On the influence of an absorption term in incompressible fluid flows

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Abstract This work is concerned with a mathematical problem derived from the Ellis model used in Fluid Mechanics to describe the response of a great variety of generalized fluid flows. For pseudoplastic fluids, it is well-known that the weak solutions to that problem extinct in a finite time. In order to obtain the same property for Newtonian and dilatant fluids, we modify the problem by introducing an absorption term in the momentum equation. The proof relies on a suitable energy method, Sobolev type interpolation inequalities and also on a generalized Korn's inequality. Then we extend our results for several cases: slip boundary conditions, anisotropic absorption and non-homogeneous fluid flows. We also discuss existence and uniqueness of weak solutions for the modified problem.

1 Introduction

In Fluid Mechanics, the most widely used constitutive relation to model the response of incompressible and homogeneous fluids is

$$\mathbf{T} = -p\mathbf{I} + \mathbf{F}(\mathbf{D}), \quad \mathbf{F}(\mathbf{D}) = \alpha_1(\mathbb{I}_{\mathbf{D}})\mathbf{D}, \quad \mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right). \tag{1}$$

The notation is well known: $\mathbf{u}$ is the velocity field, p is the pressure, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{I}$ is the unit tensor, $\mathbf{D}$ is the stretching tensor which mathematically corresponds to the symmetric part of the velocity gradient $\nabla\mathbf{u}$, usually denominated in Fluid Mechanics as the shear rate, and $\mathbb{I}_{\mathbf{D}} = 1/2|\mathbf{D}|^2$ is the first invariant of $\mathbf{D}$. In this work, we assume the extra stress tensor $\mathbf{F}$ and the stretching tensor $\mathbf{D}$ are related by the following rheological flow law

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$$\mathbf{F} = \left(\mu_0 + \mu_1 |\mathbf{D}|^{q-2}\right) \mathbf{D}\,. \tag{2}$$

Parameters μ_0, μ_1 and q are non-negative, being the first two related with the fluid viscosity and the latter characterizing the flow. Equations (1)-(2) are known in the literature as the Ellis model and describe the response of a great variety of generalized fluids. For $q >> 1$ and small values of $|\mathbf{D}|$, (1)-(2) tend to approximate the Stokes model for Newtonian fluids. On the contrary, for $q << 1$ and great values of $|\mathbf{D}|$, they approximate the Ostwald-de Waele model for power law fluids, very often used to model non-Newtonian fluids. Therefore many fluid models can be obtained from the Ellis law (1)-(2) by combining the parameters μ_0, μ_1 and q as follows:

$$\begin{cases} \text{Newtonian} & \text{if } \mu_0 > 0 \text{ and } \mu_1 = 0 \\ \text{Ostwald-de Waele} & \text{if } \mu_0 = 0 \text{ and } \mu_1 > 0 \quad \begin{cases} \text{Bingham} & \text{if } q = 1 \\ \text{pseudoplastic} & \text{if } 1 < q < 2 \\ \text{Newtonian} & \text{if } q = 2 \\ \text{dilatant} & \text{if } q > 2\,. \end{cases} \end{cases}$$

Examples of such fluids are water solutions, gasoline, vegetal and mineral oils (Newtonian), drilling muds used in petroleum industry, toothpaste and face creams (Bingham), milk fluids, varnishes, shampoo and blood fluids (pseudoplastic), polar ice, glaciers, volcano lava and sand (dilatant). It is well known that a pseudoplastic fluid is characterized by a viscosity that decreases with shear rate and a dilatant fluid by a viscosity that increases with shear rate. As a consequence, many authors rather use to denote them as shear thinning and shear thickning fluids, respectively. Bingham fluid is similar to a pseudoplastic fluid, but it exhibits a yield point. The existence of a yield point means fluid flow is prevented below a critical stress level, but flow occurs when the critical stress level is exceeded. Other names found in the literature for Bingham fluids are plastic fluids and sometimes viscoplastic fluids to avoid confusion with the word plastic as applied to solid polymers. See Schowalter [18] and the references therein for the models more often used in non-Newtonian Fluid Mechanics.

