Radial solutions of Dirichlet problems with concave-convex nonlinearities *

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Abstract

We prove the existence of a double infinite sequence of radial solutions for a Dirichlet concave-convex problem associated with an elliptic equation in a ball of $\mathbb{R}^n$. We are interested in relaxing the classical positivity condition on the weights, by allowing the weights to vanish. The idea is to develop a topological method and to use the concept of rotation number. The solutions are characterized by their nodal properties.

1 Introduction

In this paper we are concerned with the Dirichlet problem

$$\begin{cases} \Delta u(x) + q(|x|)\,|u(x)|^{\delta-1}\,u(x) + p(|x|)\,|u(x)|^{\gamma-1}\,u(x) = 0 & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is the unit ball in $\mathbb{R}^N$ with $N \geq 3$ and $p, q : [0,1] \to \mathbb{R}$ are C^1 functions. Moreover, we assume $0 < \gamma < 1 < \delta$; hence, the nonlinearity we are dealing with is of concave-convex type.

We are interested in the search of solutions of (1.1) with prescribed nodal properties. This kind of problem has been intensively studied in literature; for a quite exhaustive bibliography, we refer to [6], [9], [19], [33] and references therein.

The nonlinearity we are studying combines two different aspects; on the one hand it is superlinear at infinity, on the other hand it is sublinear near zero. As a consequence,

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the problem inherits both the qualitative features of superlinear and sublinear problems. Starting from the pioneering paper [3], many authors have studied this problem, under various assumptions on the coefficients q and p. The main question is to prove the existence of four sequences of nodal solutions of (1.1), two of them with large norm and two with small norm.

This has been proved, for instance, in [9] and [21, Theorem 1.5] when q and p are strictly positive. In the one-dimensional setting, we wish to quote, among others, the contribution of [1], [11] and [29, Theorem 4], providing a detailed study of the exact structure of the solutions in presence of positive constant weights.

The paper [1] treats also the p-strictly-negative case, showing that, in this context, no more than two sequences of solutions can be achieved. More in general, the positiveness of each weight on a subset of Ω of non-zero measure is a necessary condition to guarantee the existence of four sequences of solutions. It is still an open question whether this condition is also sufficient.

As for multiplicity of positive solutions, in the more general case of nonradial weights q and p, an answer to this question has been recently obtained in [7], [15, Corollary 2.2], [16, Theorem 4.2] and [17, Theorem 6]. These papers provide the existence of two positive solutions in the case where the weights are allowed to vanish and change sign in Ω. We wish to remark that there is an extensive literature concerning multiplicity results of positive solutions to problem (1.1) under relaxed sign assumptions on one of the weights (cf., for instance [2], [18, Theorem 6.9], [21, Theorem 1.1], [23], [27], [37], [41]). For interesting extensions to concave-convex systems, we refer to [34, Theorem 1.2 (d)].

Focusing our attention on the superlinear and sub-critical structure of problem (1.1) and taking into account the wide literature about superlinear nonlinearities at infinity (cf., among others, the classical paper [38]), we immediately deduce the existence of two sequences of large norm solutions for (1.1), provided that q is positive, regardless of the sign of the weight p. In this setting, we quote, among many others, the papers [21, Theorem 1.5 (I)-(ii)], [26], [28], [31], [36].

In the superlinear context, interesting multiplicity results have been also achieved under relaxed sign conditions on the weight q. In particular, we refer to [13], dealing with a possibly vanishing coefficient q and to [5], [39] and reference therein, dealing with nonradial and indefinite (changing sign) weights. More in detalis, in [5] q is supposed to have a "thick" zero set, while in [39] the authors assume that $\nabla q(x) \neq 0$ for every $x \in \Omega$ with $q(x) = 0$.

Analogously, privileging the sublinear aspect of (1.1), many authors have proved the existence of two sequences of small norm solutions, provided that p is positive, regardless of the sign of q (cf., among many others, [9], [22], [25], [40], [42, Theorem 3.3]).

The list of result available in literature concerned with relaxed sign conditions on the coefficient p in a sublinear context is shorter. Some existence result in presence of indefinite weights can be found in [4] and [32, Theorem 2.3]. We wish also to mention the work [30] concerned with multiple solutions to the equation (1.1) in $\mathbb{R}^n$ when p is allowed to vanish.

Our aim consists in proving the existence of changing sign solutions in the case where the coefficients q and p can vanish.
In particular, we assume that

$$q \succ 0, \qquad p \succ 0, \tag{1.2}$$

where, given a continuous function $\varphi : [0,1] \to \mathbb{R}$, by $\varphi \succ 0$ we mean that $\varphi(r) \geq 0$ for every $r \in [0,1]$ and $\varphi \not\equiv 0$.
According to [13], we also require that both q and p satisfy some regularity conditions in a neighbourhood of $r = 0$ and of their zeros. More precisely, we first suppose that one of the following alternatives holds

$$q(0) > 0 \tag{1.3}$$

or

$$\exists \beta > 0,\ h \geq 1 :\ q(r) \sim \beta r^h, \quad \text{as} \ \ r \to 0^+. \tag{1.4}$$

Analogously, we assume that

$$p(0) > 0 \tag{1.5}$$

or

$$\exists \alpha > 0,\ k \geq 1 :\ p(r) \sim \alpha r^k, \quad \text{as} \ \ r \to 0^+. \tag{1.6}$$

Finally, denoting by $\mathcal{C}_0$ and $\mathcal{C}_1$ the set of zeros of q and p in $(0,1]$, respectively, i.e.

$$\mathcal{C}_0 := \{r \in (0,1] :\ q(r) = 0\} \qquad \text{and} \qquad \mathcal{C}_1 := \{r \in (0,1] :\ p(r) = 0\},$$

we assume that

$$\mathcal{C}_0 \text{ and } \mathcal{C}_1 \text{ have a finite number of connected components,} \tag{1.7}$$

denoted respectively by $I_i = [\widetilde{r}_i, r_i]$ and $J_j = [\widetilde{s}_j, s_j]$ with $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$;

$$\begin{aligned}
q' &\leq 0 \quad \text{in a left neighborhood of } \widetilde{r}_i, && (1.8)\\
q' &\geq 0 \quad \text{in a right neighborhood of } r_i, && (1.9)\\
p' &\leq 0 \quad \text{in a left neighborhood of } \widetilde{s}_i, && (1.10)\\
p' &\geq 0 \quad \text{in a right neighborhood of } s_i. && (1.11)
\end{aligned}$$

Our purpose will be attained in two steps; first of all, we will concentrate in the search of solutions with large C^1-norm. In this context, one more assumption involving the superlinear term is needed. More precisely, we require the nonlinearity to be subcritical by setting

$$\delta < \frac{N+2}{N-2}, \qquad \text{if condition (1.3) is satisfied.} \tag{1.12}$$

Condition (1.12) can be relaxed by assuming

$$\delta < \frac{N+2+2h}{N-2}, \qquad \text{if condition (1.4) holds.} \tag{1.13}$$

Following [13], by means of a shooting approach, we are able to prove the following result

Theorem 1.1 *Consider* $p,\ q \in C^1([0,1],\mathbb{R})$ *satisfying* (1.2)*, and one of the alternatives* (1.3)-(1.6) *associated with* (1.12)-(1.13)*. Moreover, assume* (1.7)-(1.11)*. Then, there exist* $j_0 \in \mathbb{N}$ *and* $\mathcal{L} > 0$ *such that for every integer* $n \geq j_0$ *there exist two radial solutions* u_n *and* v_n *of* (1.1)*, satisfying* $u_n(0) > \mathcal{L}$ *and* $v_n(0) < -\mathcal{L}$ *and*

$$\lim_{n\to+\infty} \|u_n\|_1 = +\infty, \qquad \lim_{n\to+\infty} \|v_n\|_1 = +\infty.$$

On the other hand, in the spirit of [9], we can prove the validity of the following

Theorem 1.2 *Consider* $p,\ q \in C^1([0,1],\mathbb{R})$ *satisfying* (1.2)*, and one of the alternatives* (1.3)-(1.6) *associated with* (1.12)-(1.13)*. Moreover, assume* (1.7)-(1.11)*. Then, there exist* $j_0 \in \mathbb{N}$ *and* $\mathcal{L} > 0$ *such that for every integer* $n \geq j_0$ *there exist two radial solutions* z_n *and* w_n *of* (1.1)*, satisfying* $0 < z_n(0) < \mathcal{L}$ *and* $-\mathcal{L} < w_n(0) < 0$*.*

As a consequence, we get the existence of four sequences of solutions of (1.1).
We remark that the same result can be obtained for the more general problem

$$\begin{cases} \Delta u(x) + q(|x|)\,g(u) + p(|x|)\,f(u) = 0 & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.14}$$

Indeed, we can prove results analogous to Theorem 1.1 and Theorem 1.2 for (1.14) when $f \in C(\mathbb{R})$ satisfies

$$\lim_{u\to 0} \frac{f(u)}{u} = +\infty$$

and $g \in C(\mathbb{R})$ fulfills

$$\lim_{|u|\to+\infty} \frac{g(u)}{u} = +\infty$$

and a suitable sub-critical growth condition at infinity (see Theorem 2.1 and Theorem 2.4 in [13] for the details on the precise assumptions on g).
In the proof, for the sake of simplicity, we will consider the easier problem (1.1).

2 Preliminary notation

We focus our attention on the boundary value problem

$$\begin{cases} \left(r^{N-1}\,u'(r)\right)' + r^{N-1}\,|u(r)|^{\delta-1}u(r)\,q(r) + r^{N-1}\,|u(r)|^{\gamma-1}u(r)\,p(r) = 0, \\ u'(0) = 0 = u(1), \end{cases} \tag{2.1}$$

where $p,\, q \in C^1([0,1],\mathbb{R})$, $N \geq 3$ and $0 < \gamma < 1 < \delta$. We are interested in the search of its solutions, since, by setting $r = |x|$, it is immediate to notice that each solution to (2.1) is a radial solution of the Dirichlet problem (1.1).
To this purpose, let us concentrate on the following Cauchy problem

$$\begin{cases} \left(r^{N-1}\, u'(r)\right)' \,+\, r^{N-1}\, |u(r)|^{\delta-1}\, u(r)\, q(r) \,+\, r^{N-1}\, |u(r)|^{\gamma-1}\, u(r)\, p(r) \,=\, 0 \\ u'(0) = 0, \qquad u(0) = d, \end{cases} \tag{2.2}$$

where d varies in $\mathbb{R}$. We denote by u_d a solution of (2.2) and by $[0, \rho_{u_d}) \subset [0,1]$ its maximal interval of definition.

We devote the following sections to prove the global existence of solutions for problem (2.2), some uniqueness result and the validity of some "elastic properties".
Our idea consists in generalizing the techniques developed in [13] to the case where the presence of a sublinear term is allowed. To this aim, let us first introduce the constant $R_1 < 1$ (depending on our assumptions (1.3)-(1.6)) by the following remark

Remark 2.1 *From condition* (1.3) *combined with the regularity assumptions on* q*, we infer the existence of* $\varepsilon > 0$, $q_0 > 0$, $R_1 \in (0,1)$, *such that*

$$\frac{r\, q'(r)}{q(r)} \geq -\varepsilon \qquad \forall\, r \in (0, R_1], \tag{2.3}$$

$$q(r) \geq q_0 \qquad \forall\, r \in [0, R_1]. \tag{2.4}$$

Furthermore, from condition (1.4)*, we infer the existence of* $\varepsilon \in (0,1)$, $R_1 \in (0,1)$ *such that*

$$h - \varepsilon \leq \frac{r\, q'(r)}{q(r)} \leq h + \varepsilon, \quad \forall\, r \in (0, R_1]. \tag{2.5}$$

Analogously, according to assumption (1.5)*, it is not restrictive to choose* $\varepsilon > 0$, $p_0 > 0$, $R_1 \in (0,1)$*, such that*

$$\frac{r\, p'(r)}{p(r)} \geq -\varepsilon \qquad \forall\, r \in (0, R_1], \tag{2.6}$$

$$p(r) \geq p_0 \qquad \forall\, r \in [0, R_1]. \tag{2.7}$$

Furthermore, according to condition (1.6)*, it is not restrictive to choose* $\varepsilon \in (0,1)$, $R_1 \in (0,1)$ *such that*

$$k - \varepsilon \leq \frac{r\, p'(r)}{p(r)} \leq k + \varepsilon, \quad \forall\, r \in (0, R_1]. \tag{2.8}$$

Condition (2.5) *and* (2.8) *guarantee respectively that* q *and* p *are strictly increasing in* $(0, R_1]$. *Hence, recalling* (2.4) *and* (2.7), *we conclude that the alternatives in* (1.3)-(1.6) *ensure the existence of* $q_0 > 0$ *and* $p_0 > 0$ *such that*

$$q(r) \geq q_0 \qquad \textit{and} \qquad p(r) \geq p_0 \qquad \forall\, r \in [R_0, R_1]. \tag{2.9}$$

where $R_0 \in (0, R_1)$ *is fixed.*

We remark that the definitions of ε, q_0, R_1 *should be further refined. The additional refinements are described in the next part of this section and in the proofs of the main lemmas of Section* 5 (*cf.* (5.5)-(5.6), (5.9)-(5.10) *when* $h > k$, *and* (5.12)-(5.13)).

Note that conditions (1.12) and (1.13) are respectively equivalent to

$$\frac{N-2}{2} < \frac{N}{\delta+1} \quad \text{and} \quad \frac{N-2}{2} < \frac{N+h}{\delta+1}.$$

We point out that $\gamma < 1 < \frac{N+2}{N-2}$ or, equivalently, $\frac{N-2}{2} < \frac{N}{\gamma+1}$.
Without loss of generality, according to the previous inequalities, we can choose $\varepsilon > 0$ and $R_1 \in (0, 1)$ in Remark 2.1 satisfying also

$$\frac{N-2}{2} < \frac{N-\varepsilon}{\delta+1} \qquad \text{under assumption (1.12)} \tag{2.10}$$

$$\frac{N-2}{2} < \frac{N+h-\varepsilon}{\delta+1} \quad \text{and} \quad \frac{N-2}{2} < \frac{N-\varepsilon}{\gamma+1} \qquad \text{under assumption (1.13).} \tag{2.11}$$

Before proceeding, we need to introduce some useful notation. We define the following positive constants

$$Q := \max_{r\in[0,1]} q(r), \qquad P := \max_{r\in[0,1]} p(r) \tag{2.12}$$

$$Q' := \max_{r\in[0,1]} |q'(r)|, \qquad P' := \max_{r\in[0,1]} |p'(r)|. \tag{2.13}$$

Given $\varphi \in C([0,1], \mathbb{R})$ with $\varphi \succ 0$, let us set

$$\Phi_\varphi(r) := \int_0^r s^{N-1}\varphi(s)\, ds, \qquad \Psi_\varphi(r) := \int_0^r \frac{\int_0^s t^{N-1}\varphi(t)\, dt}{s^{N-1}}\, ds \qquad r \in (0,1]. \tag{2.14}$$

It is immediate to note that Φ_φ is nondecreasing in $[0,1]$. Moreover, if φ is positive on a right neighbourhood of 0, then $\Psi_\varphi(r)$ is striclty increasing and, consequently, invertible in $(0,1]$. Observe also that, as $r \to 0^+$,

$$\Phi_q(r) \sim \frac{q(0)}{N}\, r^N, \quad \Psi_q(r) \sim \frac{q(0)}{2N}\, r^2, \quad \text{under assumption (1.3),} \tag{2.15}$$

$$\Phi_q(r) \sim \frac{\beta\, r^{h+N}}{h+N}, \quad \Psi_q(r) \sim \frac{\beta\, r^{h+2}}{(h+N)(h+2)}, \quad \text{under assumption (1.4),} \tag{2.16}$$

$$\Phi_p(r) \sim \frac{\alpha\, r^{k+N}}{k+N}, \quad \Psi_p(r) \sim \frac{\alpha\, r^{k+2}}{(k+N)(k+2)}, \quad \text{under assumption (1.6).} \tag{2.17}$$

3 Some properties of the solutions in the ball $|x| \le R_1$

Our first purpose consists in describing some properties of the solutions to the initial value problem (2.2) in the interval $[0, R_1]$. The approach followed is based on the one adopted in [13], [10] and [19].

