CMAF, UNIVERSITY OF LISBON, PORTUGAL
PREPRINT 2009-007

ON THE BLOW-UP OF SOLUTIONS TO ANISOTROPIC PARABOLIC EQUATIONS WITH VARIABLE NONLINEARITY

S. ANTONTSEV AND S. SHMAREV

Abstract. We establish sufficient conditions of the finite time blow-up in solutions of the homogeneous Dirichlet problem for the anisotropic parabolic equations with variable nonlinearity

$$u_t = \sum_{i=1}^{n} D_i \left(a_i(x,t)|D_i u|^{p_i(x)-2} D_i u \right) + \sum_{i=1}^{K} b_i(x,t)\,|u|^{\sigma_i(x,t)-2}\,u.$$

Two different cases are studied. In the first case $a_i \equiv a_i(x)$, $p_i \equiv 2$, $\sigma_i \equiv \sigma_i(x,t)$ and $b_i(x,t) \geq 0$. We show that in this case every solution corresponding to "large" initial function blows-up in finite time if there exists at least one j for which $\min \sigma_j(x,t) > 2$ and either $b_j > 0$, or $b_j(x,t) \geq 0$ and $\displaystyle\int_\Omega b_j^{-\rho(t)}(x,t)\,dx < \infty$ with some $\rho(t) > 0$ depending on σ_j. In the case of the quasilinear equation with the exponents p_i, σ_i depending only on x we show that the solutions may blow-up if $\min \sigma_i \geq \max p_i$, $b_i \geq 0$, and there exists at least one j for which $\min \sigma_j > \max p_j$ and $b_j > 0$. We extend these results to semilinear equation with nonlocal forcing terms and quasilinear equations which combines the absorption ($b_i \leq 0$) and reaction terms.

1. Introduction

This note is a contribution to the study of the blow-up phenomenon in solutions to nonlinear parabolic equations. We study the homogeneous Dirichlet problem for the parabolic equation with variable nonlinearity

$$(1.1)\qquad \begin{cases} u_t = \sum_i D_i \left(a_i(x,t)|D_i u|^{p_i(x)-2} D_i \right) + \sum_i b_i(x,t)|u|^{\sigma(x,t)-2}u & \text{in } Q_T = \Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a domain with Lipschitz-continuous boundary Γ, and $\Gamma_T = \Gamma \times (0,T)$.

The coefficients a_i, p_i, σ are given measurable functions of their arguments. It is always assumed that these functions satisfy the conditions

$$(1.2)\qquad \begin{cases} \text{there exist finite constants } a^{\pm},\ b^{\pm} \text{ such that} \\ 0 < a^- \leq a_i(x,t) \leq a^+, \qquad 0 \leq b^- \leq b_i(x,t) \leq b^+, \end{cases}$$

$$(1.3)\qquad \begin{cases} p_i(x) \text{ and } \sigma(x,t) \text{ are continuous in } \overline{Q}_T, \\ \text{there exist finite constants } p^{\pm},\ \sigma^{\pm} \text{ such that} \\ 1 < p^- \leq p_i(x) \leq p^+, \qquad 1 < \sigma^- \leq \sigma_i(x,t) \leq \sigma^+. \end{cases}$$

Equations of the type (1.1) appear in the mathematical modelling of various physical phenomena such as flows of electro-rheological or thermo-rheological fluids [2, 7, 8, 9, 27, 28], processes of filtration through a

Date: February 28, 2009.
The first author was partially supported by the project MTM2008-06208 of the Ministerio de Ciencia e Innovacion, Spain.
The second author acknowledges the support of the research grant MTM2007-65088 (Spain).

porous medium, processing of digital images [1, 16, 22]. For a more detailed information on the applications of these models to the study of the real world processes we refer the reader to the papers [9, 27, 28] and the further references therein.

The questions of existence, uniqueness and qualitative properties of solutions for elliptic and parabolic equations with variable nonlinearity were discussed by many authors and under different conditions on the data - see, for example, [3]-[6], [8]-[12], [14, 15].

For certain ranges of the exponents, parabolic equations with variable nonlinearity display the localization (alias vanishing) properties which are intrinsic for the solutions of nonlinear equations with constant nonlinearity such as vanishing in a finite time, finite speed of propagation of disturbances from the data or the waiting time phenomena (see [10, 14, 11, 12]). In this note, we are interested in the the blow-up phenomenon in solutions of equations with variable nonlinearity. An excellent insight to the theory of this phenomenon and an overview of the known results can be found in the monographs [20, 29] (see also [17, 18, 19], [23], [25, 26], [30, 31]). To the best of our knowledge, by now only works [13, 24] have addressed the question of possible blow-up of solutions of the parabolic PDEs with nonstandard growth conditions.

Paper [24] deals with the solutions of the homogeneous Dirichlet problem for the semilinear parabolic equation

$$\begin{cases} u_t = \Delta u + f(x,u) \quad \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases} \tag{1.4}$$

where the source term is either a power,

$$f(x,u) = a(x)u^{p(x)} \quad \text{or is nonlocal: } f(x,u) = a(x)\int_\Omega u^{q(y)}(y,t)dy.$$

In the present note, we consider parabolic equations with variable and possibly anisotropic nonlinearity which may depend on x and t. In Section 2 we collect some facts from the theory of Orlicz-Sobolev spaces the solutions of problem (1.1) belong to, give the definition of weak solution to problem (1.1) and remind the existence and regularity results.

In Section 3 we study problem (1.1) for the semilinear equations with $a_i(x,t) \equiv a_i(x)$, $p_i(x) = 2$, and variable $\sigma_i(x)$, $b_i(x,t)$. These restrictions on the data allow us to apply the eigenvalue method of S. Kaplan [21] and to formulate the conditions of the finite time blow-up in terms of the first eigenvalue of the linear operator associated with the elliptic part of equation (1.1). For the sake of presentation, we study in detail the case $K = 1$. It is shown that every weak solution of problem (1.1) corresponding to "large" initial function u_0 blows-up in finite time if $\min \sigma(x,t) > 2$ and the coefficient $b_i(x,t)$ is either strictly positive, or is nonnegative and subject to the additional restriction

$$\int_\Omega \frac{dx}{b^{\rho(t)}(x,t)} < \infty \quad \text{for all } t \geq 0$$

with a power $\rho(t)$ depending on n and $\sigma(x,t)$. The case $K > 1$ follows then as a formal byproduct.

In Section 4 we discuss possible generalizations of the results obtained in Section 3. We describe the possibility to regional blow-up in solutions of semilinear problem (1.1) and study the equation with nonlocal forcing terms.

In Section 5 we establish sufficient conditions of the blow-up for solutions of the quasilinear equation (1.1), assuming that the exponents of nonlinearity $p_i(x)$ and $\sigma_i(x)$ are independent of t. The coefficients $a_i(x,t)$, $b_i(x,t)$ are assumed differentiable in t and monotone: $a_{it}(x,t) \leq 0$, $b_{it}(x,t) \geq 0$.

