Convergence of numerical schemes for interaction equations of short and long waves

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Abstract

We study numerical approximations of systems of partial differential equations modeling the interaction of short and long waves. The short waves are modeled by a nonlinear Schrödinger equation which is coupled to another equation modeling the long waves. Here, we consider the case where the long wave equation is either a hyperbolic conservation law or a Korteweg–de Vries equation. In the former case, we prove the strong convergence of a Lax–Friedrichs type scheme towards the unique entropy solution of the problem, while in the latter case we prove convergence of a finite difference scheme towards the global solution of the problem.

Keywords: Conservation law, nonlinear Schrödinger equation, entropy condition, KdV equation, finite difference scheme.

1 Introduction

1.1 Interaction equations of short and long waves

The nonlinear interaction between short waves and long waves has been studied in a variety of physical situations. In [4], D.J. Benney presents a general theory, deriving nonlinear differential systems involving both short and long waves. The short waves $u(x,t)$ are described by a nonlinear Schrödinger equation and the long waves $v(x,t)$ satisfy a quasilinear wave equation, eventually with a dispersive term. In the most general context, the interaction is described by the nonlinear system

$$\begin{cases} i\partial_t u + ic_1\partial_x u + \partial_{xx} u = \alpha u\, v + \gamma |u|^2 u \\ \partial_t v + c_2 \partial_x v + \mu \partial_x^3 v + \nu \partial_x v^2 = \beta \partial_x (|u|^2), \end{cases} \tag{1.1}$$

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where $c_1, c_2, \alpha, \beta, \gamma, \mu$ and ν are real constants.

A typical case is given by the system

$$\begin{cases} i\partial_t u + \partial_{xx} u = u\,v + \alpha\,|u|^2 u \\ \partial_t v + c\,\partial_x v = \partial_x(|u|^2), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1.2}$$

which, by a suitable gauge transformation, is found to be equivalent (see [16]) to the system

$$\begin{cases} i\partial_t u + \partial_{xx} u = u\,v + \alpha\,|u|^2 u \\ \partial_t v = \partial_x(|u|^2), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \end{cases} \tag{1.3}$$

This system was studied by M. Tsutsumi and S. Hatano in [15, 16], where the global well-posedness of (1.2), (1.3) is proved for initial data in the Sobolev spaces $H^{m+1/2} \times H^m$, with $m = 1, 2, \ldots$, and generalized by Bekiranov *et al.* [3]. A numerical study of (1.3) was considered by the present authors in [1], where the convergence (in the energy space) of a semidiscrete finite difference approximation is proved.

1.1.1 A Schrödinger–conservation law system

Motivated by Benney [4], Dias *et al.* [7] introduced the coupled system of a nonlinear Schrödinger equation coupled with a scalar conservation law,

$$i\partial_t u + \partial_{xx} u = |u|^2 u + g(v)u \tag{1.4a}$$

$$\partial_t v + \partial_x f(v) = \partial_x(g'(v)|u|^2) \tag{1.4b}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \tag{1.4c}$$

where $f \in C^2(\mathbb{R})$, $g \in C^3(\mathbb{R})$ are real functions such that $f(0) = 0$, g' has compact support, and f, g verify a standard nonlinearity condition, which is needed to apply the compensated compactness method:

$$\forall\,\kappa > 0, \quad \text{the set} \quad \{s : f''(s) - \kappa g'''(s) \neq 0\} \quad \text{is dense in } \mathbb{R}.$$

In [7], the authors proved the global existence of a weak entropy solution for the Cauchy problem (1.4a)–(1.4c).

Comparing this model to the original formulation of Benney, we remark the appearance of a general interaction function $g(v)$ in the Schrödinger equation and of its derivative $g'(v)$ in the conservation law. This new function, it is argued in [7], gives a more physically realistic coupling between the equations. The fact that g' has compact support allows the authors of [7] to obtain a uniform bound for $\|v\|_\infty$ through a sort of maximum principle. This L^∞ bound is fundamental in the analysis, and it is not clear whether a corresponding well-posedness result can be obtained for system (1.4a)–(1.4c) with $g(v) = v$, and for general f (see, however [6] and [2] for some partial well-posedness and blow-up results, respectively).

1.1.2 A Schrödinger–Kdv system

We will also consider the problem of a nonlinear Schrödinger equation coupled with a Korteweg–de Vries equation,

$$i\partial_t u + \partial_{xx} u = |u|^2 u + vu \tag{1.5a}$$

$$\partial_t v + \partial_x^3 v + \partial_x (v^2) = \partial_x |u|^2 \tag{1.5b}$$

$$u(0) = u_0, \quad v(0) = v_0. \tag{1.5c}$$

This problem describes the capillarity-gravity interaction and has been studied by M. Tsutsumi [14], Bekiranov *et al.* [3] and Corcho and Linares [5]. In [5], using a method introduced by Bourgain and strongly improved by Kenig *et al.* [10, 11], the authors proved the global well-posedness in the energy space $H^1 \times H^1$. More recently, Dias *et al.* [8], using an approximation method, proved the global well-posedness in $H^1 \times H^1$ for a Schrödinger–*generalized* KdV system, in which the term $\partial_x v^2$ is replaced by a more general term $\partial_x f(v)$. This recent result will be essential in the analysis of the numerical method in Section 3 below.

1.2 Outline of the paper

In the first part of this paper, we establish the convergence of a simple numerical scheme to approximate the problem (1.4a)–(1.4c). To this end, we use a semidiscrete Lax-Friedrichs type scheme as an approximation of the quasilinear equation (1.4b) and a standard semidiscrete finite-difference scheme for the first equation, (1.4a). This allows us to prove a convergence result towards the unique entropy solution, which had been announced in [1].

The convergence result concerning problem (1.4a)–(1.4c) (Theorem 2.2 below) relies on the compensated compactness method and on the crucial energy estimates in Lemma 2.3. Interestingly, these estimates are finer than the corresponding ones in [7], where the vanishing viscosity approach is used, and consequently our result does not require a smallness assumption on a coupling parameter α used in [7], and which we take equal to unity for simplicity.

Also, no existence of solution is assumed *a priori*, and thus our convergence proof is also a new existence proof for the problem (1.4a)–(1.4c). An interesting open question, which we plan to address in future works, is to find a more general class of first-order finite volume schemes (for instance, monotone schemes) for which convergence can be rigorously proved.

In the second part of this paper, we will consider the numerical approximation of the Schrödinger–KdV problem (1.5a)–(1.5c), for which we propose a semidiscrete finite difference discretization. The energy methods used by Tsutsumi [14] to prove the global existence of solution fail in this setting, so we propose a new approach: by an appropriate truncation, we consider a related problem satisfying stability bounds from which we obtain the convergence of a method. We then prove that this related problem can be made to reduce to the original problem.

