

Existence of bound states for the coupled  
Schrödinger-KdV system with cubic nonlinearity

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**Abstract**

We prove in this Note the existence of an infinite family of smooth positive bound states for the coupled Schrödinger-Korteweg-de Vries system (1), which decay exponentially at infinity.

Existence d'ondes solitaires pour le système couplé  
de Schrödinger-KdV avec non-linearité cubique

**Résumé**

Nous prouvons dans cette Note l'existence d'une famille infinie d'ondes solitaires régulières pour le système couplé de Schrödinger-Korteweg-de Vries (1), qui décroissent exponentiellement à l'infini.

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## 1 Introduction and main result

Let us consider the coupled Schrödinger-KdV system:

$$\begin{cases} if_t + D^2 f = \beta fg - |f|^2 f \\ g_t + D^3 g + gDg = \frac{\beta}{2} D(|f|^2), \end{cases} \quad (1)$$

where  $f = f(x, t)$  is a complex-valued function,  $g = g(x, t)$  is real-valued,  $D = \frac{\partial}{\partial x}$  represents the spatial derivative and  $\beta < 0$  is a real constant.

The coupled Schrödinger-KdV system appears in the context of interaction phenomena between long waves and short waves such as the resonant interaction between long and short capillary - gravity water waves. The global well-posedness of the Cauchy problem for the I.V.P. associated to (1) was solved recently by A.Corcho and F.Linares ([5]) in the energy space  $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ .

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Here, we are concerned in finding bound-state solutions to (1) of the form

$$(f(x, t), g(x, t)) = (e^{i\omega t} e^{ikx} \phi(x - ct), \psi(x - ct)) \quad (2)$$

where  $\phi, \psi \geq 0$ . By choosing  $c = 2k$  and putting  $c^* := k^2 + \omega$ , we obtain the system

$$\begin{cases} -\phi'' + c^*\phi = \phi^3 - \beta\phi\psi \\ -\psi'' + c\psi = \frac{1}{2}\psi^2 - \frac{\beta}{2}\phi^2. \end{cases} \quad (3)$$

In [2], an existence theorem is derived for a general system similar to (3), although the method employed cannot be exploited here due to the absence of a cubic term in the second equation. Also, in [1] and [3], the existence of bound states (and ground states) for the coupled systems

$$\begin{cases} if_t + D^2f = \alpha fg \\ g_t + \gamma \mathcal{H}Dg = \beta D(|f|^2) \end{cases} \quad \text{and} \quad \begin{cases} i(f_t + c_1 Df) + \delta_1 D^2f = \alpha fg \\ g_t + c_2 Dg + D^3g + \gamma g Dg = \beta D(|f|^2) \end{cases} \quad (4)$$

is studied (Here,  $\mathcal{H}$  denotes de Hilbert transform). Note however that the different approaches used rely heavily on the fact that the nonlinear terms are quadratic and do not seem suitable to handle the term  $-|f|^2f$  in the right-hand-side of (1).

In what follows, for  $s \in \mathbb{R}$ , we denote by  $H^s \equiv H^s(\mathbb{R})$  the usual Sobolev space with the norm  $\|\cdot\|_{H^s}$  and the  $L^p$  norm will be denoted by  $\|\cdot\|_p$ .

We end this introduction by stating our main result:

**Theorem 1.1** *There exists a family*

$$(f_n, g_n) = (e^{i\omega_n t} e^{ik_n x} \phi_n(x - c_n t), \psi_n(x - c_n t)) \quad (5)$$

*of non trivial solutions to (1) with  $\lim_{n \rightarrow \infty} c_n = +\infty$ .*

*Here,  $\phi_n, \psi_n$  are smooth positive functions which decay exponentially at infinity.*

