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REPRESENTATION OF DIVERGENCE-FREE VECTOR FIELDS

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Abstract. This paper focuses on a representation result for divergence-free vector fields. Known results are recalled, namely the representation of divergence-free vector fields as curls in two and three dimensions. The representation proposed in the present paper expresses the vector field as exterior product of gradients and stands valid in arbitrary dimension. Links to computer graphics and partial differential equations are discussed.

1. Introduction. The aim of this paper is to study properties of vector fields having zero divergence, with particular emphasis on the representation of such vector fields in terms of a potential (which may be vector-valued). More precisely, a divergence-free vector field in $\mathbb{R}^n$ is expressed as exterior product of $n-1$ gradients. The main result is stated in Theorem 7.3. This representation appears sometimes in textbooks on mechanics (especially fluid mechanics) and electromagnetism is a somewhat vage formulation; see the bibliographical comments at the end of Section 5. One motivation for seeking this type of representation result comes from computer graphics, see Section 4. Another motivation is related to elliptic partial differential equations, see Section 8.

The outline of the paper is as follows. Section 2 presents results on two-dimensional vector fields, some of which are well-known from graduate-level calculus. We begin by studying curl-free vector fields and then extend the results to divergence-free vector fields. Section 3 focuses on generalizations of these results for higher dimension. In Section 4 some remarks are made on drawing two- and three-dimensional vector fields, with particular emphasis on divergence-free vector fields. Section 5 gives an intuitive description of the representation of a three-dimensional divergence-free vector field as the exterior product of two gradients; some bibliographical comments are included. In Section 6 we fix notations about the exterior product in the spirit of multi-linear algebra. In Section 7 the main representation result is stated and proven (Theorem 7.3). Section 8 deals with possible applications of this result to the study of elliptic partial differential equations, in particular to their numerical approximation. Section 9 discusses limitations of the results here presented and points directions for future research.

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Throughout this paper, Ω will be an open subset of $\mathbb{R}^n$. If $\varphi : \Omega \to \mathbb{R}$ is a differentiable function, we shall denote as usual by $\varphi_{,i}$ its partial derivative with respect to x_i. If $\vec{g} : \Omega \to \mathbb{R}^n$ is a vector field, we shall denote by g_i its components; thus, $g_{i,j}$ will denote the derivative of the i^{th} component of $\vec{g}$ with respect to x_j.

2. Vector fields in two dimensions. It is a basic result in calculus that, in two dimensions, if φ is a $\mathcal{C}^2$ function and $\vec{g} = \nabla\varphi$, then $g_{1,2} = g_{2,1}$, simply because second order partial derivatives commute: $\varphi_{,12} = \varphi_{,21}$. In other words, any gradient has zero curl.

The converse property holds also, at least locally :

THEOREM 2.1. Let $\vec{g} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^1$ vector field of zero curl. Then, for any $x_0 \in \Omega$ there is a neighbourhood V of x_0 where $\vec{g} = \nabla\varphi$ for some $\varphi \in \mathcal{C}^2(V)$.

The above representation result can be generalized in several directions. On one hand, a global representation can be searched for, instead of the local one. This can be done by imposing certain conditions on the behaviour of $\vec{g}$ at the boundary of Ω. More precisely, the integral of the tangential component of $\vec{g}$ on each connected component of $\partial\Omega$ should vanish in order for a global potential φ to exist (defined in the entire Ω). These conditions are automatically verified if Ω is simply connected.

Another direction for generalizing Theorem 2.1 has to do with the regularity that we assume on $\vec{g}$. The regularity can be lowered, in which case the partial derivatives should be interpreted in the weak sense. If one takes $\vec{g} \in L^2(\Omega)$, the same local representation holds for a potential $\varphi \in H^1(V)$. Or, for $\vec{g} \in H^{-1}(\Omega)$, a potential is shown to exist in $L^2(V)$. See [2], section 2-6 and Corollary 2 in section 7-7.