Let us define the Banach spaces :

$$l_h^p(\mathbb{Z}) = \big\{(z_j) : z_j \in \mathbb{C},\ \|z_j\|_{p,h}^p = h\sum_{j\in\mathbb{Z}} |z_j|^p < \infty\big\}, \quad h > 0.$$

For $p = 2$ we denote the usual scalar product by $(z_j, w_j)_h = h\sum_{j\in\mathbb{Z}} z_j\bar{w}_j$. We will also use the following notations for the well known finite difference operators: for $u = (u_j)$,

$$D_+u_j = (u_{j+1} - u_j)/h, \quad D_-u_j = (u_j - u_{j-1})/h, \quad D_0u_j = (u_{j+1} - u_{j-1})/2h,$$
$$\Delta^h u_j = D_+D_-u_j = D_-D_+u_j = (u_{j+1} - 2u_j + u_{j-1})/h^2,$$
$$D^3u_j = D_0\Delta^h u_j.$$

2 A finite difference approximation of the Schrödinger–conservation law system

In this section, we introduce a semidiscrete finite difference approximation of the problem (1.4a)–(1.4c), which we repeat here for convenience,

$$i\partial_t u + \partial_{xx}u = |u|^2u + g(v)u \tag{2.1a}$$
$$\partial_t v + \partial_x f(v) = \partial_x(g'(v)|u|^2) \tag{2.1b}$$
$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)). \tag{2.1c}$$

Namely, the Schrödinger equation (2.1a) is approximated by a standard finite difference scheme while the conservation law (2.1b) is approximated by a Lax–Friedrichs type scheme:

$$i\partial_t u^h + \Delta^h u^h = |u^h|^2u^h + g(v^h)u^h \tag{2.2a}$$
$$\partial_t v^h + D_0 f(v^h) = D_0(g'(v^h)|u^h|^2) + \frac{h}{2\lambda}\Delta^h v^h + \frac{1}{2\gamma}\Big(|u^h|_+^2 D_+v^h - |u^h|_-^2 D_-v^h\Big) \tag{2.2b}$$
$$u^h(0) = u_0^h(0), \quad v^h(0) = v_0^h(0). \tag{2.2c}$$

We have set $(|u^h|_\pm^2)_j = |u|_{j\pm1/2}^2 = (|u_{j\pm1}|^2 + |u_j|^2)/2$, and $(u^h, v^h) = ((u_j), (v_j))$. Here, λ, γ are some constants ensuring the stability of the scheme via a CFL condition (see (2.11) below), and u_0^h, v_0^h are some suitable approximations of the initial data u_0, v_0.

Note that the existence (for each h) of a solution to the equations (2.2a)–(2.2c) with initial data in l_h^2 is guaranteed by a simple fixed-point argument. Also, in what follows we will use the notation u^h, v^h to denote either an element (u_j) of, say, $l_h^2(\mathbb{Z})$ or the element u^h of $L^2(\mathbb{R})$ defined by some piecewise constant interpolation such that all the relevant norms coincide.

2.1 Statement of main result and stability estimates

First, we recall from [7] the notion of entropy solution to problem (1.4a)–(1.4c).

Definition 2.1. Let $\eta(v)$ be a convex function (the *entropy*), and define the *entropy fluxes* $q_{1,2}$ by $q_1'(v) = \eta'(v)f'(v)$ and $q_2'(v) = \eta'(v)g''(v)$. We say that $(u, v) \in L^\infty_{\mathrm{loc}}(\mathbb{R} \times [0, \infty))$ is an entropy solution to the problem (1.4a)–(1.4c) if for each entropy triplet (η, q_1, q_2) we have:

1. $u \in L^\infty_{\mathrm{loc}}([0, \infty); H^1(\mathbb{R})) \cap C([0, \infty); L^2(\mathbb{R}))$, $u(0) = u_0$ in $L^2(\mathbb{R})$, and

$$\iint_{\mathbb{R}\times[0,\infty)} iu\, \partial_t\theta + \partial_x u\, \partial_x\theta - (|u|^2 u + g(v)u)\theta\, dxdt = 0$$

   for every $\theta \in C_0^\infty(\mathbb{R} \times (0, \infty))$;

2. For every non-negative $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\iint_{\mathbb{R}\times[0,\infty)} \eta(v)\partial_t\phi + \partial_x\phi(x, t)(q_1(v) - q_2(v)|u|^2) \\ - \big(\eta'(v)g'(v) - q_2(v)\big)\partial_x|u|^2\phi\, dxdt + \int_{\mathbb{R}} \eta(v_0(x))\phi(x, 0)\, dx \geq 0.$$

We now state our first convergence result, dealing with the approximation of the Cauchy problem (1.4a)–(1.4c), and which was announced in [1].

Theorem 2.2. *Let (u^h, v^h) be defined by the semidiscrete approximation* (2.2a)–(2.2c)*. Then there exist functions $u \in C([0, \infty); H^1(\mathbb{R}))$, $v \in L^\infty(\mathbb{R} \times [0, \infty))$, solutions of the Cauchy problem* (1.4a)–(1.4c) *in the sense of Definition 2.1 such that, up to a subsequence, (u^h, v^h) converge to (u, v) in $L^1_{\mathrm{loc}}(\mathbb{R} \times [0, \infty))$.*

Remark that, as was noted in the introduction, Theorem 2.2 does not rely on any prior existence of solutions. Therefore it provides a new, independent proof of existence of solution to the Cauchy problem (1.4a)–(1.4c). Moreover, our result requires no smallness assumption on a coupling parameter α appearing in [7], where the right-hand side of equation (1.4b) takes the form $\alpha\partial_x(g'(v)|u|^2)$, and the last term of (1.4a) is $\alpha g(v)u$. This fact, due to our improved estimate (2.5) below, allows us to take $\alpha = 1$ for simplicity.

The proof of Theorem 2.2, relying on the compensated compactness method, is postponed to the next section. In the remainder of this section, we prove the following crucial estimates, which play a key role in establishing the compactness properties of the approximations u^h, v^h.

Lemma 2.3. *Let (u^h, v^h) be defined by* (2.2a)–(2.2c)*. Then, under the CFL condition* (2.11)*, there exist constants $k, C, M > 0$ depending only on the initial data, and non-negative functions $a(t), b(t)$ continuous on $[0, \infty)$ such that for* *every $t > 0$ we have, uniformly in h,*

$$\|u^h(t)\|_2 \le C, \tag{2.3}$$

$$\|v^h(t)\|_\infty \le M, \tag{2.4}$$

$$\|v^h(t)\|_2^2 + k\int_0^t \sum_{j\in\mathbb{Z}} (1+|u_j^h|^2)(v_{j+1}^h - v_j^h)^2\, ds \le a(t), \tag{2.5}$$

$$\|D_+ u^h\|_2 \le b(t). \tag{2.6}$$

Proof. We focus on obtaining (2.4), (2.5) and (2.6), since (2.3) follows easily from the equation (2.2a).

We begin by the uniform L^∞ bound (2.4). As in [7], let M' be such that $\operatorname{supp} g' \subset (-M', M')$. We will prove that one may take $M = \max\{\|v^h(0)\|_\infty, M'\}$ in (2.4). First, for each fixed h, consider the perturbed problem

$$i\partial_t u^{h,\epsilon} + \Delta^h u^{h,\epsilon} = |u^{h,\epsilon}|^2 u^{h,\epsilon} + g(v^{h,\epsilon})u^{h,\epsilon} \tag{2.7}$$

$$\begin{aligned}&\partial_t v^{h,\epsilon} + D_0 f(v^{h,\epsilon}) = D_0(g'(v^{h,\epsilon})|u^{h,\epsilon}|^2)\\ &+ \frac{h}{2\lambda}\Delta^h v^{h,\epsilon} + \frac{1}{2\gamma}\Big(D_+ v^{h,\epsilon}|u^{h,\epsilon}|_+^2 - D_- v^{h,\epsilon}|u^{h,\epsilon}|_-^2\Big) - \epsilon \operatorname{sgn} v^{h,\epsilon},\end{aligned} \tag{2.8}$$

where we have added the term $-\epsilon \operatorname{sgn} v^{h,\epsilon}$ to the second equation. We take the same initial data as in the unperturbed problem (2.2a),(2.2b). We now prove that for each h, ϵ, we have $\|v^{h,\epsilon}\|_\infty < M$. Suppose by contradiction that there is a first t^* and a first j^* such that, say, $v^* \equiv v_{j^*}^{h,\epsilon}(t^*) = M$ (the case $v^* = -M$ is similar). First, after an easy calculation we find (omitting h and ϵ for simplicity),

$$\begin{aligned}&-D_0 f(v_{j^*}) + \frac{h}{2\lambda}\Delta^h v_{j^*}\\ &\quad= \frac{1}{2h}\Big((v_{j^*+1} - v_{j^*})(-f'(\theta_1) + \lambda^{-1}) + (v_{j^*-1} - v_{j^*})(f'(\theta_2) + \lambda^{-1})\Big).\end{aligned} \tag{2.9}$$