## 2 The minimization problem

For  $\alpha > 0$  and  $\mu \geq 0$ , we set

$$\mathcal{I}(\mu) = \inf\{E(u, v) : (u, v) \in X_\mu\}, \quad (6)$$

where  $X_\mu = \{(u, v) \in H^1 \times H^1 : \|u\|_2^2 + \alpha \|v\|_2^2 = \mu\}$  ( $u, v$  real-valued) and

$$E(u, v) = \int (Du)^2 + \int (Dv)^2 - \frac{1}{2} \int u^4 - \frac{1}{3} \int v^3 + \beta \int u^2 v. \quad (7)$$

We will apply the concentration-compactness method ([6],[7]) to prove the existence of a minimizer for  $\mathcal{I}(\mu)$ .

**Proposition 2.1** For all  $\mu > 0$ ,  $\mathcal{I}(\mu) > -\infty$ .

**Proof:**

Let  $(u, v) \in X_\mu$ :  $\|u\|_2^2 \leq \mu$  and  $\|v\|_2^2 \leq \frac{\mu}{\alpha}$ . By the Gagliardo-Nirenberg inequalities:

$$\|v\|_3^3 \leq C_1 \|Dv\|_2^{\frac{1}{2}} \|v\|_2^{\frac{5}{2}} \leq C_1 \left(\frac{\mu}{\alpha}\right)^{\frac{5}{4}} \|Dv\|_2^{\frac{1}{2}} \quad \text{and} \quad \|u\|_4^4 \leq C_2 \|Du\|_2 \|u\|_2^3 \leq C_2 \mu^{\frac{3}{2}} \|Du\|_2,$$

where  $C_j$  denote positive constants. Also,  $\int |v|u^2 \leq \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|u\|_4^4 \leq \frac{\mu}{2\alpha} + C_2 \frac{\mu^{\frac{3}{2}}}{2} \|Du\|_2$ . Finally, we obtain

$$\begin{aligned} E(u, v) &\geq \|Du\|_2^2 + \|Dv\|_2^2 - \frac{1}{2} \int u^4 - \frac{1}{3} \int |v|^3 - |\beta| \int u^2 |v| \\ &\geq \|Du\|_2^2 + \|Dv\|_2^2 - C_2(1 + |\beta|) \frac{\mu^{\frac{3}{2}}}{2} \|Du\|_2 - \frac{C_1}{3} \left(\frac{\mu}{\alpha}\right)^{\frac{5}{4}} \|Dv\|_2^{\frac{1}{2}} - \frac{|\beta|\mu}{2\alpha}, \end{aligned} \quad (8)$$

from where we deduce the existence of an inferior bound for  $E(u, v)$  depending exclusively on  $\mu$  and  $\alpha$ . ■

**Proposition 2.2** For all  $\mu \geq 0$ ,  $\mathcal{I}(\mu) \leq 0$ . Also, there exists  $\mu^* > 0$  such that for all  $\mu > \mu^*$ ,  $\mathcal{I}(\mu) \leq -A\mu$ , where  $A$  is a positive constant independent of  $\mu$  and  $\alpha$ .

**Proof:**

Let  $\mu \geq 0$  and  $u \in H^1$  such that  $\|u\|_2^2 = \mu$ . Then  $(u, 0) \in X_\mu$  and  $E(u, 0) \leq \int (Du)^2$ . Noticing that  $\inf\{\int (Du)^2 : \|u\|_2^2 = \mu\} = 0$ , we get  $\mathcal{I}_\mu \leq 0$ .

We now consider  $u \in H^1(\mathbb{R})$  such that  $\|u\|_2 = 1$ . For  $\theta > 0$ , we put  $u_\theta(x) = u(\theta x)$ . We have  $E(u_\theta, 0) = \theta \int |Du|^2 - \frac{1}{2\theta} \int |u|^4$ .

By choosing  $\theta = \frac{1}{\mu}$ ,  $(u_\theta, 0) \in X_\mu$  and  $E(u_\theta, 0) = \frac{1}{\mu} \int |Du|^2 - \frac{\mu}{2} \int |u|^4$ .