In two dimensions, there is a simple mapping between the operators curl and divergence. Let us define the matrix

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

representing a $90°$ rotation in the plane. If $\vec{\jmath}$ and $\vec{g}$ are two-dimensional vector fields, then

$$\operatorname{div} R\vec{g} = \operatorname{curl} \vec{g}\,,\ \operatorname{curl} R\vec{\jmath} = \operatorname{div} \vec{\jmath}.$$

See [8], section 15.3, for an intuitive presentation of the effect of a $90°$ rotation on a two-dimensional field.

Thus, in two dimensions, any result on curl-free vector fields applies to divergence-free vector fields by means of a $90°$ rotation, the converse being also true. In particular, one has :

THEOREM 2.2 ([9], Theorem 3.1). Let $\Omega \subset \mathbb{R}^2$ be a bounded open set satisfying the Lipschitz condition. Let $\vec{\jmath} : \Omega \to \mathbb{R}^2$ be a vector field whose components belong to $L^2(\Omega)$ such that $\operatorname{div} \vec{\jmath} = 0$ in Ω and

$$\int_{\Gamma_i} \langle \vec{\jmath}, \vec{n} \rangle = 0\,,\ i = 1, 2, \ldots, m\,, \tag{2.1}$$

where the integral should be understood in the sense of traces. $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are the boundaries of the m holes of Ω and $\vec{n}$ is the unitary vector normal to Γ_i. Then, there is a scalar function $\varphi \in H^1(\Omega)$ such that $\vec{\jmath} = R\,\nabla\varphi$.

3. Vector fields in higher dimensions. Many authors define the curl of a vector field $\vec{g} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ as the vector field $\operatorname{curl}\vec{g} = (g_{2,3} - g_{3,2}, g_{3,1} - g_{1,3}, g_{1,2} - g_{2,1})$. This is a convention which does not extend to dimension 4 or higher (it is a mere coincidence that the curl of a three-dimensional vector field is still a three-dimensional vector field). One can define the curl of $\vec{g} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ as being the field with $n(n-1)/2$ coordinates $g_{i,j} - g_{j,i}$ for $1 \le i < j \le n$ (note that the signs are not the same as in the above definition for three-dimensional vector fields). Or, one may find it more clear to think of the curl of $\vec{g}$ as the $n \times n$ anti-symmetric matrix of entries $g_{i,j} - g_{j,i}$ $(1 \le i, j \le n)$. Whatever the formal definition chosen, one must have in mind that $\operatorname{curl}\vec{g}$ has $n(n-1)/2$ components.

We say that a vector field $\vec{g} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is curl-free if $g_{i,j} = g_{j,i}$ $(1 \le i, j \le n)$.

It is well-known that a curl-free vector field is locally a gradient:

THEOREM 3.1. Let $\vec{g} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field of zero curl. Then, for any $x_0 \in \Omega$ there is a neighbourhood V of x_0 where $\vec{g} = \nabla\varphi$ for some $\varphi \in \mathcal{C}^2(V)$.

The divergence of an n-dimensional vector field $\vec{\jmath} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is defined as $\operatorname{div}\vec{\jmath} = \jmath_{1,1} + \jmath_{2,2} + \cdots + \jmath_{n,n}$.

The essential difference between the two-dimensional case and the case $n \ge 3$ is that in higher dimension there is no correspondence between the curl and the divergence of a vector field. As explained above, the curl of a vector field consists of $n(n-1)/2$ scalar functions, while the divergence is only one scalar function.

Thus, when one specifies that a certain vector field has zero curl, much more information is provided than for zero divergence. Consequently, Theorem 3.1 cannot be expressed in terms of divergence for dimension $n \ge 3$.

Very little is known about divergence-free vector fields in dimension $n \ge 4$. Some results are known in the literature for three-dimensional divergence-free vector fields.