Lemma 3.1 *Consider $R_0 \in (0, R_1)$ and $p, q \in C^1([0,1], \mathbb{R})$ satisfying*

$$p \succ 0, \quad q \succ 0 \quad \textit{on } [0, R_0]. \tag{3.1}$$

Then, there exists $\vartheta_0 \in (0,1)$ such that for every $d \in \mathbb{R}$ with $d > 0$ and for every solution u_d of (2.2), there exists $r_{u_d} \in (0, \min\{\rho_{u_d}, R_0\})$ such that

$$u_d(r_{u_d}) = \vartheta_0\, d \quad \textit{and} \quad u_d(r) > \vartheta_0\, d \quad \forall r \in [0, r_{u_d}). \tag{3.2}$$

Proof. Let us argue by contradiction, assuming that for every $n \in \mathbb{N} \setminus \{1\}$, there exist $d_n \in \mathbb{R}$ with $d_n > 0$ and a solution u_{d_n} of (2.2) with $d = d_n$ such that

$$u_{d_n}(r) > \left(1 - \frac{1}{n}\right) d_n \quad \forall r \in [0, \min\{\rho_{u_{d_n}}, R_0\}). \tag{3.3}$$

Let us first show that $\rho_{u_{d_n}} > R_0$.
From the equation in (2.2), we get that for every $r \in [0, \rho_{u_{d_n}})$

$$-r^{N-1}\, u'_{d_n}(r) = \int_0^r s^{N-1} |u_{d_n}(s)|^{\delta-1} u_{d_n}(s) q(s)\, ds + \int_0^r s^{N-1} |u_{d_n}(s)|^{\gamma-1} u_{d_n}(s) p(s)\, ds. \tag{3.4}$$

Hence, taking into account (3.3) and the non-negativeness of p and q, it follows that $u'_{d_n} \le 0$ on $[0, \min\{\rho_{u_{d_n}}, R_0\})$ and, consequently,

$$\left(1 - \frac{1}{n}\right) d_n < u_{d_n}(r) \le d_n \quad \forall r \in [0, \min\{\rho_{u_{d_n}}, R_0\}). \tag{3.5}$$

According to the definitions in (2.12), from (3.4) we immediately deduce that for every $r \in [0, \min\{\rho_{u_{d_n}}, R_0\})$

$$-r^{N-1}\, u'_{d_n}(r) \le d_n^\delta \int_0^r s^{N-1} q(s)\, ds + d_n^\gamma \int_0^r s^{N-1} p(s)\, ds \le \left(d_n^\delta Q + d_n^\gamma P\right) \frac{r^N}{N},$$

whence we infer that $u'_{d_n}(r) \ge -\left(d_n^\delta Q + d_n^\gamma P\right) \frac{r}{N}$. In particular,

$$-\frac{d_n^\delta Q + d_n^\gamma P}{N} \le u'_{d_n}(r) \le 0 \quad \forall r \in [0, \min\{\rho_{u_{d_n}}, R_0\}),$$

which, combined with (3.5), guarantees the uniform boundedness of u_{d_n} and u'_{d_n} on $[0, \min\{\rho_{u_{d_n}}, R_0\}]$. Thus, u_{d_n} is globally defined on $[0, R_0]$ and $\rho_{u_{d_n}} > R_0$.

From (3.3) and (3.4), we obtain that

$$u'_{d_n}(r) \le -\left(1-\frac{1}{n}\right)^{\delta} d_n^{\delta} \frac{\int_0^r s^{N-1} q(s)\, ds}{r^{N-1}} - \left(1-\frac{1}{n}\right)^{\gamma} d_n^{\gamma} \frac{\int_0^r s^{N-1} p(s)\, ds}{r^{N-1}} \qquad \forall r \in [0, R_0].$$

We now integrate and, taking into account the definitions in (2.14) and the inequality (3.3), we finally conclude that

$$\left(1-\frac{1}{n}\right) d_n - d_n \,<\, u_{d_n}(R_0) - d_n \,\le\, -\left(1-\frac{1}{n}\right)^{\delta} d_n^{\delta}\, \Psi_q(R_0) \,-\, \left(1-\frac{1}{n}\right)^{\gamma} d_n^{\gamma}\, \Psi_p(R_0),$$

whence it follows

$$\left(1-\frac{1}{n}\right)^{\delta} d_n^{\delta-1}\, \Psi_q(R_0) + \left(1-\frac{1}{n}\right)^{\gamma} d_n^{\gamma-1}\, \Psi_p(R_0) \,<\, \frac{1}{n}. \tag{3.6}$$

Our aim consists in showing that (3.6) leads to a contradiction.
The goal is easily achieved by observing that $\Psi_p(R_0) > 0$ and $\Psi_q(R_0) > 0$ from assumption (3.1), and by noticing that, up to a subsequence, d_n tends to $d_0 \in [0, +\infty]$ as $n \to +\infty$. Passing to the limit on this subsequence, the contradiction follows. In particular, if $d_0 \in (0, +\infty)$, then (3.6) becomes

$$d_0^{\delta-1}\, \Psi_q(R_0) + d_0^{\gamma-1}\, \Psi_p(R_0) \,\le\, 0,$$

an absurd. It is immediate to verify that also the cases $d_0 = 0$ and $d_0 = +\infty$ lead to a contradiction. This completes the proof. ∎

Lemma 3.1 could be easily refined into the following.

Lemma 3.2 *Consider p, $q \in C^1([0,1],\mathbb{R})$ satisfying (3.1), then there exists $\vartheta_0 \in (0,1)$ such that for every $d \in \mathbb{R}\setminus\{0\}$ and for every solution u_d of (2.2), there exists $r_{u_d} \in (0, \min\{\rho_{u_d}, R_0\})$ such that*

$$u_d(r_{u_d}) = \vartheta_0\, d \quad \text{and} \quad |u_d(r)| > \vartheta_0\, |d| \quad \forall r \in [0, r_{u_d}). \tag{3.7}$$

By a slight modification of the proofs of Theorem 4-β(v) and of its Corollay in [35], based on contraction mapping arguments, we can deduce the following uniqueness result

Lemma 3.3 *Consider p, $q \in C^1([0,1],\mathbb{R})$ satisfying (3.1). Then, for every $d \in \mathbb{R}\setminus\{0\}$, the Cauchy problem (2.2) admits a unique solution u_d in $[0, r_{u_d}]$.*

In particular, $r_{u_d} \equiv r_d$ depends only on the initial data.
According to the definitions in (2.14) and to Lemma 3.1 and Lemma 3.2, the following inequalities hold.

Remark 3.4 *Consider p, $q \in C^1([0,1],\mathbb{R})$ satisfying (3.1). Then, for every $d \in \mathbb{R}\setminus\{0\}$,*

$$\vartheta_0\, |d| \,\le\, |u_d(r)| \,\le\, |u_d(0)| \,=\, |d| \qquad \forall\, r \in [0, r_d], \tag{3.8}$$

$$|u'_d(r)| \,\le\, \frac{|d|^{\delta}\, \Phi_q(r)}{r^{N-1}} + \frac{|d|^{\gamma}\, \Phi_p(r)}{r^{N-1}} \qquad \forall\, r \in [0, r_d]. \tag{3.9}$$

Proof. As a consequence of the relations (3.7) established in the previous lemma combined with

$$-r^{N-1}\,u_d'(r) = \int_0^r s^{N-1}|u_d(s)|^{\delta-1}u_d(s)q(s)\,ds + \int_0^r s^{N-1}|u_d(s)|^{\gamma-1}u_d(s)p(s)\,ds, \quad (3.10)$$

we immediately deduce that $|u_d(\cdot)|$ is decreasing in $[0, r_d]$. This proves the validity of (3.8). Then, by combining (3.8) with (3.10), the inequalities (3.9) also follow. ■

In order to perform estimates on the C^1-norm of the solutions of the initial value problem (2.2), we need some auxiliary lemmas which provide bounds from above on r_d. Taking into account Lemma 3.2 and the notation introduced in (2.12) and (2.14), we can state the following two results

Lemma 3.5 *Consider p, $q \in C^1([0,1],\mathbb{R})$ satisfying (3.1). Then, there exists $d^* > 0$ such that for every $d \in \mathbb{R}$ with $|d| > d^*$*

$$r_d \ge \Psi_q^{-1}\left(|d|^{1-\delta}(1-\vartheta_0) - |d|^{\gamma-\delta}\frac{P}{2N}\right). \quad (3.11)$$

Proof. We set $d^* := P^{\frac{1}{1-\gamma}}\,[2N(1-\vartheta_0)]^{-\frac{1}{1-\gamma}}$. Recalling that $\gamma < 1$, we easily deduce that

$$|d|^{1-\delta}(1-\vartheta_0) - |d|^{\gamma-\delta}\frac{P}{2N} = |d|^{1-\delta}\left((1-\vartheta_0) - |d|^{\gamma-1}\frac{P}{2N}\right) > 0, \quad \text{when } |d| > d^*. \quad (3.12)$$

Let us first consider $d > d^*$. Lemma 3.1 and Remark 3.4 guarantee the existence of $r_d \in (0, \min\{\rho_{u_d}, R_0\})$ such that

$$u_d(r_d) = \vartheta_0\, d \quad \text{and} \quad u_d'(r) \ge -\frac{d^\delta\,\Phi_q(r)}{r^{N-1}} - \frac{d^\gamma P}{N}\,r \qquad \forall\, r \in [0, r_d].$$

Hence, by integrating u_d' in $[0, r_d]$ and recalling the definitions in (2.14), we conclude that

$$\vartheta_0 d - d = u(r_d) - d \ge -d^\delta \Psi_q(r_d) - \frac{d^\gamma P}{2N},$$

which leads to

$$\Psi_q(r_d) \ge d^{1-\delta}\,(1-\vartheta_0) - d^{\gamma-\delta}\,\frac{P}{2N}.$$

Due to the inequality (3.12), the relation (3.11) follows.
The case $d < -d^* < 0$ can be treated in an analogous way. ■

Lemma 3.6 *Consider p, $q \in C^1([0,1],\mathbb{R})$ satisfying (3.1). Then, there exists $d_0 \in (0, d^*)$ such that for every $d \in \mathbb{R} \setminus \{0\}$ with $|d| \le d_0$*

$$r_d \ge \Psi_p^{-1}\left(|d|^{1-\gamma}\,(1-\vartheta_0) - |d|^{\delta-\gamma}\,\frac{Q}{2N}\right). \quad (3.13)$$

Proof. Recalling that $\gamma < 1 < \delta$, we easily deduce the existence of $d_0 \in (0, d^*)$ such that

$$|d|^{1-\gamma}\left((1-\vartheta_0) - |d|^{\delta-1}\frac{Q}{2N}\right) > 0, \quad \text{when } \ 0 < |d| \le d_0. \tag{3.14}$$

Lemma 3.1 and Remark 3.4 guarantee the existence of $r_d \in (0, \min\{\rho_{u_d}, R_0\})$ such that

$$u_d(r_d) \, = \, \vartheta_0 \, d \quad \text{ and } \quad u_d'(r) \, \ge \, -\frac{d^\delta Q}{N}\, r - \frac{d^\gamma\, \Phi_p(r)}{r^{N-1}} \qquad \forall\, r \in [0, r_d].$$

Hence, by integrating u_d' in $[0, r_d]$ and recalling the definitions in (2.14), we conclude that

$$\vartheta_0 d \, - \, d \, = \, u(r_d) \, - \, d \, \ge \, -\frac{d^\delta Q}{2N} \, - \, d^\gamma\, \Psi_p(r_d),$$

which leads to

$$\Psi_p(r_d) \, \ge \, d^{1-\gamma}\,(1-\vartheta_0) \, - \, d^{\delta-\gamma}\,\frac{Q}{2N}.$$

Due to the inequality (3.14), the relation (3.13) follows.
The case $d \le -d_0 < 0$ can be treated in an analogous way. ∎

4 Global existence results

This section is devoted to present the global existence result concerning the solutions u_d of (2.2) and Cauchy problems with more general initial data. Some remarks about uniqueness conclude the section.
First of all, let us introduce some useful notation. According to (1.7), we denote by $\widetilde{r}_1$ the first positive zero of the map q. By assumption (1.8) there exist $S_1 \in (R_1, \widetilde{r}_1)$ and $S_i \in (r_{i-1}, \widetilde{r}_i)$ with $i \in \{2, \dots, n\}$ such that

$$q'(r) \le 0, \qquad \forall\, r \in [S_i, \widetilde{r}_i], \quad \forall\, i \in \{1, \dots, n\}. \tag{4.1}$$

The definition of S_i should be further refined according to the behaviour of the weight p in $\widetilde{r}_i$. Taking into account (1.7) and our assumption (1.10), we know that $p(\widetilde{r}_i) = 0$ implies that $p'(\widetilde{r}_i) \le 0$ in a left neighbourhood of $\widetilde{r}_i$ both in the case $\widetilde{r}_i = \widetilde{s}_{j_i}$ and in the case $\widetilde{r}_i \in J_{j_i} \setminus \widetilde{s}_{j_i}$, for a suitable $j_i \in \{1, \dots, m\}$. Thus, it is not restrictive to choose S_i and a positive constant p_i satisfying also

$$p(r) \ge p_i > 0 \quad \forall\, r \in [S_i, \widetilde{r}_i], \quad \forall\, i \in \{1, \dots, n\}, \qquad \text{if } p(\widetilde{r}_i) > 0, \tag{4.2}$$

$$p'(r) \le 0 \qquad \forall\, r \in [S_i, \widetilde{r}_i], \quad \forall\, i \in \{1, \dots, n\}, \qquad \text{if } p(\widetilde{r}_i) = 0. \tag{4.3}$$

Our first aim consists in showing that the solutions u_d of (2.2) are globally defined in $[0, \widetilde{r}_1]$, whenever $d \in \mathbb{R} \setminus \{0\}$.

Lemma 4.1 *Consider* p, $q \in C^1([0,1], \mathbb{R})$ *satisfying* (1.2), *one of the alternatives described in* (1.3)-(1.6), (1.7), (1.8) *and* (1.10). *Then,* $\rho_{u_d} > \widetilde{r}_1$ *for every solution* u_d *of* (2.2) *with* $d \in \mathbb{R} \setminus \{0\}$.

