Boundary value problems of second and fourth order in infinite intervals

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Abstract

Concerning second order problems, we study the existence of positive solutions for the differential equation

$$u'' = a(x)u - g\,(u)\,,$$

with $u'(0) = u(+\infty) = 0$, where a is a positive function, g satisfies some growth hypotheses (in particular the bounded case). We also deal with the problem in which the differential equation has an extra dissipative term on the left-hand side. The main motivation is to check that some well known second order results concerning the existence of homoclinics for the autonomous case (where a is constant) extend to the nonautonomous equation.

For fourth order problems, we study the existence of solution for the bvp

$$\begin{cases} u^{(4)} - c\,u'' + a(x)\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0. \end{cases}$$

Key words: Second order, Fourth order; Non-autonomous equation; Variational methods; Unbounded domains; Positive solution;

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1 Introduction

The study of existence of positive homoclinics of the ordinary differential equation

$$u''(x) = a(x)u(x) - g\,(u\,(x)) \tag{1}$$

where $g(0) = 0$ is partially motivated by the search of "standing wave" solutions $\Phi(t, x) = e^{i\omega t} u(x)$ of the Klein-Gordon type equation

$$\Phi_{tt} - \Delta\Phi + a^2\Phi = f(\Phi),$$

where $\Phi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a complex function, $a \in \mathbb{R}$ and $f(\rho\, e^{i\theta}) = f(\rho)e^{i\theta}$. In this process we are lead to the equation

$$-\Delta u + \left(a^2 - \omega^2\right) u = f(u). \tag{2}$$

The corresponding Euler-Lagrange functional is

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \left(a^2 - \omega^2\right) u^2 - 2F(u)\right) dx,$$

where $F(u) = \int_0^u f(s)\, ds$, and for this integral to be well-defined, $|u|$ needs to vanish at $+\infty$.

In this paper we study this problem under conditions different from those considered in [7], where we proved the existence of a positive solution for the problem

$$u'' = a(x)u - g(u), \quad u'(0) = u(+\infty) = 0, \tag{3}$$

where $g(u)$ satisfies the hypotheses

(H_1) setting $G(u) = \int_0^u g(s)\, ds$, there exists $q > 2$ such that

$$0 < q\, G(u) \leq u\, g(u), \quad \forall u \in]0, +\infty[,$$

(H_2) $g(u) = o(u)$ at $u = 0$,

and $a(x)$ is such that

(A_1) there exists $0 < a < A$ such that $0 < a(x) \leq A\ \forall x \geq 0$ and $\lim_{x\to+\infty} a(x) = a$,

(A_2) ${J_A}^* < 2{J_a}^*$.

Here ${J_a}^*$ is the value of the Euler-Lagrange functional associated with the autonomous problem

$$u''(x) = au(x) - g(u(x)), \quad u'(0) = u(+\infty) = 0,$$

computed at its nontrivial solution u_a, that is

$${J_a}^* = \int_0^\infty \left({u_a}'(x)^2 + a\, u_a(x)^2 - 2G(u_a)\right) dx = 2\int_0^\xi \sqrt{a\, u^2 - 2G(u)}\, du,$$

where ξ is the smallest positive root of $a\, u^2 - 2G(u)$. In this paper we prove existence of solution for a different type of nonlinearity, for the equation with a linear term in u' and we also prove existence of homoclinics in somehow similar fourth order problems.

Problems of type (3) have been studied in the last two decades, especially in the case where $g(u)$ is a superlinear power. P. Korman and A. Lazer gave a variational approach for the cases $g(u) = u^3$ in [10] and $g(u) = u^p$, where $p > 1$, in [11]. In these papers, the coefficient $a(x)$ is increasing in $[0, +\infty[$. We shall solve a sequence of boundary value problems in $[0, T]$ and if we consider an appropriate sequence of T's tending to $+\infty$, a nontrivial solution of the infinite interval problem will be found as the limit of the corresponding solutions u_T. M. Grossinho, F. Minhós and S. Tersian also gave a similar variational approach for this problem in [9], but working with two simultaneous powers in the nonlinear term. Related with these problems we also mention the papers [2], [8], [12], [14] and [15].

In Section 2 we consider a special case of the problem (3) and prove the existence of a positive solution with a "shooting" technique. Section 3 will be dedicated to the same problem, with a bounded nonlinear term $g(u)$, that of course does not satisfy condition (H_1). The following two sections deal with analogous results for the equation with a linear term in u'. Sections 6 and 7 generalize to fourth order problems some of the variational techniques used for second order problems.

2 A second order problem with $a(x)$ nondecreasing

So we consider the problem

$$\begin{cases} u'' = a(x)u - u^3 = u\left(a(x) - u^2\right) \\ u'(0) = 0, \quad u(+\infty) = 0. \end{cases} \tag{4}$$

Lemma 2.1. *Let $a(x)$ be a positive and nondecreasing function defined in $[0, +\infty[$. If $u(x)$ is a solution of (4), the energy function $E(x) \equiv \frac{u'^2}{2} + \frac{u^4}{4} - \frac{a(x)u^2}{2}$ is decreasing in $\mathbb{R}^+$.*

Proof. Let $x_1 < x_2 \in \mathbb{R}^+$. Using the Stieltjes integral, we have

$$\begin{aligned} E(x_2) - E(x_1) &= \int_{x_1}^{x_2} dE = \int_{x_1}^{x_2} \left(\frac{u'^2}{2} + \frac{u^4}{4}\right)' - \left[a(x)\frac{u^2}{2}\right]_{x_1}^{x_2} = \\ &= \int_{x_1}^{x_2} a(x)u\,u'\,dx - \left[a(x)\frac{u^2}{2}\right]_{x_1}^{x_2} = -\int_{x_1}^{x_2} \frac{u^2}{2}\,da(x) \leq 0. \end{aligned}$$

□

Positive solutions solutions of $u''(x) = a(x)u(x) - u(x)^3 = u(x)\left(a(x) - u(x)^2\right)$ are concave if $u(x) > \sqrt{a(x)}$ and convex if $u(x) < \sqrt{a(x)}$, therefore the graph of the solution of

$$\begin{cases} u''(x) = a(x)u(x) - u(x)^3 \\ u(0) = L, \quad u'(0) = 0 \end{cases} \tag{5}$$

where $L > \sqrt{a(0)}$ crosses the graph of $\sqrt{a(x)}$ at $x = c_L$ for some $c_L > 0$, and we may suppose that c_L is the minimum value with this property.

Proposition 2.2. *As L tends to $+\infty$, c_L tends to 0.*

Proof. Let us first prove the result for $a(x)$ bounded. Let d_L be the minimum value such that $u_L(d_L) = \frac{L}{2}$. Suppose towards a contradiction that $d_L \not\to 0$. This means that there exists sequence $L_n \to +\infty$ such that $d_{L_n} > k$ for some constant $k > 0$. Let $p = \frac{\pi}{2k} > \frac{\pi}{2d_{L_n}}$.

Since $a(x)$ is bounded, for n large enough we have $a(x) - u_{L_n}^2(x) \leq -p^2$ for $x \in [0, d_{L_n}]$, so the unique solution v of the initial value problem

$$\begin{cases} v''(x) = -p^2 v \\ v(0) = L, \quad v'(0) = 0 \end{cases} \tag{6}$$

is such that $v(x) \geq u_{L_n}(x)$ in the interval $[0, d_{L_n}]$. But $v(x) = \cos(p\,x)$ vanishes at $x = \frac{\pi}{2p} = k < d_{L_n}$, which contradicts $v(x) \geq u_{L_n}(x)$. Consequently we have $d_L \to 0$ and a simple geometric argument implies that $c_L \leq 2d_L$, so we conclude that $c_L \to 0$.