THEOREM 3.2 ([9], Theorem 3.2). Let $\Omega \subset \mathbb{R}^3$ be a bounded open set satisfying the Lipschitz condition. Let $\vec{\jmath} : \Omega \to \mathbb{R}^3$ be a vector field whose components belong to $L^2(\Omega)$ such that $\operatorname{div}\vec{\jmath} = 0$ in Ω and

$$\int_{\Gamma_i} \langle \vec{\jmath}, \vec{n} \rangle = 0\,,\ \ i = 1, 2, \ldots, m\,,$$

where the integral should be understood in the sense of traces. $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are the boundaries of the m holes of Ω and $\vec{n}$ is the unitary vector normal to Γ_i. Then, there is a vector field $\vec{\varphi} \in H^1(\Omega; \mathbb{R}^3)$ such that $\vec{\jmath} = \operatorname{curl}\vec{\varphi}$, that is,

$$\begin{aligned} \jmath_1 &= \varphi_{2,3} - \varphi_{3,2} \\ \jmath_2 &= \varphi_{3,1} - \varphi_{1,3} \\ \jmath_3 &= \varphi_{1,2} - \varphi_{2,1} \end{aligned} \tag{3.1}$$

It seems impossible to generalize the representation $\vec{\jmath} = \operatorname{curl} \vec{\varphi}$ for dimensions higher than three. This is one motivation for looking for different representations of divergence-free vector fields. Another motivation is presented in the next section.

4. Drawing vector fields. This topic (computer graphics) may seem unrelated to the subject of the present paper. However, when trying to draw vector fields nicely (that is, in a mathematically meaningful manner), one gets a feeling about their structure.

Drawing a vector field is a challenging task for those who do not content themselves with a fuzzy collection of arrows. The best way to represent a vector field is by drawing its trajectories, alone or tagged with tangent arrows. But choosing the right distance between trajectory lines is not easy. One often gets zones with high density of lines, and other zones too empty. When the vector field has zero divergence, there is a physically sound way of representing the magnitude of the vectors with the aid of the distance between trajectories. Zones with high intensity correspond to small interline distance, while zones where the vector field has small magnitude correspond to large distance between trajectories.

In two dimensions, this effect is easy to achieve if we remember that the vector field $\vec{\jmath}$ is the rotated gradient of a scalar potential φ (see Theorem 2.2). After computing φ, it suffices to draw its level lines (for equally spaced level values). These lines are trajectories of $\vec{\jmath}$, and there is an obvious relation between the interline distance and the magnitude of the gradient of φ, which is equal to the magnitude of $\vec{\jmath}$.

Can we generalize the above procedure for a three-dimensional vector field having zero divergence ? The physical interpretation linking the intensity of the field to the density of drawn trajectories still holds. But the representation $\vec{\jmath} = \operatorname{curl} \vec{\varphi}$ does not help to choose and draw trajectory lines. We need a different type of representation. We want a function γ whose level lines be trajectories of $\vec{\jmath}$. This means that the “potential” γ should take values in $\mathbb{R}^2$. Thus, the level sets of each component of γ would be surfaces, and intersections of these surfaces would be trajectories of $\vec{\jmath}$.

Although there is no point in drawing vector fields in dimension $n > 3$, the procedure would be the same: a “potential” γ should be constructed (taking values in $\mathbb{R}^{n-1}$), points should be chosen in $\gamma(\Omega)$, and the corresponding level lines should be drawn.

As a side note, we mention that arbitrary vector fields (having divergence not necesseraly zero) can be drawn using a similar technique. In zones where the divergence is positive, additional flow lines should start in order to increase the line density (thus decreasing the interline distance). In zones where the divergence is negative, some flow lines should be ended in order to decrease the line density. One systematic way to do this is to approximate the divergence of the vector field by a linear combination of Dirac masses, with weights of equal absolute value. At each positive Dirac mass, a new flow line should be started; near each negative Dirac mass, a flow line should be ended.

5. Divergence-free vector fields in three dimensions. The above considerations lead to the following idea : given a three-dimensional vector field $\vec{\jmath}$ with zero divergence, we are looking for two scalar functions γ_1 and γ_2 such that, at each point $x \in \Omega$, the vector $\vec{\jmath}(x)$ be tangent to the corresponding level surfaces. This means that the gradients of γ_1 and of γ_2 at x should be orthogonal to $\vec{\jmath}(x)$.