From the basic principles of Fluid Mechanics, it is well known that, in motions of incompressible fluids modeled by the Ellis law (1)-(2) (with neither inner mass sources nor sinks), the velocity field and pressure are determined by:

- the incompressibility condition

$$\operatorname{div}\mathbf{u} = \mathbf{0}; \tag{3}$$

- the conservation of mass

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0; \tag{4}$$

- the conservation of momentum

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} \right) = \rho\,\mathbf{f} - \nabla p + \operatorname{div}\left[\left(\mu_0 + \mu_1 |\mathbf{D}|^{q-2}\right) \mathbf{D} \right]. \tag{5}$$

Again in (3)-(5), ρ is the density and $\mathbf{f}$ is the forcing term. For homogeneous fluids, the density is regarded as constant. Therefore, in that case, we can replace the continuity equation (4) by its incompressible form (3). System (3)-(5) must be supplemented with boundary conditions characterizing the flow on the boundary of the domain occupied by the fluid and by initial conditions determining the initial state of the flow at the beginning of the time interval.

2 The modified Ellis problem

In this section, we shall introduce the main problem we are going to work with. Let us consider a general cylinder

$$Q_T := \Omega \times (0,T) \subset \mathbb{R}^N \times \mathbb{R}^+ ,$$

where Ω is a bounded domain whose boundary $\partial\Omega$ is assumed to be smooth enough. The boundary of Q_T is defined by

$$\Gamma_T := (0,T) \times \partial\Omega .$$

The dimensions of physical interest are $N = 2$ and $N = 3$, but the results to be presented here extend to any dimension $N \geq 2$. In this paper we shall consider the following modified Ellis problem (MEP) for homogeneous fluids

$$\operatorname{div} \mathbf{u} = 0 \tag{6}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\left[\left(\nu_0 + \nu_1 |\mathbf{D}|^{q-2}\right)\mathbf{D}\right] + \alpha|\mathbf{u}|^{\sigma-2}\mathbf{u} = \mathbf{f} - \nabla p \tag{7}$$

supplemented by the initial condition

$$\mathbf{u} = \mathbf{u}_0 \quad \text{when} \quad t = t_0 \tag{8}$$

and by the adherence boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_T . \tag{9}$$

In (7), we have set $p = p/\rho$, and $\nu_0 = \mu_0/\rho$ and $\nu_1 = \mu_1/\rho$ are non-negative parameters related to the kinematics viscosity, $\alpha = \alpha/\rho$ is a non-negative constant and, for the time being, σ is a constant such that $\sigma \geq 1$. The term $|\mathbf{u}|^{\sigma-2}\mathbf{u}$ is understood as an absorption and, as we shall see, it behaves like a sink inside the flow domain. Its motivation is purely mathematical and goes back to the works of Benilan, Brezis and Crandall, Díaz and Herrero, and Bernis (see Antontsev *et al.* [5] and Díaz [10]) Of particular importance to this work is the Dirichlet problem

$$-\text{div}\left(|\nabla u|^{q-2}\nabla u\right)+|u|^{\sigma-2}u=f \text{ in } \Omega\subset\mathbb{R}^N, \quad q\geq 1,\ \sigma>1, \\ u=h \text{ on } \partial\Omega \tag{10}$$

presented in Díaz [10] as modeling a stationary non-Newtonian fluid. When (10) is linear, *i.e.* with $p=\sigma=2$, the solution u of (10) corresponding to data, say $f\geq 0$ and $h\geq 0$, is such that $u>0$ in Ω. When (10) is nonlinear, entirely different behavior may appear. Roughly speaking, the effective power of the diffusion term $\text{div}\left(|\nabla u|^{p-2}\nabla u\right)$ and of the absorption term $|u|^{\sigma-2}u$ vary with p and σ, generating new phenomena. If $\sigma\geq p$, Ω is an unbounded open domain and f and h have compact support, then the support of the solution contains the whole domain Ω and if $\sigma<p$ the solution u has compact support and so $u=0$ in an unbounded region of Ω. With this motivation, the purpose of this work is to study the asymptotic behavior of the weak solutions to the MEP problem (6)-(9). Specifically we want to know if these solutions extinct in a finite time and if so, what is the relation between the parameter q which characterizes the flow and the absorption constants α and σ. The problem of the solutions vanishing in some space region is much more difficult and we address the reader to Antontsev *et al.* [3, 4], where 2D stationary Navier-Stokes problems were studied. It should be remarked that these issues have been studied by many authors, either in time or in space. There exists an extensive literature on decay rates for the solutions of the Navier-Stokes problem (see Oliveira [17] and the references cited therein). For the non-Newtonian setting, the literature is scarce, although many techniques of the Navier-Stokes model can be used. In Bae [6] it was proved that the L^2 norm of weak solutions to the homogeneous problem (6)-(9) with $\alpha=0$, decreases with rate $t^{-1/(q-2)}$ as t tends to infinity for the dilatant case ($q>2$) and vanishes in a finite time for the pseudoplastic case ($1<q<2$). The same problem was solved by Guo and Zhu [11] and by Ňečasová and Penel [16], but only for the dilatant case ($q>2$). As regards to the non-homogeneous problem (6)-(9), with $\alpha=0$, and to the best of our knowledge, there is no references where such kind of decays are studied.