2. Preliminaries

2.1. **Orlicz-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega$ be Lipschitz-continuous, and let $p(x)$ be log-continuous in Ω:

$$\forall\, x,y \in \Omega,\ |x-y| < \frac{1}{2} \quad |p(x)-p(y)| \leq \omega(|x-y|), \quad \overline{\lim_{\tau\to 0+}}\,\omega(\tau)\ln\frac{1}{\tau} = C < \infty. \tag{2.5}$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$A_{p(\cdot)}(f) = \int\limits_{\Omega} |f(x)|^{p(x)}\, dx < \infty.$$

The set $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{p(\cdot),\Omega} \equiv \|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{\lambda > 0 : \ A_{p(\cdot)}\left(f/\lambda\right) \leq 1\right\}$$

becomes a Banach space. The Banach space $W_0^{1,\,p(\cdot)}(\Omega)$ with $p(x) \in [p^-, p^+] \subset (1, \infty)$ is defined by

$$\left\{ \begin{array}{l} W_0^{1,\,p(\cdot)}(\Omega) = \left\{ f \in L^{p(\cdot)}(\Omega) :\, |\nabla f|^{p(x)} \in L^1(\Omega),\ u = 0 \text{ on } \partial\Omega \right\}, \\ \|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \sum\limits_i \|D_i u\|_{p(\cdot),\Omega}. \end{array} \right. \tag{2.6}$$

In what follows we will need the following properties of the functions from the spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$:

1) if condition (2.5) is fulfilled, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,\,p(\cdot)}(\Omega)$, and the space $W_0^{1,\,p(\cdot)}(\Omega)$ can be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.6) – see [32, 33];

2) if $p(x) \in C^0(\overline{\Omega})$, the the space $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive;

3) it follows directly from the definition that

$$\min\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right) \leq A_{p(\cdot)}(f) \leq \max\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right); \tag{2.7}$$

4) for all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with $p(x) \in (1, \infty)$, $p' = \dfrac{p}{p-1}$, Hölder's inequality holds in the form

$$\int\limits_{\Omega} |f\, g|\, dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \leq 2\, \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}\,. \tag{2.8}$$

In particular, for every constant $q \in (1, p^-)$

$$\|f\|_{q,\Omega} \leq C\, \|f\|_{p(\cdot),\Omega} \quad \text{with the constant} \quad C = 2\, \|1\|_{\frac{p(\cdot)}{p(\cdot)-q},\Omega}\,.$$

2.2. **Parabolic anisotropic spaces $\mathbf{W}(Q)$.** Let $p_i(z)$, $z = (x, t) \in Q$, satisfy condition (2.5) in the cylinder Q. For every fixed $t \in [0, T]$ we introduce the Banach space

$$\begin{array}{l} \mathbf{V}_t(\Omega) = \left\{ u(x) :\, u(x) \in L^2(\Omega) \cap W_0^{1,1}(\Omega), \quad |D_i u(x)|^{p(x,t)} \in L^1(\Omega) \right\}, \\ \|u\|_{\mathbf{V}_t(\Omega)} = \|u\|_{2,\Omega} + \sum\limits_i \|D_i u\|_{p_i(\cdot,t),\Omega}, \end{array}$$

and denote by $\mathbf{W}(Q)$ the anisotropic Banach space

$$\begin{array}{l} \mathbf{W}(Q) = \left\{ u :\, [0, T] \mapsto \mathbf{V}_t(\Omega) |\ u \in L^2(Q),\, |D_i u|^{p_i(z)} \in L^1(Q),\, u = 0 \text{ on } \Gamma \right\}, \\ \|u\|_{\mathbf{W}(Q)} = \sum\limits_i \|D_i u\|_{p_i(\cdot),Q} + \|u\|_{2,Q}. \end{array}$$

$\mathbf{W}'(Q)$ is the dual of $\mathbf{W}(Q)$ with respect to the scalar product in $L^2(Q)$.

2.3. **Existence and regularity of weak solutions.** Let us consider the following generalization of problem (1.1):

$$\left\{\begin{array}{l} u_t=\displaystyle\sum_{i=1}^{n} D_i\left(a(x,t,u)|D_iu|^{p_i(x)-2}D_iu\right)+d(x,t,u) \quad \text{in } Q_T, \\ u=0 \text{ on } \Gamma_T, \qquad u(x,0)=u_0(x) \text{ in } \Omega. \end{array}\right. \tag{2.9}$$

The weak solution of problem (1.1) is understood as follows:

Definition 1. *A function $u(x,t)\in \mathbf{W}(Q_T)\cap L^\infty(0,T;\,L^2(\Omega))$ is called weak solution of problem* (2.9) *if for every test-function*

$$\zeta\in\{\eta(z):\,\eta\in \mathbf{W}(Q_T)\cap L^\infty\left(0,T;L^2(\Omega)\right),\ \eta_t\in \mathbf{W}'(Q_T)\},$$

and every $t_1,t_2\in[0,T]$, the following identity holds:

$$\int_{t_1}^{t_2}\int_\Omega\left(u\zeta_t-\sum_{i=1}^{n}a_i(x,t)|D_iu|^{p_i(x)-2}D_iu\cdot D_i\zeta+d(x,t,u)\,\zeta\right)dz=\int_\Omega u\zeta dx\Big|_{t_1}^{t_2}. \tag{2.10}$$

Proposition 1. [14, Theorem 4.3] *Let assumptions* (1.3), (1.2), (2.5) *be fulfilled. Assume that the function d is subject to the growth condition*

$$|d(x,t,s)|\le d_0|s|^{\delta-1}+h(x,t),\qquad d_0=const>0.$$

Then for every $u_0\in L^\infty(\Omega)$ there exists $\theta\in(0,T]$, depending on δ, d_0, $\|u_0\|_{L^\infty(\Omega)}$, $\|h\|_{L^1(0,\theta;L^\infty(\Omega))}$ such that problem (2.9) *has at least one weak solution $u\in\mathbf{W}(Q_\theta)$ with $u_t\in\mathbf{W}'(Q_\theta)$ and $\|u\|_{\infty,Q_\theta}<\infty$. The solution can be continued to the interval $[0,T^*]$ where $T^*=\sup\{\theta>0:\ \|u\|_{\infty,Q_\theta}<\infty\}$. Moreover, in the special case $d(x,t,s)=\sum_i b_i|u|^{\sigma_i-2}u$ we have $u\ge 0$ a.e. in Q_T, provided that $u_0\ge 0$ a.e. in Ω.*

Remark 1. *Under the conditions of Proposition* 1 *the weak solution of problem* (2.9) *can be taken for the test-function in* (2.10). *Moreover, for the functions $u,v\in\mathbf{W}(Q_T)$ with $u_t,v_t\in\mathbf{W}(Q_T)$ the formula of integration by parts holds (see* [12, Lemma 4.3]*):*

$$\int_{Q_T}u\,v_t\,dxdt=\int_\Omega u\,v\,dx\Big|_{t=0}^{t=T}-\int_{Q_T}u_t\,v\,dxdt.$$

It follows that for every $\theta<T^*$ the weak solutions of problem (2.9) *satisfy the energy relation*

$$\frac{1}{2}\int_\Omega u^2\,dx\Big|_{t=0}^{t=\theta}+\int_{Q_\theta}\left(\sum_i a_i|D_iu|^{p_i}+d(x,t,u)\,u\right)dxdt=0$$

Proposition 2. *Let the exponents and coefficients in problem* (1.1) *satisfy the conditions of Proposition* 1 *with $d(x,t,u)=\sum_{i=1}^{K}b_i(x,t)|u|^{\sigma(x)-2}u$. Let us additionally assume that $|u_0|^{\sigma(x)}\in L^1(\Omega)$, $|\nabla u_0|^{p(x)}\in L^1(\Omega)$, and*

$$a_it,\ b_{it}\in C^0(\overline{Q}_T),\qquad a_{it}(x,t)\le 0,\quad b_{it}(x,t)\ge 0.$$