In case $a(x)$ is unbounded, consider the bounded auxiliar function

$$\bar{a}(x) = \begin{cases} a(x), & x \leq 1 \\ a(1), & x > 1. \end{cases}$$

Applying the result obtained for bounded functions, we have that for $L > L_0$ large enough we have $c_L < 1$ and since the result only depends on the values of x smaller than c_L, the result holds for the unbounded function $a(x)$. □

Corollary 2.3. *As L tends to $+\infty$, ${u_L}'(c_L) \to -\infty$.*

Proposition 2.4. *For $L > \sqrt{a(0)}$ large enough, the solution of* (5) *has at least one zero.*

Proof. For simplicity, let us denote c_L by c. Given $L^* > \sqrt{a(0)}$ large, let c^* be the first value such that the graph of the solution of (5) with $L = L^*$ crosses the graph of $\sqrt{a(x)}$. Taking a sufficiently large $L > L^*$, the corresponding solution u_L of (5) satisfies $u_L(c) = \sqrt{a(c)}$, for some $c < c^*$. Suppose towards a contradiction that u_L does not vanish in $[0, c^*]$. Then, there exists $\hat{c} \in [c, c^*]$ such that ${u_L}'(\hat{c}) = -\frac{\sqrt{a(c)}}{c^* - c}$, which is the slope of the line connecting $(c, \sqrt{a(c)})$ and $(c^*, 0)$, and we have

$$-\frac{\sqrt{a(c)}}{c^* - c} - u'(c) = \int_c^{\hat{c}} u''(x)\,dx \leq c^* \sqrt{a(c^*)}^3.$$

Taking in consideration last corollary, we have a contradiction. □

Proposition 2.5. *Consider the initial value problem* (5) *with $L > \sqrt{a(0)}$. If its solution u_L is positive and does not have a local minimum, then $u_L(+\infty) = 0$.*

Proof. It is obvious that the graph of u_L crosses the graph of $\sqrt{a(x)}$ with negative derivative and since the derivative does not vanish again and u_L is positive, we must have $u_L'(+\infty) = 0$ and therefore $u_L(+\infty) = k > 0$. If $k \neq 0$ then

$$u_L''(+\infty) = u_L(+\infty)\left(a(+\infty) - u_L^2(+\infty)\right) > 0$$

and therefore there would exist $c \in \mathbb{R}$ such that $u_L'(c) = 0$, which is a contradiction. □

Proposition 2.6. *If $0 < L < \sqrt{2a(0)}$ then the solution u_L of* (5) *is positive in $\mathbb{R}^+$ and attains a positive minimum m for some $x_m \geq 0$.*

Proof. Since $E(0) = \frac{L^2}{2}\left(\frac{L^2}{2} - a(0)\right) < 0$ we have $E(x) < 0$ for every $x > 0$. If there exists $x_0 > 0$ such that $u_0(x_0) = 0$, then $E(x_0) = \frac{u'(x_0)^2}{2} \geq 0$, which is a contradiction.

If u_L does not attain a positive minimum, then $u_L(+\infty) = 0$ and $u'_L(+\infty) = 0$, and therefore $E(+\infty) = 0$, which is again a contradiction. □

Proposition 2.7. *If the solution u_L of* (5) *attains a positive minimum m for some $x_m \geq 0$, then u_L is positive for $x > x_m$.*

Proof. We can conclude as in the proposition above, since $E(x_m) = \frac{m^2}{2}\left(\frac{m^2}{2} - a(x_m)\right) < 0$. □

Theorem 2.8. *Let $a(x)$ be a positive, nondecreasing function. Then the problem* (4) *has at least one positive solution.*

Proof. We use a connectedness argument appearing in the paper of H. Berestycki, P. Lions and L. Peletier [5]. Consider the following subsets of $\mathbb{R}^+$

$$A = \left\{L > \sqrt{a(0)} \colon u_L > 0 \text{ and } u_L \text{ has a positive minimum}\right\},$$

$$B = \left\{L > \sqrt{a(0)} \colon u_L(x_0) = 0 \text{ for some } x_0 > 0\right\}.$$

Both sets are nonempty, obviously disjoint, and, by the continuous dependence of the parameters, open in $\mathbb{R}$. Let $u_0 = \inf B$. Since u_0 does not belong to A or B, we must conclude that the solution of problem (5) with $L = u_0$ is positive and tends to 0 at ∞. □

3 2^{nd} order problem with bounded nonlinearity and a constant in a neighborhood of ∞

In this section we prove the existence of a positive solution of

$$u'' = a(x)u - g(u), \quad u'(0) = u(+\infty) = 0. \tag{7}$$

We will consider $a(x)$ satisfying (A_2) and a stronger hypothesis than (A_1): assume that

(A'_1) $0 < a(x) \leq A$ $\forall x \geq 0$ and there exists $x_0 > 0$ such that $a(x) \equiv a$ $\forall x \geq x_0$.

The function $g \in C([0, \infty), [0, \infty))$ will be a bounded function that satisfies (H_2), and in addition:

(H_3) The function $f(u) := a\,u^2 - 2G(u)$ has only one negative minimum attained at $u = \eta$, and hence only one zero, say ξ in $(0, \eta)$.

(H_4) $A\,u^2 - 2G(u) = 0$ has also a negative minimum.

(H_5) There exists $\alpha > 0$ such that $|f(u) - f(v)| \geq \alpha|u - v|$ $\forall u$, v in a neighborhood of ξ,

(H_6) $\int_0^\eta \frac{du}{\sqrt{f(u) - f(\eta)}} = +\infty$.

Condition (H_3) is not absolutely necessary since we could reach the same conclusions in a more general context, but we included it for simplicity of notations and calculations. Note that (H_1) does not hold. Since we look for positive solutions, *in what follows we set* $g(u) = 0$ *for* $u < 0$.

Remark 3.1. Before we deal with the problem above, let us consider a slight variation. Suppose that instead of (A_2), $a(x)$ satisfies

$$\sqrt{A}\tanh(\sqrt{A}x_0) < \sqrt{a} \tag{8}$$

If we consider the initial value problem

$$u'' = a(x)u - g(u), \quad u(0) = \zeta, \quad u'(0) = 0 \tag{9}$$

it is obvious that for ζ large the solutions must be convex and therefore larger than ζ for every $x > 0$. If $\zeta > 0$ is small enough, then by (H_2), $(u(x,\zeta),\, u'(x,\zeta)) = \zeta(v(x),\, v'(x)) + \mathrm{o}(\zeta)$ uniformly in $[0, x_0]$, where v is the solution of the linear problem

$$v'' = a(x)v, \quad v(0) = 1, \quad v'(0) = 0.$$

Since $z(x) = \frac{v'(x)}{v(x)}$ satisfies $z' + z^2 = a(x)$, an elementary comparison theorem shows that $z(x_0) \leq \sqrt{A}\tanh(\sqrt{A}x_0)$.