3 Notation and auxiliary results

The notation used throughout this paper is largely standard in Mathematical Fluid Mechanics - see, *e.g.*, Lions [14]. We distinguish vectors from scalars by using boldface letters. For functions and function spaces we will use this distinction as well. The symbol C will denote a generic constant - generally a positive one, whose value will not be specified; it can change from one inequality to another by subscripting different numbers and it can be related to an important result by subscripting initial letters of that result. The dependence of C on other constants or parameters will always be clear from the exposition. In this article, the notation Ω stands always for a domain, *i.e.*, a connected open subset of $\mathbb{R}^N$, whose compact boundary is denoted by $\partial\Omega$.

Let $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^N$ be a domain. We shall use the classical Lebesgue spaces $\mathrm{L}^p(\Omega)$, whose norm is denoted by $\|\cdot\|_{\mathrm{L}^p(\Omega)}$. $\mathrm{W}^{1,p}(\Omega)$ denotes the Sobolev space of all functions $u \in \mathrm{L}^p(\Omega)$ such that the weak derivatives $\mathrm{D}u$ exist, in the generalized sense, and are in $\mathrm{L}^p(\Omega)$. Given $T > 0$ and a Banach space X, $\mathrm{L}^p(0,T;X)$ is the Bochner space used in evolutive problems. The corresponding spaces of vector-valued functions are denoted by boldface letters. All these spaces are Banach spaces and the Hilbert framework corresponds to $p = 2$. In the last case, we use the abbreviation $\mathrm{W}^{1,2} = \mathrm{H}^1$.

To prove the main results of this paper, it will be of the utmost importance some Sobolev type inequalities established in lemmas below.

Lemma 1 (Korn). *Assume $1 < p < \infty$ and let Ω be a domain of $\mathbb{R}^N$, $N \geq 2$, with a locally compact boundary $\partial\Omega$. Then for any $\mathbf{u} \in \mathbf{W}_0^{1,p}(\Omega)$,*

$$C_K \|\nabla \mathbf{u}\|_{\mathbf{L}^p(\Omega)} \leq \|\mathbf{D}(\mathbf{u})\|_{\mathbf{L}^p(\Omega)}, \qquad C_K = C(p,\Omega). \tag{11}$$

This is the so-called second Korn's inequality and it extends to suitable unbounded domains (see *e.g.* Ladyzhenskaya *et al.* [13])

Lemma 2 (Interpolation Embedding). *Let Ω be a domain of $\mathbb{R}^N$, $N \geq 1$, with a locally compact boundary $\partial\Omega$. Assume that $u \in \mathrm{W}_0^{1,p}(\Omega)$. Then, for every fixed number $r \geq 1$ there exists a constant C_{GN} depending only on N, p, r such that*

$$\|u\|_{\mathrm{L}^q(\Omega)} \leq C_{GN} \|\nabla u\|_{\mathrm{L}^p(\Omega)}^{\theta} \|u\|_{\mathrm{L}^r(\Omega)}^{1-\theta}, \tag{12}$$

where $p,\ q \geq 1$, are linked by $\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{1}{r}\right)^{-1}$, and their admissible range is:

(1) If $N = 1$, $q \in [r,\infty]$, $\theta \in \left[0, \frac{p}{p+r(p-1)}\right]$, $C_{GN} = [1+(p-1)/pr]^{\theta}$;

(2) If $p < N$, $q \in \left[\frac{Np}{N-p}, r\right]$ if $r \geq \frac{Np}{N-p}$ and $q \in \left[r, \frac{Np}{N-p}\right]$ if $r \leq \frac{Np}{N-p}$, $\theta \in [0,1]$ and $C_{GN} = [(N-1)p/(N-p)]^{\theta}$;

(3) If $p \geq N > 1$, $q \in [r,\infty)$, $\theta \in \left[0, \frac{Np}{Np+r(p-N)}\right)$ and $C_{GN} = \max\{q(N-1)/N, 1+(p-1)pr\}^{\theta}$.