Then the weak solution of problem (1.1) *satisfies the estimate*

$$\begin{aligned}\int_\Omega\left[\sum_{i=1}^{n}\frac{a_i}{p_i}|D_iu\ |^{p_i}-\sum_{i=1}^{K}\frac{b_i}{\sigma_i}|u|^{\sigma}\right]dx\Big|_{t=0}^{t=T}+\int_{Q_T}|u_t|^2\,dz\\ =\sum_{i=1}^{n}\int_\Omega a_{it}\frac{|D_iu\ |^{p_i}}{p_i}\,dx-\sum_{i=1}^{K}\int_\Omega b_{it}\frac{|u|^{\sigma_i}}{\sigma_i}\,dx\le 0.\end{aligned} \tag{2.11}$$

Inequality (2.11) is a partial case of the estimate proved in [14, Theorem 6.1]. The proof of existence of a weak solution to problem (1.1) (in a more general setting) is performed with the Galerkin-Faedo method. The solution is obtained as the limit of the sequence of functions $u^{(k)} = \sum_1^k c_i(t)\psi_i(x)$, $\{\psi_i\}$ is the orthogonal basis of the function space $L^{p^+}(\Omega)$, which is dense in $L^{p(x)}(\Omega)$. In this approach estimates on the limit function result from the uniform in k estimates for the approximate solutions $u^{(k)}$. Let u be a sufficiently regular solution of problem (1.1) (or the approximate solution $u^{(k)}$). Estimate (2.11) follows upon multiplying the equation by u_t, integrating by parts, and plugging the obvious relations

$$\partial_t \left(a_i \frac{|D_i u\ |^{p_i}}{p_i} \right) = a_{it} \frac{|D_i u\ |^{p_i}}{p_i} + a_i \left(|D_i u|^{p_i - 2} D_i u\ D_i u_t \right), \quad \partial_t \left(\frac{b_i}{\sigma_i} |u|^{\sigma_i} \right) = b_{it} \frac{|u|^{\sigma_i}}{\sigma_i} + b_i \left(|u|^{\sigma_i - 2} u\ u_t \right).$$

3. Semilinear equation with nonlinear forcing terms

3.1. **Assumptions and results.** Let $u(x,t)$ be a nonnegative solution of the problem

$$\text{(3.12)} \qquad \begin{cases} u_t = \sum_i D_i\left(a_i(x) D_i u\right) + b(x,t) u^{\sigma(x,t)-1} & \text{in } Q_T, \\ u(x,0) = u_0(x) \geq 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases}$$

with the coefficients $a_i(x)$, $b(x,t)$, $\sigma(x,t)$ satisfying the conditions (1.2), (1.3) with $p_i \equiv 2$, $\sigma^- > 2$. According to Proposition 1 the solutions of problem (3.12) are nonnegative, provided that $u_0 \geq 0$ in Ω.

We follow the well-known eigenvalue method of Kaplan [21]. Let $\lambda > 0$ and $\phi \geq 0$ be the first eigenvalue and eigenfunction of the problem

$$\text{(3.13)} \qquad -\sum_i D_i\left(a_i(x) D_i \phi\right) = \lambda\,\phi \quad \text{in } \Omega, \qquad \phi = 0 \text{ on } \partial\Omega.$$

We normalize the function ϕ by the condition

$$\int_\Omega \phi(x)dx = 1.$$

Let us introduce the functions

$$\text{(3.14)} \qquad \sigma^-(t) = \min_\Omega \sigma(x,t), \qquad \sigma^+(t) = \max_\Omega \sigma(x,t), \qquad \mu(t) = \int_\Omega u(x,t)\phi(x)\,dx$$

Definition 2. *We say that the solution $u(x,t)$ blows-up if there exists a finite moment $t^* < \infty$ such that $\mu(t^*) = \infty$ and*

$$\mu(t) = \int_\Omega u\phi\,dx \leq \|u(\cdot,t)\|_{\infty,\Omega} \int_\Omega \phi\,dx = \|u(\cdot,t)\|_{\infty,\Omega} \to \infty \text{ as } t \to t^*.$$

Theorem 1. *Let us assume that $\sigma(x,t)$, $b(x,t)$ and $\mu(0) = \int_\Omega u_0\,\phi\,dx$ satisfy the following conditions:*

$$\text{(3.15)} \qquad \sigma(x,t) \geq \sigma^- > 2, \qquad \mu(0) > 1, \qquad \min b(x,t) \geq b^- > 0.$$

Let us assume also that there exists a constant $\delta > 0$ and a function $q(t)$ and $\delta > 0$ such that

$$\text{(3.16)} \qquad -\lambda + D\,M > \delta, \quad q(t) \in (0,1), \quad q(t)\left(\sigma^-(t) - 1\right) > 1,$$

where

$$D = 2^{1-\sigma^+} \min\{1,\ |\Omega|^{2-\sigma^+}\}, \qquad M = b^- \min_{t \geq 0} \left(\frac{|\Omega|^{\frac{q-1}{q}}}{\max\left(\phi^{\sigma-2}\right)} \right).$$

Then every solution of problem (3.12) blows-up in a finite time.

Theorem 2. *Let us assume that $\sigma(x,t)$, $b(x,t)$ and $\mu(0)$ satisfy the following conditions:*

$$\sigma(x,t) \geq \sigma^- > 2, \qquad \mu(0) > 1, \qquad b(x,t) > 0. \tag{3.17}$$

Let us assume also that there exist constants $\beta > 0$, $\gamma > 1$, $q \in (0,1)$ and $\delta > 0$ such that

$$q\,(\sigma^- - 1) > 1, \qquad \int_\Omega \frac{dx}{b^{\frac{q\gamma}{1-q}}(x,t)} = B(t) \leq \beta \tag{3.18}$$

and

$$-\lambda + D\,M' > \delta \tag{3.19}$$

with

$$D = 2^{1-\sigma^+}\min\{1,\, |\Omega|^{2-\sigma^+}\}, \qquad M' = \frac{|\Omega|^{\frac{(q-1)(\gamma-1)}{q\gamma}}}{\max\left(\phi^{\sigma(x,t)-2}\right)}.$$

Then every solution of problem (3.12) *blows-up in a finite time.*

The difference between the assertions of Theorems 1 and 2 is that in the latter the coefficient $b(x,t)$ need not be separated away from zero and for every $t > 0$ it is allowed to vanish on a set of zero measure in Ω. The admissible rate of vanishing of $b(x,t)$ is defined in terms of the exponent $\sigma(x,t)$: we claim that $\|1/b(x,t)\|_{\frac{q\gamma}{q-1},\Omega} < \infty$. For example, let

$$\Omega = \{x : |x| < 1\}, \quad b \equiv b(|x|) = |x|^{-\alpha} \quad \text{with } \alpha = const > 0.$$

In this model case

$$\int_\Omega \frac{dx}{b^{\frac{q\gamma}{q-1}}(x,t)} \sim \int_0^1 s^{n-1-\alpha\frac{q\gamma}{1-q}}\,ds \quad \text{with } 1 > q > \frac{1}{\sigma^- - 1} \text{ and some } \gamma > 1.$$

This integral is convergent if, say, $\alpha < n\left(\frac{1}{q} - 1\right)$.