As a consequence of the above considerations, it is natural to look for γ_1 and γ_2 such that

$$\vec{\jmath}(x) = \nabla\gamma_1(x) \wedge \nabla\gamma_2(x)\,,\ \forall\, x \in \Omega\,,$$

where $\wedge$ denotes the exterior product between two vectors in $\mathbb{R}^3$:

$$\vec{a} \wedge \vec{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Before discussing the existence or construction of the potential $\gamma : \Omega \to \mathbb{R}^2$, we shall make various remarks.

Note that it is a trivial matter to check that, given any two functions γ_1 and γ_2 in $\mathcal{C}^1(\Omega)$, the vector field $\nabla\gamma_1 \wedge \nabla\gamma_2$ has indeed zero divergence:

$$\begin{aligned} \operatorname{div}\,(\nabla\gamma_1 \wedge \nabla\gamma_2) = (\gamma_{1,2}\gamma_{2,3} - \gamma_{1,3}\gamma_{2,2})_{,1} + (\gamma_{1,3}\gamma_{2,1} - \gamma_{1,1}\gamma_{2,3})_{,2} + \\ + (\gamma_{1,1}\gamma_{2,2} - \gamma_{1,2}\gamma_{2,1})_{,3} = 0 \end{aligned}$$

Note also that the quantity $\nabla\gamma_1 \wedge \nabla\gamma_2$ depends on γ_1 and γ_2 in a non linear manner (unlike previous representations; for instance, curl $\vec{\varphi}$ depends linearly on $\vec{\varphi}$). This non linear character brings new difficulties, which show up when one tries to lower the regularity of the functions (suppose $\gamma_1, \gamma_2 \in H^1$, then $\nabla\gamma_1 \wedge \nabla\gamma_2 \in L^1$, not L^2), and also when one studies the problem of non uniqueness of the representation.

Divergence-free vector fields appear often in fluid mechanics textbooks (these fields are called "solenoidal"), mostly as curls of other vector fields (recall that in three dimensions any curl has zero divergence). The idea of the representation $\vec{\jmath} = \nabla\gamma_1 \wedge \nabla\gamma_2$ appears sometimes in a rather vague formulation. H. Lamb describes (in [4], Chapter VII, section 167) the structure of the solenoidal vector field given as the exterior product of the derivatives of the pressure and of the reciprocal of the density; he refers the result to V. Bjerkenes, Vid.-Selsk. Skrifter, Kristiania, 1918. In [5], Section 3.12, M. Narasimhan leaves the proof of the existence of this representation as an exercise for the reader to carry out (Problem 1).

It should be noted that the representation $\vec{\jmath} = \nabla\gamma_1 \wedge \nabla\gamma_2$ is related to the so-called Clebsch representation for velocity fields, see [4] and [6].

In [10] the author states that any solenoidal vector field $\vec{\jmath}$ can be represented locally as $\nabla\gamma_1 \wedge \nabla\gamma_2$. The proof presented is very intuitive and lacks mathematical rigour.

A purely geometric argument, based on differential forms, is given in [7]: the authors assert that any closed $n-1$ form can be written (locally) as the exterior product of $n-1$ 1-forms. The authors make the following remark: "the representation of $\vec{\jmath}$ in terms of $n-1$ arbitrary functions is the most economical one" (notations have been adapted). Unfortunately, the proof presented in [7] is wrong.

The authors of [6] make the following assertion: "This representation of a divergence-free vector field was introduced by Euler".

The paper [3] gives a thorough discussion of three-dimensional solenoidal vector fields from the point of view of differential geometry.

6. On the exterior product. This section recalls some notions from multi-linear algebra. For most readers these should be well-known, however we present them for the sake of completeness and to fix notations.