Now the positive homoclinic at the origin for the autonomous equation $u'' = a\,u - g(u)$ has an image curve in the (u, u')-plane whose slope at the origin in the half-plane $u' > 0$ is precisely $\sqrt{a}$. Hence by (8), for for ζ sufficiently small, $(u(x_0,\zeta), u'(x_0,\zeta))$ lies “inside” the homoclinic. Since for ζ large $(u(x_0,\zeta), u'(x_0,\zeta))$ is obviously “outside” the homoclinic, a connectedness argument based on the Peano phenomenon (see e. g. [13]) allows us to conclude that there exists a value ζ_0 such that $(u(x_0,\zeta), u'(x_0,\zeta))$ is a point of the homoclinic solution of the autonomous problem. Since for $x \geq x_0$ we have $a(x) = a$, there exists a positive solution of (7).

Note that estimate (8) works well only if x_0 is small.

Consider now the problem with condition $(A_1')-(A_2)-(H_2)-(H_3)-(H_4)-(H_5)-(H_6)$. Proceeding as in [7], we easily see that the boundary value problems

$$\begin{cases} u'' = a(x)\,u - g(u) \\ u'(0) = 0, \quad u(T) = 0 \end{cases} \tag{10}$$

have a positive solution u_T, because the associated modified Euler-Lagrange functionals

$$J_T(u) = \int_0^T \left(u'^2 + a(x)\,u^2 - 2G(u_+)\right)\,dx,$$

have a mountain-pass geometry relative to the local minimum $u = 0$ in the space $H_T^* \equiv \left\{H^1[0,T] \colon u(T) = 0\right\}$. The mountain-pass critical values $c_T = J_T(u_T)$ are positive, decreasing in T and therefore, for $T > 1$, we have $c_T \leq c_1$. The solution u_T must attain a maximum at a point where ${u_T}'' \leq 0$ so $\|u_T\|_\infty$ is uniformly bounded in T. The differential equation allows us to conclude that $\|{u_T}''\|_\infty$ is bounded too and consequently the same is true for $\|{u_T}'\|_\infty$.

Proposition 3.2. *${u_T}'(T) \to 0$ as $T \to +\infty$.*

Proof. If ${u_T}'(T) \not\to 0$, then there exists a sequence of T's tending to $+\infty$ such that ${u_T}'(T) \to d$ for some constant $d < 0$.

If we multiply the differential equation with $u = u_T$ by ${u_T}'$ and integrate, we get

$$ {u_T}'^2 = a\,{u_T}^2 - 2G(u_T) + K_T, \quad \forall\, x \geq x_0 \tag{11} $$

where K_T is a constant.

Consider the autonomous initial value problem

$$ \begin{cases} u'' = a\,u - g(u) \\ u'(0) = d, \quad u(0) = 0. \end{cases} \tag{12} $$

Recall that ξ is the smallest positive value such that $2G(u) - a\,u^2 = 0$ and η is the maximizer of $2G(u) - a\,u^2$. Let $d_\eta < 0$ be the value of the derivative when $u = 0$ for the trajectory that goes to $(\eta, 0)$ as $x \to -\infty$. This trajectory exists by virtue of (H_6) and is given by $u' < 0$ and

$$ u'^2 = au^2 - 2G(u) + d_\eta^2 $$

where

$$ d_\eta^2 = -a\eta^2 + 2G(\eta). $$

We will divide the proof into three cases, $d_\eta < d < 0$, $d = d_\eta$ and $d < d_\eta$:

(1) If $d_\eta < d < 0$, the correspondent solution u of the autonomous problem (12) has a largest negative zero $-c$ and $u'(-c) > 0$. For T large enough we have $T - c > x_0$, so the solutions u_T coincide with the autonomous solutions and consequently, since we have uniform convergence in compact intervals, we would have a contradiction with the positivity of the solutions u_T.

(2) If $d = d_\eta$, we will distinguish two cases: ${u_T}'(T) \to d_\eta$ from above and ${u_T}'(T) \to d_\eta$ from below. In the first situation, if there exists a local maximum point $x_T \geq x_0$ (let $u_T(x_T) \equiv \eta_T$) then $\eta_T < \eta$ and $f(\eta_T) + K_T = 0$, which implies that $\eta_T \to \eta$ as $T \to \infty$. We have

$$ \begin{aligned} J_T(u_T) &= \int_0^T \left[{u_T}'^2 + a(x)\,{u_T}^2 - 2G(u_T)\right]\,dx = \\ &= \int_0^{x_0} \left[{u_T}'^2 + a(x)\,{u_T}^2 - 2G(u_T)\right]\,dx + \int_{x_0}^{x_T} \left[2\left(a\,{u_T}^2 - 2G(u_T)\right) + K_T\right]\,dx + \\ &\quad + \int_{x_T}^{T} \left[2\left(a\,{u_T}^2 - 2G(u_T)\right) + K_T\right]\,dx \end{aligned} \tag{13} $$

The first integral is obviously uniformly bounded and making a change of variable we get for the third integral

$$ \begin{aligned} &\int_{x_T}^{T} \left[2\left(a\,{u_T}^2 - 2G(u_T)\right) + K_T\right]\,dx = \\ &= \int_0^{\eta_T} \left[\sqrt{f(u) - f(\eta_T)} + \frac{f(u)}{\sqrt{f(u) - f(\eta_T)}}\right]\,du. \end{aligned} \tag{14} $$

The first part of the integral is obviosly bounded and using Fatou's Lemma and (H_6), we have

$$+\infty = \int_{\eta-\delta}^{\eta} \frac{du}{\sqrt{f(u)-f(\eta)}} \leq \liminf \int_{\eta-\delta}^{\eta_T} \frac{du}{\sqrt{f(u)-f(\eta_T)}},$$

where $\delta > 0$ is such that $f(u) < 0$ for $u \in [\eta-\delta, \eta+\delta]$. It is easy to see that this implies that the second part of integral tends to $-\infty$ and consequently (14) also tends to $-\infty$. For the second integral in (13), we have analogously

$$\int_{x_0}^{x_T} \left[2\left(a\,{u_T}^2 - 2G(u_T)\right) + K_T\right]\,dx =$$
$$= \int_{u_T(x_0)}^{\eta_T} \left[\sqrt{a\,u^2 - 2G(u) + K_T} + \frac{a\,u^2 - 2G(u)}{\sqrt{a\,u^2 - 2G(u) + K_T}}\right]\,du,$$

and if $u_T(x_0)$ does not tend to η, we also have this integral tending to $-\infty$ (otherwise it is bounded). This implies that $J_T(u_T)$ tends to $-\infty$, which contradicts the fact that the mountain pass critical level is positive. Consider now the case where the solution u_T is decreasing for every $x \geq x_0$. In this situation we have

$$J_T(u_T) = \int_0^{x_0} \left[{u_T}'^2 + a(x)\,{u_T}^2 - 2G(u_T)\right]\,dx + \int_{x_0}^{T} \left[2\left(a\,{u_T}^2 - 2G(u_T)\right) + K_T\right]\,dx,$$

where $K_T \to 2G(\eta) - a\,\eta^2$. Setting $u_T(x_0) = \eta_T$, we have

$$\int_{x_0}^{T} \left[2\left(a\,{u_T}^2 - 2G(u_T)\right) + K_T\right]\,dx =$$
$$= \int_0^{\eta_T} \left[\sqrt{a\,u^2 - 2G(u) + K_T} + \frac{a\,u^2 - 2G(u)}{\sqrt{a(x)\,u^2 - 2G(u) + K_T}}\right]\,du,$$

and since we must have $\eta_T \to \eta$ ($T - x_0 \to \infty$ implies it), we have a contradiction of the same type as above.