Remark 2. *The above assertions remain true for the equations*

$$u_t = \sum_{i=1}^n D_i\left(a_i(x) D_i u\right) + \sum_{i=1}^K b_i(x,t)|u|^{\sigma_i(x,t)-2}u, \qquad 1 \leq K \leq n,$$

provided that $b_i(x,t) \geq 0$ for all i, and there is at least one $j \in \{1,\dots,K\}$ such that $b_j(x,t)$ and $\sigma_j(x,t)$ satisfy the conditions of Theorems 1, 2.

3.2. **Ordinary differential inequality for $\mu(t)$.** Multiplying equation (3.12) by ϕ and integrating over Ω we arrive at the relation

$$\begin{aligned}\mu'(t) &= \int_\Omega u_t \phi\,dx = \sum_i \int_\Omega u\,D_i\left(a_i(x) D_i\phi\right)\,dx + \int_\Omega b\,u^{\sigma-1}\phi\,dx \\ &= -\lambda\,\mu(t) + \int_\Omega b u^{\sigma-1}\phi\,dx \equiv -\lambda\,\mu(t) + I(t).\end{aligned} \tag{3.20}$$

To estimate $I(t)$ from below we use the inverse Hölder inequality

$$\int_\Omega |u||v|dx \geq \|u\|_{q,\Omega}\|v\|_{q',\Omega} \quad \text{with } q \in (0,1),\ q' = \frac{q}{q-1}. \tag{3.21}$$

Let us take $q(t)$ such that for all $t \geq 0$ $q(t) \in (0,1)$, $q(t)(\sigma(x,t) - 1) > 1$ in Ω, and estimate the integral $I(t)$ as follows:

$$I(t)=\int_\Omega (u\,\phi)^{\sigma-1}\left(b\,\phi^{2-\sigma}\right)\,dx \geq \left(\int_\Omega (u\phi)^{(\sigma-1)q}\,dx\right)^{\frac{1}{q}}\left(\int_\Omega (b\,\phi^{2-\sigma})^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}}.$$

Using (2.7) and (2.8), we also have that for the same q

$$\begin{aligned}\mu(t)&=\int_\Omega u\,\phi\,dx \leq C\,\|u\phi\|_{(\sigma(\cdot,t)-1)q,\Omega}\leq\\ &\leq C(t)\,\max\left\{\left(\int_\Omega |u\,\phi|^{(\sigma-1)q}\,dx\right)^{\frac{1}{q(t)(\sigma^-(t)-1)}},\left(\int_\Omega |u\,\phi|^{(\sigma-1)q}\,dx\right)^{\frac{1}{q(t)(\sigma^+(t)-1)}}\right\}\end{aligned}\tag{3.22}$$

with

$$\sigma^-(t)=\min_\Omega \sigma(x,t),\quad \sigma^+(t)=\max_\Omega \sigma(x,t),\quad C(t)=2\,\|1\|_{\frac{(\sigma-1)q}{(\sigma-1)q-1},\Omega}.$$

The function $C(t)$ is separated away from zero and infinity and can be estimated through $|\Omega|$ and σ^+:

$$2\,\min\left\{1,\,|\Omega|^{\frac{\sigma^+-2}{\sigma^+-1}}\right\}\leq C(t)\leq 2\,\max\left\{1,\,|\Omega|^{\frac{\sigma^+-2}{\sigma^+-1}}\right\}.$$

Writing inequality (3.22) in the equivalent form

$$\min\left\{\left(\frac{\mu(t)}{C(t)}\right)^{\sigma^-(t)-1},\left(\frac{\mu(t)}{C(t)}\right)^{\sigma^+(t)-1}\right\}\leq\left(\int_\Omega (u\,\phi)^{(\sigma-1)q}\,dx\right)^{\frac{1}{q}},$$

we obtain the needed estimate on $I(t)$:

$$\begin{aligned}I(t)&\geq\left(\int_\Omega (u\,\phi)^{(\sigma-1)q}\,dx\right)^{\frac{1}{q}}\left(\int_\Omega (b\,\phi^{2-\sigma})^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}}\\ &\geq D\,\min\left\{\mu^{\sigma^-(t)-1}(t),\,\mu^{\sigma^+(t)-1}(t)\right\}\left(\int_\Omega (b\,\phi^{2-\sigma})^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}}\end{aligned}$$

with $D=2^{1-\sigma^+}\min\{1,\,|\Omega|^{2-\sigma^+}\}$. In the case that $b(x,t)$ is separated away from zero (see (3.15))

$$g(t)\equiv\left(\int_\Omega (b\,\phi^{2-\sigma})^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}}\geq b^-\min_{t\geq 0}\left(\frac{|\Omega|^{\frac{q-1}{q}}}{\max_{x\in\Omega}\left(\phi^{\sigma(x,t)-2}\right)}\right)\equiv M,\tag{3.23}$$

and the differential inequality for $\mu(t)$ takes on the form

$$\mu'(t)\geq-\lambda\,\mu(t)+D\,M\,\min\left\{\mu^{\sigma^-(t)-1}(t),\,\mu^{\sigma^+(t)-1}(t)\right\}.\tag{3.24}$$

Let us specially notice that D and M are independent of u and depend only on $\sigma(x,t)$ and $|\Omega|$.

3.3. **Proof of Theorem 1: analysis of the differential inequality.** Let

$$-\lambda+D\,M\geq\delta>0\quad\text{and}\quad\mu(0)>1.\tag{3.25}$$

By virtue of (3.20) $\mu(t)$ is continuous. Writing (3.24) in the form

$$\frac{\mu'(t)}{\mu(t)}\geq-\lambda+D\,M\,\min\left\{\mu^{\sigma^-(t)-2}(t),\,\mu^{\sigma^+(t)-2}(t)\right\},\tag{3.26}$$

integrating and using continuity of $\mu(t)$, we conclude that for some $s>0$

$$\mu(t) \geq \mu_0 \exp\left(\int_0^t \left(-\lambda + D\,M\,\min\left\{\mu^{\sigma^-(s)-2}(s), \mu^{\sigma^+(s)-2}(s)\right\}\right) ds\right)$$
$$\geq \mu_0 \exp\left(\int_0^t (-\lambda + D\,M\,)\,ds\right) \geq \mu_0 e^{\delta\,t} > \mu(0) \qquad \text{for } t \in (0, s]$$

and, thus, $\mu(t) \geq \mu(0) > 1$ for all $t \in [0, s]$. Set

$$T^* = \sup\{t > 0 : \mu(t) < \infty\}.$$

Since $\mu(s) > 1$, this process can be continued as long as the function $\mu(t)$ remains bounded, which gives the estimate $\mu(t) > 1$ for all $t \in [0, T^*]$. Now we may transform (3.26) to the form

$$\begin{aligned}\mu'(t) &\geq \mu^{\sigma^-(t)-1}(t)\left(-\lambda\mu^{2-\sigma^-(t)}(t) + D\,M\right)\\ &\geq \mu^{\sigma^-(t)-1}(t)\,(-\lambda + D\,M)\\ &\geq \delta\,\mu^{\sigma^-(t)-1}(t) \geq \delta\,\mu^{\sigma^- -1}(t).\end{aligned} \tag{3.27}$$