Similarly, using that $g'(v^*) = 0$ (since, by assumption, $v^* \notin \operatorname{supp} g'$), we find

$$\begin{aligned}D_0(g'(v^*)|u^h|^2) &+ \frac{1}{2\gamma}\Big(D_+ v^*|u^h|_+^2 - D_- v^*|u^h|_-^2\Big) - \epsilon \operatorname{sgn} v^*\\ &= \frac{1}{2h}\Big(|u_{j^*+1}|^2(v_{j^*+1} - v_{j^*})(-g''(\theta_1) + (2\gamma)^{-1})\\ &\quad + |u_{j^*-1}|^2(v_{j^*-1} - v_{j^*})(g''(\theta_2) + (2\gamma)^{-1})\Big)\\ &\quad + \frac{1}{4h\gamma}|u_{j^*}|^2\big((v_{j^*+1} - v_{j^*}) + (v_{j^*-1} - v_{j^*})\big) - \epsilon \operatorname{sgn} v_{j^*}.\end{aligned} \tag{2.10}$$

Note that $v_{j^*\pm 1} - v_{j^*}$ is nonpositive by assumption, and assume the following CFL conditions:

$$\lambda \sup_{(-M,M)} |f'(\theta)| < 1, \qquad \gamma \sup_{(-M,M)} |g''(\theta)| < 1/2. \tag{2.11}$$

Thus, we find from (2.8)–(2.11) and $\operatorname{sgn} v^* = 1$

$$\partial_t v^*(t^*) \le -\epsilon < 0,$$

which is in contradiction with the assumption that $v^*(t^*)$ is a maximum. If we now prove that for each fixed h,

$$\|u^{h,\epsilon} - u^h\|_{2,h} + \|v^{h,\epsilon} - v^h\|_{2,h} \to 0 \quad \text{as} \quad \epsilon \to 0, \tag{2.12}$$

the estimate (2.4) will be proved since $l_h^2 \subset l_h^\infty$ and M does not depend on h. But (2.12) is easy to establish by integrating the equations (2.7),(2.8) on $(0,t)$ and comparing with the integrated version of (2.2a),(2.2b). This proves the uniform L^∞ bound (2.4).

We now prove (2.5) and (2.6). Take (2.2b), multiply by hv_j and sum over $j \in \mathbb{Z}$ to obtain

$$\begin{aligned}\partial_t \sum_{j\in\mathbb{Z}} hv_j^2 + \sum_{j\in\mathbb{Z}} \frac{v_j}{2}(f(v_{j+1}) - f(v_{j-1})) - \sum_{j\in\mathbb{Z}} \frac{h^2}{2\lambda} v_j \Delta^h v_j \\ - \sum_{j\in\mathbb{Z}} \frac{v_j}{2}(g'(v_{j+1})|u_{j+1}|^2 - g'(v_{j-1})|u_{j-1}|^2) \\ - \sum_{j\in\mathbb{Z}} \frac{h}{2\gamma}\Big(v^h D_+ v^h |u^h|_+^2 - v^h D_- v^h |u^h|_-^2\Big) = 0.\end{aligned} \tag{2.13}$$

If $f = F'$, then Taylor developing around v_j gives for some intermediate values $\theta_{1,2}$ which may change from line to line,

$$\begin{aligned}0 &= \sum_{j\in\mathbb{Z}} F(v_{j+1}) - F(v_{j-1}) \\ &= \sum_{j\in\mathbb{Z}} f(v_j)(v_{j+1} - v_{j-1}) + \frac{1}{2}\sum_{j\in\mathbb{Z}} (f'(\theta_1) - f'(\theta_2))(v_{j+1} - v_j)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \frac{v_j}{2}(f(v_{j+1}) - f(v_{j-1})) &= -\sum_{j\in\mathbb{Z}} \frac{f(v_j)}{2}(v_{j+1} - v_{j-1}) \\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}} \frac{1}{2}(f'(\theta_1) - f'(\theta_2))(v_{j+1} - v_j)^2.\end{aligned}$$

Adding this to the third sum on the left-hand side of (2.13) (after summation by parts), and using the CFL condition $\lambda \sup |f'| < 1$ gives for some positive constant k

$$\begin{aligned}\sum_{j\in\mathbb{Z}} v_j \frac{1}{2}(f(v_{j+1}) - f(v_{j-1})) - \sum_{j\in\mathbb{Z}} \frac{h^2}{2\lambda} v_j \Delta^h v_j \\ = \sum_{j\in\mathbb{Z}} \frac{1}{2}\Big\{\frac{1}{2}(f'(\theta_1) - f'(\theta_2)) + \frac{1}{\lambda}\Big\}(v_{j+1} - v_j)^2 \\ \ge k \sum_{j\in\mathbb{Z}} (v_{j+1} - v_j)^2.\end{aligned}$$

In turn, we obtain

$$-\sum_{j\in\mathbb{Z}}\frac{v_j}{2}(g'(v_{j+1})|u_{j+1}|^2-g'(v_{j-1})|u_{j-1}|^2)=\sum_{j\in\mathbb{Z}}\frac{1}{2}(v_{j+1}-v_{j-1})g'(v_j)|u_j|^2$$

$$=-\sum_{j\in\mathbb{Z}}\frac{g(v_j)}{2}(|u_{j+1}|^2-|u_{j-1}|^2)$$

$$+\frac{1}{2}\sum\frac{1}{2}g''(\theta_2)(v_{j+1}-v_j)^2(|u_{j+1}|^2-|u_j|^2)$$

$$-\frac{1}{2}\sum_{j\in\mathbb{Z}}\frac{1}{2}|u_j|^2(g''(\theta_2)-g''(\theta_1))(v_{j+1}-v_j)^2$$

and also

$$-\sum_{j\in\mathbb{Z}}\frac{h}{2\gamma}\Big(v^hD_+v^h|u^h|_+^2-v^hD_-v^h|u^h|_-^2\Big)$$

$$=\sum_{j\in\mathbb{Z}}\frac{1}{2\gamma}(v_{j+1}-v_j)^2|u_j|^2+\sum_{j\in\mathbb{Z}}\frac{1}{2\gamma}(v_{j+1}-v_j)^2\frac{1}{2}(|u_{j+1}|^2-|u_j|^2),$$

so that from (2.13) we have

$$-\sum_{j\in\mathbb{Z}}\frac{v_j}{2}(g'(v_{j+1})|u_{j+1}|^2-g'(v_{j-1})|u_{j-1}|^2)$$

$$-\sum_{j\in\mathbb{Z}}\frac{h}{2\gamma}\Big(v^hD_+v^h|u^h|_+^2-v^hD_-v^h|u^h|_-^2\Big)$$

$$=-\sum_{j\in\mathbb{Z}}A_j(|u_{j+1}|^2-|u_j|^2),$$

$$+\sum_{j\in\mathbb{Z}}\frac{1}{2}|u_j|^2\Big(\frac{g''(\theta_2)-g''(\theta_1)}{2}+\gamma^{-1}\Big)(v_{j+1}-v_j)^2,$$

with A_j depending on g, v^h, and γ. Now, from the CFL condition (2.11), which implies $\gamma\sup|g''|<1$, the last term is bounded below by

$$k\sum_{j\in\mathbb{Z}}|u_j|^2(v_{j+1}-v_j)^2,$$

for some positive constant k. Next, using the L^∞ bound on v^h, the conservation of the L^2 norm of u^h in (2.3), the term with A_j in the previous equality is bounded by $C\|D_+u^h\|_2$ with C independent of h. All these estimates together give, after integration on $(0,t)$,

$$\|v^h(t)\|_2^2+k\int_0^t\sum_{j\in\mathbb{Z}}(1+|u_j|^2)(v_{j+1}-v_j)^2\,ds\le c+c\int_0^t\|D_+u^h(s)\|_2ds. \quad (2.14)$$