The case where ${u_T}'(T) \to d_\eta$ from below can also be treated in a similar way, since we also must have $K_T \to 2G(\eta) - a\,\eta^2$. Setting $u_T(x_0) = \eta_T$, it follows that $\eta_T \to \eta$ and therefore we would again reach the contradiction $J_T(u_T) \to -\infty$.

(3) If $d < d_\eta$, the correspondent solution w of the autonomous problem satisfies

$$w'^2 = aw^2 - 2G(w) + d^2.$$

This shows that $w'(x) < -\sqrt{d^2 - d_\eta^2}$ for all $x < 0$, and hence w is unbounded above. Again by uniform convergence in compact intervals, u_T would take arbitrarily large values for T sufficiently large. This is a contradiction with the uniform boundness of u_T.

We can therefore conclude that ${u_T}'(T) \to 0$.

□

Corollary 3.3. *Setting l_T as the largest maximizer of u_T, we have $T - l_T \to +\infty$.*

In the following, let $J_T(u)\big|_{[m,n]} = \int_m^n \left[u'^2 + a(x)\, u^2 - 2G(u_+)\right]\, dx$. In order to show that the limit of the solutions u_T cannot be the trivial solution, we need the following

Proposition 3.4. *There exists a constant $k > 0$ such that $u_T(0) > k$ for all $T > 1$.*

Proof. Suppose towards a contradiction that there exists a sequence of $T's$ tending to $+\infty$ such that $u_T(0) \to 0$. Then $|u_T(x)| + |u_T'(x)| \to 0$ uniformly in $[0, x_0]$ as $T \to \infty$. Since for u small $G(u) = o(u^2)$, we have $J_T(u_T)\big|_{[0,x_0]} \to 0$.

Since $J_T(u_T)$ is bounded away from zero, there exists a maximizer $x_T > x_0$ (otherwise we easily would show that $J_T(u_T)$ becomes arbitrarily small). It is obvious that x_T tends to $+\infty$ with T.

Now, setting $\xi_T = u_T(x_T)$, we compute

$$J_T(u_T)\big|_{[x_0,x_T]} = \int_{u(x_0)}^{\xi_T} \left(2\sqrt{a\, u^2 - 2G(u) + K_T} - \frac{K_T}{\sqrt{a\, u^2 - 2G(u) + K_T}} \right) du,$$

with $K_T = 2G(\xi_T) - a\, \xi_T^2$, and because of Proposition 3.2, we have $\xi_T \to \xi$ and $K_T \to 0$. For simplicity we can write the second integral in the simpler form

$$\int_{u(x_0)}^{\xi_T} \frac{K_T}{\sqrt{f(u) + K_T}}\, du = \int_{u(x_0)}^{\xi} \frac{K_T}{\sqrt{f(u) + K_T}}\, du + \int_{\xi}^{\xi_T} \frac{K_T}{\sqrt{f(u) - f(\xi_T)}}\, du.$$

Since $f(u) \geq 0$ for $u \in [0, \xi]$, the first integral is smaller than $\xi\sqrt{K_T}$. The second integral has a singularity at $u = \xi_T$, but using (H_5) we easily check that there exists a constant $k > 0$ such that

$$\int_{\xi}^{\xi_T} \frac{K_T}{\sqrt{f(u) - f(\xi_T)}}\, du \leq \int_{\xi}^{\xi_T} \frac{k\, K_T}{\sqrt{\xi_T - u}}\, du,$$

it follows that this integral tends to zero as well. This implies that

$$\lim_{T\to\infty} J_T(u_T)\big|_{[x_0,x_T]} = J_a^*.$$

The same computations are valid for the integral

$$J_T(u_T)\big|_{[x_T,T]} = \int_0^{\xi_T} \left(2\sqrt{a\, u^2 - 2G(u) + K_T} - \frac{K_T}{\sqrt{a\, u^2 - 2G(u) + K_T}} \right) du.$$

Since

$$J_T(u_T) = J_T(u_T)\big|_{[0,x_0]} + J_T(u_T)\big|_{[x_0,x_T]} + J_T(u_T)\big|_{[x_T,T]}$$

and as we have seen

$$\lim_{T\to\infty} J_T(u_T)\big|_{[0,x_0]} = 0,$$

$$\lim_{T\to\infty} J_T(u_T)\big|_{[x_0,x_T]} = \lim_{T\to\infty} J_T(u_T)\big|_{[x_T,T]} = J_a^*,$$

it follows that:

$$\lim_{T\to\infty} J_T(u_T) = 2J_a^*. \tag{15}$$

Let $J_{A,T}(u) = \int_0^T u'^2 + A\, u^2 - 2G(u_+)\, dx$.
<u>*Claim*</u> We have

$$J_T(u_T) \leq J_{A,T}(z_T),$$

where z_T is a solution to

$$\begin{cases} z'' = A\, z - g(z) \\ z'(0) = 0, \quad z(T) = 0, \quad z > 0 \quad \text{in } [0, T). \end{cases} \tag{16}$$

Proof of Claim Let $\alpha > 0$ be such that $A\, \alpha^2 - 2G(\alpha) < 0$. Consider the function

$$\overline{u} = \begin{cases} \alpha, & 0 \leq x \leq L \\ \alpha(L + 1 - x), & L \leq x \leq L + 1 \\ 0, & x \geq L + 1 \end{cases} \tag{17}$$

We have $J_{A,T}(\overline{u}) < 0$. It is obvious that for all $u \in H_T^*$ we have

$$J_T(u) \leq J_{A,T}(u),$$

so $J_T(\overline{u})$ is also negative. Defining $\Gamma_T = \{\gamma(\tau) \colon [0,1] \to H_T^* \colon \gamma(0) = 0, \gamma(1) = \overline{u}\}$, we may assume that

$$J_T(u_T) = \inf_{\gamma \in \Gamma_T} \max_{\tau \in [0,1]} J_T\left(\gamma(\tau)\right) \quad \text{and} \quad J_{A,T}(z_T) = \inf_{\gamma \in \Gamma_T} \max_{\tau \in [0,1]} J_{A,T}\left(\gamma(\tau)\right).$$

For a given $\gamma \in \Gamma_T$, we obviously have

$$\max_{\tau \in [0,1]} J_T\left(\gamma(\tau)\right) \leq \max_{\tau \in [0,1]} J_{A,T}\left(\gamma(\tau)\right)$$

and taking the infimum of both sides of the inequality, the claim follows.
By arguments already used in the proof, we easily see that this solution z_T is given by

$$z_T'^2 = A\, z_T^2 - 2G(z_T) + d_T^2$$

where $d_T = z_T'(T) \to 0$ as $T \to \infty$. Therefore $z_T(0) \to \bar{\xi}$ as $T \to \infty$, where $\bar{\xi}$ is the smallest positive root of $Au^2 - 2G(u)$. We conclude that

$$\lim_{T \to \infty} J_T(u_T) \leq J_A^*,$$

contradicting (15) and (A_2). □

Theorem 3.5. *Let a and g satisfy* $(A_1') - (A_2) - (H_3) - (H_4) - (H_5) - (H_6)$. *Then the problem* (7) *has at least one positive solution.*