Integrating this inequality we find that the function $\mu(t)$ becomes infinite in a finite time: in the case that $\sigma^-(t) \geq \sigma^- = const > 2$ we have

$$\mu^{\sigma^- -2}(t) \geq \frac{\mu^{\sigma^- -2}(0)}{1 - \delta\,(\sigma^- - 2)\,\mu^{\sigma^- -2}(0)\,t} \to \infty \quad \text{as} \quad t \to t^* = \frac{\mu^{2-\sigma^-}(0)}{\delta\,(\sigma^- - 2)}.$$

3.4. **Proof of Theorem 2.** In the case that $b^- = 0$ estimate (3.23) on $g(t)$ ceases to be true. Let us estimate $g(t)$ from below in a different way: by Hölder's inequality with $\gamma > 1$

$$\begin{aligned}\frac{1}{g(t)} &= \left(\int_\Omega |b\,\phi^{2-\sigma}|^{\frac{q}{q-1}}\,dx\right)^{\frac{1-q}{q}} \leq \|b^{\frac{q}{q-1}}\|_{\gamma,\Omega}^{\frac{1-q}{q}}\,\|\phi^{\frac{q(2-\sigma)}{q-1}}\|_{\gamma',\Omega}^{\frac{1-q}{q}}\\ &\leq |\Omega|^{\frac{1-q}{q}\frac{\gamma-1}{\gamma}}\,\|b^{\frac{q}{q-1}}\|_{\gamma,\Omega}^{\frac{1-q}{q}}\,\max_{Q_T}\left(\phi^{\sigma(x,t)-2}\right)\\ &= K\left(\int_\Omega \frac{dx}{b^{\frac{q\gamma}{1-q}}(x,t)}\right)^{\frac{1-q}{q\gamma}} \equiv K\,B(t), \qquad K \equiv K(|\Omega|, \phi, \sigma, \gamma).\end{aligned}$$

According to assumption (3.18) $B(t) \leq \beta < \infty$, which gives the needed estimate from below: $g(t) \geq \dfrac{1}{K\,\beta} \equiv M'$. This leads to the differential inequality (3.24) with the constant M' instead of M, and the conclusion follows.

4. Generalizations

4.1. **Regional blow-up.** The conclusions about the blow-up remain true if instead of the whole domain Ω we restrict the study to a subdomain. Let $u(x,t)$ be a nonnegative solution of problem (3.12) and $D \subset \Omega$, $\operatorname{meas} D > 0$ be a subdomain with a sufficiently smooth boundary ∂D. Let $\phi > 0$ in D and λ be the first eigenfunction and the corresponding eigenvalue of the problem

$$-\sum_i D_i\left(a_i(x)D_i\phi\right) = \lambda\,\phi \quad \text{in } D, \qquad \phi = 0 \text{ on } \partial D. \tag{4.28}$$

We introduce the function

$$\mu(t) = \int_D u\,\phi\,dx$$

and use the relation

$$\mu'(t) = \int_D u_t \phi dx = -\lambda\mu - \int_{\partial D} u \sum_i a_i D_i \phi \, n_i \, dx + \int_D a \, u^{\sigma(x,t)} \phi dx, \tag{4.29}$$

where $\mathbf{n} = (n_1, \dots, n_n)$ denotes the outward normal to ∂D. Since ϕ is nonnegative in D, its conormal derivative is nonpositive on ∂D and for the nonnegative solution u we obtain

$$-\int_{\partial D} u \sum_i a_i \, D_i \phi \, n_i \, dx \geq 0.$$

It follows that $\mu(t)$ satisfies the differential inequality

$$\mu'(t) \geq -\lambda\mu + \int\limits_D a \, u^{\sigma(x,t)} \phi \, dx,$$

which can be studied exactly like inequality (3.20).

4.2. **Quasilinear equations of general form.** Let us consider a nonnegative solution $u(x,t)$ of the problem

$$\begin{cases} u_t = Lu + f(x,t,u) \quad \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases} \tag{4.30}$$

where L is the linear elliptic operator

$$Lu = D_i \left(a_{ij}(x) D_j u + a_i(x) u \right) + a_0(x) u,$$

and the right-hand side has the form

$$f(x,t,u) = \sum_{k=1}^{N} b_k(x,t) u^{\sigma_k(x,t)-1} + \sum_{i=N}^{K} c_i(x,t) \int_\Omega d_i(s,t) u^{\sigma_i(s,t)-1} ds,$$

with $b_k \geq 0$, $c_i \geq 0$, $d_i \geq 0$. Let us denote by L^* the conjugate operator

$$\begin{cases} L^*\phi \equiv -D_j \left(a_{ij}(x) D_i \phi \right) - a_i(x) D_{x_i} \phi + a_0(x) \phi, \\ (u, L^*\phi)_\Omega = (\phi, Lu)_\Omega \quad \forall \, u, \, \phi \in W_0^{1,2}(\Omega), \end{cases}$$

and assume that there exist a nonnegative function $\phi \geq 0$ and a positive number $\lambda > 0$ such that

$$L^*\phi = -\lambda\phi \text{ in } \Omega, \qquad \phi = 0 \text{ on } \Gamma.$$

Multiplying (4.30) by u and integrating over Ω we arrive at the relation (cf. with (3.20))

$$\mu'(t) = -\lambda\mu + I_1 + I_2, \tag{4.31}$$

where

$$I_1 = \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k(x,t)-1} \right) \phi \, dx, \quad I_2 = \int_\Omega \left(\sum_{i=N}^{K} c_i(x,t) \int_\Omega d_i(s,t) u^{\sigma_i(s,t)-1} ds \right) \phi \, dx.$$

I_1, I_2 are estimated from below in the following way:

$$
\begin{aligned}
I_1 &\geq \int_{\Omega\cap(u\geq 1)} \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k(x,t)-1}\right) \phi dx \geq \int_{\Omega\cap(u\geq 1)} \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k^-(x,t)-1}\right) \phi dx \\
&= \int_{\Omega} \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k^-(x,t)-1}\right) \phi dx - \int_{\Omega\cap(u<1)} \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k^-(x,t)-1}\right) \phi dx \\
&\geq \int_{\Omega} \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k^-(x,t)-1}\right) \phi(x) dx - \int_{\Omega} \left(\sum_{k=1}^{N} b_k(x,t)\right) \phi(x) dx \\
&\geq \sum_{k=1}^{N} \left(\int_{\Omega} b_k^{-\frac{1}{\sigma_k^-(x,t)-2}}(x,t)\phi(x)dx\right)^{-\frac{1}{\sigma_k^-(x,t)-2-}} \left(\int_{\Omega} u(x,t)\phi(x)dx\right)^{\sigma_k^-(x,t)-1} \\
&\quad - \int_{\Omega} \left(\sum_{k=1}^{N} b_k(x,t)\right) \phi dx = \sum_{k=1}^{N} \alpha_k(t) \mu^{\sigma_k^-(t)-1} - \beta(t),
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\geq \sum_{i=N+1}^{K} \int_{\Omega} c_i(x,t)\phi(x,t) \left(\int_{\Omega} u(s,t)\phi(s)ds\right)^{\sigma_i^- -1} \left(\int_{\Omega} d_i^{\frac{1}{2-\sigma_i^-(t)}}(s,t)\phi^{\frac{\sigma_i-1}{\sigma_i-2}}(s,t)ds\right)^{2-\sigma_i^-} dx \\
&- \sum_{i=N+1}^{K} \int_{\Omega} \left(c_i(x,t)\int_{\Omega} d_i(s,t)ds\right) \phi dx = \sum_{i=N+1}^{K} \mu^{\sigma_i^- -1}(t)\theta_i(t) \int_{\Omega} c_i(x,t)\phi(x)dx - \beta(t) \\
&= \sum_{i=N+1}^{K} \alpha_i(t)\mu^{\sigma_i^- -1} - \beta(t),
\end{aligned}
$$