Here and in what follows, c denotes a generic constant which may change from one occurrence to the next. If we now establish (2.6), that is, if

$$\|D_+u^h(t)\|_2 < b(t)$$

for some continuous function $b(t)$, then (2.5) will follow and the lemma will be proved. In order to prove (2.6), we begin by deducing an energy inequality for u^h. From the equations (2.2a),(2.2b) we derive by methods similar to [7] (which involve multiplying the first equation by $\partial_t \bar{u}^h$, algebraic manipulations and using the second equation),

$$\begin{aligned}
&\frac{d}{dt}\Big\{\|D_+u^h\|_2^2 + \frac{1}{2}\|u\|_4^4 + (g(v)u,u)_h - (F(v^h),1)_h\Big\}\\
&\quad = -\frac{h}{2\lambda}\Big(f'(\theta)D_+v^h, D_+v^h\Big)_h - \frac{h}{2\gamma}\Big(f'(\theta)(D_+v^h)^2, |u^h|_+^2\Big)_h\\
&+ \frac{h}{2\lambda}\Big((D_+v^h)^2, g''(\theta)|u^h|^2\Big)_h + \frac{h}{2\lambda}\Big(\tau_+g'(v^h)D_+v^h, D_+|u^h|^2\Big)_h\\
&+ \frac{h}{2\gamma}\Big(\tau_+g'(v^h)D_+v^h, |u^h|_+^2 D_+|u^h|^2\Big)_h + \frac{h}{2\gamma}\Big(g''(\theta)(D_+v_j)^2, |u^h|^2|u^h|_+^2\Big)_h,
\end{aligned}$$

where θ is some intermediate value which may change from one occurrence to the next, and $\tau_+a_j = a_{j+1}$. From $f(0) = 0$ we get

$$|(F(v^h),1)_h| \le c\|v\|_2^2,$$

and thus we find after integration in $(0,t)$, and using (2.14), (2.3) and (2.4),

$$\begin{aligned}
&\|D_+u^h\|_2^2 \le c + c\int_0^t \|D_+u^h\|_2 ds + ch\int_0^t \|D_+v^h\|_2^2\|u^h\|_\infty^2 ds\\
&+ ch\int_0^t \|D_+v^h\|_2\|D_+u^h\|_2\|u^h\|_\infty ds + ch\int_0^t \big|(|u^h|_+^2 D_+v^h, D_+|u^h|^2)_h\big| ds\\
&+ ch\int_0^t \big|((D_+v^h)^2, |u^h|^4)_h\big| ds.
\end{aligned} \tag{2.15}$$

We focus now on the last two integrals, since they are the hardest to estimate. We will use the estimate (2.14) and the Gagliardo–Niremberg inequalities: if $\phi_j \in l_h^2(\mathbb{Z})$, then

$$\|\phi\|_\infty \le C\|\phi\|_{2,h}^{1/2}\|D_+\phi\|_{2,h}^{1/2} \tag{2.16}$$

$$\|\phi\|_{4,h} \le C\|\phi\|_{2,h}^{3/4}\|D_+\phi\|_{2,h}^{1/4}. \tag{2.17}$$

Also, we will denote by $a^*(t)$ the supremum of a function a over $(0,t)$. We have

$$\begin{aligned}
h\int_0^t ((D_+v^h)^2, |u^h|^4)_h ds &= \int_0^t \sum_{j\in\mathbb{Z}} (v_{j+1}-v_j)^2|u_j|^4 ds \\
&\le \int_0^t \|u^h\|_\infty^2 \sum_{j\in\mathbb{Z}} (v_{j+1}-v_j)^2|u_j|^2 ds \\
&\le c\|D_+u^h\|_2^* \int_0^t \sum_{j\in\mathbb{Z}} (v_{j+1}-v_j)^2|u_j|^2 ds \\
&\le \|D_+u^h\|_2^* \Big(c + c\int_0^t \|D_+u^h\|_2 ds \Big).
\end{aligned}$$

Next, we have

$$\begin{aligned}
&h\int_0^t (|u^h|^2 D_+v^h, uD_+u^h)_h ds = \int_0^t \sum_{j\in\mathbb{Z}} (v_{j+1}-v_j)|u_j|^2(\overline{u}_{j+1}-\overline{u}_j)\overline{u}_j ds \\
&\le \int_0^t \Big(\sum_{j\in\mathbb{Z}} (v_{j+1}-v_j)^2|u_j|^4\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}} |u_{j+1}-u_j|^2|u_j|^2\Big)^{1/2} ds \\
&\le \int_0^t \|u^h\|_\infty \Big(\sum_{j\in\mathbb{Z}} (v_{j+1}-v_j)^2|u_j|^2\Big)^{1/2} 2\|u^h\|_\infty^{1/2}\Big(\sum_{j\in\mathbb{Z}} |u_{j+1}-u_j||u_j|^2\Big)^{1/2} ds \\
&\le c(\|D_+u^h\|_2^{1/2})^* \int_0^t \Big(\sum_{j\in\mathbb{Z}} (v_{j+1}-v_j)^2|u_j|^2\Big)^{1/2} \|D_+u^h\|_2^{1/4} \\
&\qquad \cdot \Big(\sum_{j\in\mathbb{Z}} h|u_{j+1}-u_j|^2/h^2\Big)^{1/4}\Big(\sum_{j\in\mathbb{Z}} h|u_j|^4\Big)^{1/4} ds
\end{aligned}$$

and so

$$\begin{aligned}
&h\int_0^t (D_+v^h|u^h|^2, uD_+u^h)_h ds \\
&\quad \le c\|D_+u^h\|_2^* \Big(\int_0^t \sum_{j\in\mathbb{Z}} (v_{j+1}-v_j)^2|u_j|^2 ds\Big)^{1/2}\Big(\int_0^t \|D_+u^h\|_2 ds\Big)^{1/2} \\
&\quad \le c\|D_+u^h\|_2^* \Big(c + c\int_0^t \|D_+u\|_2 ds \Big).
\end{aligned}$$

The remaining terms in (2.15) are estimated using similar techniques and yield similar terms, each one bounded by

$$c\|D_+u^h\|_2^* \Big(c + c\int_0^t \|D_+u\|_2 ds \Big).$$

The desired estimate (2.6) now follows from (2.15) and the previous estimates by a Gronwall argument with the function $\sup_{(0,t)}(1+\|D_+u^h\|_2^2)^{1/2}$. This completes the proof of Lemma 2.3. □

2.2 Proof of convergence

In this section we prove Theorem 2.2, relying on the compensated compactness method [12, 13], adapted to the present case in [7]. According to this method, the strong compactness of a sequence of approximate solutions (v^h) is a consequence of the following property:

$$\partial_t \eta(v^h) + \partial_x\big(q_1(v^h) - |u^h|^2 q_2(v^h)\big) \in \{\text{ compact of } W^{-1,2}_{\text{loc}}\}, \tag{2.18}$$

where, we recall, $\eta(v)$ is a convex function (the *entropy*), and the *entropy fluxes* $q_{1,2}$ verify $q_1'(v) = \eta'(v)f'(v)$ and $q_2'(v) = \eta'(v)g''(v)$. In practice, one may use the following result to establish (2.18): If $1 < q < 2 < r \leq \infty$, then

$$\{\text{ compact of } W^{-1,q}_{\text{loc}}\} \cap \{\text{ bounded in } W^{-1,r}_{\text{loc}}\} \subset \{\text{ compact of } W^{-1,2}_{\text{loc}}\}. \tag{2.19}$$

Theorem 2.2 is an immediate consequence of the following result.