Proof. Applying the classical diagonal method, we know that there exists a sequence of T's and $u \in C^2[0, +\infty)$ such that $u_T \to u$ C^1-uniformly in compact intervals. Applying the arguments of the previous proposition, if there exists a maximizer $x_T > x_0$ of u_T, then these maximizers must be bounded from above and we must have $u_T(x_T) \to \xi$. It follows that $u \to 0$ as $T \to +\infty$ and consequently u must be a branch of the well known homoclinic solution u_a of the autonomous problem for $x \geq x_0$. Since $[0, x_0]$ is a compact interval, we conclude that u must be a solution of (7). □

4 The second order autonomous problem with a dissipative term

In this section we prove the existence of a positive nonincreasing solution of the autonomous problem

$$\begin{cases} u'' + c\,u' = f(u) \\ u'(0) = 0, \quad u(+\infty) = 0, \end{cases} \tag{18}$$

where c is a positive constant, $f(u)$ is a continuous function in $[0, b]$ such that $f(0) = f(b) = 0$ for some $b > 0$ and $f(u) > 0$ for $u \in]0, b[$. We consider in addition that $\liminf_{u\to+\infty} \frac{f(u)}{u} = -\infty$. We follow the approach of [3] (p.133) to reduce the order of this problem.

Remark 4.1. The function $f(u) = u - u^p$, where $p > 1$ satisfies the conditions above.

Lemma 4.2. *The derivative of a nonincreasing positive solution u of* (18) *does not vanish for any value $x_1 > 0$.*

Proof. A nonincreasing positive solution u of (18) satisfies $u''(0) < 0$ (the uniqueness of the initial value problem rules out the case $u''(0) = 0$) and consequently $u(0) > b$. Suppose towards a contradiction that there exists $x_1 > 0$ such that $u'(x_1) = 0$ (we assume that it is the smallest positive value with that property). Rolle's Theorem would provide us the existence of a value $x_2 \in]0, x_1[$ such that $u''(x_2) = 0$ (using the differential equation, we have $u(x_2) > b$) and therefore $u''(x_1) \geq 0$. If $u''(x_1) > 0$, then x_1 would the a strict local minimum, which contradicts the fact that the solution in nonincreasing. If $u''(x_1) = 0$ then $u(x_1) = b$ and by the uniqueness of the initial value problem we would have $u(x) = b$, which contradicts the condition $u(+\infty) = 0$. □

Let $U(x)$ be a nonincreasing solution of the differential equation in (18) defined in the maximal interval $[0, x_+[$ where $U > 0$. Since $U'(x) < 0$ for $x \in]0, x_+[$ we can consider the inverse function $x(u)$ of $U(x)$ and define $\varphi(u) = U'(x(u))$. We have $\varphi'\varphi + c\varphi = f(u)$, and setting $\psi(u) = \varphi(u)^2$ (noting that $\varphi(u) = -\sqrt{\psi(u)}$), we have

$$\psi' = 2c\sqrt{\psi} + 2f(u), \quad \psi(0) = 0. \tag{19}$$

Let M be the maximum of $f(u)$ for $u \in]0, b[$ and consider the initial value problem

$$\hat{\psi}' = 2c\sqrt{\hat{\psi}} + 2M, \quad \hat{\psi}(0) = 0. \tag{20}$$

The solution of this problem is given implicitly by the expression

$$\frac{\sqrt{\hat{\psi}}}{c} - \frac{M}{c^2}\ln\left|c\sqrt{\hat{\psi}} + M\right| = x - \frac{M}{c^2}\ln(M).$$

By a well known comparison theorem we have $\psi < \hat{\psi}$ and consequently, $\psi(u) \leq ku^2$ for some positive constant k. Hence $\psi'(u) \leq \tilde{k}u + 2f(u)$ for some constant $\tilde{k}$, from which we infer that $\lim_{u\to+\infty} \psi'(u) = -\infty$ and therefore ψ vanishes at some positive value u^*.

Since there exists a solution ψ of (19) that vanishes vanishes at some positive value u^*, following the argument used in [3], we can conclude that $u(x)$ defined by

$$\begin{cases} u' = -\sqrt{\psi} \\ u(0) = u^* \end{cases} \tag{21}$$

is a solution of the differential equation in (18) in the interval $[0, x_+[$, where $x_+ = \int_0^{u^*} \frac{du}{\sqrt{\psi}}$. An easy computation gives $x_+ = +\infty$ and consequently we have proved the following

Theorem 4.3. *The autonomous boundary value problem* (18) *has a positive decreasing solution.*

5 The second order non-autonomous problem with a dissipative term

In this section we focus on finding a positive solution of the problem

$$\begin{cases} u'' + c\,u' = a(x)\,u - g(u) \\ u'(0) = 0, \quad u(+\infty) = 0, \end{cases} \tag{22}$$

where $a(x) > \delta > 0$ for all $x \geq 0$ and $g(u)$ satisfies the assumptions

(H_1) There exists $q > 2$ such that

$$0 < q\,G(u) \leq u\,g(u), \quad \forall u \in]0, +\infty[$$

(H_2) $g(u) = o\,(u)$ at $x = 0$.

A simple example of functions satisfying these assumptions are the powers $g(u) = u^p$ where $p > 1$.

As in Section 3, we will find a solution of (22) as the limit of the solutions of the boundary value problems

$$\begin{cases} u'' + c\,u' = a(x)\,u - g(u) \\ u'(0) = 0, \quad u(T) = 0. \end{cases} \tag{23}$$

Let us consider the associated Euler-Lagrange functional

$$J_T(u) = \int_0^T e^{c\,x} \left(u'^2 + a(x)u^2 - 2G(u_+)\right)\,dx,$$

defined in the functional space $H_T^c \equiv \left\{H^1(0,T)\colon\ \int_0^T e^{c\,x} u'^2 < +\infty,\ u(T) = 0\right\}$, with the norm $\|u\| = \left(\int_0^T e^{c\,x} u'^2\right)^{1/2}$. We have $J_T(0) = 0$, and, for $\epsilon > 0$ small enough, if $\|u\| = \epsilon$, then $J_T(u) > \delta(\epsilon) > 0$. The Palais-Smale condition is satisfied and, setting $u_\lambda = \lambda(1-x^2)$, it is easy to see that $J_{a,T}(u_\lambda^+) < 0$ for $\lambda > 0$ large enough (independent of $T > 1$). The Mountain-Pass Theorem allows us to conclude that the boundary value problems (23) have a positive solution. Let c_T be the mountain-pass critical value of J_T, that is, $c_T = J_T(u_T)$. Defining $\Gamma_T = \left\{\gamma(\tau)\colon [0,1] \to H_T^*\colon \gamma(0) = 0, \gamma(1) = u_\lambda^+\right\}$, we know that

$$c_T = \inf_{\gamma\in\Gamma_T} \max_{\tau\in[0,1]} J_T\left(\gamma(\tau)\right).$$

Since $\Gamma_{T_1} \subseteq \Gamma_{T_2}$ for $T_1 < T_2$, we have $c_T \le c_1$ for $T \ge 1$.

