Theorem 2.1. We have*

$$\|v_y(t)\| \le c(t), \qquad t \in [0, \tau_*], \tag{3.10}$$

for some given positive function $c \in C([0, \infty))$ depending only on the data.

Proof. We follow an idea in [13] recast as in [26]. Set $\beta := u - \varepsilon \frac{v_y}{v} = u - \varepsilon(\log v)_y$. Using (1.21) and (1.22), we find

$$\beta_t = -p(v)_y + \alpha(g'(v)h(|w|^2))_y = (-p'(v) + \alpha g''(v)h(|w|^2))\frac{v}{\varepsilon}(u-\beta) + \alpha g'(v)h'(|w|^2)(|w|^2)_y.$$

Multiplying this equality by β, we obtain

$$\frac{1}{2}(\beta^2)_t + (-p'(v) + \alpha g''(v)h(|w|^2))\frac{v}{\varepsilon}\beta^2 = (-p'(v) + \alpha g''(v)h(|w|^2))\frac{v}{\varepsilon}u\beta + \alpha g'(v)h'(|w|^2)(|w|^2)_y\beta.$$

Using that $(-p'(v) + \alpha g''(v)h(|w|^2))\frac{v}{\varepsilon} > \delta$, for some $\delta > 0$, and Young's inequality, we obtain

$$\frac{1}{2}(\beta^2)_t + \frac{\delta}{2}\beta^2 \le c(u^2 + |w_y|^2),$$

which clearly implies (3.10).

□

We finally have the following a priori estimate for $\|u_y(t)\|$ and $\int_0^t \|u_{yy}(s)\|^2\,ds$.

Lemma 3.4. *Let (w,u,v) be the solution of (1.20)–(2.1) obtained by Theorem 2.1. We have*

$$\|u_y(t)\|^2 + \int_0^t \|u_{yy}\|^2\,ds \le c(t), \qquad t \in [0,\tau_*], \tag{3.11}$$

for some given positive function $c \in C([0,\infty))$ depending only on the data.

Proof. We first proceed formally assuming all functions sufficiently smooth. In this way, we multiply (1.21) by u_{yy}, integrate the resulting equation and use integration by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|u_y\|^2 + \int_{\mathbb{R}} \frac{\varepsilon}{v}u_{yy}^2\,dy = -\int_{\mathbb{R}} p(v)_y u_{yy}\,dy + \int_{\mathbb{R}} \alpha(h(|w|^2)g'(v))_y u_{yy}\,dy,$$

from which, using that $\frac{\varepsilon}{v} > \delta$, for some $\delta > 0$, and Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_y\|^2 + \frac{\delta}{2}\|u_{yy}\|^2 \le c(\|v_y\|^2 + \|w_y\|^2), \tag{3.12}$$

for some constant $c > 0$ depending only on the data, and this together with Lemmas 3.1 and 3.3 implies the assertion.

We can make the above argument rigorous by replacing v and w in (1.21) by smooth approximations v_n, w_n, we then obtain an inequality like (3.12) with u, v, w replaced by u_n, v_n, w_n, where u_n is the solution of the equation obtained from (1.21) when v and w are replaced by v_n and w_n, respectively. We then take the limit as $n \to \infty$ and use uniqueness to arrive at the desired conclusion.

□

Finally, from Lemmas 3.1–3.4 it follows immediately the global existence result below.

Theorem 3.1. *There exists a unique solution (w,u,v) of (1.20)-(2.1) satisfying*

$$\begin{gathered} w \in C([0,\infty); W^{1,2}(\mathbb{R})),\ v - v_* \in C([0,\infty); W^{1,2}(\mathbb{R})), \\ m \le v \le M, \quad u \in C([0,\infty); W^{1,2}(\mathbb{R})) \cap L^2_{loc}([0,\infty); W^{2,2}(\mathbb{R})), \\ v_t, u_t \in L^2_{loc}([0,\infty); L^2(\mathbb{R})), \end{gathered} \tag{3.13}$$

for certain constants $0 < m < M$ depending only on the data.

4. The vanishing viscosity and interaction coefficient limit.

In this section we assume that the interaction coefficient α satisfies $\alpha = O(\varepsilon^{1/2})$ and we want to analyze the vanishing viscosity process, $\varepsilon \to 0$. When there is no coupling between nonlinear Schrödinger equation describing the short wave propagation and the Navier-Stokes system describing the compressible fluid flow, so that $\alpha = 0$, we are left with only the Navier-Stokes system depending on the viscosity ε. The problem of the convergence of the vanishing viscosity sequence of solutions of the Navier-Stokes system was only solved recently by Chen and Perepelitsa [8]. We will base our analysis here on the framework developed in [8].

Since it is not possible to avoid that the solutions of the Navier-Stokes system approach the vacuum state as $\varepsilon \to 0$, the vanishing viscosity process is better understood writing the Navier-Stokes system in Eulerian coordinates, as usually done in the case of the Euler system for compressible flows with vanishing artificial (non-physical) viscosity (see, e.g., [10, 6, 20, 7] and references therein).