The interpolation inequality (12) is known in the literature as Gagliardo-Nirenberg inequality. This result is valid whether the domain Ω is bounded or not and notice the constant C_{GN} does not depend on Ω (see *e.g.* Ladyzhenskaya *et al.* [13]). Notice also that in the particular case of $\theta = 1$, (12) reduces to the well-known Sobolev's inequality and, in this case, the constant will be denoted by C_S.

4 Weak formulation

The mathematical analysis of incompressible fluid problems is commonly done in the context of divergence free function spaces. Working in this context, we introduce the following function spaces:

$$\mathscr{V} := \{\mathbf{v} \in \mathbf{C}_0^\infty(\Omega) : \operatorname{div}\mathbf{v} = 0\};$$

$$\mathbf{H} := \text{closure of } \mathscr{V} \text{ in } \mathbf{L}^2(\Omega);$$

$$\mathbf{V}_q := \text{closure of } \mathscr{V} \text{ in } \mathbf{W}^{1,q}(\Omega).$$

It is worth to recall that $\mathbf{H}$ is endowed with the $\mathbf{L}^2(\Omega)$ inner product and norm, and $\mathbf{V}_q$ is a Banach space whose norm is inherited from the Sobolev space $\mathbf{W}^{1,q}(\Omega)$. For a bounded domain Ω, $\mathbf{V}_2 \subset \mathbf{V}_q$ if $q < 2$ and $\mathbf{V}_q \subset \mathbf{V}_2$ if $q > 2$. Below we define the notion of weak solutions we shall work with.

Definition 1. Let Ω be a bounded domain in $\mathbf{R}^N$, $N \geq 2$, with a Lipschitz boundary $\partial\Omega$, and let $q,\ \sigma > 1$. A vector field $\mathbf{u}$ is a weak solution to the MEP problem (6)-(9), if:
1. $\mathbf{u} \in \mathrm{L}^2(0,T;\mathbf{V}) \cap \mathrm{L}^q(0,T;\mathbf{V}_q) \cap \mathrm{L}^\sigma(Q_T) \cap \mathrm{L}^\infty(0,T;\mathbf{H})$;
2. The following identity

$$-\int_{Q_T} \mathbf{u}\cdot\varphi_t\,d\mathbf{z} - \int_{Q_T} \mathbf{u}\otimes\mathbf{u} : \nabla\varphi\,d\mathbf{z} + \int_{Q_T} \left(\mu_0 + \mu_1|\mathbf{D}(\mathbf{u}|^{q-2}\right)\mathbf{D}(\mathbf{u}) : \nabla\varphi\,d\mathbf{z}$$
$$+\alpha\int_{Q_T} |\mathbf{u}|^{\sigma-2}\mathbf{u}\cdot\varphi\,d\mathbf{z} = \int_{Q_T} \mathbf{f}\cdot\varphi\,d\mathbf{x} + \int_\Omega \mathbf{u}_0\cdot\varphi(0)\,d\mathbf{x}$$

holds for all $\varphi \in \mathbf{C}^\infty(Q_T)$ with $\operatorname{div}\varphi = 0$ and $\operatorname{supp}\varphi \subset\subset \Omega\times[0,T)$, and where $\mathbf{z} = (\mathbf{x},t)$.