Lemma 2.4. *Let (u^h, v^h) be defined by the semidiscrete approximation* (2.2a)–(2.2c)*. Then, the compactness property in* (2.18) *is valid. Moreover, the (strong) limits (u, v) of (u^h, v^h) are the unique entropy solution of the problem under consideration.*

Proof. Let ϕ be continuous and compactly supported on $\mathbb{R} \times (0, \infty)$. In what follows, $\phi_j = \phi(x_j, t)$, $I_j = [x_j, x_{j+1}]$, $\overline{\phi}_j = \frac{1}{h}\int_{I_j} \phi(y, t)dy$, and θ_j denotes various intermediate values. Let $J = J(h) \in \mathbb{N}, t' > 0$ be such that $[x_{-J}, x_J] \times [0, t'] \supset \operatorname{supp} \phi$. We have

$$\begin{aligned}
&-\iint_{\mathbb{R}\times[0,\infty)} \phi(x,t)\partial_t\eta(v^h) + \phi(x,t)\partial_x(q_1(v^h) - q_2(v^h)|u^h|^2)\,dxdt \qquad (2.20)\\
&= -\int_0^\infty h\sum_{j\in\mathbb{Z}} \overline{\phi}_j \eta'(v_j)\partial_t v_j dt - \int_0^\infty \sum_{j\in\mathbb{Z}} \phi_{j+1}\big(q_1(v_{j+1}) - q_1(v_j)\big)dt\\
&\quad + \int_0^\infty \sum_{j\in\mathbb{Z}} \phi_{j+1}\big(q_2(v_{j+1})|u_{j+1}|^2 - q_2(v_j)|u_j|^2\big)dt.
\end{aligned}$$

From the definition of v^h, (2.2b), we find (recall that $|u|^2_{j\pm1/2} = (|u_{j\pm1}|^2 + |u_j|^2)/2$)

$$\begin{aligned}
-\int_0^\infty h\sum_{j\in\mathbb{Z}} \overline{\phi}_j \eta'(v_j)\partial_t v_j dt = \int_0^\infty \sum_{j\in\mathbb{Z}} \overline{\phi}_j \bigg\{ &\frac{\eta'(v_j)}{2}(f(v_{j+1}) - f(v_{j-1}))\\
&- \frac{\eta'(v_j)}{2}\big(g'(v_{j+1})|u_{j+1}|^2 - g'(v_{j-1})|u_{j-1}|^2\big)\\
&- \eta'(v_j)\frac{h^2}{2\lambda}\Delta^h v_j - \eta'(v_j)\frac{h}{2\gamma}\Big(D_+ v_j|u|^2_{j+1/2} - D_- v_j|u|^2_{j-1/2}\Big)\bigg\}.
\end{aligned}$$

Now, from (2.20), we find after some calculation

$$\int_0^\infty \sum_{j\in\mathbb{Z}} \overline{\phi}_j \frac{\eta'(v_j)}{2}\Big(f(v_{j+1}) - f(v_{j-1}) - \frac{h^2}{\lambda}\Delta^h v_j\Big) - \phi_{j+1}\big(q_1(v_{j+1}) - q_1(v_j)\big)dt$$

$$\leq \int_0^{t'} \sum_{|j|\leq J} |D_j(\phi)A_j(u^h, v^h)|dt \tag{2.21}$$

$$- \int_0^{t'} \sum_{|j|\leq J} \phi_{j+1} \int_{v_j}^{v_{j+1}} \Big(\frac{1}{2}(f_j + f_{j+1}) - f(s) - \frac{1}{2\lambda}(v_{j+1} - v_j)\Big)\eta''(s)\, ds\, dt,$$

where $D_j(\phi)$ is either $\phi_{j+1} - \phi_j$ or $\overline{\phi}_j - \phi_j$, and $A_j(u^h, v^h)$ verifies $|A_j| \leq c|v_{j+1} - v_j|$. Following [9], suppose that ϕ is α-Hölder continuous, for some $\alpha \in (1/2, 1)$. We find

$$\begin{aligned}
\int_0^{t'} \sum_{|j|\leq J} |D_j(\phi)A_j(u^h, v^h)|dt &\leq Ch^\alpha \|\phi\|_{0,\alpha} \int_0^{t'} \sum_{|j|\leq J} |v_{j+1} - v_{j-1}|\, dt \\
&\leq Ch^{\alpha-1/2} \|\phi\|_{0,\alpha} \int_0^{t'} \big(\sum_{|j|\leq J} |v_{j+1} - v_{j-1}|^2\big)^{1/2} \big(\sum_{|j|\leq J} h\big)^{1/2} dt \\
&\leq Ch^{\alpha-1/2} \|\phi\|_{0,\alpha} \Big(\int_0^{t'} \sum_{|j|\leq J} |v_{j+1} - v_{j-1}|^2 dt\Big)^{1/2} \Big(\int_0^{t'} \sum_{|j|\leq J} h dt\Big)^{1/2} \\
&\leq C'h^{\alpha-1/2} \|\phi\|_{0,\alpha} \sqrt{t'a(t')}.
\end{aligned}$$

We have used (2.5) before the last line. Since $W^{1,q'} \subset C^{0,\alpha}$, with compact embedding, for $q' \geq 2/(1-\alpha) > 4$ (and thus $q \in (1, 4/3)$), we see that this term is compact in $W^{-1,q}_{\text{loc}}$ for $q \in (1, 4/3) \subset (1, 2)$. For the other term in (2.21) we obtain

$$\begin{aligned}
\int_0^{t'} \sum_{|j|\leq J} \phi_{j+1} \int_{v_j}^{v_{j+1}} \Big|\frac{1}{2}(f_j + f_{j+1}) - f(s) - \frac{1}{2\lambda}(v_{j+1} - v_j)\Big|\eta''(s)\, ds\, dt \\
\leq C\|\phi\|_\infty \int_0^{t'} \sum_{|j|\leq J} |v_{j+1} - v_j|^2 dt \leq C\|\phi\|_\infty a(t'),
\end{aligned}$$

showing that this term is bounded in the space $\mathcal{M}$ of bounded Radon measures (on the support of ϕ), which is compactly embedded in $W^{-1,q}(\operatorname{supp}\phi)$ for any $q \in [1, 2)$.

Similarly, we obtain after some manipulation and using the uniform L^∞

bounds on u^h, v^h,

$$
\begin{aligned}
-\int_0^{t'} \sum_{|j|\le J} \overline{\phi}_j \frac{\eta'(v_j)}{2} \bigg(& g'(v_{j+1})|u_{j+1}|^2 - g'(v_{j-1})|u_{j-1}|^2 \\
& - \frac{h}{\gamma}\Big(D_+ v_j |u|^2_{j+1/2} - D_- v_j |u|^2_{j-1/2} \Big) \bigg) dt \\
& + \int_0^{t'} \sum_{|j|\le J} \phi_{j+1} \Big(q_2(v_{j+1})|u_{j+1}|^2 - q_2(v_j)|u_j|^2 \Big) dt \\
\le & \int_0^{t'} \sum_{|j|\le J} |D_j(\phi)| \big| A_j(u^h, v^h) + B_j(u^h) \big| dt \qquad (2.22) \\
& + \int_0^{t'} \sum_{|j|\le J} \phi_{j+1} \big| C_j(u^h, v^h) + E_j(v^h) \big| dt,
\end{aligned}
$$

where $D_j(\phi)$ and A_j are as in (2.21), and

$$
|B_j(u^h)| \le c\big||u_{j+1}|^2 - |u_j|^2\big|, \quad |C_j(u^h, v^h)| \le c\big||u_{j+1}|^2 - |u_j|^2\big||v_{j+1} - v_j|,
$$