Proof. To prove this result we use analogous arguments to the ones developed in Lemma 3.3 of [13]. Let us introduce the function energy $\mathcal{E}_{u_d}$ by setting

$$\mathcal{E}_{u_d}(r) := \frac{1}{2}\, u_d'(r)^2 \,+\, \frac{q(r)}{\delta+1}\, |u_d(r)|^{\delta+1} \,+\, \frac{p(r)}{\gamma+1}\, |u_d(r)|^{\gamma+1} \qquad \forall\, r \in [0, \rho_{u_d}). \tag{4.4}$$

Observe that the equation in (2.2) can be equivalently written in the form

$$u''(r) \,=\, -|u(r)|^{\delta-1}\, u(r)\, q(r) \,-\, |u(r)|^{\gamma-1}\, u(r)\, p(r) \,-\, \frac{N-1}{r}\, u'(r), \tag{4.5}$$

whenever $r \in (0, \rho_{u_d})$. According to (4.5), we obtain that for every $r \in (0, \rho_{u_d})$

$$\begin{aligned} \mathcal{E}_{u_d}'(r) &:= u_d''(r)\, u_d'(r) + \frac{q'(r)}{\delta+1}|u_d(r)|^{\delta+1} + q(r)\, |u_d(r)|^{\delta-1}\, u_d(r)\, u_d'(r) + \\ &\quad + \frac{p'(r)}{\gamma+1}|u_d(r)|^{\gamma+1} + p(r)|u_d(r)|^{\gamma-1} u_d(r)\, u_d'(r) = \\ &= -\frac{N-1}{r}\, u_d'(r)^2 + \frac{q'(r)}{\delta+1}|u_d(r)|^{\delta+1} + \frac{p'(r)}{\gamma+1}|u_d(r)|^{\gamma+1}. \end{aligned} \tag{4.6}$$

According to (2.13) and (3.8), it immediately follows that

$$\mathcal{E}_{u_d}'(r) \,\leq\, \frac{Q'}{\delta+1}\, |d|^{\delta+1}\, \frac{P'}{\gamma+1}\, |d|^{\gamma+1} =: \widetilde{\alpha}_d \qquad \forall\, r \in [0, r_d]. \tag{4.7}$$

We are now interested in estimating $\mathcal{E}_{u_d}'$ in the interval $(r_d, \min\{\rho_{u_d}, S_1\})$, where we refer to the beginning of this section for the definition of S_1.
Setting $A_{u_d} := \{r \in (r_d, \min\{\rho_{u_d}, S_1\}) : |u_d(r)| \leq 1\}$ and $B_{u_d} := (r_d, \min\{\rho_{u_d}, S_1\}) \setminus A_{u_d}$, from (4.6) combined with (2.13) we deduce that

$$\mathcal{E}_{u_d}'(r) \,\leq\, \frac{Q'}{\delta+1} \,+\, \frac{P'}{\gamma+1} =: \widehat{\alpha}, \qquad \forall\, r \in A_{u_d}. \tag{4.8}$$

Let us define

$$q_d \,:=\, \min_{r \in [r_d, S_1]} q(r),$$

which is positive since $[r_d, S_1] \subset (0, \widetilde{r}_1)$, for every $d \neq 0$. Hence, recalling that $\gamma < \delta$ and taking into account (2.13), (4.4), (4.6), we easily obtain the following inequality

$$\mathcal{E}_{u_d}'(r) \,\leq\, \widetilde{\beta}_d\, \frac{q(r)}{\delta+1}\, |u_d(r)|^{\delta+1} \,\leq\, \widetilde{\beta}_d\, \mathcal{E}_{u_d}(r), \qquad \forall\, r \in B_{u_d}, \tag{4.9}$$

where $\widetilde{\beta}_d := \frac{1}{q_d}\left(Q' + \frac{\delta+1}{\gamma+1}\, P'\right)$.
It remains to provide estimates in the interval $[S_1, \min\{\rho_{u_d}, \widetilde{r}_1\}]$, in case $\rho_{u_d} > S_1$. Note that from (4.1) and (4.6), it follows

$$\mathcal{E}_{u_d}'(r) \,\leq\, \frac{p'(r)}{\gamma+1}\, |u_d(r)|^{\gamma+1}, \qquad \forall\, r \in [S_1, \min\{\rho_{u_d}, \widetilde{r}_1\}].$$

Thus, by one of the alternatives in (4.2)-(4.3), we infer the existence of $p_1 > 0$ such that

$$\mathcal{E}'_{u_d}(r) \le \frac{P'}{p_1} \frac{p(t)}{\gamma+1} |u_d(r)|^{\gamma+1} \le \frac{P'}{p_1} \mathcal{E}_{u_d}(r), \qquad \forall\, r \in [S_1, \min\{\rho_{u_d}, \widetilde{r}_1\}]. \tag{4.10}$$

Finally, by combining (4.7)-(4.10), we conclude that

$$\mathcal{E}'_{u_d}(r) \le \alpha_d + \beta_d \mathcal{E}_{u_d}(r) \qquad \forall\, r \in [0, \min\{\rho_{u_d}, \widetilde{r}_1\})\,,$$

for suitable positive constants α_d, β_d. Now, the Gronwall Lemma ensures that

$$\mathcal{E}_{u_d}(r) \le \left(\mathcal{E}_{u_d}(0) + \frac{\alpha_d}{\beta_d}\right) e^{\beta_d r} - \frac{\alpha_d}{\beta_d} \le \left(\frac{q(0)}{\delta+1} |d|^{\delta+1} + \frac{p(0)}{\gamma+1} |d|^{\gamma+1} + \frac{\alpha_d}{\beta_d}\right) e^{\beta_d} =: c_d,$$

for every $r \in [0, \min\{\rho_{u_d}, \widetilde{r}_1\})$. It easily follows that $|u'_d(r)| \le \sqrt{2c_d}$ for every $r \in [0, \min\{\rho_{u_d}, \widetilde{r}_1\}]$, and, consequently,

$$|u_d(r)| \le |d| + r\sqrt{2c_d} \le |d| + \sqrt{2c_d} \qquad \forall\, r \in [0, \min\{\rho_{u_d}, \widetilde{r}_1\}]\,.$$

Thus, u_d is globally defined on $[0, \widetilde{r}_1]$ and $\rho_{u_d} > \widetilde{r}_1$, since the solution proceeds further $\widetilde{r}_1$ simply by Peano theorem about local existence. ■

As a second step, we extend the global continuability result to the interval $[0, \widetilde{r}_2]$.

Lemma 4.2 *Consider p, $q \in C^1([0,1], \mathbb{R})$ satisfying* (1.2), *one of the alternatives described in* (1.3)-(1.6), (1.7), (1.8) *and* (1.10). *Then, each solutions u_d of* (2.2) *with $d \in \mathbb{R} \setminus \{0\}$ is globally defined in $[0, \widetilde{r}_2]$.*

Proof. Let u_d be a solution of (2.2), with $d \neq 0$. The previous lemma guarantees that $\rho_{u_d} > \widetilde{r}_1$.

• **We first claim that u_d is globally defined on $[0, r_1]$.**
We are interested in studying the behaviour of the solution in the interval $I_1 := [\widetilde{r}_1, r_1]$, provided that this set does not degenerate to a single point.
Since $q = q' = 0$ in $(\widetilde{r}_1, r_1)$, the energy expression (4.4) and its derivative reduce to

$$\mathcal{E}_{u_d}(r) = \frac{1}{2} u'_d(r)^2 + \frac{p(r)}{\gamma+1} |u_d(r)|^{\gamma+1}, \quad \mathcal{E}'_{u_d}(r) = -\frac{N-1}{r} u'_d(r)^2 + \frac{p'(r)}{\gamma+1} |u_d(r)|^{\gamma+1}, \tag{4.11}$$

for every $r \in (\widetilde{r}_1, \min\{\rho_{u_d}, r_1\})$. Our aim consists in estimating the energy derivative according to the behaviour of the weight p. For this reason, we will treat the intervals $\widehat{J}_j := J_j \cap [\widetilde{r}_1, r_1]$ where p vanishes and, separately, the intervals $\widehat{Z}_j := [s_j, \widetilde{s}_{j+1}] \cap [\widetilde{r}_1, r_1]$, in whose interior part p is positive. Denote by $\widehat{Z}_j := [s_j, \widetilde{s}_{j+1}] \cap [\widetilde{r}_1, r_1]$ the closure of these intervals in $[\widetilde{r}_1, r_1]$. We refer to (1.7) for more details concerning the notation.
We argue iteratively by showing, roughly speaking, that the boundedness of the energy can be extended from an interval to the next one. More precisely, we claim that the existence of a constant $c_{u_d} > 0$ such that

$$\mathcal{E}_{u_d}(r) \le c_{u_d} \qquad \forall\, r \in [0, s_j] \;\text{ or }\; \forall\, r \in [0, \widetilde{s}_j], \tag{4.12}$$

implies the existence of $\widetilde{c}_{u_d} > 0$ such that

$$\mathcal{E}_{u_d}(r) \,\leq\, \widehat{c}_{u_d} \qquad \forall\, r \in \widehat{Z}_j \;\text{ or }\; \forall\, r \in \widehat{J}_{j+1}, \tag{4.13}$$

respectively.

***Suppose first the validity of* (4.12) *in* $[0, s_j]$.**
As a consequence, $\rho_{u_d} > s_j$ and $|u_d(s_j)| \leq c^*_{u_d}$, for a suitable constant $c^*_{u_d} > 0$ depending on c_{u_d}. Therefore, we can consider $\widehat{r}_{u_d} \in (s_j, \min\{\rho_{u_d}, \widetilde{s}_{j+1}\})$ such that $|u_d(r)| \leq c^*_{u_d} + 1$ for each $r \in [s_j, \widehat{r}_{u_d}]$. From (2.13) and (4.11), it immediately follows that

$$\mathcal{E}'_{u_d}(r) \,\leq\, \frac{P'}{\gamma+1}\left(c^*_{u_d}+1\right)^{\gamma+1}, \qquad \forall\, r \in [s_j, \widehat{r}_{u_d}]\,. \tag{4.14}$$

Moreover, assumption (1.10) ensures the existence of $\widehat{S}_j$ such that $p' \leq 0$ in $[\widehat{S}_j, \widetilde{s}_{j+1}]$. Define

$$p_{u_d} \,:=\, \min_{r \in [\widehat{r}_{u_d}, \widehat{S}_j]} p(r).$$

According to (2.13) and (4.11), we conclude that

$$\mathcal{E}'_{u_d}(r) \leq \frac{P'}{p_{u_d}}\,\frac{p(t)}{\gamma+1}\,|u_d(r)|^{\gamma+1} \leq \frac{P'}{p_{u_d}}\,\mathcal{E}_{u_d}(r) \qquad \forall\, r \in [\widehat{r}_{u_d}, \min\{\rho_{u_d}, \widetilde{s}_{j+1}\}],$$

which, combined with (4.14), allows us to apply the Gronwall Lemma and, consequently, to guarantee the validity of (4.13) in $\widehat{Z}_j$. In particular, $\rho_{u_d} > \widetilde{s}_{j+1}$.

***Suppose now the validity of* (4.12) *in* $[0, \widetilde{s}_j]$.**
Recalling that $p = 0$ in $\widehat{J}_{j+1}$, we immediately observe that

$$\mathcal{E}'_{u_d}(r) \,\leq\, \frac{p'(r)}{\gamma+1}|u_d(r)|^{\gamma+1} \,\leq\, 0, \qquad \forall\, r \in \widehat{J}_{j+1},$$

from which (4.13) in $\widehat{J}_{j+1}$ follows. In particular, $\rho_{u_d} > s_{j+1}$.

Summarizing, we have proved that each solution u_d proceeds from one interval to the next one in I_1. Since, by assumption (1.7) I_1 can be covered by a finite number of closed intervals of the form $\widehat{J}_j$ and $\widehat{Z}_j$, according to the conclusions of Lemma 4.1, we can iteratively deduce the existence of $\bar{c}_{u_d}$ such that

$$\mathcal{E}_{u_d}(r) \,\leq\, \bar{c}_{u_d} \qquad \forall\, r \in [0, r_1]. \tag{4.15}$$

This ensures that $\rho_{u_d} > r_1$ and u_d is globally defined on $[0, r_1]$, proving our claim.

• **Let us now claim that u_d is globally defined on** $[0, \tilde{r}_2]$.
The estimates on the energy we are interested in exhibiting can be achieved by means of the previous lemma arguments. Recalling the definition of S_2 given at the beginning of this section, by (4.15) we are allowed to take $\mathtt{r}_{u_d} \in (r_1, \min\{\rho_{u_d}, S_2\})$ such that

$$|u_d(r)| \,\leq\, |d| + \sqrt{2\bar{c}_{u_d}} + 1, \qquad \forall\, r \in [r_1, \mathtt{r}_{u_d}]\,. \tag{4.16}$$

Set $A^2_{u_d} := \{r \in (\mathtt{r}_{u_d}, \min\{\rho_{u_d}, S_2\}) : |u_d(r)| \le 1\}$ and $B^2_{u_d} := (\mathtt{r}_{u_d}, \min\{\rho_{u_d}, S_2\}) \setminus A^2_{u_d}$. Exactly as in proof of the previous lemma, according to (2.13), (4.7) and (4.16), we deduce that there exists $\alpha_{u_d} > 0$ such that

$$\mathcal{E}'_{u_d}(r) \le \alpha_{u_d}, \qquad \forall\, r \in [r_1, \mathtt{r}_{u_d}] \cup A^2_{u_d}. \tag{4.17}$$

As in the previous lemma, we define

$$q_{\mathtt{r}_{u_d}} := \min_{r \in [\mathtt{r}_{u_d}, S_2]} q(r)$$

to guarantees the existence of $\widetilde{\beta}_{\mathtt{r}_{u_d}} = \widetilde{\beta}(q_{\mathtt{r}_{u_d}})$ such that

$$\mathcal{E}'_{u_d}(r) \le \widetilde{\beta}_{\mathtt{r}_{u_d}} \frac{q(r)}{\delta+1} |u_d(r)|^{\delta+1} \le \widetilde{\beta}_{\mathtt{r}_{u_d}} \mathcal{E}_{u_d}(r), \qquad \forall\, r \in B^2_{u_d}. \tag{4.18}$$

The estimates on the set $[S_2, \min\{\rho_{u_d}, \widetilde{r}_2\}]$, in case $\rho_{u_d} > S_2$, follow by assumptions (1.8) and (1.10) and their respective consequences (4.1) and (4.2)-(4.3). In particular, as before we deduce the existence of $p_2 > 0$ such that

$$\mathcal{E}'_{u_d}(r) \le \frac{p'(r)}{\gamma+1} |u_d(r)|^{\gamma+1} \le \frac{P'}{p_2} \frac{p(t)}{\gamma+1} |u_d(r)|^{\gamma+1} \le \frac{P'}{p_2} \mathcal{E}_{u_d}(r), \quad \forall\, r \in [S_2, \min\{\rho_{u_d}, \widetilde{r}_2\}]. \tag{4.19}$$

Finally, by combining (4.17)-(4.19), we obtain the existence of $\beta_{\mathtt{r}_{u_d}}$

$$\mathcal{E}'_{u_d}(r) \le \alpha_{u_d} + \beta_{\mathtt{r}_{u_d}} \mathcal{E}_{u_d}(r) \qquad \forall\, r \in [r_1, \min\{\rho_{u_d}, \widetilde{r}_2\}).$$

Now, by combining the Gronwall Lemma with the definition of $\bar{c}_{u_d}$ in (4.15), we conclude that

$$\mathcal{E}_{u_d}(r) \le \left(\mathcal{E}_{u_d}(r_1) + \frac{\alpha_{u_d}}{\beta_{\mathtt{r}_{u_d}}} \right) e^{r\beta_{\mathtt{r}_{u_d}}} - \frac{\alpha_{u_d}}{\beta_{\mathtt{r}_{u_d}}} \le \left(\bar{c}_{u_d} + \frac{\alpha_{u_d}}{\beta_{\mathtt{r}_{u_d}}} \right) e^{\beta_{\mathtt{r}_{u_d}}},$$

for every $r \in [r_1, \min\{\rho_{u_d}, \widetilde{r}_2\})$. This allows us to conclude that u_d is globally defined on $[0, \widetilde{r}_2]$ and $\rho_{u_d} > \widetilde{r}_2$. The thesis is achieved. ■

The arguments of the previous lemma can be finally adopted to extend the global continuability result to the whole interval $[0,1]$.