To the best of our knowledge, the MEP problem (6)-(9) is new and there are no references to it in the literature. On the other hand, for the problem (6)-(9) with $\alpha = 0$ there are some important works. The first results on existence and uniqueness of weak solutions were achieved by Ladyzhenskaya [12] for $N = 2$ and $N = 3$. Lions [14] has extended her results to a general dimension $N \geq 2$. Combining monotone operator theory and compactness arguments, he has proved the existence of weak solutions for $q > (3N+2)/(N+2)$ and their uniqueness for $q \geq (N+2)/2$. For the success of those proofs, it was of the utmost importance the embedding

$$\mathrm{L}^q(0,T;\mathbf{V}_q) \cap \mathrm{L}^\infty(0,T;\mathbf{H}) \hookrightarrow \mathrm{L}^{q\frac{N+2}{N}}(Q_T)$$

to prove that $\mathbf{u}\otimes\mathbf{u} : \mathbf{D}(\varphi) \in \mathrm{L}^1(Q_T)$ for all $\mathbf{u}$, $\varphi \in \mathrm{L}^q(0,T;\mathbf{V}_q) \cap \mathrm{L}^\infty(0,T;\mathbf{H})$ (see Lions [14]). For $2 \leq q \leq 11/5$ and $N = 3$, the existence of weak solutions has been proved by Málek *et al.* [15] under the restrictive assumption that $\partial\Omega \in \mathrm{C}^3$. Moreover, they have proved a uniqueness result for any $q > 20/9$. Without any restriction on Ω, Wolf [19] has proved the existence of weak solutions in the space $\mathrm{L}^q(0,T;\mathbf{V}_q) \cap \mathrm{C}_w(0,T;\mathbf{H})$ for $q > 2(N+1)/(N+2)$. For the existence results of Málek *et al.* [15] and Wolf [19], were very important the q-coercivity condition

$$\mathbf{F} : \mathbf{D} \geq C|\mathbf{D}|^q, \quad C = constant > 0 \tag{13}$$

and the q-growth condition

$$|\mathbf{F}| \leq C\,(1+|\mathbf{D}|)^{q-1}\,, \quad C = constant > 0\,.$$

Although the results of Wolf [19] extend to some values $q < 2$, so far it is an open problem to prove the existence of weak solutions for all $q > 1$. The main obstacle is the fact that one is unable to construct the pressure as a measurable function in $(\mathbf{x},t) \in Q_T$, but only as a distribution with regard to $t \in (0,T)$. On the contrary, for the space periodic problem, the full existence result is proved for any $q > 1$ (see Bae and Choe [7]).

Theorem 1. *Assume that $q > 2N/(N+2)$, $\sigma > 1$ and $\boldsymbol{u}_0 \in \boldsymbol{H}$. Then, there exists, at least, a weak solution to the MEP problem (6)-(9) in the sense of Definition 1.*

The proof can be adapted from Wolf [19] by using the arguments of Bernis [9] to deal with the absorption term if $\sigma > 2$. For $1 < \sigma \leq 2$, we use the Sobolev embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^\sigma(\Omega)$ valid for $1 \leq \sigma \leq 2N/(N-2)$ if $N \geq 3$ and for any $\sigma \geq 1$ if $N = 2$. In this case and in order to prove the convergence of the approximate solutions, it is also important the inequality (14), established in the Lemma below, to deal with the absorption term, where we have to take $p = \sigma$ and $\delta = 2-\sigma$, which in turn implies $\sigma \leq 2$.

Lemma 3. *For all $p \in (1,\infty)$ and $\delta \geq 0$, there exist constants C_1 and C_2, depending on p and N, such that for all ξ, $\eta \in \mathbb{R}^N$, $N \geq 1$,*

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leq C_1|\xi-\eta|^{1-\delta}\,(|\xi|+|\eta|)^{p-2+\delta} \tag{14}$$

and

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi-\eta) \geq C_2|\xi-\eta|^{2+\delta}\,(|\xi|+|\eta|)^{p-2-\delta} \tag{15}$$

Proof. See Barret and Liu [8]. □

Below we present a result about uniqueness which, for $N = 3$, can be improved by using the results of the same kind of Málek *et al.* [15].

Theorem 2. *Assume that $q > 2N/(N+2)$, $\sigma > 1$, $\boldsymbol{f} \in \mathrm{L}^{q'}(Q_T)$ and $\boldsymbol{u}_0 \in \boldsymbol{H}$. Then the weak solution of the MEP problem (6)-(9) is unique.*

For $q \geq (N+2)/2$, the uniqueness result can be derived from Lions [14]. Here, the problem lies in proving that the difference of absorptions from two weak solutions is non-negative. But this follows directly from (15) by taking $p = \sigma$ and $\delta = 0$.