Multiplying the differential equation by $e^{c\,x}$ and then by u and integrating, we get

$$-\int_0^T e^{c\,x} u_T'^2\,dx = \int_0^T e^{c\,x}\left(a(x)u_T^2 - u_T\,g(u_T)\right)\,dx,$$

and consequently, using (H_1), we have

$$J_T(u_T) \ge \left(1-\frac{2}{q}\right)\int_0^T e^{c\,x}\left(u_T'^2 + a(x)\,u_T^2\right)\,dx = \left(1-\frac{2}{q}\right)\int_0^T e^{c\,x} g(u_T)u_T\,dx. \qquad (24)$$

Extending u_T to $[0,+\infty[$ by $u_T(x) = 0$ for $x \ge T$, and considering the functional space

$$H_c(0,+\infty) \equiv \left\{H^1_{loc}[0,+\infty[\colon \int_0^{+\infty} e^{c\,x}u'^2 < +\infty,\ u(+\infty) = 0\right\}$$

with the norm $\|u\| = \left(\int_0^{+\infty} e^{c\,x}u'^2\right)^{1/2}$, it follows:

Proposition 5.1. *We have uniform estimates for the $H_c(0,+\infty)$ norms of the solutions u_T (for $T \ge 1$).*

Proof. Since $J_T(u_T) \le c_1$ for all $T > 1$, (24) allows us to conclude the result. □

In [1], it is proved the following result:

Lemma 5.2. *For $u \in H_c(0,+\infty)$ we have*

$$\|u\|_{L^\infty(s,+\infty)} \le \frac{e^{-\frac{c\,s}{2}}}{\sqrt{c}}\,\|u\|.$$

Corollary 5.3. *There exists $k > 0$ such that, for all $T > 1$,*

$$|u_T|,\ |u_T'|,\ |u_T''| \le k \quad \forall x \in [0,T].$$

Proof. The previous result implies the uniform estimate for u_T. Setting $v = u_T'$, from the differential equation it follows that $v' + cv$ is bounded with $v(0) = 0$ and this implies that v is bounded. Again using the differential equation, we conclude the uniform boundness of u_T''. □

As a consequence, using the diagonal argument, we can pick up a sequence of values $T \to +\infty$ such that $u_T \to u$ C^1-uniformly in compact intervals and $u_T' \rightharpoonup u'$ weakly in $L^2(0,+\infty)$.

With this it is easy to prove the following

Lemma 5.4. *Given an arbitrary positive constant ϵ, there exists x_ϵ such that for all $T \ge 1$ and all $x > x_\epsilon$ we have $u_T(x) \le \epsilon$.*

Proof. By the previous lemma, for x_ϵ large enough we have

$$|u_T(x)| \le \frac{e^{-\frac{c\,x_\epsilon}{2}} c_1}{\sqrt{c}} \le \epsilon \quad \forall x > x_\epsilon,\ T \ge 1.$$

□

In order to show that the limit is not the trivial solution, we need the following

Proposition 5.5. *There exists a constant* $c_0 > 0$ *such that* $u_T(0) > c_0$ *for all* $T \ge 1$.

Proof. Suppose towards a contradiction that there exists a sequence of T's tending to infinity such that $u_T(0) \to 0$. We have $a(x) \ge \delta > 0$ and for $u_T(0)$ small enough we have $u''_T(0) > 0$, so the solutions u_T must have a local maximizer $l_T \to +\infty$. Using the differential equation it is easy to see that $\frac{g(u_T(l_T))}{u_T(l_T)} \ge a(l_T) > \delta$, and since for u close enough to 0 we have $\frac{g(u)}{u} < \delta$, we can conclude that $u_T(l_T)$ is bounded from below by a positive constant k_δ. Taking $\epsilon < k_\delta$, we have a contradiction with the fact that $u_T(l_T) < \epsilon$ when $l_T > x_\epsilon$. □

We are now able to prove

Theorem 5.6. *The boundary value problem* (22) *has a positive solution.*

Proof. Using Proposition 5.5 and Lemma 5.4 we have $u_T(x) \to u(x)$ C^1-uniformly in compact intervals, with $u(x)$ a positive solution of (22). □

Remark 5.7. If instead of a positive constant c we take a continuous function $c(x)$ with $0 < c_1 \le c(x) \le c_2$, the arguments used above are still valid for the differential equation $u'' + c(x)u' = a(x)u - g(u)$.

6 The fourth order autonomous problem

In this section we prove the existence of a nontrivial solution of the problem

$$\begin{cases} u^{(4)} - c\,u'' + a\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0, \end{cases} \tag{25}$$

where a and c are positive constants and $p > 1$. The solution of (25) will be found as a limit of solutions of the boundary value problems

$$\begin{cases} u^{(4)} - c\,u'' + a\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(T) = u'(T) = 0, \end{cases} \tag{26}$$

by taking $T \to +\infty$.

Proposition 6.1. *The boundary value problem* (26) *has a nontrivial solution.*

Proof. Consider the functional

$$J_T(u) = \int_0^T u''^2 + c\, u'^2 + a\, u^2\, dx$$

defined in $H^2(0, T)$ and let us minimize it in the manifold

$$M_T = \left\{ u \in H^2(0,T) \colon u'(0) = u(T) = u'(T) = 0,\ \int_0^T |u|^{p+1}\, dx = 1 \right\}.$$

Since the interval is bounded there exists such a minimum u_T, and by the Lagrange multipliers theory, there exists $\lambda_T \in \mathbb{R}$ and $u_T \in M_T$ such that ${u_T}'''(0) = 0$ and

$$\int_0^T {u_T}'' v'' + c\, {u_T}' v' + a\, u_T\, v\, dx = \lambda_T \int_0^T |u_T|^{p-1}\, u_T\, v \quad \forall v \in M_T,$$

that is, u_T is a solution of

$$u^{(4)} - c\, u'' + a\, u = \lambda_T\, |u|^{p-1}\, u \quad u'(0) = u'''(0) = 0, \quad u(T) = u'(T) = 0.$$

By taking $v = u_T$ we get

$$\int_0^T {u_T}''^2 + c\, {u_T}'^2 + a\, {u_T}^2\, dx = \lambda_T \int_0^T |u_T|^{p+1}\, dx = \lambda_T,$$

and since $\int_0^T |u_T|^{p+1}\, dx = 1$, u_T cannot be the trivial solution. □

Remark 6.2. If we take a sequence of values T tending to $+\infty$, the corresponding sequence λ_T is decreasing (if $T_1 < T_2$, then $M_{T_1} \subseteq M_{T_2}$). Considering the obvious extensions of the functions $u \in H^2(0, T)$, it is obvious that $J_T(u)$ is an equivalent norm of $H^2(0, +\infty)$, therefore u_T is a bounded sequence in $H^2(0, +\infty)$ and, consequently, there exists a constant $k > 0$ such that $\|u_T\|_\infty \leq k$. We have

$$1 = \int_0^T |u_T|^{p+1}\, dx \leq k^{p-1} \int_0^T {u_T}^2\, dx \leq K\, J_T(u_T) \text{ with } K = k^{p-1} \frac{\lambda_T}{a} > 0,$$

so the sequence λ_T tends to a strictly positive value.

Proposition 6.3. *There exists a constant $c > 0$ such that $u_T(0) > c$ for all $T > 1$.*

Proof. Suppose towards a contradiction that there exists a sequence of $T's$ tending to infinity such that $u_T(0) \to 0$.