Lemma 4.3 *Consider p, $q \in C^1([0,1], \mathbb{R})$ satisfying* (1.2)*, one of the alternatives described in* (1.3)-(1.6)*,* (1.7)*,* (1.8) *and* (1.10)*. Then, each solution u_d of* (2.2) *with $d \in \mathbb{R} \setminus \{0\}$ is globally defined in* $[0,1]$.

Sketch of the proof. The proof follows by repeating iteratively the arguments of Lemma 4.2 in each interval covering $[r_1, 1]$. We can proceed by estimating the energy in the intervals $I_i = [\widetilde{r}_i, r_i]$, $i \in \{2, \dots, n\}$ where q vanishes and, separately, in the intervals in $I_i^c := [r_i, \widetilde{r}_{i+1}]$, in whose interior part the weight q is positive.
The intervals I_i, $i \ge 2$, can be handled as I_1 in the proof of Lemma 4.2. According to the

sign of p, we divide each I_i in a finite number of sub-intervals, and proceed iteratively. If p vanishes in the sub-interval, then the estimates are immediately achieved since the energy decreases. Otherwise, if p is positive in the interior part of the interval, we proceed further the left extreme of the sub-interval by the classical Peano local existence result. Then, we easily estimate the energy derivative in a sub-interval where p is strictly positive. Finally, when p monotonically decreases to 0 in a left neighborhood of each s_j (as assumed in (1.10)) the energy still decresases, and the complete estimates follow.
The remaining intervals I_i^c can be studied by using the same argument of Lemma 4.1 and of the second claim of Lemma 4.2. In these intervals, the behaviour of q has the priority on the one of p. The basic idea consists again in proceeding further r_i by the Peano local existence result. Again, we can easily estimate the energy derivative in a sub-interval where q is strictly positive. Finally, when q monotonically decreases to 0 in the left neighborhood of $\widetilde{r}_{i+1}$ defined by $[S_{i+1}, \widetilde{r}_{i+1}]$ the energy can be easily estimated by (4.1)-(4.3), which take into account also the behaviour of p in such a neighborhood. Since the number of intervals I_i and I_i^c covering $[0,1]$ is finite by assumption (1.7), we achieve the global continuability of the solutions in $[0,1]$ through a finite number of iterative steps. ■

The global existence results can be extended to the solutions of the Cauchy problem

$$\begin{cases} \left(r^{N-1}\, u'(r)\right)' + r^{N-1}\,|u(r)|^{\delta-1}\, u(r)\, q(r) + r^{N-1}\,|u(r)|^{\gamma-1}\, u(r)\, p(r) = 0, \\ (u'(\tau), u(\tau)) = \mathbf{d} \in \mathbb{R}^2. \end{cases} \tag{4.20}$$

Lemma 4.4 *Fix $\tau \in [R_1, 1]$, $\mathbf{d} \in \mathbb{R}^2$, and consider $p,\, q \in C^1([0,1], \mathbb{R})$ satisfying* (1.7)-(1.10)*. Then, each solution of* (4.20) *is globally defined in* $[R_1, 1]$.

Sketch of the proof. Arguing exactly as in the previous lemma, we extend each solution of (4.20) to the interval $[\tau, 1]$.
Thus, we only need to prove the global existence of solutions in $[R_1, \tau]$. To achieve this goal, we follow an analogous approach, based on the iterative study of suitable sub-intervals in which both the weights q and p vanish, in which one of the weights is positive and in which at least one of the weights reaches the zero in the extreme point of the sub-interval given by $\widetilde{r}_i, r_i, \widetilde{s}_j, s_j$. Again we need to estimate the energy derivative (4.6) to apply the Gronwall Lemma, with the difference that now we are interested in providing lower bounds to the expression in (4.6). Notice that the first addendum in (4.6) is easily estimated by the following

$$-\frac{N-1}{r}\, u'(r)^2 \ge -\frac{N-1}{R_1}\, u'(r)^2 \ge -\frac{2(N-1)}{R_1}\, \mathcal{E}_u(r) \quad \forall\, r \in [R_1, \tau].$$

On the other hand, the lower bounds on the second and third addendum can be obtained by means of the same techniques developed in the proof of the previous lemmas. In particular, note that the solution immediately extends to the left side of $\widetilde{r}_i$ and $\widetilde{s}_j$ by the classical Peano local existence result. Observe, moreover, that the lower estimates on the energy derivative in right neighbourhoods of r_i and s_j should be now based on assumptions (1.9) and (1.11), instead of (1.8) and (1.10) employed in the previous lemmas and to extend our solution to $[\tau, 1]$. ■

We end the Section by presenting some uniqueness results. First of all, taking into account Lemma 3.3 and [35, Theorem 4-δ(ii)], we immediately deduce the following.

Remark 4.5 *Consider p, $q \in C^1([0,1],\mathbb{R})$ satisfying the assumptions of Lemma 4.3. Then, for every $d \in \mathbb{R} \setminus \{0\}$, the Cauchy problem (2.2) admits a unique solution u_d in $[0, \widetilde{s}_1)$.*

Consider now a solution u_d of (2.2) such that $u_d(\tau) \neq 0$, for a fixed $\tau \in [\widetilde{s}_1, 1]$. Then, Theorem 4-$\beta$(v) and its Corollary in [35] guarantee that the extension $u_d(r)$ remains unique as long as $u_d(r) \neq 0$.
Analogously, by [35, Theorem 2] each solution u_d of (2.2) with $u_d'(\tau) \neq 0$ admits a unique extension as long as $u_d'(r) \neq 0$.
Summarizing these remarks, we can state the following

Remark 4.6 *Consider p, $q \in C^1([0,1],\mathbb{R})$ satisfying the assumptions of Lemma 4.3. Then, for every $d \in \mathbb{R} \setminus \{0\}$, the solution u_d the Cauchy problem (2.2) can lose its uniqueness only in the values $\tau_0 \in [\widetilde{s}_1, 1]$ in which both u_d and u_d' vanish.
According to [35, Theorem 4-δ(ii)], it is easy to verify that $\tau_0 \in \{\widetilde{s}_1, \ldots, \widetilde{s}_m\}$.
Moreover, by the classical continuous dependence results (cf., among others, [20] and [35]), as long as u_d preserves its uniqueness, it depends continuously in the C^1-norm on the initial datum d.*

With the same arguments, analogous results for the solutions of (4.20) can be deduced.

Remark 4.7 *Consider p, $q \in C^1([0,1],\mathbb{R})$ satisfying (1.2), one of the alternatives in (1.3)-(1.6), and (1.7)-(1.10). Then, for every $\mathbf{d} \in \mathbb{R}^2$, the solution u of (4.20) with $\tau \in [s_{j_1}, s_{j_1+1}) \cup (\widetilde{s}_{j_2-1}, \widetilde{s}_{j_2}]$ can lose its uniqueness only in the values $\tau_0 \in [0,1]$ in which both u and u' vanish, which, in turns, implies that $\tau_0 \in \{0, s_1, \ldots, s_{j_1}, \widetilde{s}_{j_2}, \ldots, \widetilde{s}_m\}$.*

In Section 6.2 we show that the solutions corresponding to a suitable range of initial data are uniquely determined.

5 Small norm solutions of the Cauchy problems in $[0, R_1]$

This section is devoted to provide upper bounds on the C^1-norm of the solutions of the Cauchy problem (2.2) in $[0, R_1]$ when the initial data are suitably small.

Lemma 5.1 *Consider p, $q \in C^1([0,R_1],\mathbb{R})$ satisfying (1.2) and one of the alternatives described in (1.3)-(1.6). Then, for every $\sigma > 0$ there exists $d_\sigma \in (0, d_0)$ such that for every solution u_d of (2.2)*

$$\|u_d\|_{1_{|[0,R_1]}} \leq \sigma \qquad \forall\, d \in \mathbb{R} \setminus \{0\} : \ |d| \leq d_\sigma.$$

Proof. Fix $d \in \mathbb{R} \setminus \{0\}$ and consider a solution u_d of (2.2).
By Remark 3.4, we observe that for every $\sigma > 0$ there exists $\widetilde{d}_\sigma > 0$ such that

$$\|u_d\|_{1_{|[0,r_d]}} < \frac{\sigma}{2} \qquad \forall\, d \in \mathbb{R} \setminus \{0\} : \ |d| \le \widetilde{d}_\sigma. \tag{5.1}$$

Our aim consists now in estimating $\|u_d\|_1$ in $[r_d, R_1]$.
We analyze separately the different alternatives of our assumptions (1.3)-(1.6).
Assume first the validity of (1.4) and (1.6).
Let us recall the energy definition (4.4) and its derivative calculated in (4.6). According to (2.5) and (2.8), we obtain

$$\mathcal{E}'_{u_d}(r) \le \frac{h+\varepsilon}{r}\,\frac{q(r)}{\delta+1}\,|u_d(r)|^{\delta+1} + \frac{k+\varepsilon}{r}\,\frac{p(r)}{\gamma+1}\,|u_d(r)|^{\gamma+1} \qquad \forall\, r \in [0, R_1],$$

which implies that

$$\mathcal{E}'_{u_d}(r) \le \frac{\max\{h,k\}+\varepsilon}{r}\,\mathcal{E}_{u_d}(r) \qquad \forall\, r \in [r_d, R_1] \subset [0, R_1].$$

Define

$$c := \max\{h,k\} + \varepsilon. \tag{5.2}$$

Then, by applying the Gronwall Lemma, we deduce that

$$\mathcal{E}_{u_d}(r) \le \mathcal{E}_{u_d}(r_d)\ e^{\int_{r_d}^{r} \frac{c\,ds}{s}} = \mathcal{E}_{u_d}(r_d)\ e^{c(\log r - \log r_d)} = \mathcal{E}_{u_d}(r_d)\ \frac{r^c}{r_d^c}, \tag{5.3}$$

for every $r \in [r_d, R_1]$. Taking into account Lemma 3.2 and Remark 3.4, we infer that for every $r \in [r_d, R_1]$

$$\mathcal{E}_{u_d}(r) \le \frac{1}{2}\,\frac{\left(|d|^\delta\,\Phi_q(r_d) + |d|^\gamma\,\Phi_p(r_d)\right)^2}{r_d^{(2N-2)}\,r_d^c} + \frac{q(r_d)}{\delta+1}\,\frac{|d|^{\delta+1}}{r_d^c} + \frac{p(r_d)}{\gamma+1}\,\frac{|d|^{\gamma+1}}{r_d^c}. \tag{5.4}$$

According to Remark 2.1, to assumptions (1.4), (1.6), respectively, and to their straightforward consequences (2.16), (2.17), it is not restrictive to choose $\varepsilon > 0$ and $R_1 \in (0,1)$ satisfying the further conditions

$$q(r) \le (\beta+\varepsilon)\,r^h, \quad \Phi_q(r) \le \frac{\beta+\varepsilon}{h+N}\,r^{h+N}, \qquad \forall\, r \in (0, R_1] \tag{5.5}$$

$$p(r) \le (\alpha+\varepsilon)\,r^k, \quad \Phi_p(r) \le \frac{\alpha+\varepsilon}{k+N}\,r^{k+N} \qquad \forall\, r \in (0, R_1]. \tag{5.6}$$

Thus, we deduce the existence of five positive constants $c_i = c_i(\alpha, \beta, h, k, \varepsilon, \delta, \gamma, N)$, $i \in \{1, \ldots, 5\}$ such that for every $r \in [r_d, R_1]$

$$\mathcal{E}_{u_d}(r) \le c_1\,r_d^{2h-c+2}\,|d|^{2\delta} + c_2\,r_d^{2k-c+2}\,|d|^{2\gamma} + c_3\,r_d^{h+k-c+2}\,|d|^{\delta+\gamma} + c_4\,r_d^{h-c}\,|d|^{\delta+1} + c_5\,r_d^{k-c}\,|d|^{\gamma+1}. \tag{5.7}$$

We have exhibited an upper bound on $\mathcal{E}_{u_d}$ which depends on the initial data, but does not depend on the particular solution u_d. Our aim consists in showing that this upper bound tends to zero as $d \to 0$.
To this purpose, we need to establish a relation between r_d and d. Let us first recall that Lemma 3.6 ensures that

$$r_d \geq \Psi_p^{-1}\left(|d|^{1-\gamma}(1-\vartheta_0) - |d|^{\delta-\gamma}\frac{Q}{2N}\right) \quad \forall d \in \mathbb{R}\setminus\{0\}: \ |d| \leq d_0.$$

Combining this estimate with the relation

$$\Psi_p^{-1}(r) \sim \left(\frac{(k+2)(k+N)}{\alpha}\right)^{\frac{1}{k+2}} r^{\frac{1}{k+2}} \quad \text{when } r \to 0^+,$$

obtained by (2.17), we deduce that there exist $\tilde{c} = \tilde{c}(k, N, \alpha, \vartheta_0, Q, \delta, \gamma) > 0$ and $\tilde{d}_0 = \tilde{d}_0(k, N, \alpha, \vartheta_0, Q, \delta, \gamma) \in (0, d_0)$ such that

$$\frac{1}{r_d} \leq \frac{\tilde{c}}{|d|^{\frac{1-\gamma}{k+2}}} \qquad \forall d \in \mathbb{R}\setminus\{0\}: \ |d| \leq \tilde{d}_0. \tag{5.8}$$

This relation allows us to easily estimate the upper bound in (5.7) when $h \leq k$. Let us first concentrate on this easiest case.

• Assume that $h \leq k$.
Recalling the definition of c in (5.2) and the fact that $r_d < R_1 \leq 1$, it is immediate to verify that the third addendum in (5.7) tends to zero as $d \to 0$.
Since $h \leq k$, we know that $c = k + \varepsilon$. Therefore, also the second addendum in (5.7) tends to zero as $d \to 0$.
As far as the fourth and fifth addendum are concerned, from (5.8) it follows

$$\frac{c_4\ |d|^{\delta+1}}{r_d^{k-h+\varepsilon}} + \frac{c_5\ |d|^{\gamma+1}}{r_d^{\varepsilon}} \leq \tilde{c}_4 \frac{|d|^{\delta+1}}{|d|^{\frac{(1-\gamma)(k-h+\varepsilon)}{k+2}}} + \tilde{c}_5 \frac{|d|^{\gamma+1}}{|d|^{\frac{\varepsilon(1-\gamma)}{k+2}}} \to 0 \quad \text{as } d \to 0,$$

where $\tilde{c}_4, \tilde{c}_5$ are suitable positive constants.
As for the first addendum, we immediately note that it converges to zero when $2h - k - \varepsilon + 2 \geq 0$. Otherwise, according to (5.8), we infer that

$$\frac{c_1\ |d|^{2\delta}}{r_d^{k+\varepsilon-2h-2}} \leq \tilde{c}_1 \frac{|d|^{2\delta}}{|d|^{\frac{(1-\gamma)(k+\varepsilon-2h-2)}{k+2}}} \to 0 \quad \text{as } d \to 0,$$

where $\tilde{c}_1$ is a suitable positive constant. Our goal is achieved.

• Assume now that $h > k$.
In this case, we cannot use relation (5.8) to prove that the second and the fifth addenda in (5.7) tend to zero as $d \to 0$. To complete the proof, some more steps are required.