Shortly, existence and uniqueness of weak solutions to the MEP problem (6)-(9) hold for any $\sigma > 1$ as far as the same results hold for the problem (6)-(9) with $\alpha = 0$. If one can improve these results for any $q > 1$, then the same results for the MEP problem (6)-(9) can be obtained.

5 Extinction in time

In this section, we are interested in weak solutions for the MEP problem (6)-(9) such that

$$E(t)+\int_{\Omega}\left(|\nabla\mathbf{u}(t)|^2+|\nabla\mathbf{u}(t)|^q\right)d\mathbf{x}<\infty,\quad E(t):=\frac{1}{2}\int_{\Omega}|\mathbf{u}(t)|^2\,d\mathbf{x}. \tag{16}$$

We denote by $E(t)$ the energy associated with the MEP problem (6)-(9), which in Fluid Mechanics is usually denoted as the kinetic energy.

Theorem 3. *Assume that* $1<\sigma<2$, $\boldsymbol{u}_0\in\boldsymbol{H}$ *and let* $\boldsymbol{u}$ *be a weak solution to the MEP problem (6)-(9) in the sense of Definition 1.*
1. If $\mathbf{f}=\mathbf{0}$ a.e. in Q_T, then there exists $t^*>0$ such that $\mathbf{u}(\mathbf{x},t)=\mathbf{0}$ a.e. in Ω and for almost all $t\geq t^*$.
2. Let $\mathbf{f}\neq\mathbf{0}$ and assume that there exist positive constants ε and θ and that there exists a positive time $t_{\mathbf{f}}$ such that, for almost all $t\in[0,T]$,

$$\|\mathbf{f}(t)\|_{\mathbf{L}^{q'}(\Omega)}\leq\varepsilon\left(1-\frac{t}{t_{\mathbf{f}}}\right)_+^{\theta}\quad\text{if}\quad\frac{Nq}{N-q}\leq q<2\,,\quad(\theta\text{ is given by (30)}), \tag{17}$$

or

$$\|\mathbf{f}(t)\|_{\mathbf{L}^{q'}(\Omega)}\leq\varepsilon\left(1-\frac{t}{t_{\mathbf{f}}}\right)_+^{\theta}\quad\text{if}\quad q>2\,,\quad(\theta\text{ is given by (38)}). \tag{18}$$

Then there exists a constant $\varepsilon_0>0$ (defined by (30) for (17) and by (39) for (18)) such that $\mathbf{u}=\mathbf{0}$ a.e. in Ω and for almost all $t\geq t_{\mathbf{f}}$ provided $0<\varepsilon\leq\varepsilon_0$.

Notice that, although the property is the same, the constants ε, θ, $t_{\mathbf{f}}$ and ε_0 can be distinct in the different cases (17) and (18). The notation u_+ means the positive part of u, *i.e.* $u=\max(u,0)$.

Proof. We formally multiply (7) by $\mathbf{u}$, a weak solution to the MEP problem (6)-(9), integrate over Ω and use (6), (9) and the symmetry of $\mathbf{D}$. Then we use Korn's inequality (11) to obtain the following energy relation

$$\begin{aligned}&\frac{d}{dt}E(t)+\int_{\Omega}\left(C_0|\nabla\mathbf{u}(t)|^2+C_1|\nabla\mathbf{u}(t)|^q+\alpha|\mathbf{u}(t)|^{\sigma}\right)d\mathbf{x}\\&\leq\int_{\Omega}\mathbf{u}(t)\cdot\mathbf{f}(t)\,d\mathbf{x}\quad\text{for a.a. }t\in[0,T]\,,\end{aligned} \tag{19}$$

where $C_0=C_K^2\nu_0$ and $C_1=C_K^q\nu_1$, being C_K the Korn's inequality constant. Applying successively Schwarz's, Young's and Sobolev's (12) inequalities, the latter with $p=q$, we obtain