Consider the differential equation with $u = u_T$ and multiply it by ${u_T}'$. By simply integrating we get

$$ {u_T}''' {u_T}' - \frac{{u_T}''^2}{2} - c\, \frac{{u_T}'^2}{2} + a \frac{{u_T}^2}{2} - \frac{|u_T|^{p+1}}{p+1} = c_T, \tag{27}$$

for some constant c_T. If we integrate in $[0, T]$ we get

$$\int_0^T -\frac{3}{2} {u_T''}^2 - c\, \frac{{u_T}'^2}{2} + a \frac{{u_T}^2}{2} - \frac{|u_T|^{p+1}}{p+1}\, dx = c_T\, T.$$

Since

$$\int_0^T \left| -\frac{3}{2} {u''_T}^2 - c\,\frac{{u_T'}^2}{2} + a\frac{{u_T}^2}{2} - \frac{|u_T|^{p+1}}{p+1}\,dx \right|$$

is clearly bounded and λ_T is decreasing in T, we conclude that c_T must tend to 0 as T tends to infinity. Considering the initial sequence of T's, (27) implies that $u''_T(0) \to 0$ (we already know that $u_T(0) \to 0$, $u'_T(0) = u'''_T(0) = 0$). As a consequence, by the continuous dependence of parameters, the solutions u_T have their last maximizer m_T tending to $+\infty$ (we may assume it is a maximizer since $-u_T$ is also a solution with the same properties). We have

$$1 = \int_0^T |u_T|^{p+1}\ dx \le \|u_T\|_\infty^{p-1} \int_0^T {u_T}^2\,dx \le \frac{\lambda_T}{a}\,\|u_T\|_\infty^{p-1}\,,$$

so we can conclude that $\|u_T\|_\infty \ge \left(\frac{a}{\lambda_T}\right)^{\frac{1}{p-1}}$. We know that $\|u_T\|_\infty$ is bounded independently of T.

<u>*Claim*</u> All the derivatives u'_T, u''_T, u'''_T and $u_T^{(4)}$ are bounded, independently of T.

Proof of Claim Setting $w = u''_T$, we have, taking (27) into account

$$w'' - cw \quad \text{is bounded,} \quad w'(0) = 0 \quad \text{and} \quad w(T) \quad \text{is bounded.}$$

Therefore u''_T is also bounded independently of T, and considering the differential equation, we have that ${u_T}^{(4)}$ is also bounded. As a consequence, all the intermediate derivatives are bounded too, independently of T.

Let us now consider two auxiliar functions v_T and w_T defined in the following way:

$$v_T(x) = \begin{cases} u_T(x+m_T) & x \in [0, T-m_T] \\ 0, & x \in [T-m_T, T], \end{cases} \qquad , \qquad w_T(x) = \begin{cases} u_T(m_T - x), & x \in [0, m_T] \\ \rho(x), & x \in [m_T, m_T+\eta] \\ 0 & x \in [m_T+\eta, T], \end{cases}$$

where $\rho(x) = \frac{u_T(0)}{2}(\cos(\frac{\pi}{\eta}(x-m_T)) + 1)$. Since $u_T(m_T) \ge \left(\frac{a}{2}\right)^{\frac{1}{p-1}}$ and u_T' is bounded, we can take a constant η small enough such that $m_T + \eta < T$ for T large. Let $\alpha_T = \int_0^T |v_T|^{p+1}\,dx$ and $\beta_T = \int_0^T |w_T|^{p+1}\,dx$. The uniform boundness of u_T' implies that each of these integrals cannot be arbitrarily small. We have $\alpha_T + \beta_T = 1 + \delta(T)$, where $\delta(T) = \frac{u_T(0)}{2}^{p+1} \int_0^\eta \left(cos(\frac{\pi\,x}{\eta}) + 1\right)^{p+1} dx$. If $T \to +\infty$ then $\delta \to 0$. For all $z \in H^2(0,T)$ such that $z'(0) = z(T) = z'(T) = 0$ we have

$$J_T(z) \ge \lambda_T\,\|z\|^2_{L^{p+1}(0,T)}\,,$$

and since

$$J_T(u_T) = J_T(v_T) + J_T(w_T) - \delta_1(T),$$

where $\delta_1(T) = \frac{u_T(0)}{2}^2 \int_0^\eta \left(cos(\frac{\pi\,x}{\eta}) + 1\right)^2 dx$, we can conclude that

$$J_T(u_T) \ge \lambda_T\left(\|v_T\|^2_{L^{p+1}(0,T)} + \|w_T\|^2_{L^{p+1}(0,T)}\right) - \delta_1(T) = \lambda_T\left({\alpha_T}^{\frac{2}{p+1}} + {\beta_T}^{\frac{2}{p+1}}\right) - \delta_1(T).$$

The fact that α_T and β_T do not tend to 0 and

$$\alpha_T + \beta_T = 1 + \delta(T)$$

with δ small enough, implies that

$$\left(\alpha_T{}^{\frac{2}{p+1}} + \beta_T{}^{\frac{2}{p+1}}\right) > K > 1,$$

where K is independent of T. It follows that for T large

$$J_T(u_T) > \lambda_T,$$

which is a contradiction. $\square$

Using the diagonal argument, we can pick up a sequence of values $T \to +\infty$ such that $u_T \to u$ C^3-uniformly in compact intervals and $u(x)$ is a solution of

$$\begin{cases} u^{(4)} - c\,u'' + a\,u = \lambda\,|u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0, \end{cases} \tag{28}$$

where $\lambda = \lim_{T\to+\infty} \lambda_T$. Since $u_T(0) > c > 0$ for all $T > 1$, the solution $u(x)$ cannot be the trivial solution and we can conclude the main result of this section:

Theorem 6.4. *There exists a nontrivial solution of* (25)*.*

Corollary 6.5. *The equation in* (25) *has a nontrivial homoclinic at* $u = 0$*.*

Proof. The function $u^*(x) = \lambda^{\frac{1}{p-1}} u(x)$ solves the half-line problem (25). Since $u^*(-x)$ is also a solution of the differential equation and $u'(0) = u'''(0) = 0$, the conclusion follows. $\square$

If we consider the manifold

$$M_T^+ = \left\{ u \in H^2(0,T) \colon u'(0) = u(T) = u'(T) = 0,\ \int_0^T u_+{}^{p+1}\,dx = 1 \right\},$$

where $u_+ = \max(0,u)$, the arguments used above will still provide a solution of

$$\begin{cases} u^{(4)} - c\,u'' + a\,u = u_+{}^p \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0. \end{cases} \tag{29}$$

The following lemma allows us to prove that for c large enough, this solution is positive.