First of all, from (1.4) and (1.6), it is not restrictive to choose $\varepsilon \in (0,1)$, $R_1 \in (0,1)$ in Remark 2.1 such that

$$q'(r) \leq \frac{(\beta+\varepsilon)\,h}{\alpha}\,\frac{r^{h-1}}{r^k}\,p(r), \quad \forall\, r \in (0, R_1], \tag{5.9}$$

$$h > k + 2\varepsilon, \qquad R_1^{h-k} \leq \frac{\alpha\,\varepsilon\,(\delta+1)}{(\beta+\varepsilon)\ h\ (\gamma+1)}. \tag{5.10}$$

Moreover, without loss of generality, we can choose $\sigma < 1$ and $\widetilde{d_\sigma} \in (0, \tilde{d}_0)$ (cf. (5.1) and (5.8) for the definitions of $\widetilde{d_\sigma}$ and $\tilde{d}_0$, respectively).
In the spirit of Lemma 2.3 in [9] and according to (5.1), we consider the maximal value $R_d \in (r_d, R_1]$ in which the following inequalities are satisfied

$$|u_d(r)| \leq \frac{\sigma}{2}, \quad |u_d'(r)| \leq \frac{\sigma}{2} \quad \forall\, r \in [0, R_d], \quad 0 < |d| \leq \widetilde{d_\sigma}. \tag{5.11}$$

From (4.6), (5.9) and (2.8), we deduce that for every $r \in [r_d, R_d]$

$$\begin{aligned} \mathcal{E}_{u_d}'(r) &\leq \frac{q'(r)\,|u_d(r)|^{\delta+1}}{\delta+1} + \frac{p'(r)\,|u_d(r)|^{\gamma+1}}{\gamma+1} \leq \\ &\leq \left(\frac{(\beta+\varepsilon)\,h\,r^{h-k}}{\alpha\,r}\,\frac{\gamma+1}{\delta+1} + \frac{k+\varepsilon}{r}\right)\frac{p(r)\,|u_d(r)|^{\gamma+1}}{\gamma+1}. \end{aligned}$$

Moreover, (5.10) guarantees that

$$\frac{(\beta+\varepsilon)\,h\,r^{h-k}}{\alpha\,r}\,\frac{\gamma+1}{\delta+1} \leq \frac{\varepsilon}{r} \qquad \forall\, r \in (0, R_1].$$

As a consequence,

$$\mathcal{E}_{u_d}'(r) \leq \frac{k+2\varepsilon}{r}\,\frac{p(r)\,|u_d(r)|^{\gamma+1}}{\gamma+1} \leq \frac{k+2\varepsilon}{r}\,\mathcal{E}_{u_d}(r) \qquad \forall\, r \in [r_d, R_d].$$

As before, we can now get (5.3), (5.4) and (5.7) for every $r \in [r_d, R_d]$, where $c := k + 2\varepsilon$. In particular, for every $r \in [r_d, R_d]$, the energy $\mathcal{E}_{u_d}(r)$ is less than or equal to

$$c_1\,r_d^{2h-k+2-2\varepsilon}\,|d|^{2\delta} + c_2\,r_d^{k+2-2\varepsilon}\,|d|^{2\gamma} + c_3\,r_d^{h+2-2\varepsilon}\,|d|^{\delta+\gamma} + c_4\,r_d^{h-k-2\varepsilon}\,|d|^{\delta+1} + c_5\,r_d^{-2\varepsilon}\,|d|^{\gamma+1}.$$

Taking into account (5.8), (5.10) and recalling that $R_d \leq R_1 < 1$, we infer the existence of $\bar{c}_5 > 0$ such that for every $d \in \mathbb{R} \setminus \{0\}$, with $|d| \leq \widetilde{d_\sigma}$

$$\mathcal{E}_{u_d}(r) \leq c_1\,|d|^{2\delta} + c_2\,|d|^{2\gamma} + c_3\,|d|^{\delta+\gamma} + c_4\,|d|^{\delta+1} + \bar{c}_5\,\frac{|d|^{\gamma+1}}{|d|^{\frac{2\varepsilon(1-\gamma)}{k+2}}} \to 0 \quad \text{as } d \to 0.$$

This guarantees the existence of $d_\sigma \in (0, \widetilde{d_\sigma})$ such that

$$|u_d(r)| < \frac{\sigma}{2}, \quad |u_d'(r)| < \frac{\sigma}{2} \quad \forall\, r \in [0, R_d], \quad 0 < |d| \leq d_\sigma.$$

By the maximality of the choice of R_d satisfying (5.11), we conclude that $R_d = R_1$ for every $d \in \mathbb{R} \setminus \{0\}$ with $|d| \le d_\sigma$, from which it follows

$$\|u_d\|_{1_{|[0,R_1]}} \le \sigma \qquad \forall\, d \in \mathbb{R} \setminus \{0\} : \ |d| \le d_\sigma.$$

This completes the proof of the lemma under the assumptions (1.4) and (1.6).

Assume secondly the validity of (1.3) and (1.6).

Taking into account the energy derivative expression (4.6), from (2.4), (2.8) and (2.13), we obtain

$$\mathcal{E}'_{u_d}(r) \le \frac{Q'}{q_0}\,\frac{q(r)}{\delta+1}\,|u_d(r)|^{\delta+1} + \frac{k+\varepsilon}{r}\,\frac{p(r)}{\gamma+1}\,|u_d(r)|^{\gamma+1} \qquad \forall\, r \in [0, R_1].$$

According to assumption (1.3), it is not restrictive to choose $\varepsilon > 0$, $q_0 > 0$ and $R_1 \in (0,1)$ in Remark 2.1 satisfying also the following inequality

$$\frac{Q'}{q_0} \le \frac{k+\varepsilon}{r}, \qquad \forall\, r \in (0, R_1]. \tag{5.12}$$

Hence, recalling the energy definition (4.4) we infer that

$$\mathcal{E}'_{u_d}(r) \le \frac{k+\varepsilon}{r}\ \mathcal{E}_{u_d}(r) \qquad \forall\, r \in [r_d, R_1] \subset (0, R_1].$$

Exactly as in the previous case, by Gronwall Lemma we get (5.4) with $c := k+\varepsilon$.
According to condition (2.15), it is not rescrictive to require the positive costants ε, R_1 in Remark 2.1 to satisfy the further inequality

$$\Phi_q(r) \le \frac{q(0)+\varepsilon}{N}\, r^N, \quad \forall\, r \in (0, R_1]. \tag{5.13}$$

By combining (5.4) with (5.6) and (5.13), we deduce the existence of five positive constants $\bar c_i = \bar c_i(\alpha, k, \varepsilon, \delta, \gamma, N, Q, q(0))$, $i \in \{1, \dots, 5\}$ such that for every $r \in [r_d, R_1]$

$$\mathcal{E}_{u_d}(r) \le \bar c_1\, r_d^{2-k-\varepsilon}\, |d|^{2\delta} + \bar c_2\, r_d^{k-\varepsilon+2}\, |d|^{2\gamma} + \bar c_3\, r_d^{-\varepsilon+2}\, |d|^{\delta+\gamma} + \bar c_4\, r_d^{-k-\varepsilon}\, |d|^{\delta+1} + \bar c_5\, r_d^{-\varepsilon}\, |d|^{\gamma+1}.$$

As before, we immediately notice that both the second and the third addendum tend to zero as $d \to 0$. The estimates of the other addenda are based on (5.8). In particular, from (5.8) we deduce the existence of two positive constants $\tilde c_4, \tilde c_5$ such that

$$\frac{\bar c_4\ |d|^{\delta+1}}{r_d^{k+\varepsilon}} + \frac{\bar c_5\ |d|^{\gamma+1}}{r_d^{\varepsilon}} \le \tilde c_4\, \frac{|d|^{\delta+1}}{|d|^{\frac{(1-\gamma)(k+\varepsilon)}{k+2}}} + \tilde c_5\, \frac{|d|^{\gamma+1}}{|d|^{\frac{\varepsilon(1-\gamma)}{k+2}}} \to 0 \quad \text{as } d \to 0.$$

If $k+\varepsilon \le 2$, it is immediate to note that the first addendum converges to zero. Otherwise, (5.8) ensures the existence of $\tilde c_1 > 0$ such that

$$\frac{\bar c_1\ |d|^{2\delta}}{r_d^{k+\varepsilon-2}} \le \tilde c_1\, \frac{|d|^{2\delta}}{|d|^{\frac{(1-\gamma)(k+\varepsilon-2)}{k+2}}} \to 0 \quad \text{as } d \to 0.$$

This completes the proof of the lemma under the assumptions (1.3) and (1.6).

Finally, assume the validity of (1.5) combined with the assumption $q \geq 0$. Define

$$F(r,x) := \frac{q(r)}{\delta+1}\,|x|^{\delta+1} + \frac{p(r)}{\gamma+1}\,|x|^{\gamma+1}, \qquad \forall\,(r,x) \in [0,R_1]\times\mathbb{R}.$$

According to (2.7), observe that for every $(r,x) \in [0,R_1]\times[-1,1]$

$$\left|\frac{\partial F(r,x)}{\partial r}\right| \leq \frac{|q'(r)|}{\delta+1}\,|x|^{\delta+1} + \frac{|p'(r)|}{\gamma+1}\,|x|^{\gamma+1} \leq \varphi(r)\,\frac{p(r)}{\gamma+1}\,|x|^{\gamma+1} \leq \varphi(r)\,F(r,x),$$

where $\varphi(r) := \left(\frac{|q'(r)|(\gamma+1)}{p_0(\delta+1)} + \frac{|p'(r)|}{p_0}\right)$. The goal follows by a straightforward application of Lemma 2.3 in [9]. ■

6 Elastic property

The aim of this section consists in showing some qualitative properties of the solutions to (2.2) with large initial data. In particular, we are concerned with the validity of Elastic property, which ensures arbitrarily large norms of the solutions to (2.2) for suitably large values of $|d|$.

6.1 Elastic property in the ball $|x| \leq R_1$

Let us focus our attention on the Elastic property in the interval $[0,R_1]$.

Proposition 6.1 *Under the assumptions of Theorem* 1.1*, for every $\mu > 0$, $\eta > 0$ and $L_1 > 0$ there exists $L_2 > 0$ such that for every $d \in \mathbb{R}$ with $|d| \geq L_2$*

$$\mu\,u_d(r)^2 + \eta\,u_d'(r)^2 \geq L_1, \qquad \forall\,r \in [0,R_1]. \tag{6.1}$$

Due to Lemma 3.2, we can easily verify the validity of the Elastic property in $[0,r_d)$, whenever d is chosen large enough. Indeed, for every $\mu > 0$ and $\eta > 0$,

$$\mu\,u_d(r)^2 + \eta\,u_d'(r)^2 \geq \mu\,u_d(r)^2 \geq \mu\,\vartheta_0^2\,d^2, \qquad \forall\,r \in [0,r_d], \tag{6.2}$$

whence (6.1) restricted to the interval $[0,r_d]$ immediately follows.

Taking into account (2.10) and (2.11), we now take a positive constant a satisfying

$$\frac{N-2}{2} < a < \frac{N-\varepsilon}{\delta+1} < \frac{N-\varepsilon}{\gamma+1} \quad \text{under assumption (1.12)} \tag{6.3}$$

$$\frac{N-2}{2} < a < \min\left\{\frac{N+h-\varepsilon}{\delta+1}, \frac{N-\varepsilon}{\gamma+1}\right\} \quad \text{under assumption (1.13).} \tag{6.4}$$

Hence, we can deduce the existence of a suitably small $\hat{\varepsilon} = \hat{\varepsilon}(N, \delta, \gamma, \varepsilon, q) > 0$ satisfying

$$\frac{N-\varepsilon}{\delta+1} - a > \hat{\varepsilon} \qquad \text{under assumption (1.12)}, \tag{6.5}$$

$$\frac{N+h-\varepsilon}{\delta+1} - a > \hat{\varepsilon} \qquad \text{under assumption (1.13)}. \tag{6.6}$$

According to this choice and to Remark 4.5 ensuring the uniqueness of u_d in $[0, R_1] \subset [0, \widetilde{s}_1)$, we can set for every $r \in [0, R_1]$

$$\mathtt{E}_d(r) := \frac{1}{2}\, r^N\, u_d'(r)^2 \,+\, r^N\, \frac{q(r)}{\delta+1}\, |u_d(r)|^{\delta+1} \,+\, r^N\, \frac{p(r)}{\gamma+1}\, |u_d(r)|^{\gamma+1} + a\, r^{N-1}\, u_d'(r)\, u_d(r).$$

Proposition 6.2 *Assume* (1.2) *and one of the alternatives* (1.5)-(1.6). *Moreover, suppose either* (1.3) *and* (1.12) *or* (1.4) *and* (1.13). *Then, there exists a positive constant* $C := \hat{\varepsilon}\, \vartheta_0^{\delta+1}$, *such that for every* $d \in \mathbb{R}$ *with* $|d| \geq d^*$

$$\mathtt{E}_d(r) \geq C\, |d|^{\delta+1}\, \Phi_q \left(\Psi_q^{-1} \left(|d|^{1-\delta}(1-\vartheta_0) \,-\, |d|^{\gamma-\delta} \frac{P}{2N} \right) \right) \qquad \forall\, r \in [r_d, R_1]. \tag{6.7}$$

Proof. From (4.5), it is immediate to verify that for every $r \in [0, R_1]$

$$\begin{aligned} \mathtt{E}_d'(r) \;\;&=\;\; r^{N-1}\, u_d'(r)^2 \left(a - \frac{N-2}{2} \right) + r^{N-1} |u_d(r)|^{\delta+1} \left(r \frac{q'(r)}{\delta+1} + \frac{N\, q(r)}{\delta+1} - a q(r) \right) + \\ &+\;\; r^{N-1} |u_d(r)|^{\gamma+1} \left(r \frac{p'(r)}{\gamma+1} + \frac{N\, p(r)}{\gamma+1} - a p(r) \right). \end{aligned} \tag{6.8}$$

By both (6.3) and (6.4), we deduce that

$$\begin{aligned} \mathtt{E}_d'(r) \;\;&\geq\;\; r^{N-1} |u_d(r)|^{\delta+1} \left(\frac{r q'(r) + N q(r)}{\delta+1} - a q(r) \right) + \\ &+\;\; r^{N-1} |u_d(r)|^{\gamma+1} \left(\frac{r p'(r) + N\, p(r)}{\gamma+1} - a p(r) \right). \end{aligned} \tag{6.9}$$

Moreover, according to (2.6) and (2.8) (consequence of our assumptions (1.5) and (1.6), respectively), it follows that for every $r \in [0, R_1]$

$$\mathtt{E}_d'(r) \geq r^{N-1} |u_d(r)|^{\delta+1} \left(\frac{r q'(r) + N q(r)}{\delta+1} - a q(r) \right) + r^{N-1} |u_d(r)|^{\gamma+1} p(r) \left(\frac{N-\varepsilon}{\gamma+1} - a \right). \tag{6.10}$$

Taking into account (2.3), (2.5) and (6.5)-(6.6), we immediately obtain

$$\left(\frac{r q'(r) + N q(r)}{\delta+1} - a q(r) \right) \geq q(r) \left(\frac{N-\varepsilon}{\delta+1} - a \right) \geq q(t)\, \hat{\varepsilon}, \qquad \text{if (1.12) holds,}$$

$$\left(\frac{r q'(r) + N q(r)}{\delta+1} - a q(r) \right) \geq q(r) \left(\frac{N+h-\varepsilon}{\delta+1} - a \right) \geq q(t)\, \hat{\varepsilon}, \quad \text{if (1.13) holds.}$$

Thus, using one of (6.3)-(6.4) we infer from (6.10) that

$$\mathtt{E}_d'(r) \ \geq \ r^{N-1}\,|u_d(r)|^{\delta+1}\,q(t)\,\hat{\varepsilon} \qquad \forall\, r \in [0, R_1]. \tag{6.11}$$

Our aim consists in proving the validity of (6.7). According to (6.11), observe that

$$\mathtt{E}_d(r) = \int_0^r \mathtt{E}_d'(s)\,ds = \int_0^{r_d} \mathtt{E}_d'(s)\,ds + \int_{r_d}^r \mathtt{E}_d'(s)\,ds \geq \int_0^{r_d} \mathtt{E}_d'(s)\,ds, \qquad \forall\, r \in [r_d, R_1].$$

By combining (3.8) with (6.11) and recalling the definition of Φ_q in (2.14), it follows that for every $r \in [r_d, R_1]$

$$\mathtt{E}_d(r) \ \geq \ \hat{\varepsilon} \int_0^{r_d} s^{N-1}\,|u_d(s)|^{\delta+1}\,q(s)\,ds \ \geq \ \hat{\varepsilon}\ \vartheta_0^{\delta+1}\,|d|^{\delta+1}\ \Phi_q(r_d).$$

Recalling now that Φ_q is nondecreasing, from Lemma 3.5 we infer that for every $d \in \mathbb{R}$ with $|d| \geq d^*$ and for every $r \in [r_d, R_1]$

$$\mathtt{E}_d(r) \ \geq \ \hat{\varepsilon}\ \vartheta_0^{\delta+1}\,|d|^{\delta+1}\ \Phi_q\left(\Psi_q^{-1}\left(|d|^{1-\delta}(1-\vartheta_0) \ - \ |d|^{\gamma-\delta}\frac{P}{2N}\right)\right),$$

whence the thesis follows. ∎

Proposition 6.3 *Assume* (1.2) *and one of the alternatives* (1.5)-(1.6)*. Moreover, suppose either* (1.3) *and* (1.12) *or* (1.4) *and* (1.13)*. Then, for every $K > 0$ there exists $d_K \geq d^*$ such that for every $d \in \mathbb{R}$ with $|d| > d_K$*

$$\mathtt{E}_d(r) \geq K, \qquad \forall\, r \in [r_d, R_1].$$

Proof. We analyze separately the two alternatives of our assumptions.