$$\left|\int_{\Omega}\mathbf{u}(t)\cdot\mathbf{f}(t)\,d\mathbf{x}\right|\leq\varepsilon C_S^q\|\nabla\mathbf{u}(t)\|^q_{\mathbf{L}^q(\Omega)}+C(\varepsilon)\|\mathbf{f}(t)\|^{q'}_{\mathbf{L}^{q'}(\Omega)}\ \text{for a.a. }t\in[0,T]\,, \tag{20}$$

where C_S is the Sobolev inequality constant. Choosing $\varepsilon : 0<\varepsilon<C_1/C_S^q$, we obtain from (19) and (20)

$$\begin{aligned} &\frac{d}{dt}E(t)+\left(\int_\Omega \left(C_0|\nabla \mathbf{u}(t)|^2+C_2|\nabla \mathbf{u}(t)|^q+\alpha|\mathbf{u}(t)|^\sigma\right)\,d\mathbf{x}\right) \\ &\leq C_3\int_\Omega |\mathbf{f}(t)|^{q'}\,d\mathbf{x} \quad \text{for a.a. } t\in[0,T]\,, \end{aligned} \tag{21}$$

where $C_2=C_1-\varepsilon C_S^q$ and $C_3=C(\varepsilon)$. To be precise, (21) is first established for the approximations of the weak solutions to the MEP problem (6)-(9) and then is shown to be true for the limit function.

Case $2N/(N+2)\leq q<2$. Using a vector version of Sobolev's inequality (12), we obtain

$$E(t)\leq C_4\left(\int_\Omega |\nabla \mathbf{u}(t)|^q\,d\mathbf{x}\right)^{\frac{2}{q}} \quad \text{for a.a. } t\in[0,T]\,,\quad q\geq\frac{2N}{N+2}\,, \tag{22}$$

where $C_4=C_S^2/2$. Taking into account that C_0 and α are non-negative constants, we obtain from (19) and (22) that

$$\frac{d}{dt}E(t)+C_5E(t)^{\frac{q}{2}}\leq C_3\int_\Omega |\mathbf{f}(t)|^{q'}d\mathbf{x} \quad \text{for a.a. } t\in[0,T]\,, \tag{23}$$

where $C_5=C_2C_4^{-q/2}$. If $\mathbf{f}=\mathbf{0}$, then we obtain the following homogeneous ordinary differential inequality for the energy function $E(t)$

$$\frac{d}{dt}E(t)+C_5E(t)^{\frac{q}{2}}\leq 0 \quad \text{for a.a. } t\in[0,T]. \tag{24}$$

Knowing that $2N/(N+2)\leq q<2$, an explicit integration of (24) between $t=0$ and t leads us to

$$E(t)\leq\left(E(0)^{\frac{2-q}{2}}-\frac{2-q}{2}C_3t\right)^{\frac{2}{2-q}} \quad \text{for a.a. } t\in[0,T]. \tag{25}$$

The right-hand side of (25) vanishes for $t\geq t^*:=2E(0)^{(2-q)/2}/[C_5(2-q)]$ and the first assertion is then proved.

If $\mathbf{f}$ satisfies to (17), then from (23), we obtain the following non-homogeneous ordinary differential inequality

$$\frac{d}{dt}E(t)+C_5E(t)^{\frac{q}{2}}\leq C_3\varepsilon^{q'}\left(1-\frac{t}{t_{\mathbf{f}}}\right)_+^{q'\theta}. \tag{26}$$

To analyze (26), we need the following result.

Lemma 4. *Let $\delta > 0$ such that $\big(t_f - \delta, t_f + \delta\big) \subset [0,T]$ and assume $E \in \mathrm{W}^{1,1}\big(t_f - \delta, t_f + \delta\big)$ satisfies the differential inequality*

$$\frac{d}{dt}E(t) + \varphi(E(t)) \leq F\left(\left(1 - \frac{t}{t_f}\right)_+\right) \quad \textit{a.e. in } \big(t_f - \delta, t_f + \delta\big), \tag{27}$$

where φ is a continuous non-decreasing function such that

$$\varphi(0) = 0 \quad \textit{and} \quad \int_{0^+} \frac{ds}{\varphi(s)} < \infty, \tag{28}$$

and the function F satisfies, for some $\overline{k} \in (0,1)$, to

$$F(s) \leq (1 - \overline{k})\varphi(\eta_{\overline{k}}(s)) \quad \textit{in } (0, t_f), \tag{29}$$

where

$$\eta_k(s) = \theta_k^{-1}(s) \quad \textit{and} \quad \theta_k(s) = \int_0^s \frac{d\tau}{k\varphi(\tau)}.$$

Then $E(t) = 0$ for all $t \geq t_f$.