We then consider the inverse Lagrangian transformation $X(y,t) = (x(y,t),t)$, defined by

$$\frac{\partial x}{\partial y} = v, \qquad \frac{\partial x}{\partial t} = u \tag{4.1}$$

$$x(y,0) = \int_0^y v_0(z)\,dz. \tag{4.2}$$

We denote, as usual, $\rho(x,t) = 1/(v \circ Y(x,t))$, where $Y = X^{-1}$ is the Lagrangian transformation, and write the system (1.20)-(1.22) in the form

$$iw_t + w_{yy} = |w|^2 w + \alpha g(v) h'(|w|^2) w, \tag{4.3}$$

$$\rho_t + (\rho u)_x = 0, \tag{4.4}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x = \alpha\big(g'(\frac{1}{\rho}) h(|w \circ Y|^2)\big)_x + \varepsilon u_{xx}, \tag{4.5}$$

$$p(\rho) = \kappa \rho^\gamma, \qquad \kappa = \frac{(\gamma-1)^2}{4\gamma},\ \gamma > 1.$$

We rewrite the initial data for the transformed system (4.3)–(4.5)

$$(w(y,0), u(x,0), \rho(x,0)) = (w_0^\varepsilon(y), u_0^\varepsilon(x), \rho_0^\varepsilon(x)), \tag{4.6}$$

which satisfy

$$w_0^\varepsilon \in W^{1,2}(\mathbb{R};\mathbb{C}), \qquad u_0^\varepsilon \in W^{1,2}(\mathbb{R}), \tag{4.7}$$

and there exist $\rho_*, \rho_1^\varepsilon, \rho_2^\varepsilon$ such that $\lim_{|x|\to\infty} \rho_0^\varepsilon = \rho_*$, and

$$\rho_0^\varepsilon - \rho_* \in W^{1,2}(\mathbb{R}), \quad 0 < \rho_1^\varepsilon < \rho_0^\varepsilon(x) < \rho_2^\varepsilon,\ x \in \mathbb{R}. \tag{4.8}$$

Since X and Y are smooth with bounded derivatives $X_t, Y_t, X_y, Y_x, X_{yy}, Y_{xx}$ we can restate Theorem 3.1 in the following form.

Theorem 4.1. *For each $\varepsilon > 0$ there exists a unique solution $(w^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ of (4.3)-(4.6) satisfying*

$$\begin{gathered} w^\varepsilon \in C([0,+\infty); W^{1,2}(\mathbb{R})),\ \rho^\varepsilon - \rho_* \in C([0,+\infty); W^{1,2}(\mathbb{R})), \\ \sigma_1^\varepsilon \leqslant \rho^\varepsilon \leqslant \sigma_2^\varepsilon, \quad u^\varepsilon \in C([0,+\infty); W^{1,2}(\mathbb{R})), \\ \rho_t^\varepsilon, u_t^\varepsilon, u_{xx}^\varepsilon \in L^2_{loc}([0,+\infty); L^2(\mathbb{R})), \end{gathered} \tag{4.9}$$

for certain $0 < \sigma_1^\varepsilon < \sigma_2^\varepsilon$ depending only on the data.

For the purpose of analyzing the limit as $\varepsilon \to 0$ and $\alpha \to 0$, let us now briefly consider the limit problems, namely, the nonlinear Schrödinger equation

$$iw_t + w_{yy} = |w|^2 w, \tag{4.10}$$

with initial data

$$w(y,0) = w_0(y), \qquad w_0 \in W^{1,2}(\mathbb{R}), \tag{4.11}$$

and the Euler equations for isentropic gas dynamics

$$\rho_t + (\rho u)_x = 0, \tag{4.12}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \tag{4.13}$$

$$p(\rho) = \kappa \rho^\gamma, \qquad \kappa = \frac{(\gamma-1)^2}{4\gamma},\ \gamma > 1,$$

with initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \qquad (\rho_0, u_0) \in L^\infty(\mathbb{R};\mathbb{R}^2),\ \rho_0 \ge 0. \tag{4.14}$$

As usual we denote $m = \rho u$, which represents the the momentum.

We recall that the mechanical energy and the mechanical energy flux form a convex entropy-entropy flux pair given by

$$\eta^*(\rho,m) = \frac{1}{2}\frac{m^2}{\rho} + e(\rho), \qquad q^*(\rho,m) = \frac{1}{2}\frac{m^3}{\rho^2} + me'(\rho), \tag{4.15}$$

where

$$e(\rho) = \frac{\kappa}{\gamma-1}\rho^\gamma.$$

Let us also introduce the total mechanical energy for (4.12),(4.13)

$$E[\rho,u](t) := \int_{\mathbb{R}} \left(\eta^*(\rho,m) - \eta^*(\rho_*,0) - \nabla\eta^*(\rho_*,0)\cdot(\rho-\rho_*,m)\right)\,dx \ge 0.$$

We have

$$E[\rho,u](t) = \int_{\mathbb{R}} \left(\frac{1}{2}\rho(t,x)|u(t,x)|^2 + e^*(\rho(t,x))\right)\,dx$$

where

$$e^*(\rho) = e(\rho) - e(\rho_*) - e'(\rho_*)(\rho-\rho_*).$$

The following concept was introduced by LeFloch and Westdickenberg [16], who proved the convergence of the vanishing artificial viscosity sequence of solutions to the multidimensional isentropic Euler equations with spherical or nozzle flow symmetry.

Definition 4.1. Let (ρ_0, u_0) be given initial data with such that $E[\rho_0, u_0] \le E_0 < \infty$. A pair $(\rho, u) : \mathbb{R}\times[0,\infty) \to [0,\infty)\times\mathbb{R}$ is called a finite-energy entropy solution of the Cauchy problem (4.12)-(4.14) if the following holds:

- There is a locally bounded function $C(E,t) \ge 0$ such that, for a.e. $t > 0$,
$$E[\rho,u](t) \le C(E_0,t);$$
- (ρ,u) satisfies (4.12) and (4.13) in the sense of distributions and
$$\eta^*(\rho,u)_t + q^*(\rho,u)_x \le 0,$$
also in the sense of distributions.
- The initial data (ρ_0,u_0) is attained in the sense of distributions.