where for $k = 1, \ldots, N$

$$
\alpha_k(t) = \left(\int_{\Omega} b_k^{\frac{1}{2-\sigma_k^-(t)}}(x,t)\phi(x)dx\right)^{2-\sigma_k^-(t)}, \qquad \beta(t) = \int_{\Omega} \left(\sum_{k=1}^{N} b_k(x,t)\right) \phi dx,
$$

and for $k = N+1, \ldots, K$

$$
\alpha_k(t) = \int_{\Omega} c_i \phi \, dx \left(\int_{\Omega} d_i^{\frac{1}{2-\sigma_i^-}} \phi^{\frac{\sigma_i-1}{\sigma_i-2}} \, dx\right)^{2-\sigma_i^-}, \quad \beta(t) = \sum_{i=N+1}^{K} \int_{\Omega} \left(c_i \int_{\Omega} d_i ds\right) \phi \, dx.
$$

Gathering these formulas we arrive at the nonlinear ODI of the type (3.20)

$$
\mu'(t) \geq -\lambda\mu + \sum_{i=1}^{K} \alpha_i(t)\mu^{\sigma_i^-(t)-1}(t)(t) - \beta(t).
$$

Remark 3. *The above arguments are applicable to equations of higher order. Let us consider the problem*

$$
\begin{cases}
u_t = Lu + f(x,t,u) \quad \text{in } Q_T, \\
B_k u = 0 \text{ on } \Gamma_T,\ k = 1, \ldots, m, \\
u(x,0) = u_0(x) \text{ in } \Omega,
\end{cases}
$$

where L is an operator of order $2m$, with the coefficients independent of t and such that eigenvalue problem for the conjugate operator

$$
L^*\phi = \lambda\phi \text{ in } x \in \Omega, \qquad B_k^*\phi = 0 \text{ on } \Gamma,\ k = 1, \ldots, m
$$

has a nonnegative solution ϕ with positive λ. Proceeding in the standard way we reduce the study of sufficient conditions of the blow-up behavior to the analysis of the ordinary differential inequality for the function $\mu(t) = \displaystyle\int_{\Omega} u(x,t)\,\phi(x)\,dx$.

5. Evolutional anisotropic $p(x)$-Laplacian with nonlinear reaction term

5.1. **Assumptions and result.** Let $u(x,t)$ be a weak solution of the parabolic equation

$$\left\{\begin{array}{l} u_t = \sum_{i=1}^n D_i\left(a_i(x,t)|D_i u|^{p_i(x)-2}\right) + \sum_{i=1}^K b_i(x,t)|u|^{\sigma_i(x)-2}u \quad \text{in } Q_T, \quad 1 \le K \le n, \\ u(x,0) = u_0(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T. \end{array}\right. \tag{5.32}$$

It is assumed that the coefficients a_i, b_i and the exponents p_i, σ_i satisfy the following conditions:

$$\left\{\begin{array}{l} \text{there exist constants } a_i^{\pm} \in (0,\infty),\ b_i^{\pm} \in [0,\infty) \text{ such that} \\ a_i(x,t) \in (a_i^-, a_i^+), \quad b_i(x,t) \in (b_i^-, b_i^+), \end{array}\right. \tag{5.33}$$

$$\left\{\begin{array}{l} \text{the exponents } p_i(x),\ \sigma_i(x) \text{ are continuous with logarithmic module of continuity (see (2.5))} \\ \text{there exists constants } p_i^{\pm},\ \sigma_i^{\pm} \in (1,\infty) \text{ such that} \\ p_i(x) \in (p_i^-, p_i^+), \quad \sigma_i(x) \in (\sigma_i^-, \sigma_i^+). \end{array}\right. \tag{5.34}$$

For the sake of notation, in the case that $1 \le K < n$ we simply set $b_i \equiv 0$ for $i = K+1, \dots, n$. Let us define the functions

$$f(t) = \frac{1}{2}\int_0^t \int_{\Omega} |u(x,t)|^2\,ds, \tag{5.35}$$

$$E(t) = \sum_{i=1}^n \int_{\Omega} \left(\frac{a_i(x,t)}{p_i(x)}\,|D_i u|^{p(x)} - \frac{b_i(x,t)}{\sigma_i(x)}\,|u|^{\sigma_i(x)}\right) dx \tag{5.36}$$

and assume that

$$E(0) = \sum_{i=1}^n \int_{\Omega} \left(\frac{a_i(x,0)}{p_i(x)}\,|D_i u_0|^{p(x)} - \frac{b_i(x,0)}{\sigma_i(x)}\,|u_0|^{\sigma_i(x)}\right) dx \le 0. \tag{5.37}$$

Theorem 3. *Let conditions* (5.33), (5.34), (5.37) *be fulfilled. Assume that* $|u_0|^{\sigma(x)} \in L^1(\Omega)$, $|\nabla u_0|^{p(x)} \in L^1(\Omega)$,

$$a_{it},\, b_{it} \in C^0(\overline{Q}_T), \qquad a_{it} \le 0, \quad b_{it} \ge 0.$$

If there exists a constant $\lambda > 0$ such that

$$\left\{\begin{array}{l} \dfrac{1}{p_i^+} \ge \lambda \ge \dfrac{1}{\sigma_i^-} \text{ for all } i = 1, \dots, K, \\ \dfrac{1}{p_j^+} \ge \lambda > \dfrac{1}{\sigma_j^-} \text{ and } b_j^- > 0 \text{ for at least one } j \in \{1, \dots, n\}, \end{array}\right. \tag{5.38}$$

then every solution of problem (5.32) *blows-up in a finite time: there exists $t^* < \infty$ such that $\|u(\cdot,t)\|_{\infty,\Omega} \to \infty$ as $t \to t^*$.*

Remark 4. *The assertion of Theorem* 3 *is applicable, for example, to the equations*

$$\text{(a)}\quad u_t = \sum_{i=1}^{n} D_i(|D_i u|^{p_i(x)-2} D_i u) + |u|^{\sigma(x)-2}u, \qquad \text{(b)}\quad u_t = \Delta u + \sum_{i=1}^{K} |u|^{\sigma_i(x)-2}u.$$

For the former equation, the solutions blow-up in a finite if $p_i(x) > 1$ for all $i = 1, \dots, n$ and $p_j^+ < \sigma^+$ for some $j \in \{1, \dots, n\}$. For the latter equation the restrictions on the exponents read: $\sigma_i(x) \geq 2$, $\sigma_j^- > 2$ for at least one $j \in \{1, \dots, K\}$. It is worth noting here that to equation (b) *Theorem* 1 *is applicable.*

Remark 5. *A revision of the proof of Theorem* 3 *given below shows that the condition $b_i \geq 0$ for all $i = 1, \dots, K$ can be relaxed, and the assertion remains valid if condition* (5.38) *is substituted by the following one: there exists a constant $\lambda > 0$ such that*

$$\text{(5.39)}\qquad \begin{cases} \dfrac{1}{p_j^+} \geq \lambda \text{ and } \left(\lambda - \dfrac{1}{\sigma_i^-}\right) b_i(x,t) \geq 0 \text{ for all } i = 1, \dots, K, \\ \dfrac{1}{p_j^+} \geq \lambda > \dfrac{1}{\sigma_j^-} \text{ and } b_j^- > 0 \text{ for at least one } j \in \{1, \dots, n\}. \end{cases}$$

An example of such a situation is furnished by the equation with constant exponents of nonlinearity

$$\begin{cases} u_t = u_{xx} + u_{yy} + (|u_z|^{p_3-2}u_z) + u - |u|^{-1/2}u + |u|u, \\ p_1 = p_2 = 2, \quad p_3 \in (1,2], \quad \sigma_1 = 2, \quad \sigma_2 = 3/2, \quad \sigma_3 = 3, \\ b_1 = 1, \quad b_2 = -1, \quad b_3 = 1. \end{cases}$$

In this case we may take $\lambda = 1/2$.