Lemma 6.6. *Let $y \in C^2(0,+\infty)$ be a bounded function such that $y'(0) = 0$ and $\mu > 0$ a constant. Then, if $y'' - \mu\,y = h(x) \geq 0$, we have $y \leq 0$.*

Proof. If $y(0) > 0$, then $y''(0) > 0$ and since $y'(0) = 0$, we must have $y(x) > y(0)$ for $x > 0$ close to 0. It is then obvious that y is a convex function and stays above a line of positive slope. This is a contradiction because y is bounded. If $y(0) < 0$ and there exists a value $x_0 > 0$ such that $u(x_0) = 0$, then we could apply the same argument as above and reach a contradiction. $\square$

Theorem 6.7. *If $c^2 \geq 4\,a$, then the boundary value problem* (25) *has a positive solution.*

Proof. Consider the solution u of (29). Let μ_1 and μ_2 be the solutions of $x^2 - c\,x + a = 0$. Since $c^2 > 4\,a$, these values are positive and we can write the differential equation in the form

$$\left(D^2 - \mu_1\right)\left(D^2 - \mu_2\right) u = u_+(x)^p = h(x) \geq 0.$$

Setting $y(x) = \left(D^2 - \mu_2\right) u$, we have $y'(0) = 0$ and $y'' = \mu_1 y + h(x)$. Since u is a solution of (29), we know (by the arguments of the *Claim* in the proof of Proposition 6.3) that u and u'' are bounded and therefore y is bounded. Applying the previous lemma we have $y \leq 0$. Applying the same lemma to $-u$ we conclude that $u \geq 0$ and therefore is also a solution of (25). $\square$

7 The fourth order non-autonomous problem with $a(x) \to a$

Consider the boundary value problem

$$\begin{cases} u^{(4)} - c\,u'' + a(x)\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0 \end{cases} \tag{30}$$

where $a(x)$ is a nondecreasing function with $\lim_{x\to+\infty} a(x) = a \in \mathbb{R}^+$, c is a positive constant and $p > 1$. We will follow the approach of the previous section, so let u_T be defined as above (with $a(x)$ substituted for a).

Proposition 7.1. *There exists a constant $c > 0$ such that $|u_T(0)| + |u_T''(0)| > c$ for all $T > 1$.*

Proof. Suppose towards a contradiction that there exists a sequence of $T's$ tending to infinity such that $|u_T(0)| + |u_T''(0)| \to 0$. Let $v_T(x)$ and $w_T(x)$ be defined in the proof of Proposition 6.3. We have

$$\begin{aligned} J_T(v_T) &= \int_{m_T}^{T} \left[{u_T}''(x)^2 + c\,{u_T}'(x)^2 + a(x - m_T) u_T(x)^2\right] dx \leq \\ &\leq \int_{m_T}^{T} \left[{u_T}''(x)^2 + c\,{u_T}'(x)^2 + a(x) u_T(x)^2\right] dx, \end{aligned} \tag{31}$$

and

$$J_T(w_T) = \int_0^{m_T} \left[{u_T}''(x)^2 + c\,{u_T}'(x)^2 + a(m_T - x) u_T(x)^2\right] dx + \int_{m_T}^{m_T+\eta} \left[\rho''^2 + c\rho'^2 + a(x)\rho^2\right] dx. \tag{32}$$

Given $\epsilon > 0$, there exists $x_0(\epsilon)$ such that $a - a(x) < \epsilon$ if $x \geq x_0$.

By continuous dependence of parameters, for T large enough, we have $|u_T(x)| < \delta$ for all $x \in [0, x_0]$, therefore

$$\int_0^{x_0} a(m_T - x) u_T(x)^2\, dx \leq a\, x_0\, \delta^2. \tag{33}$$

By the uniform boundness in T of the $L^2[0,T]$ norms (let K be such bound), we can conclude that

$$\int_{x_0}^{\frac{m_T}{2}} \left(a(m_T - x) - a(x)\right) u_T(x)^2\, dx \leq \epsilon \int_{x_0}^{\frac{m_T}{2}} u_T(x)^2\, dx \leq K\epsilon \tag{34}$$

Since $a(m_T - x) \leq a(x)$ for $x \in [\frac{m_T}{2}, m_T]$ we have

$$\int_{\frac{m_T}{2}}^{m_T} a(m_T - x) u_T(x)^2 \, dx \leq \int_{\frac{m_T}{2}}^{m_T} a(x) u_T(x)^2 \, dx$$

and therefore

$$\int_0^{m_T} a(m_T - x) u_T(x)^2 \, dx \leq a \, x_0 \, \delta^2 + \int_{x_0}^{m_T} a(x) u_T(x)^2 \, dx + K\epsilon. \tag{35}$$

We can now make the following estimates:

$$\begin{aligned}
J_T(u_T) \geq & \int_0^{m_T} \left[{u_T}''^2 + c \, {u_T}'^2 \right] dx + \int_0^{x_0} a(x) {u_T}^2 \, dx + \int_{x_0}^{m_T} a(x) {u_T}^2 \, dx + J_T(v_T) \geq \\
\geq & \int_0^{m_T} \left[{u_T}''^2 + c \, {u_T}'^2 \right] dx + \int_0^{x_0} a(x) {u_T}^2 \, dx + \\
& + \int_0^{m_T} a(m_T - x) {u_T}^2 \, dx - a \, x_0 \, \delta^2 - K\epsilon + J_T(v_T) \geq \\
\geq & \, J_T(v_T) + J_T(w_T) - \int_{m_T}^{m_T + \eta} \left[\rho''^2 + c\rho'^2 + a(x)\rho^2 \right] dx - a \, x_0 \, \delta^2 - K\epsilon
\end{aligned}$$

(we have used (31) in the first inequality, (35) in the second and (32) in the third). The terms of negative sign can be taken arbitrarily small, so we can repeat the arguments from the previous section and reach a contradition. ☐

Theorem 7.2. *Let $a(x)$ be a nondecreasing function with $\lim_{x \to +\infty} a(x) = a \in \mathbb{R}$, $c > 0$ and $p > 1$. Then the problem* (30) *has a solution.*

Remark 7.3. Let us now consider the boundary value problem (30), but now assuming that $a(x)$ is a positive function in $\mathbb{R}^+$ such that $\lim_{x \to \infty} a(x) = +\infty$. We will prove the existence of a nontrivial solution by proving that the functional

$$J(u) = \int_0^{+\infty} \left[u''^2 + c \, u'^2 + a \, u^2 \right] dx$$

defined in $H^2(0, +\infty)$ has a minimum in the manifold

$$M = \left\{ u \in H^2(0, +\infty) \colon u'(0) = 0, \int_0^T \frac{u^{p+1}}{p+1} \, dx = 1 \right\}.$$

Let m be the infimum of $J(u)$ in M ($m \geq 0$) and consider a sequence u_n, with $n \in \mathbb{N}$ such that $J(u_n) \to m$. Obviously, $J(u_n)$ is bounded and u_n is bounded in $L^\infty(0, +\infty)$ (since it is bounded in $H^2(0, +\infty)$), so we have

$$p + 1 = \int_0^{+\infty} {u_n}^{p+1} \, dx \leq \|u_n\|_\infty{}^{p-1} \int_0^{+\infty} {u_n}^2 \, dx,$$

and hence, we have $\int_0^{+\infty} {u_n}^2 \, dx \geq c_1$.

On the other hand, for all positive L, there exists $x_0 > 0$ such that $a(x) > L$ for $x > x_0$, so

$$L \int_{x_0}^{+\infty} {u_n}^2 \, dx \leq \int_{x_0}^{+\infty} a(x) {u_n}^2 \, dx \leq c_2,$$

where c_2 is the upper bound of $J(u_n)$, and consequently $\int_{x_0}^{+\infty} {u_n}^2 \, dx \leq \frac{c_2}{L}$. Considering L large enough, we have $\int_{x_0(L)}^{+\infty} {u_n}^2 \, dx \leq \frac{c_1}{2}$, and therefore $\int_0^{x_0(L)} {u_n}^2 \, dx \geq \frac{c_1}{2}$, which implies that the limit u of the convergent subsequence of u_n cannot be the trivial solution.

Theorem 7.4. *Let $a(x)$ be a positive function in $\mathbb{R}^+$ such that $\lim_{x\to\infty} a(x) = +\infty$ and c a positive constant. Then the problem* (30) *has a solution.*

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