We also recall that $w : \mathbb{R}^2_+ \to \mathbb{C}$ is called a (H^{-1})weak solution of (4.10), (4.11) if $w \in L^\infty([0,\infty); W^{1,2}(\mathbb{R}))\cap W^{1,\infty}((0,\infty), H^{-1}(\mathbb{R}))$, if (4.10) is satisfied in $H^{-1}(\mathbb{R})$ for each $t>0$, and $w(0) = w_0$ in $H^{-1}(\mathbb{R})$. Existence and uniqueness of a weak solution to (4.10),(4.11) is a well known result (see [14, 5]).

We assume that $\alpha = O(\varepsilon^{1/2})$ and

(S1) $w_0^\varepsilon \rightharpoonup w_0$ as $\varepsilon \to 0$ in $W^{1,2}(\mathbb{R})$.

As to ρ_0^ε and u_0^ε we assume the following: There exist $E_0, E_1, M_0 > 0$ such that

(E1) $\rho_0^\varepsilon(x) \ge c_0^\varepsilon > 0$, $\int \rho_0^\varepsilon(x)|u_0^\varepsilon(x)|\,dx \le M_0 < \infty$;
(E2) $E[\rho_0^\varepsilon, u_0^\varepsilon] \le E_0 < \infty$;
(E3) $\varepsilon^2 \int \dfrac{|\rho_{0,x}^\varepsilon(x)|^2}{\rho_0^\varepsilon(x)^3}\,dx \le E_1 < \infty$;
(E4) $(\rho_0^\varepsilon(x), \rho_0^\varepsilon(x)u_0^\varepsilon(x)) \rightharpoonup (\rho_0(x), \rho_0(x)u_0(x))$ in the sense of distributions as $\varepsilon \to 0$, with $\rho_0(x) \ge 0$ a.e..

Conditions (E1)–(E4) are taken from [8]. By adapting the compactness framework developed in [8], for the vanishing viscosity limit of the compressible isentropic Navier-Stokes equations, we establish the following result.

Theorem 4.2. *Let $(w^\varepsilon(y,t), \rho^\varepsilon(x,t), u^\varepsilon(x,t))$ be the solution of (4.3)–(4.8) given by Theorem 3.1. Assume that $\alpha = O(\varepsilon^{1/2})$ and that (S1) and (E1)–(E4) hold as $\varepsilon \to 0$. Then, as $\varepsilon \to 0$, $w^\varepsilon(y,t)$ converges in $L^2_{loc}(\mathbb{R}^2_+)$ to the unique weak solution of (4.10),(4.11), $w(x,t)$, and we may extract a subsequence from $(\rho^\varepsilon(x,t), \rho^\varepsilon(x,t)u^\varepsilon(x,t))$ converging in $L^1_{loc}(\mathbb{R}^2_+)$ to $(\rho(x,t), \rho(x,t)u(x,t))$ such that the pair $(\rho(x,t), u(x,t))$ is an entropy finite-energy solution of (4.12)–(4.14).*

Proof. Since $w^\varepsilon(y,t)$ is uniformly bounded in $L^\infty([0,\infty); W^{1,2}(\mathbb{R}))$, an immediate application of Aubin's lemma (see, e.g., [17]) implies the compactness of $w^\varepsilon(y,t)$ in $L^2_{\rm loc}(\mathbb{R}^2)$, and the uniqueness of the weak solution of (4.10),(4.11) implies the convergence of the whole sequence $w^\varepsilon(y,t)$ to the weak solution of that problem.

As to the precompactness of $(\rho^\varepsilon(x,t), \rho^\varepsilon(x,t)u^\varepsilon(x,t))$ in $L^1_{\rm loc}(\mathbb{R}^2_+)$, we basically apply the compactness framework developed in [8], as already mentioned. We sketch the steps of this framework as follows, leaving for the Appendix the details of the adaptations in the computations in [8] that need to be made.

First, as usual, we may write (4.12),(4.13) in the standard form of a nonlinear hyperbolic system of conservation laws

$$U_t + F(U)_x = 0, \tag{4.16}$$

with $U = (\rho, m)^\top$ and $F(U) = (m, \frac{m^2}{\rho} + p(\rho))$. We recall that a pair of functions $(\eta, q) : \mathbb{R}^2 \to \mathbb{R}^2$ is called an entropy pair of system (4.16) if (η, q) satisfy

$$\nabla q(U) = \nabla\eta(U)\nabla F(U),$$

which reduces to a 2×2 hyperbolic system in the case of system (4.12),(4.13). An entropy pair for (4.12),(4.13) is said convex if $\nabla^2\eta(\rho, m) \ge 0$ and $\eta(\rho, m)$ is called a weak entropy if

$$\lim_{\substack{\rho\to0\\ \frac{m}{\rho}=\text{const.}}} \eta(\rho, m) = 0.$$

An example of a weak convex entropy pair for (4.12),(4.13) in the region $\rho \ge 0$ is provided by the mechanical energy η^* and the mechanical energy flux q^* defined in (4.15).