Let V be a real vector space of finite dimension n. Let $\Sigma(V)$ be the set of all multi-linear skew-symmetric applications

$$\omega : V \times V \times \cdots \times V = V^n \to \mathbb{R}$$

It is known that this set, together with the usual operations between applications, is a one-dimensional vector space. We shall call "volume form on V" to an arbitrary fixed non-zero element ω of $\Sigma(V)$. Note that $\omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = 0$ if and only if the n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent in V.

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}$ be $n-1$ elements of a vector space V endowed with a volume form ω. One defines then the exterior product of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}$ as the 1-form $\ell \in V'$ defined as

$$\ell(\vec{w}) = \omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}, \vec{w}) , \quad \forall\, \vec{w} \in V$$

Note that ℓ is zero if and only if the $n-1$ vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}$ are linearly dependent in V.

Provided V has also a scalar product, one can identify ℓ with an element $\vec{\ell} \in V$ characterized by

$$\langle \vec{\ell}, \vec{w} \rangle = \omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}, \vec{w}) , \quad \forall\, \vec{w} \in V$$

We shall use the notation $\vec{\ell} = \vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_{n-1}$.

When a scalar product is available, a typical choice of volume form ω is simply the determinant of the matrix of coordinates of its n arguments, computed in some orthonormal basis of V (an orientation must be chosen). In particular, in $\mathbb{R}^2$, the exterior product of a vector $\vec{v}$ (alone) is a vector of the same length and orthogonal to $\vec{v}$ (that is, $\vec{v}$ rotated at $90°$). In $\mathbb{R}^3$, we recover the usual exterior product between two vectors $\vec{v}_1$ and $\vec{v}_2$ (a vector $\vec{\ell}$ orthogonal to both $\vec{v}_1$ and $\vec{v}_2$ and having length equal to the area of the parallelogram defined by $\vec{v}_1$ and $\vec{v}_2$).

Taking into account the above remarks, one can see that the representation $\vec{\jmath} = \nabla\gamma_1 \wedge \nabla\gamma_2$ of three-dimensional solenoidal vector fields is a natural generalization of $\vec{\jmath} = R\nabla\varphi$ for two-dimensional div-free vector fields.

7. The main representation result. We begin by presenting a result on arbitrary vector fields (having divergence not necessarily zero).

Theorem 7.1. Let $\vec{v} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^k$, $k \geq 1$. Let $x_0 \in \Omega$ be a point such that $\vec{v}(x_0) \neq \vec{0}$. There exists a neighbourhood V of x_0 and scalar functions $\alpha \in \mathcal{C}^{k-1}(V)$ and $S_2, S_3, \ldots, S_n \in \mathcal{C}^k(V)$ such that

$$\vec{v}(x) = \alpha(x)\, \nabla S_2(x) \wedge \nabla S_3(x) \wedge \cdots \wedge \nabla S_n \,, \ \forall\, x \in V \,.$$

Proof. By a classical result of ordinary differential equations (rectification of a vector field, see [1], section 7, Theorem 1 and Corolary 10), there exists a diffeomorphism S of class $\mathcal{C}^k$ between a neighbourhood V of x_0 and an open subset of $\mathbb{R}^n$ of the form $I \times W$

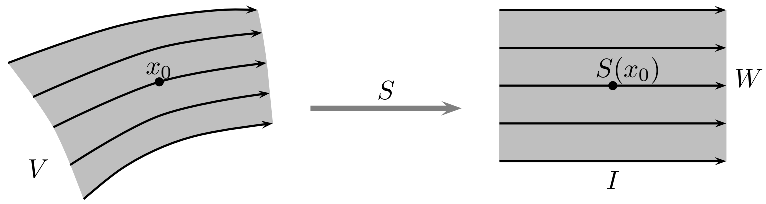

Fig. 1. Rectification of a smooth vector field

(where I is an open interval in $\mathbb{R}$ and W is an open subset of $\mathbb{R}^{n-1}$, see Figure 1) with the property

$$DS(x)\vec{v}(x) = \vec{e}_1 \,, \quad \forall\, x \in V\,.$$

In coordinate notation:

$$\frac{\partial S_i}{\partial x_j}\, v_j = \nabla S_i \cdot \vec{v} = \delta_{1i} \,, \quad i = 1, 2, \ldots, n\,.$$