$$
|E_j(v^h)| \le c|v_{j+1} - v_j|^2.
$$

We have from $\|u^h\|_\infty \le C\|D_+ u^h\|_2^{1/2}$ and from (2.6),

$$
\begin{aligned}
\int_0^{t'} \sum_{|j|\le J} |D_j(\phi)||B_j(u^h)| dt &\le C h^\alpha \|\phi\|_{0,\alpha} \int_0^{t'} \sum_{|j|\le J} |u_{j+1} - u_j| dt \\
&\le C h^\alpha \|\phi\|_{0,\alpha} \int_0^{t'} h^{-1/2} \Big(\sum_{|j|\le J} |u_{j+1} - u_j|^2 \Big)^{1/2} \Big(\sum_{|j|\le J} h \Big)^{1/2} dt \\
&\le C(t') h^\alpha \|\phi\|_{0,\alpha},
\end{aligned}
$$

and so this term tends to zero with h in $W^{1,q'}_{\mathrm{loc}}$ for suitable q'. Next,

$$
\begin{aligned}
\int_0^{t'} \sum_{|j|\le J} \phi_{j+1} |C_j(u^h, v^h)| dt &\le C\|\phi\|_\infty \int_0^{t'} \sum_{|j|\le J} |u_{j+1} - u_j|\, |v_{j+1} - v_j| dt \\
&\le C\|\phi\|_\infty \int_0^{t'} \Big(\sum_{|j|\le J} |u_{j+1} - u_j|^2 \Big)^{1/2} \Big(\sum_{|j|\le J} |v_{j+1} - v_j|^2 \Big)^{1/2} dt \\
&\le C\|\phi\|_\infty \int_0^{t'} h^{1/2} \Big(\sum_{|j|\le J} |v_{j+1} - v_j|^2 \Big)^{1/2} dt \\
&\le C\|\phi\|_\infty \Big(\int_0^{t'} h \sum_{|j|\le J} |v_{j+1} - v_j|^2 dt \Big)^{1/2} t'^{1/2},
\end{aligned}
$$

which tends to zero with h in view of (2.5). In particular, this term is uniformly bounded in $\mathcal{M}$ and compact in $W^{-1,q}_{\text{loc}}$ for suitable q. The remaining terms with A_j and E_j have been treated before. Finally, from (2.20), taking $\phi \in W^{1,1}$, it is immediate that the left-hand side is bounded in, say, $W^{-1,\infty}_{\text{loc}}$. The desired compactness property (2.18) now follows from (2.19).

It remains to show that the limit pair (u, v) of the approximate solutions (u^h, v^h) is a weak entropy solution of the equations (see Definition 2.1). We focus on the treatment of the second equation, since the first equation is more straightforward and very similar to the treatment in [7] and [1]. Thus, we must prove that for every non-negative smooth function with compact support, one has

$$-\iint_{\mathbb{R}\times[0,\infty)} \phi(x,t)\partial_t\eta(v) + \phi(x,t)\partial_x(q_1(v) - q_2(v)|u|^2)$$
$$- \phi(x,t)\big(\eta'(v)g'(v) - q_2(v)\big)\partial_x|u|^2\,dxdt + \int_{\mathbb{R}} \eta(v_0(x))\phi(x,0)\,dx \geq 0.$$

Computing from (2.20) as in (2.21), we find (omitting the term at $t = 0$, whose treatment is straightforward, for brevity)

$$\int_0^{t'} \sum_{|j|\leq J} |D_j(\phi)A_j(u^h, v^h)|dt \leq h^{1/2}\|\phi'\|_\infty C(t).$$

Note that this gives the familiar $h^{1/2}$ rate of convergence for conservation laws in one space dimension. Still from (2.21), we find the term

$$-\int_0^{t'} \sum_{|j|\leq J} \phi_{j+1} \int_{v_j}^{v_{j+1}} \Big(\frac{1}{2}(f_j + f_{j+1}) - f(s) - \frac{1}{2\lambda}(v_{j+1} - v_j)\Big)\eta''(s)\,ds\,dt,$$

This term is non-negative, the proof of which is classical and follows from the CFL condition. Similarly, calculating as in (2.22), we find using the estimates of Lemma 2.3,

$$\begin{aligned}
&\text{L.H.S. of (2.22)} \\
&= \int_0^{t'} \sum_{|j|\leq J} \phi_{j+1}|u_j|^2 \int_{v_j}^{v_{j+1}} \Big(\frac{1}{2}(g'_j + g'_{j+1}) - g'(s) + \frac{1}{2\gamma}(v_{j+1} - v_j)\Big)\eta''(s)\,ds\,dt \\
&\quad + \mathcal{O}(h^{1/2}) + A,
\end{aligned}$$

with $A \to \iint_{\mathbb{R}\times[0,\infty)} \phi(x,t)\big(\eta'(v)g'(v) - q_2(v)\big)\partial_x|u|^2\,dxdt$ as $h \to 0$. From the CFL condition, we also find that the first term above is non-negative. Thus, passing to the limit as $h \to 0$ in (2.20), we see that the entropy inequality is verified. This completes the proof of Lemma 2.4. $\square$

3 A finite difference approximation of a Schrödinger–Korteweg–de Vries coupled system

3.1 Statement of the convergence result

In this section, we consider the Schrödinger–Korteweg–de Vries system (1.5a)–(1.5c), which we rewrite here for convenience,

$$i\partial_t u + \partial_{xx} u = |u|^2 u + vu \tag{3.1a}$$

$$\partial_t v + \partial_x^3 v + \partial_x(v^2) = \partial_x |u|^2 \tag{3.1b}$$

$$u(0) = u_0, \quad v(0) = v_0, \tag{3.1c}$$

arising in the study of capillarity-gravity interaction.

We propose the following semidiscrete finite difference approximation to the equations (3.1a)–(3.1c),

$$i\partial_t u^h + \Delta^h u^h = |u^h|^2 u^h + v^h u^h \tag{3.2}$$

$$\partial_t v^h + D^3 v^h + D_0 (v^h)^2 = D_0 |u^h|^2 \tag{3.3}$$

$$u^h(0) = u_0^h, \quad v^h(0) = v_0^h \tag{3.4}$$

(recall that $D^3 = D_0 D_+ D_-$). Let $\mathbf{P}_1^h$ denote the piecewise linear and continuous interpolator. Our main result in this section establishes the convergence of the approximations (3.2)–(3.4) towards the unique global solution of problem (3.1a)–(3.1c).

Theorem 3.1. *Let (u^h, v^h) be the global solutions of the discretized problem* (3.2)–(3.4) *with initial data (u_0^h, v_0^h), such that $\mathbf{P}_1^h u_0^h \rightharpoonup u_0$ and $\mathbf{P}_1^h v_0^h \rightharpoonup v_0$ weakly in $H^1(\mathbb{R})$ as $h \to 0$. Then, up to a subsequence,*

$$\mathbf{P}_1^h u^h \overset{*}{\rightharpoonup} u, \quad \mathbf{P}_1^h v^h \overset{*}{\rightharpoonup} v \qquad in \quad L^\infty\big([-T,T]; H^1(\mathbb{R})\big),$$

$$\mathbf{P}_1^h u^h \to u, \quad \mathbf{P}_1^h v^h \to v \qquad in \quad L^\infty\big([-T,T]; L^2_{\mathrm{loc}}(\mathbb{R})\big),$$

with (u^h, v^h) the unique strong solution of the Schrödinger–KdV system (3.1a)–(3.1c) *in $(H^1(\mathbb{R}))^2$.*

Let us outline the proof of Theorem 3.1, before providing detailed arguments. The global existence proof of Tsutsumi [14] relies on energy methods which we could not carry over to the finite difference framework. In the continuous case, no *a priori* L^∞ bound is needed to prove global existence, although one knows, *a posteriori,* that the solutions will be uniformly bounded (since they are in $H^1(\mathbb{R})$). However, in the semidiscrete case the analysis relies heavily on an *a priori* L^∞ bound. To deal with this difficulty, we consider an auxiliary problem (see Lemmas 3.2 and 3.3 below) admitting uniform L^∞ bounds and we use the fact that, under the right conditions, this problem reduces to the original one.