Assume first the validity of (1.3) and (1.12). Condition (2.15) guarantees that

$$\Phi_q(\Psi_q^{-1}(r)) \sim N^{\frac{N-2}{2}}\,q(0)^{-\frac{N-2}{2}}\,2^{\frac{N}{2}}\,r^{\frac{N}{2}} =: \chi\, r^{\frac{N}{2}}, \qquad \text{as} \ \ r \to 0^+.$$

As a consequence, there exists $\rho_0 > 0$ such that

$$\Phi_q(\Psi_q^{-1}(r)) \geq (1-\varepsilon)\,\chi\,r^{\frac{N}{2}} \qquad \forall\, r \in [0, \rho_0].$$

Recalling (3.12) and taking into account that $\gamma < 1 < \delta$, we infer the existence of $\hat{d}^* > d^*$ such that for every $d \in \mathbb{R}$ with $|d| \geq \hat{d}^*$

$$0 \leq |d|^{1-\delta}(1-\vartheta_0) \ - \ |d|^{\gamma-\delta}\frac{P}{2N} \leq \rho_0. \tag{6.12}$$

Hence, Proposition 6.2 ensures that for every $d \in \mathbb{R}$ with $|d| \geq \hat{d}^*$

$$\mathtt{E}_d(r) \geq C\,|d|^{\delta+1}\,(1-\varepsilon)\,\chi\left(|d|^{1-\delta}(1-\vartheta_0) - |d|^{\gamma-\delta}\frac{P}{2N}\right)^{\frac{N}{2}}, \quad \forall\, r \in [r_d, R_1],$$

which leads to

$$\mathtt{E}_d(r) \ge \widetilde{C}\,|d|^{\delta+1}\,|d|^{\frac{(1-\delta)N}{2}} \left((1-\vartheta_0) - |d|^{\gamma-1}\frac{P}{2N}\right)^{\frac{N}{2}}, \quad \forall\, r \in [r_d, R_1],$$

where we set $\widetilde{C} := C(1-\varepsilon)\chi > 0$. Passing to the limit as $|d| \to +\infty$ and taking into account that (1.12) can be equivalently written in the form

$$\delta + 1 + \frac{(1-\delta)N}{2} \;>\; 0,$$

we can deduce the existence of $d_K \ge \hat{d}^*$ such that for every $d \in \mathbb{R}$ with $|d| \ge d_K$

$$\mathtt{E}_d(r) \ge K, \qquad \forall\, r \in [r_d, R_1].$$

Assume now the validity of (1.4) and (1.13). The condition (2.16) guarantees that

$$\Phi_q(\Psi_q^{-1}(r)) \sim\; (h+N)^{\frac{N-2}{h+2}}\,\beta^{-\frac{N-2}{h+2}}\,(h+2)^{\frac{h+N}{h+2}}\,r^{\frac{h+N}{h+2}} := \chi\, r^{\frac{h+N}{h+2}} \qquad \text{as} \;\; r \to 0^+. \quad (6.13)$$

As a consequence, there exists $\rho_0 > 0$ such that

$$\Phi_q(\Psi_q^{-1}(r)) \ge (1-\varepsilon)\,\chi\, r^{\frac{h+N}{h+2}} \qquad \forall\, r \in [0, \rho_0].$$

As before, by (3.12) we infer the existence of $\hat{d}^* > d^*$ such that (6.12) holds for every $d \in \mathbb{R}$ with $|d| \ge \hat{d}^*$. By combining Proposition 6.2 with (6.12), we deduce that for every $d \in \mathbb{R}$ with $|d| \ge \hat{d}^*$

$$\mathtt{E}_d(r) \ge C\,|d|^{\delta+1}\,(1-\varepsilon)\,\chi \left(|d|^{1-\delta}(1-\vartheta_0) - |d|^{\gamma-\delta}\frac{P}{2N}\right)^{\frac{h+N}{h+2}}, \quad \forall\, r \in [r_d, R_1],$$

which leads to

$$\mathtt{E}_d(r) \ge \widetilde{C}\,|d|^{\delta+1}\,|d|^{\frac{(1-\delta)(h+N)}{(h+2)}} \left((1-\vartheta_0) - |d|^{\gamma-1}\frac{P}{2N}\right)^{\frac{h+N}{h+2}}, \quad \forall\, r \in [r_d, R_1],$$

where $\widetilde{C} := C(1-\varepsilon)\chi$. Passing to the limit as $|d| \to +\infty$ and taking into account that (1.13) can be equivalently written in the form

$$\delta + 1 + \frac{(1-\delta)(h+N)}{(h+2)} \;>\; 0,$$

we can conclude that there exists $d_K \ge \hat{d}^*$ such that for every $d \in \mathbb{R}$ with $|d| \ge d_K$

$$\mathtt{E}_d(r) \ge K, \qquad \forall\, r \in [r_d, R_1].$$

This completes the proof. ∎

Recalling the definition of $\mathtt{E}_d$, we can easily infer from Proposition 6.3 that for every $\mu > 0$, $\eta > 0$ and $L_1 > 0$ there exists $L_2 > 0$ such that for every $d \in \mathbb{R}$ with $|d| \ge L_2$

$$\mu\, u_d(r)^2 \,+\, \eta\, u_d'(r)^2 \;\ge\; L_1, \qquad \forall\, r \in [r_d, R_1].$$

Taking into account (6.2), we can conclude the proof of Proposition 6.1.

6.2 Elastic property in the annulus $R_1 \leq |x| \leq 1$

We are interested in proving that there exists $\mathcal{L} > 0$ such that, for every $d \in \mathbb{R}$ with $|d| \geq \mathcal{L}$, u_d can be uniquely extended to the whole interval $[0, 1]$.
According to Remark 4.6, this result is obviously ensured by the validity of the Elastic property in the whole interval $[0, 1]$, on which we are going to devote our attention.

First of all, we show that the interval of validity of the Elastic property could be extended as long as the solutions of the Cauchy problems remain unique. More precisely, setting $\widetilde{s}_0 := R_1$ and according to the notation in (1.7) we can state the following result.

Proposition 6.4 *Consider p, $q \in C^1([0,1], \mathbb{R})$ satisfying (1.2), one of the alternatives in (1.3)-(1.6), and (1.7)-(1.10). Fix $j \in \{0, 1, \ldots, m-1\}$, $\mathcal{T} \in (\widetilde{s}_j, \widetilde{s}_{j+1})$, $\nu > 0$ with $\widetilde{s}_j + \nu < \mathcal{T}$. Then,*

$$\forall\, L_1 > 0 \quad \exists\, L_2 > 0 : \; u_d(r)^2 + u_d'(r)^2 \geq L_1, \quad \forall\, r \in [0, \widetilde{s}_j + \nu], \quad |d| \geq L_2 \tag{6.14}$$

$$\Downarrow$$

$$\forall\, L > 0 \quad \exists\, d_L > 0 : \; u_d(r)^2 \,+\, u_d'(r)^2 \,\geq\, L, \quad \forall\, r \in [\widetilde{s}_j + \nu, \mathcal{T}], \quad |d| \geq d_L.$$

Proof. Let us concentrate on the case $d > 0$. The negative case can be treated in an analogous way. Arguing by contradiction, assume that there exists $L > 0$ such that for every $n \in \mathbb{N}$ there exists $d_n \geq n$ and $t_n \in [\widetilde{s}_j + \nu, \mathcal{T}]$ such that

$$u_{d_n}(t_n)^2 \,+\, u_{d_n}'(t_n)^2 \,<\, L.$$

Setting $\mathbf{v}_n := (u_{d_n}(t_n), u_{d_n}'(t_n))$, we note that, up to a subsequence, $t_n \to \tau \in [\widetilde{s}_j + \nu, \mathcal{T}]$ and $\mathbf{v}_n \to \mathbf{d}$ with $|\mathbf{d}| \leq \sqrt{L}$. Consider now the solution u_0 of the Cauchy problem (4.20). Lemma 4.4 and Remark 4.7 guarantee that u_0 is defined in $[\widetilde{s}_j + \nu, \mathcal{T}]$, and it is unique too. Thus, by classical continuous dependence results (cf., for instance, [20]) we infer that $u_{d_n} \to u_0$ in $C^1([\widetilde{s}_j + \nu, \mathcal{T}], \mathbb{R})$ as $n \to +\infty$. Fixed $\epsilon > 0$, from (6.14) we conclude that

$$\exists\, L_2 > 0 : \; u_{d_n}(\widetilde{s}_j + \nu)^2 + u_{d_n}'(\widetilde{s}_j + \nu)^2 \geq u_0(\widetilde{s}_j + \nu)^2 + u_0'(\widetilde{s}_j + \nu)^2 + \epsilon \quad \forall\, n \in \mathbb{N},\, n \geq L_2,$$

a contradiction with the convergence of u_{d_n} to u_0. This completes the proof. ∎

The next step consists in extending the Elastic Property from a certain $\mathcal{T}_0 \in (\widetilde{s}_j, \widetilde{s}_{j+1})$ to $\widetilde{s}_{j+1}$, and even a bit further than $\widetilde{s}_{j+1}$. To this aim, we need to introduce some notation. According to (2.12), we set

$$K \,:=\, \left(\frac{3}{2} + \frac{Q}{\delta + 1}\right)^{\frac{1}{\gamma+1}}. \tag{6.15}$$

Moreover, define

$$\sigma \in \left(0, \frac{R_1}{2}\right) \qquad \text{and} \qquad b = b(\sigma) > 0 : \qquad b \leq \min\left\{\sigma^{\frac{1}{\gamma+1}}, \frac{1}{2}\right\}. \tag{6.16}$$

Fix $j \in \{0,1,\dots,m-1\}$. Since $p(\widetilde{s}_{j+1}) = p'(\widetilde{s}_{j+1}) = 0$ by (1.7), according also to (2.13) and (6.16), we deduce the existence of $\mathcal{T}_0 \in (\widetilde{s}_j, \widetilde{s}_{j+1})$ and $\nu_0 \in (\widetilde{s}_{j+1}, \widetilde{s}_{j+2})$ such that

$$rp'(r) + N\,p(r) + Nb \,>\, 0 \quad \forall\, r \in [\mathcal{T}_0, \widetilde{s}_{j+1} + \nu_0], \tag{6.17}$$

$$\frac{p(r)+b}{\gamma+1} \,\le\, \frac{1}{2} \qquad \forall\, r \in [\mathcal{T}_0, \widetilde{s}_{j+1} + \nu_0], \tag{6.18}$$

$$\frac{R_1}{2}\; e^{-\max\left\{\frac{N-2}{R_1}+1+2K,1+\frac{2Q'}{\delta+1}\right\}(\widetilde{s}_{j+1}+\nu_0-\mathcal{T}_0)} \,>\, \sigma. \tag{6.19}$$

For the above defined $\mathcal{T}_0$ and ν_0, the following result holds.

Proposition 6.5 *Under the assumptions of Proposition* 6.4*,*

$$\forall\, L_1 > 0 \quad \exists\, L_2 > 0 :\; u_d(\mathcal{T}_0)^2 + u_d'(\mathcal{T}_0)^2 \ge L_1, \quad \forall\, d \in \mathbb{R} \;\textit{ with }\; |d| \ge L_2 \tag{6.20}$$

$$\Downarrow$$

$$\forall\, L > 0 \quad \exists\, d_L > 0 :\; u_d(r)^2 + u_d'(r)^2 \ge L, \quad \forall\, r \in [\mathcal{T}_0, \widetilde{s}_{j+1} + \nu_0], \quad |d| \ge d_L. \tag{6.21}$$

Proof. According to the previous notation, let us define

$$\mathtt{E}_{u_d,b}(r) := r^N \left[\frac{1}{2}\left(u_d'(r)^2 \,+\, u_d(r)^2\right) \,+\, \frac{q(r)}{\delta+1}\,|u_d(r)|^{\delta+1} \,+\, \frac{p(r)+b}{\gamma+1}\,|u_d(r)|^{\gamma+1}\right].$$

We point out that

$$\mathtt{E}_{u_d,b}(r) \,\le\, \frac{R_1}{2} \qquad \Longrightarrow \qquad u_d(r) \,\le\, 1, \quad u_d'(r) \,\le\, 1 \quad \forall\, r \in [\mathcal{T}_0, 1] \subset [R_1, 1]. \tag{6.22}$$

From (4.5), it is immediate to verify that for every $r \in [0,1]$

$$\mathtt{E}'_{u_d,b}(r) = r^{N-1}\left[-\frac{N-2}{2}u_d'(r)^2 + \frac{N}{2}u_d(r)^2 + r u_d(r) u_d'(r)\left(1 + b|u_d(r)|^{\gamma-1}\right) + \right.$$

$$\left. + \frac{|u_d(r)|^{\delta+1}}{\delta+1}\left(rq'(r) + N\,q(r)\right) + \frac{|u_d(r)|^{\gamma+1}}{\gamma+1}\left(rp'(r) + N\,p(r) + Nb\right)\right].$$

Assumption (6.20) guarantees the existence of $d_{R_1} > 0$ such that $\mathtt{E}_{u_d,b}(\mathcal{T}_0) > R_1$ for every $d \in \mathbb{R}$ with $|d| \ge d_{R_1}$.
We claim that

$$\mathtt{E}_{u_d,b}(r) \,>\, \sigma, \qquad \forall\, r \in [\mathcal{T}_0, \widetilde{s}_{j+1} + \nu_0], \quad |d| \ge d_{R_1}. \tag{6.23}$$

Suppose, by contradiction, that there exist t_0, $l_0 \in (\mathcal{T}_0, \widetilde{s}_{j+1}+\nu_0]$ with $t_0 < l_0$ and $d_0 \in \mathbb{R}$ with $|d_0| \ge d_{R_1}$ such that

$$\sigma \,<\, \mathtt{E}_{u_{d_0},b}(r) \,<\, \frac{R_1}{2}, \qquad \forall r \in (t_0, l_0),$$

$$\mathtt{E}_{u_{d_0},b}(t_0) = \frac{R_1}{2} \qquad \text{and} \qquad \mathtt{E}_{u_{d_0},b}(l_0) = \sigma. \tag{6.24}$$