Let  $\epsilon > 0$  such that  $\int |u|^4 > \epsilon$ :  $\mathcal{I}_\mu \leq E(u_\theta, 0) = \frac{1}{\mu} \int |Du|^2 - \frac{\epsilon}{2}\mu - \frac{\mu}{2} \left( \int |u|^4 - \epsilon \right)$ .

The lemma holds by taking  $\mu^*$  such that  $\frac{1}{\mu^*} \int |u|^2 - \frac{\epsilon}{2}\mu^* < 0$  and  $A = \frac{1}{2} \left( \int |u|^4 - \epsilon \right)$ . ■

**Remark 2.3** It is well known that for  $f \in H^1(\mathbb{R})$  real valued,  $\|D|f|\|_{L^2} \leq \|Df\|_{L^2}$ . For a pair  $(u, v) \in X$ ,

$$E(|u|, |v|) \leq E(u, v).$$

Hence, there exists a minimizing sequence  $(u_j, v_j)$  for problem (6) with  $u_j, v_j \geq 0$ .

**Lemma 2.4** Let  $\mu > \mu^*$ . For all  $\theta > 1$ ,  $\mathcal{I}(\theta\mu) < \theta\mathcal{I}(\mu)$ .

**Proof:** Consider a positive minimizing sequence  $(u_j, v_j) \in X_\mu$  for problem (6). We have

$$\begin{aligned} E(\sqrt{\theta}u_j, \sqrt{\theta}v_j) &= \theta E(u_j, v_j) - \frac{1}{2}(\theta^2 - \theta) \int u_j^4 + (\theta^{\frac{3}{2}} - \theta) \left( \beta \int u_j^2 v_j - \frac{1}{3} \int v_j^3 \right) \\ &\leq \theta E(u_j, v_j) + \max\{\theta - \theta^2, \theta - \theta^{\frac{3}{2}}\} \left( \frac{1}{2} \int u_j^4 + |\beta| \int u_j^2 v_j + \frac{1}{3} \int v_j^3 \right). \end{aligned}$$

Since  $(u_j, v_j)$  is a minimizing sequence,  $\frac{1}{2} \int u_j^4 + |\beta| \int u_j^2 v_j + \frac{1}{3} \int v_j^3 \geq \delta$  for some  $\delta > 0$ . Otherwise there would exist a subsequence - still denoted  $(u_j, v_j)$  - such that  $\lim E((u_{\phi(j)}, v_{\phi(j)})) \geq 0$ , which is absurd since  $\mathcal{I}(\mu) < 0$ . Hence

$$E(\sqrt{\theta}u_j, \sqrt{\theta}v_j) \leq \theta E(u_j, v_j) - \delta(\theta^2 - \theta) \quad \text{for all } \theta > 1.$$

Since  $\|\sqrt{\theta}u_j\|_2^2 + \alpha\|\sqrt{\theta}v_j\|_2^2 = \theta\|u_j\|_2^2 + \alpha\theta\|v_j\|_2^2 = \theta\mu$ , we obtain  $\mathcal{I}(\theta\mu) < \theta\mathcal{I}(\mu)$ . ■

From this Lemma it is straightforward to prove that  $\mathcal{I}$  is a non-increasing function of  $\mu$  and therefore there exists  $\mu_1 \geq 0$  such that  $\mathcal{I}(\mu) < 0 \Leftrightarrow \mu > \mu_1$ . Arguing as in [8] (Lemma 2.3), these facts are sufficient to prove the following key result:

**Corollary 2.5** *Sub-additivity*

Let  $\mu > \mu_1$  and  $0 < \Omega < \mu$ . Then  $\mathcal{I}(\mu) < \mathcal{I}(\Omega) + \mathcal{I}(\mu - \Omega)$ .