We conclude that, for each $x \in V$, the $n{-}1$ vectors $\nabla S_2(x), \nabla S_3(x), \ldots, \nabla S_n(x)$ are orthogonal to $\vec{v}(x)$. As S is a diffeomorphism, we know that the vectors $\nabla S_i(x)$, $i = 1, 2, \ldots, n$, are linearly independent. Hence, $\vec{v}(x)$ is colinear with the non-zero vector $\nabla S_2(x) \wedge \nabla S_3(x) \wedge \cdots \wedge \nabla S_n(x)$, so there is a scalar (which we denote by $\alpha(x)$) such that

$$\vec{v}(x) = \alpha(x)\, \nabla S_2(x) \wedge \nabla S_3(x) \wedge \cdots \wedge \nabla S_n(x)\,, \ \forall\, x \in V. \tag{7.1}$$

The proof is almost completed, it remains only to prove that $\alpha \in \mathcal{C}^{k-1}(V)$. Consider the equality $\nabla S_1 \cdot \vec{v} = 1$ and replace $\vec{v}$ by (7.1) in order to obtain

$$1 = \nabla S_1 \cdot \big[\alpha\, \nabla S_2 \wedge \nabla S_3 \wedge \cdots \wedge \nabla S_n\big] = \alpha \,\det DS$$

We deduce that

$$\alpha = \frac{1}{\det DS}$$

which provides the desired regularity for a and concludes the proof. □

Lemma 7.2. Let $W = [a_2, b_2] \times [a_3, b_3] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^{n-1}$ and let $c \in \mathcal{C}^k(W)$, $k \geq 1$. Then, there exists an application $\Phi : W \to \mathbb{R}^{n-1}$ of class $\mathcal{C}^k$ such that $\det D\Phi = c$.

The above result says that it is possible to deform a domain W obeying to a prescribed area deformation c. It is easy to generalize the result for any convex domain W (and even for more general domains).

Proof of Lemma 7.2. We denote by y points in W and by y' points in $[a_3, b_3] \times \cdots \times [a_n, b_n]$; thus, $y = (y_2, y') \in W$.

There are many possible ways of constructing Φ; we choose here to deform W in the y_2 direction only. That is, we define $\Phi_i(y) = y_i$ for $i \geq 3$ and then build Φ_2 by integrating c in y_2:

$$\Phi_2(y_2, y') = \int_{a_2}^{y_2} c(s, y')\mathrm{d}s$$

It is easy to compute the derivatives of Φ:

$$\frac{\partial \Phi_i}{\partial y_j} = \delta_{ij} \quad \text{for } i \geq 3, j \geq 2$$

$$\frac{\partial \Phi_2}{\partial y_2} = c$$

$$\frac{\partial \Phi_2}{\partial y_j}(y_2, y') = \int_{a_2}^{y_2} \frac{\partial c}{\partial y_j}(s, y')\mathrm{d}s \quad \text{for } j \geq 3$$

and this concludes the proof. ☐

Now we turn our attention to divergence-free vector fields:

THEOREM 7.3. Let $\vec{\jmath} \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^k$ vector field having zero divergence ($k \geq 2$). Let $x_0 \in \Omega$ be a point such that $\vec{\jmath}(x_0) \neq \vec{0}$. Then there exists a neighbourhood V of x_0 and $n{-}1$ scalar functions $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \in \mathcal{C}^{k-1}(V)$ such that

$$\vec{\jmath}(x) = \nabla\gamma_1(x) \wedge \nabla\gamma_2(x) \wedge \cdots \wedge \nabla\gamma_{n-1}(x)\,, \ \forall\, x \in V \tag{7.2}$$