5.2. **Proof of Theorem 3.**

5.2.1. *Ordinary differential inequality for $f(t)$.*

Lemma 1. *Under the conditions of Theorem* 3 *the solution of problem* (5.32) *satisfies the estimate*

$$\text{(5.40)}\qquad \frac{1}{2}\int_\Omega |u(x,t)|^2\,dx + \sum_{i=1}^{n}\int_0^t \int_\Omega (a_i|D_i u|^{p_i} - b_i|u|^{\sigma_i})\,dx = \frac{1}{2}\int_\Omega |u_0|^2\,dx$$

Proof. Equality (5.40) follows from (2.10) if we take the solution u for the test-function. □

The second energy estimate is a byproduct of Proposition 2. It is easy to see that inequality (2.11) can be written in the form

$$\text{(5.41)}\qquad \forall\,\text{a.e.}\, t > 0 \qquad E(t) + \int_0^t \int_\Omega |u_t|^2\,dxdt \leq E(0).$$

Inequality (5.40) can be written in the form

$$\text{(5.42)}\qquad f'(t) \equiv \frac{1}{2}\int_\Omega |u(\cdot,t)|^2\,dx = \frac{1}{2}\int_\Omega |u_0|^2 + \sum_{i=1}^{n}\int_0^t \int_\Omega (-a_i\,|D_i u|^{p_i} + b_i\,|u|^{\sigma_i})\,dx \geq 0,$$

whence

$$\text{(5.43)}\qquad f''(t) \equiv \int_\Omega u\,u_t\,dx = \sum_{i=1}^{n}\int_\Omega (-a_i\,|D_i u|^{p_i} + b_i\,|u|^{\sigma_i})\,dx,$$

and, by virtue of (5.40),

$$0 \leq f'(t) = \frac{1}{2}\int_\Omega u_0^2\,dx + f''(t).$$

Multiplying (5.43) by a positive constant λ, adding to (5.41), and using the condition $E(0) \leq 0$, we arrive at the inequality

$$E(t) + \lambda \sum_i \int_\Omega \left(-a_i\,|D_i u|^{p_i} + b_i\,|u|^{\sigma_i}\right)\,dx + \int_0^t \int_\Omega u_t^2 ds \leq \lambda f''(t),$$

which we write in the form

$$\sum_{i=1}^n \int_\Omega \left(\left(\frac{1}{p_i} - \lambda\right) a_i\,|D_i u|^{p_i} + \left(\lambda - \frac{1}{\sigma_i}\right) b_i\,|u|^{\sigma_i}\right)\,dx + \int_0^t \int_\Omega u_t^2\,dx \leq \lambda f''(t). \tag{5.44}$$

By assumption (5.38) we may choose $\lambda > 0$ in such a way that

$$\forall\, i = 1, \ldots, n \quad \frac{1}{\sigma_i^-} \leq \lambda \leq \frac{1}{p_i^+} \quad \text{and for at least one } j \in \{1, \ldots, n\} \quad \frac{1}{\sigma_j^-} < \lambda \leq \frac{1}{p_j^+}. \tag{5.45}$$

Keeping on the left-hand side the term corresponding to the sub-index j and dropping the others, we reduce the differential inequality to the form

$$\left(\lambda - \frac{1}{\sigma_j^-}\right) b_j^- \int_\Omega |u|^{\sigma_j}\ dx + \int_0^t \int_\Omega u_t^2\,dx \leq \lambda f''(t). \tag{5.46}$$

5.3. **Analysis of the differential inequality.**

5.3.1. *Case 1:* $p_j^+ > 2$. Dropping the first nonnegative term on the left-hand side of (5.46), we have

$$0 < \int_0^t \int_\Omega u_t^2\,dx \leq \lambda f''(t). \tag{5.47}$$

Let us denote by t^* the time of existence of the solution u:

$$t^* = \sup\{t > 0 :\ \|u\|_{\infty,\Omega} < \infty \text{ for } t < t^*\}.$$

Using Hölder's inequality, we obtain the chain of relations

$$\begin{aligned}
\left(f'(t) - f'(0)\right)^2 &= \left(\int_0^t \frac{d}{dt}\left(\frac{1}{2}\int_\Omega u^2\,dx\right)\,dt\right)^2 = \left(\int_0^t \int_\Omega u u_t\,dx\right)^2 \\
&\leq \left(\int_0^t \|u_t\|_{2,\Omega}\|u\|_{2,\Omega}\,dt\right)^2 \leq \|u_t\|^2_{2,\Omega\times(0,t)}\|u\|^2_{2,\Omega\times(0,t)} \\
&\leq \lambda f''(t)\int_0^t \int_\Omega u^2\,dxdt \leq 2\lambda f''(t) f(t),
\end{aligned}$$

which gives the second-order nonlinear ordinary differential inequality for the function $f(t)$:

$$\left(f'(t) - f'(0)\right)^2 \leq 2\lambda f''(t) f(t) < \frac{2}{p_+} f''(t) f(t) \tag{5.48}$$

We want to prove that the function $f(t)$ becomes unbounded at a finite moment. Let us argue by contradiction and assume that $t^* = \infty$. Since f, f', f'' are nonnegative, it is necessary that $f(t) \nearrow \infty$ as $t \to \infty$. It follows that there exists a moment t_0 and a constant $1 < \nu < \frac{p_j^+}{2}$ such that $(f'(t) - f'(0))^2 \geq \frac{2\nu}{p_j^+}(f'(t))^2$ for $t \geq t_0$. This allows one to continue (5.48) as follows:

$$\nu(f'(t))^2 \le \frac{p_j^+}{2}\left(f'(t)-f'(0)\right)^2 \le f''(t)f(t) \qquad \text{for } t \ge t_0.$$

Integration of this inequality gives

$$f^{\nu-1}(t) \ge \frac{f^{\nu-1}(t_0)}{1-t(\nu-1)\frac{(f'(t_0))^{\nu-1}}{f(t_0)}} \to \infty \quad \text{as } t \to T = \frac{f(t_0)}{(\nu-1)\left(f'(t_0)\right)^{\nu-1}},$$

which contradicts the assumption $t^* = \infty$ because

$$\infty > \frac{1}{2}|\Omega| T \|u\|^2_{\infty,\Omega} \ge \frac{1}{2}\int_0^t |u|^2\,dxdt \equiv f(t) \to \infty \quad \text{as } t \nearrow T.$$

This completes the proof of Theorem 3 in the case $p_j^+ > 2$.