Next, we prove the existence of minimizers:

**Proposition 2.6** Let  $\mathcal{M}_\mu = \{(u, v) \in X_\mu : \mathcal{I}_\mu = E(u, v)\}$ . For  $\mu > \mu_1$ ,  $\mathcal{M}_\mu \neq \emptyset$ .

**Sketch of the Proof:** Let us consider a positive minimizing sequence  $(u_j, v_j) \in X_\mu$  for problem (6). We will apply the concentration-compactness lemma to the sequence  $\rho_j = |u_j|^2 + \alpha v_j^2$ . Using the notations in [6], we introduce the concentration function of  $\rho_j$ :

$$Q_j(t) = \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} \rho_j, \quad \text{and we set } \Omega = \lim_{t \rightarrow \infty} Q(t).$$

We now have three alternatives: vanishing ( $\Omega = 0$ ), dichotomy ( $0 < \Omega < \mu$ ) and compactness ( $\Omega = \mu$ ). The latter implies the relatively compactness of the sequence  $(u_j, v_j)$  up to translations.

First, we rule out vanishing. Indeed, if  $\Omega = 0$ ,  $\lim_{j \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} u_j^2 = \lim_{j \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} v_j^2 = 0$ .

Since  $(u_j)$  and  $(v_j)$  are bounded in  $H^1(\mathbb{R})$  (as seen in the proof of Proposition 2.1), a classical lemma (see [7], Lemma I.1) yields  $\|u_j\|_p \rightarrow 0$  and  $\|v_j\|_p \rightarrow 0$  for all  $p > 2$ .

It results that

$$\mathcal{I}(\mu) = \lim_{j \rightarrow \infty} E(u_j, v_j) = \lim_{j \rightarrow \infty} \int |Du_j|^2 + \int Dv_j^2 - \frac{1}{2} \int |u_j|^4 - \frac{1}{3} \int v_j^3 + \beta \int |u_j|^2 v_j \geq 0,$$

which is absurd by Proposition 2.2.

The Corollary 2.5 easily rules out dichotomy. Indeed, in the present situation it is standard to construct for all  $\epsilon > 0$  two sequences  $(u_j^{(i)}, v_j^{(i)})$ ,  $i = 1, 2$ , such that

$$\|u_j^{(1)}\|_2^2 + \alpha\|v_j^{(1)}\|_2^2 - \Omega < \epsilon, \quad \|u_j^{(2)}\|_2^2 + \alpha\|v_j^{(2)}\|_2^2 - (\mu - \Omega) < \epsilon$$

and

$$E(u_j, v_j) \geq E(u_j^{(1)}, v_j^{(1)}) + E(u_j^{(2)}, v_j^{(2)}) - C(\epsilon), \quad \lim_{\epsilon \rightarrow 0} C(\epsilon) = 0.$$

This leads to  $\mathcal{I}_\mu \geq \mathcal{I}_\Omega + \mathcal{I}_{\mu-\Omega}$ , which is in contradiction with Corollary 2.5.

Hence, we have compactness: extracting once again a subsequence, there exists  $\{y_j\}$  such that

$$(\tilde{u}_j = u_j(\cdot - y_j), \tilde{v}_j = v_j(\cdot - y_j)) \rightarrow (\phi, \psi) \text{ in } L^2(\mathbb{R}).$$

Furthermore, the sequence  $(\tilde{u}_j, \tilde{v}_j)$  converges to  $(\phi, \psi)$  in  $H^1(\mathbb{R})$  weak. Hence,  $(\tilde{u}_j, \tilde{v}_j) \rightarrow (\phi, \psi)$  in  $L^p$  for all  $p \geq 2$ :  $\|\tilde{u}_j\|_4 \rightarrow \|\phi\|_4$ ,  $\|\tilde{v}_j\|_3 \rightarrow \|\psi\|_3$ ,  $\int \tilde{u}_j^2 \tilde{v}_j \rightarrow \int \phi^2 \psi$  and

$$\mathcal{I}_\mu \leq E(\phi, \psi) \leq \underline{\lim} E(\tilde{u}_j, \tilde{v}_j) = \mathcal{I}_\mu.$$

Finally,  $(\phi, \psi) \in \mathcal{M}_\mu \neq \emptyset$ .