Any weak entropy pair (η, q) can be represented by

$$\begin{cases} \eta^\psi(\rho, \rho u) = \int_{\mathbb{R}} \chi(\rho; s-u)\psi(s)ds \\ q^\psi(\rho, \rho u) = \int_{\mathbb{R}} (\theta s + (1-\theta)u)\chi(\rho; s-u)\psi(s)ds, \theta = \frac{\gamma-1}{2}, \end{cases} \tag{4.17}$$

for any continuous function $\psi(s)$, where $\chi(\rho, u; s) = \chi(\rho; s-u)$ is determined by

$$\begin{cases} \chi_{\rho\rho} - \frac{p'(\rho)}{\rho^2}\chi_{uu} = 0 \\ \chi(0, u; s) = 0, \quad \chi_\rho(0, u; s) = \delta_{u=s} \end{cases} \tag{4.18}$$

where $\delta_{u=s}$ is the Dirac mass concentrated at $u = s$.

For the γ-law case, the weak entropy kernel χ is given by

$$\chi(\rho; s-u) = [\rho^{2\theta} - (s-u)^2]_+^\lambda, \quad \lambda = \frac{3-\gamma}{2(\gamma-1)} > -\frac{1}{2}, \tag{4.19}$$

and the corresponding weak entropy pairs are given by

$$\begin{cases} \eta^\psi(\rho, m) = \eta^\psi(\rho, \rho u) = \rho \int_{-1}^1 \psi(u + \rho^\theta s)[1-s^2]_+^\lambda ds \\ q^\psi(\rho, m) = q^\psi(\rho, \rho u) = \rho \int_{-1}^1 (u + \theta\rho^\theta s)\psi(u + \rho^\theta s)[1-s^2]_+^\lambda ds. \end{cases} \tag{4.20}$$

If we take, following the notation in [8], $\psi(s) = \psi_\sharp(s) := \frac{1}{2}s|s|$, the corresponding entropy pair $(\eta^\sharp, q^\sharp) = (\eta^{\psi_\sharp}, q^{\psi_\sharp})$ verifies, for a certain constant $C > 0$, depending only on $\gamma > 1$,

$$\left|\eta^\sharp(\rho, m)\right| \le C(\rho|u|^2 + \rho^\gamma), \qquad q^\sharp(\rho, m) \geqslant C^{-1}(\rho|u|^3 + \rho^{\gamma+\theta}) \tag{4.21}$$

$$\left|\eta^\sharp_m(\rho, m)\right| \le C(|u| + \rho^\theta), \qquad \left|\eta^\sharp_{mm}(\rho, m)\right| \leqslant C\rho^{-1} \tag{4.22}$$

and, regarding $\eta^\sharp_m$ in the coordinates (ρ, u),

$$\left|\eta^\sharp_{mu}(\rho, \rho u)\right| \le C, \qquad \left|\eta^\sharp_{m\rho}(\rho, \rho u)\right| \le C\rho^{\theta-1}, \tag{4.23}$$

for $\rho \geqslant 0$ and $u \in \mathbb{R}$ (cf. [21] and also [8]).

The following lemma of [8] plays an important role in the proof of the precompactness of the sequence $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$.

Lemma 4.1 (Chen and Perepelitsa [8], Lemma 2.1). *For a C^2 function $\psi : \mathbb{R} \to \mathbb{R}$, compactly supported on the interval $[a, b]$, we have*

$$\operatorname{supp} \eta^\psi, \ \operatorname{supp} q^\phi \ \subseteq \ \{(\rho, m) \, : \, \rho^\theta + u \ge a, \ u - \rho^\theta \le b\}.$$

Furthermore, there exists a constant $C_\psi > 0$ such that, for any $\rho \ge 0$, and $u \in \mathbb{R}$, we have

(i) *For $\gamma \in (1, 3]$,*
$$|\eta^\psi(\rho, m)| + |q^\psi(\rho, m)| \le C_\psi \rho;$$

(ii) *For $\gamma > 3$,*
$$|\eta^\psi(\rho, m)| \le C_\psi \rho, \qquad |q^\psi(\rho, m)| \le C_\psi \rho \max\{1, \rho^\theta\};$$

(iii) *If η^ψ is considered as a function of (ρ, m), then*
$$|\eta^\psi_m(\rho, m)| + |\rho\eta^\psi_{mm}(\rho, m)| \le C_\psi;$$
and, if η^ψ_m is considered as a function of (ρ, u), then
$$|\eta^\psi_{mu}(\rho, \rho u)| + |\rho^{1-\theta}\eta^\psi_{m\rho}(\rho, \rho u)| \le C_\psi.$$

The proof of the precompactness of $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ is based on the compensated compactness method introduced by Tartar [27] and Murat [22] and developed for applications to systems of hyperbolic conservation laws by DiPerna [10, 11]. More specifically, it extends the results of DiPerna [10], Chen [6], Lions, Perthame and Tadmor [21] and Lions, Perthame and Souganidis [20] proving the precompactness of the vanishing artificial viscosity sequence of solutions, for the Euler equations for isentropic gas flows with artificial viscosity, in which case the sequence $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ is uniformly bounded in $L^\infty(\mathbb{R}^2_+)$. However, for the vanishing viscosity sequence of solutions to the Navier-Stokes equations for isentropic gas flows, the sequence $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ is *not* uniformly bounded in $L^\infty(\mathbb{R}^2_+)$, and one needs to make highly nontrivial improvements in the framework developed through the cited works [10, 6, 21, 20], which was achieved recently in the work by Chen and Perepelitsa [8]. In this connection, we mention that the similar difficulty in the case of the system of nonlinear elasticity with physical viscosity was overcome by Serre and Shearer in [25].