From (6.22), we deduce that $u_{d_0}(r) \leq 1$ and $u'_{d_0}(r) \leq 1$ for every $r \in [t_0, l_0]$. As a consequence, according to (2.13) and (6.17), we get

$$\mathtt{E}'_{u_{d_0},b}(r) \geq -r^N \left[\frac{N-2+R_1}{2R_1} u'_{d_0}(r)^2 + \frac{1}{2} u_{d_0}(r)^2 + b|u'_{d_0}(r)||u_{d_0}(r)|^\gamma + \frac{u_{d_0}(r)^2}{\delta+1} Q'\right].$$

Note that

$$|u'_{d_0}(r)| \leq |u_{d_0}(r)| \implies |u'_{d_0}(r)||u_{d_0}(r)|^\gamma \leq |u_{d_0}(r)|^{\gamma+1}. \tag{6.25}$$

Consider now the case $|u_{d_0}(r)| \leq |u'_{d_0}(r)|$. Taking into account (6.18) and the energy definition, we observe that

$$\sigma \leq \mathtt{E}_{u_{d_0},b}(r) \leq \frac{1}{2} u'_{d_0}(r)^2 + \left(1 + \frac{Q}{\delta+1}\right) |u_{d_0}(r)|^{\gamma+1}, \quad \forall\, r \in [t_0, l_0],$$

from which it follows

$$\sigma \leq \frac{1}{2} u'_{d_0}(r)^2 + \left(1 + \frac{Q}{\delta+1}\right) |u'_{d_0}(r)|^{\gamma+1} \leq \left(\frac{3}{2} + \frac{Q}{\delta+1}\right) |u'_{d_0}(r)|^{\gamma+1}.$$

Recalling the definition in (6.15), we have so proved that for every $r \in [t_0, l_0]$

$$|u_{d_0}(r)| \leq |u'_{d_0}(r)| \implies |u'_{d_0}(r)||u_{d_0}(r)|^\gamma \leq |u'_{d_0}(r)| \leq \frac{K\,|u'_{d_0}(r)|^2}{\sigma^{\frac{1}{\gamma+1}}},$$

which combined with (6.25) leads to

$$\mathtt{E}'_{u_{d_0},b}(r) \geq -r^N \left[\left(\frac{N-2+R_1}{2R_1} + \frac{b\,K}{\sigma^{\frac{1}{\gamma+1}}}\right) u'_{d_0}(r)^2 + \left(\frac{1}{2} + \frac{Q'}{\delta+1}\right) u_{d_0}(r)^2 + b|u_{d_0}(r)|^{\gamma+1}\right],$$

for every $r \in [t_0, l_0]$. Recalling the energy definition, we infer

$$\mathtt{E}'_{u_{d_0},b}(r) \geq -\max\left\{\frac{N-2+R_1}{R_1} + \frac{2b\,K}{\sigma^{\frac{1}{\gamma+1}}}, 1 + \frac{2Q'}{\delta+1}, \gamma+1\right\} \mathtt{E}_{u_{d_0},b}(r), \quad \forall\, r \in [t_0, l_0].$$

Taking into account (6.16) and our assumption $N \geq 3$, by Gronwall Lemma we deduce

$$\mathtt{E}_{u_{d_0},b}(l_0) \geq \mathtt{E}_{u_{d_0},b}(t_0)\; e^{-\max\left\{\frac{N-2}{R_1}+1+2K, 1+\frac{2Q'}{\delta+1}\right\}(l_0-t_0)},$$

which, according to (6.24) and (6.19), implies

$$\sigma \geq \frac{R_1}{2}\; e^{-\max\left\{\frac{N-2}{R_1}+1+2K, 1+\frac{2Q'}{\delta+1}\right\}(l_0-t_0)} \geq \frac{R_1}{2}\; e^{-\max\left\{\frac{N-2}{R_1}+1+2K, 1+\frac{2Q'}{\delta+1}\right\}(\widetilde{s}_{j+1}+\nu_0-\mathcal{T}_0)} > \sigma,$$

a contradiction. Thus, the claim is proved and (6.23) is satisfied. In particular, there exists $\sigma_1 > 0$ such that

$$u_d(r)^2 + u'_d(r)^2 > \sigma_1, \qquad \forall\, r \in [\mathcal{T}_0, \widetilde{s}_{j+1} + \nu_0], \quad |d| \geq d_{R_1}. \tag{6.26}$$

The last step consists in proving that the Elastic property (6.21) follows by combining (6.20) with (6.26).
We argue exactly as in Proposition 6.4, focusing on the case $d > 0$, and assuming, by contradiction, the existence of $L > 0$ such that for every $n \in \mathbb{N}$ with $n \geq d_{R_1}$ there exists $d_n \geq n$ and $t_n \in [\mathcal{T}_0, \widetilde{s}_{j+1} + \nu_0]$ such that

$$\sigma_1 \, < \, u_{d_n}(t_n)^2 \, + \, u'_{d_n}(t_n)^2 \, < \, L.$$

Setting $\mathbf{v}_n := (u_{d_n}(t_n), u'_{d_n}(t_n))$, notice that, up to a subsequence, $t_n \to \tau \in [\mathcal{T}_0, \widetilde{s}_{j+1}+\nu_0]$ and $\mathbf{v}_n \to \mathbf{d}$ with $|\mathbf{d}| \leq \sqrt{L}$. Taking into account Remark 4.7, let u_0 be the unique solution of the Cauchy problem (4.20), defined in $[\mathcal{T}_0, \widetilde{s}_{j+1} + \nu_0]$. Thus, by the classical continuous dependence results, $u_{d_n} \to u_0$ in $C^1([\mathcal{T}_0, \widetilde{s}_{j+1} + \nu_0], \mathbb{R})$ as $n \to +\infty$. Fixed $\epsilon > 0$, from (6.20) we conclude that

$$\exists \, L_2 > 0 : \quad u_{d_n}(\mathcal{T}_0)^2 + u'_{d_n}(\mathcal{T}_0)^2 \geq u_0(\mathcal{T}_0)^2 + u'_0(\mathcal{T}_0)^2 + \epsilon, \qquad \forall \, n \in \mathbb{N}, \;\; n \geq L_2,$$

a contradiction. The Elastic property (6.21) is achieved. ■

By applying a finite number of times Proposition 6.4 and Proposition 6.5, combined with Proposition 6.1 we can conclude that

Proposition 6.6 *Under the assumptions of Theorem* 1.1, *for every* $L_1 > 0$ *there exists* $L_2 > 0$ *such that, for every* $d \in \mathbb{R}$ *with* $|d| \geq L_2$, u_d *is the unique solution of* (2.2) *and*

$$u_d(r)^2 \, + \, u'_d(r)^2 \geq L_1, \qquad \forall \, r \in [0, 1].$$

7 Rotation number and multiplicity of solutions

This section is devoted to define the rotation number associated with the solution of the Cauchy problem (2.2), and to show how some estimates on the rotation number lead to multiplicity results.
Observe first that (2.2) can be equivalently written in the form

$$\begin{cases} r^{N-1} u'(r) \, = \, y(r) \\ y'(r) \, = \, - \, r^{N-1} \, |u(r)|^{\delta-1} \, u(r) \, q(r) \, - \, r^{N-1} \, |u(r)|^{\gamma-1} \, u(r) \, p(r) \\ (u(0), y(0)) \, = \, (d, 0). \end{cases} \tag{7.1}$$

For any $\mu > 0$, introduce the polar coordinates (ϑ, ρ) by setting $\mu u = \rho \sin \vartheta$, $y = \rho \cos \vartheta$. Hence, for each interval $[s, t] \subset [0, 1]$ and for each solution u_d of (2.2) defined on $[s, t]$ and such that

$$u_d(r)^2 + u'_d(r)^2 > 0 \quad \forall \, r \in [s, t], \tag{7.2}$$

we can define the rotation number

$$\mathrm{Rot}_{[s,t]}(u_d) := \frac{\vartheta(t) - \vartheta(s)}{\pi},$$

which counts the number of half-turns of the vector $\overrightarrow{((0,0),(u,u'))}$ as r moves from s to t. According to [14], the rotation number can be expressed by the following relation

$$\mathrm{Rot}_{[s,t]}(u_d) = \frac{\mu}{\pi}\int_s^t \frac{r^{N-1}\, u_d'(r)^2 + r^{N-1}\,|u_d(r)|^{\delta+1}\, q(r) \,+\, r^{N-1}\,|u_d(r)|^{\gamma+1}\, p(r)}{r^{2(N-1)}\, u_d'(r)^2 \,+\, \mu^2 u_d(r)^2}\, dr. \tag{7.3}$$

Note that for every $j \in \mathbb{N}$

$$\mathrm{Rot}_{[0,1]}(u_d) = \frac{1}{2} + j \quad \Longleftrightarrow \quad u_d(1) = 0 \text{ and } u_d \text{ has exactly } j \text{ zeros in } (0,1). \tag{7.4}$$

This relation is independent from the choice of $\mu > 0$. For this reason we are allowed to estimate the rotation numbers with values in $\frac{1}{2} + \mathbb{N}$ by choosing $\mu > 0$ according to our convenience.
Moreover, from (1.2) observe that if $\mathrm{Rot}_{[s,t]}(u_d)$ is well defined, then

$$[s_*, t_*] \subset [s,t] \subset [0,1] \quad \Longrightarrow \quad \mathrm{Rot}_{[s_*,t_*]}(u_d) \,\le\, \mathrm{Rot}_{[s,t]}(u_d). \tag{7.5}$$

Taking into account Remark 4.6 and Proposition 6.6, we know that in general the validity of condition (7.2) is not guaranteed for $t \ge \widetilde{s}_1$ and small values of d. For this reason, fixed a positive constant c, we introduce, inspired by [14], the map $\mathrm{Rot}_c^* : \{u_d \in C^1([0,1],\mathbb{R}) : u_d$ solves (2.2) with $d \neq 0\} \to \mathbb{R}$ by setting

$$\mathrm{Rot}_c^*(u_d) := \begin{cases} \mathrm{Rot}_{[0,1]}(u_d) & \text{if (7.2) holds in } [0,1] \text{ and } \mathrm{Rot}_{[0,1]}(u_d) \le c \\ c & \text{otherwise.} \end{cases}$$

The continuity of this function follows by combining the continuity of the Volterra integral operator associated to (7.1), the uniqueness result in Remark 4.6 with the following proposition

Proposition 7.1 *Under the assumptions of Lemma* 4.3*, fix* $d \neq 0$*. Assume the existence of* $\widetilde{s}_j \in (0,1]$ *such that*

$$u_d(r)^2 \,+\, u_d'(r)^2 \,>\, 0, \quad \forall\, r \in [0, \widetilde{s}_j) \qquad \textit{and} \qquad u_d(\widetilde{s}_j)^2 + u_d'(\widetilde{s}_j)^2 \,=\, 0. \tag{7.6}$$

Then,

$$\mathrm{Rot}_{[0,t]}(u_d) \to +\infty \qquad \textit{as} \quad t \to \widetilde{s}_j^{\,-}.$$

Proof. As observed in (7.5), the map $t \mapsto \mathrm{Rot}_{[0,t]}(u_d)$ is nondecreasing. Hence, $\mathrm{Rot}_{[0,t]}(u_d)$ admits a limit $l \in (0, \overline{\mathbb{R}}]$ as $t \to \widetilde{s}_j^{\,-}$. Assume, by contradiction, that $l \in \mathbb{R}$. Then, u_d and u_d' have a finite numer of zeros in $[0, \widetilde{s}_j)$. In particular, from (7.6), we deduce the existence of $r_0 \in (0, \widetilde{s}_j)$ such that

$$u_d(r)\, u_d'(r) \,<\, 0, \qquad \forall\, r \in [r_0, \widetilde{s}_j). \tag{7.7}$$

Thus, according to (4.5) we infer that for every $r \in [r_0, \widetilde{s}_j)$

$$\big(u_d'(r)^2\big)' = 2u_d'(r)u_d''(r) = 2|u_d(r)|^\delta\, |u_d'(r)|\, q(r) + 2|u_d(r)|^\gamma\, |u_d'(r)|\, p(r) - \frac{2(N-1)}{r}\, u_d'(r)^2,$$

which leads to

$$\left(u_d'(r)^2\right)' \geq -\frac{2\,(N-1)}{r_0}\,u_d'(r)^2, \qquad \forall\, r \in [r_0, \widetilde{s}_j).$$

Finally, Gronwall Lemma and (7.7) enable us to conclude that

$$u_d'(\widetilde{s}_j)^2 \,\geq\, e^{-\frac{2\,(N-1)}{r_0}(\widetilde{s}_j-\widetilde{t})}\,u_d'(r_0)^2 \,>\, 0,$$

contradicting (7.6). This completes the proof. ∎

We now present two theorems which allow us to provide the existence of solutions of the two-point boundary value problem (2.1) when uniqueness of the solutions of the associated Cauchy problem is not guaranteed. The first Theorem has been proved in [12], by applying a variant of the classical "Théorème Fondamental" of Leray and Schauder [24].

Theorem 7.2 [12] *Let $F : [a,b] \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfy the Carathéodory conditions. Let $J \subset \mathbb{R}$ be a (non-degenerate) interval, let $\tau : J \to [a,b]$ and $\eta : J \to \mathbb{R}^2$ be two continuous functions such that, for each $d \in J$, all the (non-continuable) solutions to the Cauchy problem*

$$\begin{cases} \dot{x} = F(r,x) \\ x(\tau(d)) = \eta(d) \end{cases} \tag{7.8}$$

are defined on $[a,b]$. Then, there is a closed connected set $\mathcal{C} \subset J \times C([a,b],\mathbb{R}^2)$ such that the projection Pr_1 of $\mathcal{C}$ to J covers J and, for every $(d,u) \in \mathcal{C}$, $u(\cdot)$ is a solution of (7.8) defined on $[a,b]$.

Arguing on the same lines of [12], this theorem allows us to obtain, by means of estimates on rotation numbers, solutions to (2.1) characterized by their nodal properties.

Theorem 7.3 *Under the assumptions of Lemma 4.3, suppose that there exist $j \in \mathbb{N}$, $c \in \mathbb{R}$ with $c > j+1$ and two positive numbers $\mathcal{S} < \mathcal{R}$ such that*

$$\mathrm{Rot}_c^*(u_d) > \frac{1}{2} + j \quad \left(\textit{resp.} \;\; < \frac{1}{2} + j\right), \quad \textit{for each solution } u_d \textit{ of } (2.2) \textit{ with } |d| = \mathcal{S},$$

as well as

$$\mathrm{Rot}_c^*(u_d) < \frac{1}{2} + j \quad \left(\textit{resp.} \;\; > \frac{1}{2} + j\right), \quad \textit{for each solution } u_d \textit{ of } (2.2) \textit{ with } |d| = \mathcal{R}.$$

Then, there are at least two solutions u_j and v_j of (2.1) with $u_j(0) \in (\mathcal{S},\mathcal{R})$ and $v_j(0) \in (-\mathcal{R},-\mathcal{S})$ having exactly j zeros in $(0,1)$.