Note that we have obtained  $\underline{\lim} \int D\tilde{u}_j^2 + D\tilde{v}_j^2 = \int D\phi^2 + D\psi^2$ , hence the convergence takes place in  $H^1$  strong. ■

### 3 End of the Proof of Theorem 1.1

Let  $(\phi, \psi) \in \mathcal{M}_\mu$ . There exists a Lagrange multiplier  $\lambda = \lambda(\alpha, \mu) \in \mathbb{R}$  such that

$$\begin{cases} -\phi'' - \lambda\phi = \phi^3 - \beta\phi\psi \\ -\psi'' - \alpha\lambda\psi = \frac{1}{2}\psi^2 - \frac{\beta}{2}\phi^2 \end{cases} \quad (9)$$

**Lemma 3.1** *There exists a constant  $A > 0$  such that for all  $\alpha$  and for all  $\mu > \mu^*$ ,*

$$\lambda(\alpha, \mu) \leq -A.$$

**Proof:**

Multiplying equations (9) by  $\phi$  and  $\psi$  respectively and integrating leads to

$$\int D\phi^2 + \int D\psi^2 - \lambda\mu = \int \phi^4 - \frac{3\beta}{2} \int \phi^2\psi + \frac{1}{2} \int \psi^3.$$

Since  $\mathcal{I}_\mu = E(\phi, \psi) = \int \phi'^2 + \int \psi'^2 - \frac{1}{2} \int \phi^4 - \frac{1}{3} \int \psi^3 + \beta \int \phi^2\psi$ , we get

$$\lambda\mu = \mathcal{I}_\mu - \frac{1}{2} \int \phi^4 + \frac{\beta}{2} \int \phi^2\psi - \frac{1}{6} \int \psi^3 \leq \mathcal{I}_\mu.$$

By Proposition 2.2,  $\lambda \leq -A$ . ■

The proof of the main theorem is complete by choosing a sequence  $\alpha_n \rightarrow \infty$  and setting  $c_n = -\lambda_n \alpha_n$ ,  $k_n = -\frac{1}{2} \lambda_n \alpha_n$  and  $\omega_n = -\lambda_n - k_n^2$ .

Note that the functions  $\phi, \psi$  are positive since they are the limit in  $H^1$  of a positive minimizing sequence. Also, a classical bootstrap argument proves the regularity of  $\phi$  and  $\psi$ . Finally, since  $c_n^* = k_n^2 + \omega_n = -\lambda_n > 0$ , the argument used in the proof of Theorem 8.1.1 in [4] (see also [1], Theorem 2.1) easily proves the existence of  $\epsilon_1, \epsilon_2 > 0$  such that  $e^{\epsilon_1|x|}\phi(x), e^{\epsilon_2|x|}\psi(x) \in L^\infty$ , which results in the exponential decreasing of  $\phi$  and  $\psi$  at infinity.

**Remark 3.2** *In the particular case where  $0 > \beta > -\frac{1}{6}$  and  $c = 4c^* - \frac{1}{12}\beta(1 + 6\beta)$ , it is possible to exhibit explicit solutions: noticing that  $D^2\text{sech} = \text{sech} - 2\text{sech}^3$  and  $D^2\text{sech}^2 = 4\text{sech}^2 - 6\text{sech}^4$ , it is easy to verify that the system (3) possesses the exact solution*

$$\phi(x) = \frac{\sqrt{2c^*(1 + 6\beta)}}{\cosh(\sqrt{c^*}x)}, \quad \psi(x) = \frac{12c^*}{\cosh^2(\sqrt{c^*}x)}.$$

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