The following result is the analogue of Proposition 4.1 of [8], which is a key point in the framework developed therein for the proof of the precompactness of the sequence $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$.

Proposition 4.1. *Let $(w^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ be the solution of (4.3)–(4.6) given by Theorem 4.2, with $\alpha = O(\varepsilon^{1/2})$, and $m^\varepsilon = \rho^\varepsilon u^\varepsilon$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be any compactly supported C^2 function. Let (η^ψ, q^ψ), and consider the weak entropy pair (η^ψ, q^ψ) generated by ψ. Then, for $0 < \varepsilon \le \varepsilon_0$, the entropy dissipation measures*

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \quad \textit{belong to a compact of } W^{-1,2}_{loc}(\mathbb{R}^2_+). \tag{4.24}$$

The proof of Proposition 4.1 follows the same lines as the proof of Proposition 4.1 of [8]. As for the latter, it is based on 4 energy estimates which correspond to Lemmas 3.1, 3.2, 3.3 and 3.4 of [8]. Using the equations (4.3)–(4.5), it is easy to see that the entropy dissipation measures satisfy

$$\begin{aligned}&\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \\ &= \alpha\eta^\psi_m\big(g'(\frac{1}{\rho^\varepsilon})h(|w^\varepsilon \circ Y^\varepsilon|^2)\big)_x + \varepsilon(\eta^\psi_m(\rho^\varepsilon, m^\varepsilon)u^\varepsilon_x)_x - \varepsilon\eta^\psi_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2 - \varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x,\end{aligned} \tag{4.25}$$

where $\eta^\psi_{m\rho}(\rho, \rho u) = \partial_\rho\big(\eta^\psi_m(\rho, \rho u)\big)$ and $\eta^\psi_{mu}(\rho, \rho u) = \partial_u\big(\eta^\psi_m(\rho, \rho u)\big)$. As in [8], the aim of the 4 energy estimates is to prove that each term on the left-hand side of (4.25) is bounded in $W^{-1,q^*}_{\text{loc}}(\mathbb{R}^2_+)$ uniformly in ε, for some $q^* > 2$, and that each term on the right-hand side of (4.25) belongs to a compact in $W^{-1,q_*}_{\text{loc}}(\mathbb{R}^2_+)$, independent of ε, for some $1 < q_* < 2$, and then apply the usual interpolation argument (see, e.g., [10]) to conclude (4.24).

The first energy estimate follows directly from the one-dimensional version of (1.18) whose rigorous proof proceeds exactly through the same lines as the formal proof given in the Introduction for the multidimensional case. For simplification of notation, we will denote $\int = \int_{\mathbb{R}}$, $(w, \rho, u) = (w^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ and $C > 0$ will be a universal constant independent of ε.

Lemma 4.2. *Assume that*

$$E[\rho_0, u_0] + \int |w_{0,y}|^2 dy + \int |w_0|^2 dy \le E_0 < +\infty, \quad E_0 \textit{ independent of} \varepsilon.$$

Then there exists $C_0 = C_0(E_0, t, \rho_*)$ such that , for $\varepsilon \le 1$,

$$\sup_{\tau\in[0,t]} E[\rho, u](\tau) + \int |w_x|^2 dy + \frac{1}{2}\int |w|^4 dy + \varepsilon\int_0^t \int u_x^2 dx\, d\tau \le C_0. \tag{4.26}$$

The Lemma 4.2 together with the boundedness of η^ψ_m given by Lemma 4.1 imply immediately that the second term on the right-hand side of (4.25) converges to 0 in $W^{-1,2}_{\text{loc}}(\mathbb{R}^2_+)$ as $\varepsilon \to 0$.

We leave to the Appendix the outline of the proof of the other three main estimates analogous to those in Lemmas 3.2, 3.3 and 3.4 in [8]. These estimates prove the boundedness in $W^{-1,q^*}_{\text{loc}}(\mathbb{R}^2_+)$ of the left-hand side of (4.25), for some $q^* > 2$, as well as that the first, third and fourth terms on the right-hand side of (4.25) belong to a compact in $W^{-1,q_*}_{\text{loc}}(\mathbb{R}^2_+)$, for some $q_* \in (1, 2)$.

Because of the presence now of the interaction term $\alpha\big(g'(\frac{1}{\rho})h(|w \circ Y|^2)\big)_x$ in (4.5), giving rise to the first term on the right-hand side of (4.25), for the proof of Proposition 4.1 we also need to observe that for any entropy pair (η^ψ, q^ψ) we have that

$$\alpha\eta^\psi_m\,\big(g'(\frac{1}{\rho^\varepsilon})h(|w^\varepsilon \circ Y^\varepsilon|^2)\big)_x \text{ is bounded in } L^1_{\text{loc}}(\mathbb{R}^2_+) \text{ uniformly w.r.t. } \varepsilon. \tag{4.27}$$

This follows from (4.26), the boundedness of η_m given by Lemma 4.1, the properties of g and h, the fact that $\alpha = O(\varepsilon^{1/2})$, and the analogue of Lemma 3.2 of [8] (see Appendix below). In particular, (4.27) implies that

$$\alpha\eta^\psi_m\,\big(g'(\frac{1}{\rho^\varepsilon})h(|w^\varepsilon \circ Y^\varepsilon|^2)\big)_x \text{ belongs to a compact in } W^{-1,q}_{\text{loc}}(\mathbb{R}^2_+)\ , \tag{4.28}$$

for some $1 < q < 2$.