5.3.2. *Case 2:* $1 < p_j^+ \le 2$. Applying Hölder's inequality and inequality (2.7), we continue (5.46) as follows:

$$\min\left\{(f'(t))^{\frac{\sigma_j^+}{2}},\, (f'(t))^{\frac{\sigma_j^-}{2}}\right\} = \min\left\{\|u\|^{\sigma_j^+}_{2,\Omega},\, \|u\|^{\sigma_j^-}_{2,\Omega}\right\} \le C \int_\Omega |u|^{\sigma_j}\,dx \le C' f''(t).$$

In our conditions on the data the functions f, f' and f'' are strictly positive for $t > 0$, whence there exists some t_0 such that $f'(t) > 1$ for all $t \ge t_0$. It follows that

$$(f'(t))^{\frac{\sigma_j^-}{2}} \le C' f''(t) \quad \text{for } t \ge t_0$$

and the straightforward integration of this inequality shows that there exists a finite t^* such that $f'(t) \to \infty$ as $t \to t^*$. This leads to the desired conclusion because

$$|\Omega|\,\|u\|^2_{\infty,\Omega} \ge \int_\Omega u^2\,dx \equiv 2\,f'(t) \to \infty \quad \text{as } t \to \infty.$$

The proof of Theorem 3 is completed.

References

[1] R. Aboulaicha, D. Meskinea, and A. Souissia, *New diffusion models in image processing*, Computers and Mathematics with Applications, 56 (2008), pp. 874–882.

[2] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal., 164 (2002), pp. 213–259.

[3] E. Acerbi, G. Mingione, and G. Seregin, *Regularity results for paraboliqc systems related to a class of non-Newtonian fluids*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 21(01) (2004), pp. 25–60.

[4] Y. Alkhutov, S. N. Antontsev, and V. Zhikov, *Parabolic equations with variable order of nonlinearity*, Collection of works of the Institute of Mathematics NAS of Ukraine, 6 (2009), pp. 23–50.

[5] S. Antontsev, M. Chipot, and Y. Xie, *Uniqueness results for equations of the $p(x)$-Laplacian type*, Adv. Math. Sci. Appl., 17 (2007), pp. 287–304.

[6] S. Antontsev and V. Zhikov, *Higher integrability for parabolic equations of $p(x,t)$-Laplacian type*, Adv. Differential Equations, 10 (2005), pp. 1053–1080.

[7] S. N. Antontsev and J. F. Rodrigues, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara, Sez.,VII. Sci. Mat., 52 (2006), pp. 19–36.

[8] S. N. Antontsev and S. I. Shmarev, *Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties of solutions*, Journal Nonlinear Analysis, 65 (2006), pp. 722–755.

[9] ———, *Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions*, Elsevier, 2006. Handbook of Differential Equations. Stationary Partial Differential Equations, Elsevier, Vol. 3, Chapter 1, pp.1-100.

[10] ———, *Parabolic equations with anisotropic nonstandard growth conditions*, in Internat. Ser. Numer. Math. 154, Birkhäuser, Verlag Basel/Switzerland, 2006, pp. 33–44.

[11] ———, *Extinction of solutions of parabolic equations with variable anisotropic nonlinearities*, Proceedings of the Steklov Institute of Mathematics, Moscow, Russia, 268 (2008), pp. 2289–2301.

[12] ———, *Vanishing solutions of anisotropic parabolic equationbs with variable nonlinearity*, Preprint 2008-021, CMAF, University of Lisbon, Portugal, downloaable at :http://www.ptmat.fc.ul.pt/preprints/pdf/2008, (2008), pp. 1–27.

[13] ———, *Blow up of solutions to parabolic equations with nostandard growth conditions*, Preprint 2009-002, CMAF, University of Lisbon, Portugal, downloaable at :http://www.ptmat.fc.ul.pt/preprints/pdf/2009, (2009), pp. 1–16.

[14] ———, *Anisotropic parabolic equations with variable nonlinearity*, Publicacions Matematique[1]. Univ. Barcelona, in press.
[15] ———, *Localized solutions of anisotropic parabolic equations*, Nonlinear Anal., Theory, Methods, Applications, (2009, in press), pp. 1–18.
[16] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math, 66 (2006), pp. 1383–1406.
[17] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I, 13 (1966), pp. 109–124 (1966).
[18] ———, *On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations*, in Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 105–113.
[19] V. A. Galaktionov and S. I. Pohozaev, *Blow-up and critical exponents for parabolic equations with non-divergent operators: dual porous medium and thin film operators*, J. Evol. Equ., 6 (2006), pp. 45–69.
[20] V. A. Galaktionov and J. L. Vázquez, *A stability technique for evolution partial differential equations*, Progress in Nonlinear Differential Equations and their Applications, 56, Birkhäuser Boston Inc., (2004).
[21] S. Kaplan, *On the growth of the solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math., 16 (1963), pp. 305–330.
[22] S. Levine, Y. Chen, and J. Stanich, *Image restoration via nonstandard diffusion*, Technical Report ♯ 04-01, Dept. of Mathematics and Computer Science, Duquesne University, 2004.
[23] E. Mitidieri and S. Pohozaev, *Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on* $\mathbb{R}^n$, J.evol.equ., 1 (2001), pp. 189–220.
[24] J. P. Pinasco, *Blow-up for parabolic and hyperbolic problems with variable exponents*, J. Nonlinear Analysis: Theory, Methods and Applications, (to appear).
[25] S. I. Pohozaev and A. Tesei, *Blow-up of nonnegative solutions to quasilinear parabolic inequalities*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 11 (2000), pp. 99–109.
[26] S. I. Pokhozhaev and A. Teseĭ, *Critical exponents for the absence of solutions for systems of quasilinear parabolic inequalities*, Differ. Uravn., 37 (2001), pp. 521–528, 575.
[27] K. Rajagopal and M. Růžička, *Mathematical modelling of electro-rheological fluids*, Cont. Mech. Therm., 13 (2001), pp. 59–78.
[28] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Springer, Berlin, 2000. Lecture Notes in Mathematics, 1748.
[29] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-up in quasilinear parabolic equations*, Walter de Gruyter & Co., Berlin, 1995. Translated from the 1987 Russian original by Michael Grinfeld and revised by the authors.
[30] M. Tsutsumi, *Existence and nonexistence of global solutions for nonlinear parabolic equations*, Publ. Res. Inst. Math. Sci., 8 (1972), pp. 211–229.
[31] ———, *Existence and nonexistence of global solutions of the first boundary value problem for a certain quasilinear parabolic equation*, Funkcial. Ekvac., 17 (1974), pp. 13–24.
[32] V. Zhikov, *On Lavrentiev's effect*, Dokl., Ross. Akad. Nauk, 345 (1995), pp. 10–14.
[33] V. V. Zhikov, *On Lavrentiev's phenomenon*, Russian J. Math. Phys., 3 (1994), pp. 249–269.

CMAF, Universidade de Lisboa, Portugal
E-mail address: `anton@ptmat.fc.ul.pt, antontsevsn@mail.ru`

Departamento de Matemáticas, Universidad de Oviedo, España
E-mail address: `shmarev@orion.ciencias.uniovi.es`

[1]Preprint 2007-013, CMAF, Univ. Lisbon, Downloadable at: http://www.ptmat.fc.ul.pt/preprints/preprints.html