Proof. Consider the representation given in Theorem 7.1

$$\vec{\jmath}(x) = \alpha(x)\,\nabla S_2(x) \wedge \nabla S_3(x) \wedge \cdots \wedge \nabla S_n\,.$$

The idea of the proof is to show that, for divergence-free vector fields, it is possible to choose the functions $S_2, S_3, \ldots, S_n$ in such a way that the function α become equal to 1. One has:

$$\begin{aligned} \operatorname{div} \vec{\jmath} &= \operatorname{div}\big(\alpha\,\nabla S_2 \wedge \nabla S_3 \wedge \cdots \wedge \nabla S_n\big) = \\ &= \nabla\alpha \cdot \nabla S_2 \wedge \nabla S_3 \wedge \cdots \wedge \nabla S_n + \alpha \operatorname{div}\big(\nabla S_2 \wedge \nabla S_3 \wedge \cdots \wedge \nabla S_n\big) \end{aligned}$$

But the exterior product of any $n{-}1$ gradients has zero divergence, thus

$$\operatorname{div} \vec{\jmath} = \nabla\alpha \cdot \nabla S_2 \wedge \nabla S_3 \wedge \cdots \wedge \nabla S_n = \frac{1}{\alpha}\,\nabla\alpha \cdot \vec{\jmath}$$

(α is non-zero by construction). So, $\operatorname{div} \vec{\jmath} = 0$ implies $\nabla\alpha \perp \vec{\jmath}$ in each point of V. This means that the function α is constant along trajectories of $\vec{\jmath}$. Transporting this information through the diffeomorphism S, we conclude that the function $\alpha \circ S^{-1} \in \mathcal{C}^{k-1}(I \times W)$ is constant along I; it depends only on $y \in W$. We introduce the function $c \in \mathcal{C}^{k-1}(W)$ defined as $c(y) = \alpha(S^{-1}(\zeta, y))$ for every $y \in W$ and for arbitrary $\zeta \in I$. Now apply Lemma 2 (one can choose a smaller W which is the product of $n{-}1$ intervals) in order to obtain $\Phi : W \to \mathbb{R}^{n-1}$ of class $\mathcal{C}^{k-1}$ such that $\det D\Phi = c$. Denote by $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ the components of $\Phi \circ S$ and compute:

$$\begin{aligned} \nabla\gamma_1 \wedge \nabla\gamma_2 \wedge \cdots \wedge \nabla\gamma_{n-1} &= (\det D\Phi){\circ}S\ \nabla S_2 \wedge \nabla S_3 \wedge \cdots \wedge \nabla S_n \\ &= \alpha\,\nabla S_2 \wedge \nabla S_3 \wedge \cdots \wedge \nabla S_n = \vec{\jmath} \end{aligned}$$

The proof is complete. ☐

In the above proof, one can see the non-unique character of the representation (7.2). Let us denote by $U = \Phi(W)$ the domain where the function $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ takes values. It is an open set (as the vectors $\nabla\gamma_1(x_0), \nabla\gamma_2(x_0), \ldots, \nabla\gamma_{n-1}(x_0)$ are linearly independent – one can choose a smaller V if necessary). If $\Psi : U \to \mathbb{R}^{n-1}$ is a $\mathcal{C}^k$ application such that $\det D\Psi$ is constantly equal to 1 (a volume-preserving map) then the components of $\Psi \circ \Phi \circ S$ still enjoy the same property (the exterior product of their gradients equals $\vec{\jmath}$). That is, we can characterize the non-uniqueness of γ as follows: one may compose γ with an arbitrary volume-preserving application in $\mathbb{R}^{n-1}$.

8. Applications. Representing a divergence-free vector field as exterior product of gradients may have interesting applications for the study of partial differential equations.