3.2 Proof of Theorem 3.1

Here, we prove Theorem 3.1. First, we must consider a suitably truncated problem. For this we consider the following functions, which are simply truncations of the functions v and v^2 appearing in (1.5a),(1.5b). Thus, for each $M > 0$ we define some C^∞ functions f^M, g^M satisfying

$$f^M(v) = \begin{cases} v^2, & \text{if } |v| \leq M, \\ |v|, & \text{if } |v| > M^2 + 1, \end{cases}$$

and

$$g^M(v) = \begin{cases} v, & \text{if } |v| \leq M, \\ \pm C, & \text{if } |v| > 2M. \end{cases}$$

Here, the constant $C = C(M)$ is chosen to ensure the following property,

$$|(f^M)'|_\infty + |g^M|_\infty \leq C(M), \quad |(g^M)'|_\infty \leq 1. \tag{3.5}$$

We define also $F(v) = \int_0^v f(s)ds$ (with $f = f^M$).

We may now state our first auxiliary result. Its proof follows along the lines of [8] and so we omit it.

Lemma 3.2. *Let $M > 0$. Then, there exists a strong global solution $(u^M, v^M) \in \big(C([-T,T]; H^1(\mathbb{R}))\big)^2$ of the truncated problem*

$$i\partial_t u + \partial_{xx} u = |u|^2 u + g^M(v)u \tag{3.6}$$

$$\partial_t v + \partial_x^3 v + \partial_x f^M(v) = \partial_x((g^M)'(v)|u|^2) \tag{3.7}$$

$$u(0) = u_0, \quad v(0) = v_0. \tag{3.8}$$

Moreover, the L^2-norm of u is conserved, and the following energy estimate is valid:

$$E(t) \equiv 2\int_{\mathbb{R}} |u_x|^2 + \int_{\mathbb{R}} |u|^4 + 2\int_{\mathbb{R}} g(v)|u|^2 + \int_{\mathbb{R}} (v_x)^2 + \frac{2}{3}\int_{\mathbb{R}} F(v) = E(0). \tag{3.9}$$

Next, we make the following crucial observation: the solution (u, v) of the original problem (1.5a)–(1.5c) verifies energy bounds [14] which imply, in particular,

$$|u|_\infty + |v|_\infty \leq C(|u|_2^2 + |u_x|_2^2 + |v|_2^2 + |v_x|_2^2) \leq \bar{C}(u_0, v_0). \tag{3.10}$$

Thus, if we choose $M > \bar{C}$, then by uniqueness of solutions, and by the definition of f, g above, we deduce that problems (1.5a)–(1.5c) and (3.6)–(3.8) are equivalent.

Consider now the semidiscrete finite difference approximation of equations (3.6)–(3.8):

$$i\partial_t u^h + \Delta^h u^h = |u^h|^2 u^h + g^M(v^h)u^h \tag{3.11}$$

$$\partial_t v^h + D^3 v^h + D_0 f^M(v^h) = D_0((g^M)'(v^h)|u^h|^2) \tag{3.12}$$

$$u^h(0) = u_0^h, \quad v^h(0) = v_0^h. \tag{3.13}$$

We have:

Lemma 3.3. *For each $M > 0$, let $(u^{h,M}, v^{h,M})$ be the solution of* (3.11)–(3.13). *Then, up to a subsequence,*

$$\mathbf{P}_1^h u^{h,M} \overset{*}{\rightharpoonup} u^M, \quad \mathbf{P}_1^h v^{h,M} \overset{*}{\rightharpoonup} v^M \qquad in \quad L^\infty\big([-T,T]; H^1(\mathbb{R})\big),$$

$$\mathbf{P}_1^h u^{h,M} \to u^M, \quad \mathbf{P}_1^h v^{h,M} \to v^M \qquad in \quad L^\infty\big([-T,T]; L^2_{\text{loc}}(\mathbb{R})\big),$$

with

$$(u^M, v^M) \in C([-T,T]; H^1(\mathbb{R})) \times C([-T,T]; L^2(\mathbb{R})) \tag{3.14}$$

a strong solution of (3.6)–(3.8).

We postpone the proof until the next section. Once Lemma 3.3 is proved, we see that by taking M large enough, namely $M > \bar{C}$ (see (3.10)), then $\mathbf{P}_1^h u^{h,M}$ and $\mathbf{P}_1^h v^{h,M}$ will converge in $L^\infty([-T,T]; H^1(\mathbb{R}))$ weak * and strongly in $L^\infty([-T,T]; L^2(\mathbb{R}))$ to

$$(u^M, v^M) \equiv (u,v) \in C([-T,T]; H^1(\mathbb{R})) \times C([-T,T]; L^2(\mathbb{R})),$$

a strong solution of the problem (1.5a)–(1.5c), and so, by the uniqueness of the strong local solutions, the original problem (1.5a)–(1.5c) (cf. [5]), we conclude that (u,v) is the unique strong solution of that problem in $\big(C([-T,T]; H^1(\mathbb{R}))\big)^2$.

Moreover, we have

$$\begin{aligned}\limsup_{h\to 0}(|u^{h,M}|_\infty + |v^{h,M}|_\infty) \\ &\le C \limsup_{h\to 0}(|u^{h,M}|_2^2 + |u_x^{h,M}|_2^2 + |v^{h,M}|_2^2 + |v_x^{h,M}|_2^2) \\ &\le C(|u|_2^2 + |u_x|_2^2 + |v|_2^2 + |v_x|_2^2) \le \bar{C}(u_0, v_0) < M.\end{aligned}$$

Thus, the approximate solutions $u^{h,M}, v^{h,M}$ for sufficiently small h are bounded in L^∞ by M. in view of the definition of the functions f^M, g^M, we conclude that $u^{h,M}, v^{h,M}$ verify the equations (3.2)–(3.4). By uniqueness of discrete solution, we deduce finally that the solutions of (3.2)–(3.4) converge to the solutions of the original Schrödinger–KdV system (1.5a)–(1.5c), which concludes the proof of Theorem 3.1. □

3.3 Proof of Lemma 3.3

It remains to prove Lemma 3.3. We first remark that for $h > 0$ fixed and for each data $(u_0^h, v_0^h) \in (l_h^2(\mathbb{Z}))^2$, there exists a unique global solution $(u^h(t), v^h(t)) \in \big(C(\mathbb{R}; l_h^2(\mathbb{Z}))\big)^2$ of (3.11)–(3.13). This is an easy consequence of the classical fixed-point theorem, which gives a local solution, and of (3.17) below.

In the remainder of the proof, we omit the superscript M. We begin by multiplying equation (3.12) by hv_j and summing over $j \in \mathbb{Z}$. Summing by parts, we find

$$\partial_t |v_j|_2^2 = \sum_{j\in\mathbb{Z}} hf(v_j) D_0 v_j - \sum_{j\in\mathbb{Z}} hg'(v_j)|u_j|^2 D_0 v_j.$$

Now observe that from $f(v_j) = f(v_j) - f(0) = f'(\theta_j)v_j$ and $|g'| \le 1$ we find

$$\partial_t |v^h|_2^2 \le C(M)(|D_0 v^h|_2^2 + |v^h|_2^2) + |u^h|_4^4.$$

Integrating on $(0,t)$ and using Gronwall's Lemma gives

$$|v^h(t)|_2^2 \le a(t,M) \int_0^t |D_0 v^h(s)|_2^2 + |u^h(s)|_4^4 ds \tag{3.15}$$

for some continuous function a which may change from one occurrence to the next. On the other hand, an energy estimate which is obtained in the same way as (3.9) gives

$$|D_0 u_j|_2^2 + |D_0 v_j|_2^2 + |u_j|_4^4 \le C(u_0, v_0) + \sum_{j\in\mathbb{Z}} h|g(v_j)||u_j|^2 + \sum_{j\in\mathbb{Z}} h|F(v_j)|. \tag{3.16}$$