Proof. Let us study the following problem

$$\begin{cases} u'(r) \,=\, r^{-(N-1)}\,y(r) \\ y'(r) \,=\, -\,r^{N-1}\,|u(r)|^{\delta-1}\,u(r)\,q(r) \,-\, r^{N-1}\,|u(r)|^{\gamma-1}\,u(r)\,p(r) \\ (u,y)(r_d) \,=\, \eta(d), \end{cases} \tag{7.9}$$

where r_d is introduced in Lemma 3.2 and $\eta : [\mathcal{S}, \mathcal{R}] \longrightarrow \mathbb{R}^2 \setminus \{0\}$ is the function defined by $\eta(d) = (u_d(r_d), r_d^{N-1} u_d'(r_d))$, with u_d solution to (2.2). Due to the uniqueness of u_d in $[0, r_d]$ stated by Lemma 3.3 combined with the classical results on continuous dependence from initial data (cf., for instance, [20]), we easily deduce the continuity of the map $\tau : d \mapsto r_d$ and of the map η. Note that $\tau([\mathcal{S}, \mathcal{R}])$ is a closed interval contained in $(0, R_1)$. Denote by a its left extreme. Lemma 4.3 ensures that all the solutions to (7.9) are defined in $[a, 1]$. Thus, we can apply Theorem 7.2 to deduce the existence of a closed connected set $\mathcal{C} \subset [\mathcal{S}, \mathcal{R}] \times C([a, 1], \mathbb{R}^2)$ such that $Pr_1(\mathcal{C}) = [\mathcal{S}, \mathcal{R}]$ and, for every $(d, z) \in \mathcal{C}$, $z = (u, w)$ is a solution of (7.9) defined on $[a, 1]$. Lemma 3.3 guarantees that this solution $z = (u, w)$ can be extended to the whole interval $[0, 1]$, and its first component $u = u_d$ solves the Cauchy problem (2.2).
As a consequence, there exist two solutions $u_{\mathcal{S}}$ and $u_{\mathcal{R}}$ whose rotation numbers satisfy, by assumption,

$$\mathrm{Rot}_c^*(u_{\mathcal{S}}) > \frac{1}{2} + j \quad \left(\text{resp.} < \frac{1}{2} + j\right), \qquad \mathrm{Rot}_c^*(u_{\mathcal{R}}) < \frac{1}{2} + j \quad \left(\text{resp.} > \frac{1}{2} + j\right). \quad (7.10)$$

Note that $Pr_2(\mathcal{C})$ is a connected set in $C([a, 1], \mathbb{R}^2) = C([a, 1], \mathbb{R}) \times C([a, 1], \mathbb{R})$. Hence, $Pr_2'(\mathcal{C}) := \{u_d \in C^1([a, 1], \mathbb{R}) : u_d \text{ solves (2.2)}\}$ is connected too. From Lemma 3.3, we easily deduce that

$$\begin{aligned} \widetilde{\mathcal{C}}_2 &:= \{u_d \in C^1([0, 1], \mathbb{R}) : u_{d|_{[a,1]}} \in Pr_2'(\mathcal{C})\} \subset \\ &\subset \{u_d \in C^1([0, 1], \mathbb{R}) : u_d \text{ solves (2.2)}, \ d \in [\mathcal{S}, \mathcal{R}]\} \end{aligned}$$

is still a connected set. Taking into account the continuity of the operator $\mathrm{Rot}_c^*(\cdot)$ in $\widetilde{\mathcal{C}}_2$, the fact that $u_{\mathcal{S}}, u_{\mathcal{R}} \in \widetilde{\mathcal{C}}_2$ and satisfy (7.10), by the intermediate value theorem we infer the existence of a solution $u_d \in \widetilde{\mathcal{C}}_2$ with $d \in (\mathcal{S}, \mathcal{R})$ and such that $\mathrm{Rot}_c^*(u_d) = \frac{1}{2} + j$. Since $\frac{1}{2} + j < c$, we get $\mathrm{Rot}_{[0,1]}(u_d) = \frac{1}{2} + j$. Thus, (7.4) ensures that u_d solves the two-point boundary value problem (2.1) and it has exactly j zeros in $(0, 1)$. Furthermore, $u_d(0) = d \in (\mathcal{S}, \mathcal{R})$.
To complete the proof, it remains to prove the existence of at least a solution v_j to (2.1) with $v_j(0) < 0$, having exactly j zeros in $(0, 1)$. To this aim, it is sufficient to redefine the map η in $[-\mathcal{R}, -\mathcal{S}]$, and, arguing exactly as in the previous step, we easily achieve the thesis. ∎

8 Estimates on the rotation numbers

Proposition 6.6 ensures the existence of $L_1 > 0$, $\mathcal{L} > 0$ and $j_0 \in \mathbb{N}$ such that $\|u_d\|_1^2 \geq L_1$ for every $|d| \geq \mathcal{L}$, $\mathrm{Rot}_{[0,1]}(u_{\pm\mathcal{L}})$ is well defined, and

$$\mathrm{Rot}_{[0,1]}(u_{\pm\mathcal{L}}) < \frac{1}{2} + j_0. \quad (8.1)$$

Observe that formula (8.1) holds for every choice of $\mu > 0$ in the definition of rotation number.

The following propositions provide lower bounds on the rotation numbers of the solutions to (2.2) when $|d|$ is both sufficiently large and sufficiently small.

Proposition 8.1 *Under the assumptions of Theorem* 1.1*, let consider $j \in \mathbb{N}$ with $j \geq j_0$. Then, there exists $\mathcal{R} > \mathcal{L}$ such that for every $d \in \mathbb{R}$ with $|d| \geq \mathcal{R}$ and for every solution u_d of* (2.2)*, $\mathrm{Rot}_{[0,1]}(u_d)$ is well defined and*

$$\mathrm{Rot}_{[0,1]}(u_d) > \frac{1}{2} + j.$$

Proof. The rotation number is well defined due to Proposition 6.6.
Consider now $R_0 \in (0, R_1)$ as in Remark 2.1, and fix

$$\mu = \mu(j) > \left(\frac{1}{2} + j\right) \frac{\pi\, R_1^{2(N-1)}}{R_0^{N-1}\,(R_1 - R_0)}. \tag{8.2}$$

According to (7.5), (1.2) and (2.9), we obtain

$$\mathrm{Rot}_{[0,1]}(u_d) \geq \mathrm{Rot}_{[R_0,R_1]}(u_d) \geq \frac{\mu\, R_0^{N-1}}{\pi} \int_{R_0}^{R_1} \frac{u_d'(r)^2 + |u_d(r)|^{\delta+1}\, q_0}{R_1^{2(N-1)}\, u_d'(r)^2 + \mu^2 u_d(r)^2}\, dr.$$

Since $\delta > 1$, there exists $H_\mu > 0$ such that

$$R_1^{2(N-1)}\, |u_d(r)|^{\delta+1}\, q_0 \geq \mu^2\, u_d(r)^2 - H_\mu.$$

In particular, taking into accout (8.2), we can find $\epsilon > 0$ such that

$$\begin{aligned}\mathrm{Rot}_{[0,1]}(u_d) &\geq \frac{\mu\, R_0^{N-1}\,(R_1 - R_0)}{\pi\, R_1^{2(N-1)}} - \frac{\mu\, R_0^{N-1}}{\pi\, R_1^{2(N-1)}} \int_{R_0}^{R_1} \frac{H_\mu}{R_1^{2(N-1)}\, u_d'(r)^2 + \mu^2 u_d(r)^2}\, dr \geq \\ &> \frac{1}{2} + j + \epsilon - \frac{\mu\, R_0^{N-1}}{\pi\, R_1^{2(N-1)}} \int_{R_0}^{R_1} \frac{H_\mu}{R_1^{2(N-1)}\, u_d'(r)^2 + \mu^2 u_d(r)^2}\, dr.\end{aligned}$$

We now apply Proposition 6.1 to conclude that there exists $\mathcal{R} = \mathcal{R}(\mu, j) > \mathcal{L}$ such that for every $d \in \mathbb{R}$ with $|d| \geq \mathcal{R}$

$$\frac{\mu\, R_0^{N-1}}{\pi\, R_1^{2(N-1)}} \int_{R_0}^{R_1} \frac{H_\mu}{R_1^{2(N-1)}\, u_d'(r)^2 + \mu^2 u_d(r)^2}\, dr \leq \epsilon,$$

from which the thesis follows. ∎

Proposition 8.2 *Under the assumptions of Theorem* 1.2*, let consider $j \in \mathbb{N}$ with $j \geq j_0$ and $c \in \mathbb{R}$ with $c > j + 1$. Then, there exists $\mathcal{S} \in (0, \mathcal{L})$ such that for every $d \in \mathbb{R} \setminus \{0\}$ with $|d| \leq \mathcal{S}$ every solution u_d of* (2.2) *is such that*

$$\mathrm{Rot}_c^*(u_d) > \frac{1}{2} + j. \tag{8.3}$$

Proof. Due to the definition of Rot^*_c, note that, for every $d \in \mathbb{R} \setminus \{0\}$, (8.3) is satisfied whenever the rotation number of u_d is not defined in $[0, 1]$.
Thus, let us focus on the solutions u_d which verify

$$u_d(r)^2 + u_d'(r)^2 > 0 \quad \forall\, r \in [0, 1]. \tag{8.4}$$

To complete the proof, it remains to show that there exists $\mathcal{S} \in (0, \mathcal{L})$ such that for every $d \in \mathbb{R} \setminus \{0\}$ with $|d| \leq \mathcal{S}$ and for every solution u_d satisfying (8.4),

$$\mathrm{Rot}_{[0,1]}(u_d) \, > \, \frac{1}{2} + j.$$

To this aim, consider again $R_0 \in (0, R_1)$ as in Remark 2.1 and μ satisfying (8.2). According to (7.5), (1.2) and (2.9), we obtain

$$\mathrm{Rot}_{[0,1]}(u_d) \, \geq \, \mathrm{Rot}_{[R_0,R_1]}(u_d) \, \geq \, \frac{\mu\, R_0^{N-1}}{\pi} \int_{R_0}^{R_1} \frac{u_d'(r)^2 \, + \, |u_d(r)|^{\gamma+1}\, p_0}{R_1^{2(N-1)}\, u_d'(r)^2 \, + \, \mu^2 u_d(r)^2}\, dr.$$

Recalling that $\gamma < 1$, we can apply Lemma 5.1 which guarantees the existence of $\mathcal{S} = \mathcal{S}(\mu, j) \in (0, \mathcal{L})$ such that

$$R_1^{2(N-1)}\, |u_d(r)|^{\gamma+1}\, p_0 \, \geq \, \mu^2\, u_d(r)^2 \qquad \forall\, r \in [0, R_1], \;\; \forall\, d \in \mathbb{R} \setminus \{0\} : \; |d| \leq \mathcal{S}.$$

In particular, according to (8.2) we infer that for each $d \in \mathbb{R} \setminus \{0\}$ with $|d| \leq \mathcal{S}$

$$\mathrm{Rot}_{[0,1]}(u_d) \, \geq \, \frac{\mu\, R_0^{N-1}\, (R_1 - R_0)}{\pi\, R_1^{2(N-1)}} \, > \, \frac{1}{2} \, + \, j,$$

from which the thesis follows. ∎

9 The main results

This section is devoted to prove our main results. First, we concentrate on the proof of Theorem 1.1. To this aim, we wish to demonstrate the following result

Theorem 9.1 *Under the assumption of Theorem* 1.1*, there exists $j_0 \in \mathbb{N}$ such that for every integer $j \geq j_0$ there exist two radial solutions u_j and v_j of* (2.1) *having exactly j zeros in* $(0, 1)$*, satisfying $u_j(0) > 0 > v_j(0)$ and*

$$\lim_{j \to +\infty} \|u_j\|_1 = +\infty, \qquad \lim_{j \to +\infty} \|v_j\|_1 = +\infty. \tag{9.1}$$

Proof. Consider L_1, $\mathcal{L}$ and $j_0 \in \mathbb{N}$ as in (8.1). Take $j \in \mathbb{N}$ with $j \geq j_0$ and $c \in \mathbb{R}$ with $c > j + 1$. By (8.1), we observe that

$$\mathrm{Rot}^*_c(u_d) \, = \, \mathrm{Rot}_{[0,1]}\,(u_d) \, < \, \frac{1}{2} + j_0 \, \leq \, \frac{1}{2} + j, \qquad \text{as } |d| = \mathcal{L}. \tag{9.2}$$

From Proposition 8.1 combined with the definition of Rot_c^*, we deduce the existence of $\mathcal{R} > \mathcal{L}$ such that

$$\mathrm{Rot}_c^*(u_d) > \frac{1}{2} + j, \qquad \text{as } |d| = \mathcal{R}.$$

We apply Theorem 7.3 with $\mathcal{S} = \mathcal{L}$ to conclude the existence of at least two solutions u_j and v_j of (2.1) with $u_j(0) > \mathcal{L} > 0 > -\mathcal{L} > v_j(0)$ having exactly j zeros in $(0, 1)$.
In particular, $\|u_j\|_1^2 \geq L_1$, $\|v_j\|_1^2 \geq L_1$ and $\mathrm{Rot}_{[0,1]}(u_j) = \mathrm{Rot}_{[0,1]}(v_j) = j + 1/2$. Combining these relations with the definition of rotation number given in (7.3), we deduce (9.1). ■

We are finally interested in proving Theorem 1.2, whose statement can be rewritten as follows.

Theorem 9.2 *Under the assumption of Theorem* 1.2*, there exists* $j_0 \in \mathbb{N}$ *such that for every integer* $j \geq j_0$ *there exist two radial solutions* z_j *and* w_j *of* (2.1) *having exactly* j *zeros in* $(0, 1)$*, satisfying* $z_j(0) > 0 > w_j(0)$.

Proof. Consider L_1, $\mathcal{L}$ and $j_0 \in \mathbb{N}$ as in (8.1). Take $j \in \mathbb{N}$ with $j \geq j_0$ and $c \in \mathbb{R}$ with $c > j + 1$. By (8.1), we deduce (9.2). Moreover, from Proposition 8.2, we get the existence of $\mathcal{S} \in (0, \mathcal{L})$ such that

$$\mathrm{Rot}_c^*(u_d) > \frac{1}{2} + j, \qquad \text{as } |d| = \mathcal{S}.$$

We apply Theorem 7.3 with $\mathcal{R} = \mathcal{L}$ to conclude the existence of at least two solutions z_j and w_j of (2.1) with $z_j(0) > \mathcal{S} > 0 > -\mathcal{S} > w_j(0)$ having exactly j zeros in $(0, 1)$.
Notice that these solutions differ from the ones obtained by the previous theorem, since Theorem 7.3 establishes that $z_j(0) < \mathcal{L}$ and $w_j(0) > -\mathcal{L}$. ■

Remark 9.3 *For the sake of simplicity we have assumed* (1.3)-(1.6) *to describe the local behaviour of the functions* q *and* p *in a right neighbourhood of* 0*. We point out that different assumptions on the behaviour of* q *and* p *next to* 0 *can be equivalently considered to obtain our main results. For instance, the weights* q *and* p *can be also required to respectively satisfy one of the following alternatives*

$$\exists\, h > 1,\ \beta > 0: \qquad q(r) \sim -\beta\, r^h \log r \quad \text{or} \quad q(r) \sim \beta\, e^{-\frac{1}{r}} \qquad \text{as } r \to 0^+, \tag{9.3}$$
$$\text{and}$$
$$\exists\, k > 1,\ \alpha > 0: \qquad p(r) \sim -\alpha\, r^k \log r \quad \text{or} \quad p(r) \sim \alpha\, e^{-\frac{1}{r}} \qquad \text{as } r \to 0^+.$$

Condition (1.4) *can be replaced with the first alternative in* (9.3) *in the statements of our main theorems.*
On the other hand, if the second alternative in (9.3) *holds* (*i.e. if* $q(r) \sim \beta\, e^{-\frac{1}{r}}$ *as* $r \to 0^+$) *no upper bounds on* δ *should be imposed to get our multiplicity results* (*neither assumption* (1.12) *nor* (1.13) *are needed in the statements of our main results*).

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