In the two-dimensional case, consider the problem

$$\begin{cases} -\mathrm{div}\,(a\nabla u) = 0 & \text{in } \Omega \\ a\nabla u\,\vec{n} = g & \text{on } \Gamma_N \\ u = u_0 & \text{on } \Gamma_D \end{cases} \tag{8.1}$$

In the above, Ω is a domain in $\mathbb{R}^2$ whose boundary is partitioned into a Neumann part and a Dirichlet part, $\partial\Omega = \Gamma_N \cup \Gamma_D$, and $a \in L^\infty(\Omega)$, $g \in L^2(\Gamma_N)$, $u_0 \in L^2(\Gamma_D)$. Suppose Ω is simply connected.

By applying the representation result in Theorem 2.2, one can re-write the state equation $-\mathrm{div}\,(a\nabla u) = 0$ as $a\nabla u = -R\nabla\gamma$ for some $\gamma \in H^1(\Omega)$. The Neumann condition $a\nabla u\,\vec{n} = g$ can now be written as $\nabla\gamma\,\vec{\tau}$ in γ_N, where $\vec{\tau}$ is the unit vector tangent to Γ_N. By integrating this equality along Γ_N one obtains a Dirichlet condition on γ: $\gamma = G$ on Γ_N. Here, G is a primitive of g along Γ_N (in the sense that $\nabla G\,\vec{\tau} = g$) and contains an arbitrary additive constant. Thus, problem (8.1) becomes : find u and γ in $H^1(\Omega)$ such that

$$\begin{cases} a\nabla u = -R\nabla\gamma & \text{in } \Omega \\ \gamma = G + c & \text{on } \Gamma_N \\ u = u_0 & \text{on } \Gamma_D \end{cases}$$

Note that γ contains the additive unknown parameter c.

The above problem can be discretized by using a finite element space V_h which approximates $H^1(\Omega)$. One minimizes

$$\|a\nabla u_h + R\nabla\gamma_h\|_{L^2(\Omega)}$$

in the unknowns $u_h, \gamma_h \in V_h$, $c \in \mathbb{R}$, subject to the constraints $\gamma_h = G + c$ on Γ_N and $u_h = u_0$ on Γ_D.

It is not easy to generalize this technique to three-dimensional problems. The state equation $-\mathrm{div}\,(a\nabla u) = 0$ does not imply the representation

$$a\nabla u = \nabla\gamma_1 \wedge \nabla\gamma_2 \tag{8.2}$$

because of the lack of regularity of $a\nabla u$. Besides, it is difficult to guarantee that $a\nabla u$ does not vanish in Ω. If we admit that (8.2) holds globally in Ω, then the Neumann condition $a\nabla u\,\vec{n} = g$ becomes $(\nabla\gamma_1 \wedge \nabla\gamma_2)\cdot\vec{n} = g$ on Γ_N. This is equivalent to imposing a Dirichlet condition on γ_1 and γ_2 up to an area-preserving application θ defined on Γ_N. Thus, the original partial differential problem can be re-written as follows: given a, u_0

and G, find u, γ and θ such that

$$\begin{cases} a\nabla u = -\nabla\gamma_1 \wedge \nabla\gamma_2 & \text{in } \Omega \\ \gamma = \theta \circ G & \text{on } \Gamma_N \\ \theta \text{ preserves area} & \\ u = u_0 & \text{on } \Gamma_D \end{cases}$$

9. Future work and challenges. The main limitation of the result stated in Theorem 7.3 is the non-degeneracy hypothesis $\vec{\jmath}(x_0) \neq \vec{0}$. Unfortunately, it seems that this hypothesis cannot be eliminated – see the counterexample presented in [6]. The necessity of this hypothesis disourages also any attempt of proving a global representation result.

Another possible generalization of Theorem 7.3 is to lower the regularity of the functions involved. For instance, it is not clear what kind of representation one should expect for vector fields $\vec{\jmath}$ whose components are in L^1 or in L^2. Again, the hypothesis $\vec{\jmath}(x_0) \neq \vec{0}$ looks disouraging since such a vector field is defined almost everywhere. A first step could be to state and prove some estimates on the potentials γ_i, in an appropriate norm.

Finally, the numerical treatment of partial differential equations, as described in the previous section, is an important challenge for the future.

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