We have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} h|F(v_j)| &= \sum_{j\in\mathbb{Z}} h|F(v_j) - F(0)| = \sum_{j\in\mathbb{Z}} h|f(\theta_j)v_j| \\ &\le \sum_{j\in\mathbb{Z}} hf(\theta_j)^2 + |v^h|_2^2\end{aligned}$$

for some θ_j between 0 and v_j. Now,

$$\sum_{j\in\mathbb{Z}} hf(\theta_j)^2 = \sum_{v_j \le M^2+1} hf(\theta_j)^2 + \sum_{v_j > M^2+1} hf(\theta_j)^2.$$

For j such that $v_j \le M^2+1$, we have $f(\theta_j)^2 \le v_j^4 \le (M^2+1)v_j^2$. For j such that $v_j > M^2+1$, we have $f(\theta_j)^2 \le v_j^2$. Thus we obtain

$$\sum_{j\in\mathbb{Z}} h|F(v_j)| \le C(M)|v^h|_2^2.$$

Similarly, since the L^2-norm of u^h is conserved (as is easily seen), we find

$$\sum_{j\in\mathbb{Z}} h|g(v_j)||u_j|^2 \le |g|_\infty |u^h|_2^2 \le C(M)|u_0^h|_2^2.$$

These estimates together with (3.15) and (3.16) give

$$\begin{aligned}|D_0 u^h|_2^2 + |D_0 v^h|_2^2 + |u^h|_4^4 &\le C(u_0^h, v_0^h, M)|v^h|_2^2 \\ &\le C(u_0^h, v_0^h, t, M) \int_0^t |D_0 v^h(s)|_2^2 + |u^h(s)|_4^4 ds,\end{aligned}$$

so that a Gronwall argument, (3.16) and the conservation of $|u^h|_2^2$ give

$$|D_0 u^h|_2^2 + |D_0 v^h|_2^2 + |u^h|_2^2 + |v^h|_2^2 \le C(u_0^h, v_0^h, t, M). \tag{3.17}$$

To establish the convergence of u^h, v^h towards the solution of (3.6)–(3.8), we apply the piecewise linear interpolator $\mathbf{P}_1^h$ to the equations (3.11)–(3.13):

$$i\partial_t \mathbf{P}_1^h u^h + \Delta^h \mathbf{P}_1^h u^h = \mathbf{P}_1^h(|u^h|^2 u^h) + \mathbf{P}_1^h(g^M(v^h)u^h) \tag{3.18}$$

$$\partial_t \mathbf{P}_1^h v^h + D^3 \mathbf{P}_1^h v^h + D_0 \mathbf{P}_1^h f^M(v^h) = D_0 \mathbf{P}_1^h((g^M)'(v^h)|u^h|^2). \tag{3.19}$$

From the estimate (3.17) we deduce that there exists (u, v) such that (up to a subsequence),

$$\mathbf{P}_1^h u^h \overset{*}{\rightharpoonup} u, \quad \mathbf{P}_1^h v^h \overset{*}{\rightharpoonup} v \qquad \text{in} \quad L^\infty\big([-T,T]; H^1(\mathbb{R})\big),$$

$$\mathbf{P}_1^h u^h \to u, \quad \mathbf{P}_1^h v^h \to v \qquad \text{in} \quad L^\infty\big([-T,T]; L^2_{\mathrm{loc}}(\mathbb{R})\big).$$

Now, we consider the piecewise constant interpolator $\mathbf{P}_0^h$, which commutes with the nonlinearity. Since

$$\mathbf{P}_1^h f(v^h) - \mathbf{P}_0^h f(v^h) \to 0 \qquad \text{in } L^\infty\big([-T,T]; L^2(\mathbb{R})\big)$$

and

$$\mathbf{P}_0^h f(v^h) = f(\mathbf{P}_0^h v^h) \to f(v) \qquad \text{in } L^\infty\big([-T,T]; L^2_{\mathrm{loc}}(\mathbb{R})\big),$$

we deduce that

$$\mathbf{P}_1^h f(v^h) \overset{*}{\rightharpoonup} f(v) \qquad \text{in} \quad L^\infty\big([-T,T]; H^1(\mathbb{R})\big)$$

and so

$$D_0 \mathbf{P}_1^h f(v^h) \overset{*}{\rightharpoonup} \partial_x f(v) \qquad \text{in} \quad L^\infty\big([-T,T]; H^1(\mathbb{R})\big).$$

Similarly,

$$\mathbf{P}_1^h(|u^h|^2 u^h) \overset{*}{\rightharpoonup} |u|^2 u \qquad \text{in} \quad L^\infty\big([-T,T]; H^1(\mathbb{R})\big),$$

$$\mathbf{P}_1^h(g(v^h)u^h) \overset{*}{\rightharpoonup} g(v)u \qquad \text{in} \quad L^\infty\big([-T,T]; H^1(\mathbb{R})\big),$$

$$D_0 \mathbf{P}_1^h(g'(v^h)|u^h|^2) \overset{*}{\rightharpoonup} \partial_x(g'(v)|u|^2) \qquad \text{in} \quad L^\infty\big([-T,T]; L^2(\mathbb{R})\big).$$

By taking the limit $h \to 0$ in equations (3.18),(3.19), we obtain a strong solution

$$(u, v) \in \big(L^\infty\big([-T,T]; H^1(\mathbb{R})\big)\big)^2$$

of (3.6)–(3.8) in the space

$$C([-T,T]; H^{-1}(\mathbb{R})) \times C([-T,T]; H^{-2}(\mathbb{R})).$$

It remains to prove the regularity property (3.14). Let $U_S(t)$ and $U_K(t)$ be the free Schrödinger and KdV unitary groups defined by

$$\begin{aligned} U_S(t) &= e^{it\partial_{xx}} = \mathcal{F}^{-1} e^{-it|\xi|^2} \mathcal{F} \\ U_K(t) &= e^{it\partial_x^3} = \mathcal{F}^{-1} e^{-it|\xi|^3} \mathcal{F} \end{aligned}, \quad t \in \mathbb{R}.$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and the Fourier transform in x, respectively. The solution

$$(u,v) \in \left(L^\infty\left([-T,T];H^1\right)\right)^2 \cap \left(C\left([-T,T];H^{-1}\right) \times C\left([-T,T];H^{-2}\right)\right)$$

satisfies the integral system:

$$u(t) = U_S(t)u_0 + \int_0^t U_S(t-s)J_1(u(s),v(s))\,ds,$$

$$v(t) = U_K(t)v_0 + \int_0^t U_K(t-s)J_2(u(s),v(s))\,ds,$$

with

$$\begin{aligned} J_1(u,v) &= |u|^2u + g^M(v)u, \\ J_2(u,v) &= -\partial_x f^M(v) + \partial_x((g^M)'(v)|u|^2). \end{aligned}$$

A standard computation in semigroup theory leads to the expressions:

$$u(t+h) - u(t) = (U_S(h) - I)\,u(t) + \int_t^{t+h} U_S(t+h-s)J_1(u(s),v(s))\,ds,$$

$$v(t+h) - v(t) = (U_K(h) - I)\,v(t) + \int_t^{t+h} U_K(t+h-s)J_2(u(s),v(s))\,ds,$$

for all $h \in \mathbb{R}$. Since

$$\|J_1(u,v)\|_{H^1} \leq C\left(\|u_0\|_{H^1}, \|v_0\|_{H^1}\right), \quad \|J_2(u,v)\|_2 \leq C\left(\|u_0\|_{H^1}, \|v_0\|_{H^1}\right),$$

we deduce that

$$\|u(t+h) - u(t)\|_{H^1} \overset{h\to 0}{\longrightarrow} 0, \quad \|v(t+h) - v(t)\|_2 \overset{h\to 0}{\longrightarrow} 0,$$

and therefore

$$(u,v) \in C\left([-T,T];H^1\right) \times C\left([-T,T];L^2\right).$$

This completes the proof of Lemma 3.3. □

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