Another fundamental ingredient in the compensated compactness theory is the parameterized family of Young measures, that is, a weakly measurable mapping from $\mathbb{R}_+^2$ into the space of probability measures on some compact space where all functions of the sequence, whose precompactness is under study, take their values. In the case of the Euler equations with artificial viscosity studied in [10, 6, 21, 20], for any $\varepsilon > 0$, $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ assumes values in a compact of $\mathcal{H} = \{(\rho, u) \in \mathbb{R}^2 : \rho \ge 0\}$ independent of ε, due to the existence of a positively invariant domain. In the present case, as in [8] for the vanishing viscosity sequence of solutions to the Navier-Stokes equations, this is no longer true and it is necessary to define a compactification of $\mathcal{H}$, denoted $\bar{\mathcal{H}}$, which will clearly contain the values of the sequence $(\rho^\varepsilon, u^\varepsilon)$, through the natural imbedding $\mathcal{H} \hookrightarrow \bar{\mathcal{H}}$, and at which all entropy pairs used in the reduction argument can be defined after extension. The compactification $\bar{\mathcal{H}}$ is defined as the completion of $\mathcal{H}$ in the weak topology generated by the family Λ of continuous functions $\phi \in C(\mathcal{H})$ satisfying

(1) $\phi(\rho, u)$ is constant on the vacuum line $\{\rho = 0\}$;
(2) the map $(\rho, u) \mapsto \lim_{s\to\infty} \phi(s\rho, su)$ belongs to $C(\mathbb{S}^1) \cap \mathcal{H}$,

where $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is the unit circle. We notice that the space $\bar{\mathcal{H}}$ so defined is clearly a separable compact space, since any function in Λ is the uniform limit of a sequence of bounded continuous functions on $\mathcal{H}$, each of which coinciding with a radially symmetric function outside some ball $B(0; r)$, with $r > 0$. It is also a Hausdorff space modulo the identification of the points in the vacuum line $\{\rho = 0\}$ to a single point. In particular, by the classical Stone's theorem we may see $\mathcal{H}$ as densely imbedded in $\bar{\mathcal{H}}$ and Λ is isometrically isomorphic to $C(\bar{\mathcal{H}})$.

Then we can use the well known existence of Young measures theorem (see, e.g., [27, 1, 3]) to obtain that, given any sequence of measurable functions $(\rho^\varepsilon, u^\varepsilon) : \mathbb{R}_+^2 \to \bar{\mathcal{H}}$, there exists a subsequence, still denoted $(\rho^\varepsilon, u^\varepsilon)$, and a weakly measurable mapping from $\mathbb{R}_+^2$ to $\mathrm{Prob}(\bar{\mathcal{H}})$, $(x,t) \mapsto \nu_{x,t}$, such that for all $\phi \in C(\bar{\mathcal{H}})$,

$$\phi(\rho^\varepsilon(x,t), u^\varepsilon(x,t)) \overset{*}{\rightharpoonup} \int_{\bar{\mathcal{H}}} \phi(\rho, u)\, d\nu_{x,t} \qquad \text{in } L^\infty(\mathbb{R}_+^2). \tag{4.29}$$

For our sequence $(\rho^\varepsilon, u^\varepsilon)$ the proof that $\nu_{x,t}$ satisfies

$$\nu_{x,t} = \delta_{(\rho(x,t), \rho(x,t)u(x,t))} \tag{4.30}$$

follows by repeating line by line the analysis carried out in sections 5, 6 and 7 of [8] to which we refer. Finally, (4.30) implies in a standard way the precompactness of $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ in $L^1_{\mathrm{loc}}(\mathbb{R}_+^2)$ concluding the proof of Theorem 4.2.

□

Appendix A. Energy estimates

In this appendix we briefly describe the adaptations that should be made in the computations in [8] in order to get the analogues of Lemmas 3.2, 3.3 and 3.4 of [8].

We first show how to adapt Lemma 3.2 in [8] to our system.

Lemma A.1. *Let w_0, u_0, ρ_0 be such that*

$$\varepsilon^2 \int \frac{|\rho_{0,x}|^2}{\rho_0^3}\, dx \leqslant C_1 < +\infty,$$

when C_1 is independent of ε. Then there exists $C = C(E_0, C_1, \gamma, t) > 0$ independent of ε such that for any $t > 0$ and for $\varepsilon \le \varepsilon_0$, independent of t,

$$\varepsilon^2 \int \frac{|\rho_y(t,x)|^2}{\rho(t,x)^3} dx + \varepsilon \int_0^t \int \rho^{\gamma-3} |\rho_x|^2 dx\, d\tau \le C.$$

Proof. By abuse of notation, we set $v(x,t) = 1/\rho(x,t)$. As in the proof of Lemma 3.2 in [8], we derive from (4.4), (4.5),

$$2v_x u_{xx} = \frac{2}{\varepsilon} v_x[\rho_{1,x} + (\rho u)_t + (\rho u^2)_x] - \frac{2\alpha}{\varepsilon} v_x (g'(v)h(|w|^2))_x$$

and by the hypothesis on g and h, we obtain, using that $\alpha = O(\varepsilon^{1/2})$, for $\varepsilon \le \varepsilon_0$,

$$\varepsilon^2 \left| \frac{2\alpha}{\varepsilon} \int_0^t \int v_x (g''(v) v_x h(|w|^2)) dx\, d\tau \right| \le \frac{\varepsilon}{8}(\gamma-1)^2 \int_0^t \int \rho^{\gamma-3} |\rho_x|^2 dx\, d\tau$$

and

$$\begin{aligned} &\varepsilon^2 \left| \frac{2\alpha}{\varepsilon} \int_0^t \int v_x (g'(v) h'(|w|^2) Re(w \bar{w}_x)) dx\, d\tau \right| \leqslant \\ &\leqslant \frac{\varepsilon}{8}(\gamma-1)^2 \int_0^t \int \rho^{\gamma-3} |\rho_x|^2 dx\, d\tau + C \int_0^t \int |w_x|^2 dx\, d\tau \\ &\leqslant \frac{\varepsilon}{8}(\gamma-1)^2 \int_0^t \int \rho^{\gamma-3} |\rho_x|^2 dx\, d\tau + C_0 t, \end{aligned}$$

by (4.26) in Lemma 4.2 . The remainder of the proof is similar to the proof of Lemma 3.2 in [8]. □

The following two Lemmas are the correspondent to Lemmas 3.3 and 3.4 in [8] on the higher integrability of the solutions.

Lemma A.2. *Under the hypothesis of Lemma A.1, for any $-\infty < a < b < +\infty$ and all $t > 0$, there exists $C = C(a, b, E_0, \gamma, \rho_*, t) > 0$ independent of ε such that, for $\varepsilon \leqslant \varepsilon_1$, independent of a, b and t,*

$$\int_0^t \int_a^b \rho(t,x)^{\gamma+1} dx\, d\tau \leqslant C.$$

Proof. If $\vartheta(x) \in \mathcal{D}(\mathbb{R})$, $0 \leqslant \vartheta(x) \leqslant 1$, $\vartheta(x) = 1$ in $[a,b]$ and multiply the equation (4.5) by $\vartheta(x)$. Integrating over $(-\infty, x)$ and multiplying both members by $\rho\vartheta$, we obtain, as in the proof of Lemma 3.3 in [8], by using (4.4),

$$\begin{aligned} \rho p_1 \vartheta^2 =& \varepsilon \rho u_x \vartheta^2 + \alpha \rho \vartheta^2 g'(v) h(|w|^2) - \\ &- \left(\rho\vartheta \int_{-\infty}^x \rho u \vartheta dz \right)_t - \left(\rho u \vartheta \int_{-\infty}^x \rho u \vartheta dz \right)_x + \\ &+ \rho u \vartheta_x \int_{-\infty}^x \rho u \vartheta dz + \rho\vartheta \int_{-\infty}^x [(\rho u^2 + p_1)\vartheta_x - \varepsilon u_x \vartheta_x] dz - \\ &- \alpha \rho \vartheta \int_{-\infty}^x g'(v) h(|w|^2) dz. \end{aligned}$$

If we integrate this equation over $(0,t) \times \mathbb{R}$, we derive

$$\begin{aligned} \int_0^t \int \rho p_1 \vartheta^2 dx\, d\tau =& \varepsilon \int_0^t \int \rho u_x \vartheta^2 dx\, d\tau + \alpha \int_0^t \int \rho \vartheta^2 g'(v) h(|w|^2) dx\, d\tau - \\ &- \int \rho\vartheta \left(\int_{-\infty}^x \rho u \vartheta dz \right) dx + \int \rho_0 \vartheta \left(\int_{-\infty}^x \rho_0 u_0 \vartheta dz \right) dx + \\ &+ r_1(t) - \alpha \int_0^t \int \rho\vartheta \left(\int_{-\infty}^x g'(v) h(|w|^2) \vartheta_x dz \right) dx\, d\tau, \end{aligned}$$

where

$$r_1(t) = \int_0^t \int \rho u \vartheta_x \left(\int_{-\infty}^x \rho u \vartheta dz \right) dx\, d\tau +$$
$$+ \int_0^t \int \rho \vartheta \left(\int_{-\infty}^x [(\rho u^2 + p_1)\vartheta_x - \varepsilon u_x \vartheta_x] dz \right) dx\, d\tau.$$

We have, for fixed $\delta > 0$, and $\varepsilon \leqslant 1$,

$$\left| \alpha \int_0^t \int \rho \vartheta^2 g'(v) h(|w|^2) dx\, d\tau \right| \leqslant \delta \int_0^t \int \rho^2 \vartheta^4 dx\, d\tau + + C(\delta) \int_0^t \int |(g'(v)h(|w|^2))^2 v|\, dx\, d\tau \leqslant$$
$$\leqslant \delta \int_0^t \int (1 + \rho^{\gamma+1}) \vartheta^2 dx\, d\tau + C(\delta) \int_0^t \int |w|^4 dx\, d\tau.$$

Moreover, we have

$$\left| \alpha \int_0^t \int \rho \vartheta \left(\int_{-\infty}^x g'(v) h(|w|^2) \vartheta_x dz \right) dx\, d\tau \right| \leqslant$$
$$\leqslant C\alpha \int_0^t \int \rho \vartheta (\int |w|^2 dx) dx\, d\tau \leqslant$$
$$\leqslant C\alpha \int_0^t \int \rho \vartheta dx\, d\tau \leqslant C\alpha \int_0^t \int \rho^2 \vartheta^2 dx\, d\tau +$$
$$+ C \int_0^t \int_{\operatorname{supp} \vartheta} 1 dx\, d\tau \leqslant \delta \int_0^t \int (1 + \rho^{\gamma+1}) \vartheta^2 dx\, d\tau + Ct, \text{ for } \varepsilon \leqslant \varepsilon(\delta).$$

The estimates concerning $r_1(t)$ are described in the proof of Lemma 3.3 in [8]. The Lemma follows from (4.26) and an appropriate choice of δ. ☐

Finally we prove the following result corresponding to the Lemma 3.4 in [8] with a proof which is a variant of this Lemma:

Lemma A.3. *Assume the hypothesis of Lemmas A.1 and A.2 and that there exists $M_0 > 0$, independent of ε, such that*

$$\int \rho_0 |u_0| dx \leqslant M_0 < +\infty.$$

Then for any compact set $K \subseteq \mathbb{R}$ and $t > 0$, there exists $C > 0$ independent of ε such that for $\varepsilon \leqslant \varepsilon_2$, independent of K and t,

$$\int_0^t \int_K (\rho |u|^3 + \rho^{\gamma+\theta}) dx\, dz \leqslant C,$$

where $\theta = \frac{\gamma-1}{2}$.

Proof. We recall that the weak entropy pair $\eta^\sharp(\rho, m)$, $q^\sharp(\rho, m)$ satisfies the estimates (4.21)-(4.23) and $\eta^\sharp(\rho, m) = \alpha_1 \rho^{\theta+1} u + r_2(\rho, \rho u)$, with $|r_2(\rho, \rho u)| \leqslant C\rho |u|^2$.

If we multiply (4.4) by $\eta^\sharp_\rho$ and (4.5) by $\eta^\sharp_m$, we derive by adding and integrating the result over $(0,t)\times(-\infty,x)$,

$$\begin{aligned}
&\int_{-\infty}^{x}(\eta^\sharp(\rho,m)-\eta^\sharp(\rho_0,m_0)dz+\int_0^t q^\sharp(\rho,\rho u)d\tau-tq^\sharp(\rho^*,0)-\\
&-\varepsilon\int_0^t \eta^\sharp_m u_x d\tau-\alpha\int_0^t \eta^\sharp_m(g'(v)h(|w|^2))d\tau+\\
&+\varepsilon\int_0^t\int_{-\infty}^{x}\eta^\sharp_{mu}|u_x|^2dz\,d\tau+\alpha\int_0^t\int_{-\infty}^{x}\eta^\sharp_{mu}u_x(g'(v)h(|w|^2))dz\,d\tau\\
&+\varepsilon\int_0^t\int_{-\infty}^{x}\eta^\sharp_{m\rho}\rho_x u_x dz\,d\tau+\alpha\int_0^t \eta^\sharp_{m\rho}\rho_x(g'(v)h(|w|^2))dz\,d\tau=\\
&=0.
\end{aligned}$$

By (4.22), (4.23) and (4.26), we derive, since $\alpha=0(\varepsilon^{\frac12})$ and for $\varepsilon\leqslant\varepsilon_2$,

$$\begin{aligned}
&\left|\alpha\int_0^t\int_{-\infty}^{x}\eta^\sharp_{mu}u_x(g'(v)h(|w|^2))dz\,d\tau\leqslant\right.\\
&\leqslant\varepsilon\int_0^t\int|u_x|^2dx\,d\tau+C\int_0^t\int|w|^4dx\,d\tau\leqslant C(t)\\
&\left|\alpha\int_0^t\int\eta^\sharp_{m\rho}\rho_x(g'(v)h(|w|^2))dz\,d\tau\leqslant\right.\\
&\leqslant\alpha\int_0^t\int\rho^{\frac{\gamma-3}{2}}|\rho_x|(g'(v)h(|w|^2))dx\,d\tau\leqslant\\
&\leqslant\varepsilon\int_0^t\rho^{\gamma-3}|\rho_x|^2dx\,d\tau+C\int_0^t\int|w|^4dx\,d\tau\leqslant C(t)
\end{aligned}$$

by (4.26) and Lemma A.2.

Moreover we have, for the compact set K,

$$\begin{aligned}
&\alpha\int_0^t\int_K\left|\eta^\sharp_m(g'(v)h(|w|^2))\right|\,dx\,d\tau\leqslant\\
&\leqslant\alpha C\int_0^t\int_K\left|v^{\frac12}g'(v)h(|w|^2)(\rho^{\frac12}|u|+\rho^{\frac12}\rho^\theta)\right|\,dx\,d\tau\leqslant\\
&\leqslant C\int_0^t\int|w|^4dx\,d\tau+\varepsilon\int_0^t\int_K(\rho|u|^2+\rho^\gamma)dx\,d\tau.
\end{aligned}\tag{A.1}$$

We have

$$\begin{aligned}
&\int_K\rho|u|^2dx\leqslant C,\\
&\int_K\rho^\gamma dx\leqslant C\int_K e^*(\rho)dx+C\int_K\rho dx+C
\end{aligned}$$

and

$$C\int_K\rho dx\leqslant\frac12\int_K\rho^\gamma dx+C.$$

Hence,

$$\int_K \rho^\gamma dx \leqslant C \int e^*(\rho)dx + C$$

and so, by (4.26), we deduce from (A.1)

$$\alpha \int_0^t \int_K \left|\eta_m^\sharp(g'(v)h(|w|^2)\right)dx\,d\tau \leqslant C(t)\ .$$

The remainder of the proof follows the line of the proof of Lemma 3.4 in [8]. ☐

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