Proof. See Antontsev *et al.* [5]. □

In order to read (26) in the form (27), we define

$$\varphi(s) := C_5 s^{\frac{q}{2}} \quad \text{and} \quad F(s) := C_3 \varepsilon^{q'} s^{\theta q'}.$$

Then clearly (28) is satisfied and we have

$$\theta_k(s) = \frac{2}{kC_5(2-q)} s^{\frac{2-q}{2}} \quad \text{and} \quad \eta_k(s) = \left(\frac{kC_5(2-q)}{2} s\right)^{\frac{2}{2-q}}.$$

Moreover, (29) is satisfied if

$$\theta := \frac{q-1}{2-q} \quad \text{and} \quad \varepsilon \leq \varepsilon_0 := \frac{(1-k)C_5}{C_3}\left(\frac{kC_5(2-q)}{2}\right)^{\frac{q}{2-q}} \tag{30}$$

for a certain $k \in (0,1)$, *e.g.* $k = q/2$. Then Lemma 4 proves the second assertion.

Case $q \geq 2$. Using a vector version of the interpolation embedding inequality (12) with $q = 2$, $p = q$ and $r = \sigma$, and the algebraic inequality $A^{\alpha}B^{\beta} \leq (A+B)^{\alpha+\beta}$ valid for every α, $\beta \in \mathbb{R}$ and every A, $B \geq 0$, we obtain

$$E(t) \leq C_6\left(\int_{\Omega} \left(|\nabla \mathbf{u}(t)|^q + |\mathbf{u}(t)|^{\sigma}\right) d\mathbf{x}\right)^{\mu} \quad \text{for a.a. } t \in [0,T], \tag{31}$$

where $C_6 = C_{GN}^2/2$, being C_{GN} the interpolation embedding inequality constant, and

$$\mu := 1 + \frac{q(2-\sigma)}{q(N+\sigma)-N\sigma}. \tag{32}$$

The analysis of (32) shows us that

$$q \geq 2 \quad \Rightarrow \quad \mu > 1 \text{ iff } \sigma < 2. \tag{33}$$

Taking into account that C_0 is a non-negative constant, we obtain from (21) and (31),

$$\frac{d}{dt}E(t) + C_7 E(t)^{\frac{1}{\mu}} \leq C_3 \int_\Omega |\mathbf{f}(t)|^{q'}\, d\mathbf{x} \quad \text{for a.a. } t \in [0,T], \tag{34}$$

where $C_7 = \min(C_2, \alpha) C_6^{-1/\mu}$ and C_3 is given in (21). If $\mathbf{f} = \mathbf{0}$, then (34) leads us to the homogeneous ordinary differential inequality

$$\frac{d}{dt}E(t) + C_7 E(t)^{1/\mu} \leq 0. \tag{35}$$

An explicit integration of (35) between $t = 0$ and t, where we use (32)-(33), leads us to

$$E(t) \leq \left(E(0)^{\frac{\mu-1}{\mu}} - \frac{C_7(\mu-1)}{\mu} t \right)^{\frac{\mu}{\mu-1}} \tag{36}$$

and $E(t)$ vanishes for $t \geq t^* := \mu E(0)^{\frac{\mu-1}{\mu}} / [C_7(\mu-1)]$. This proves the first assertion.

Assume now that (18) is satisfied. Then, using (18) and (34), we obtain the following non-homogeneous ordinary differential inequality

$$\frac{d}{dt}E(t) + C_7 E(t)^{1/\mu} \leq C_3 \varepsilon^{q'} \left(1 - \frac{t}{t_\mathbf{f}}\right)_+^{\theta} \qquad \text{for a.a. } t \in [0,T], \tag{37}$$

where C_7 is given in (34) and C_3 in (21). In order to use Lemma 4, let

$$\varphi(s) = C_7 s^{\frac{1}{\mu}} \quad \text{and} \quad F(s) = C_3 \varepsilon^{q'} s^{\frac{1}{\mu-1}}.$$

Then clearly (28) is satisfied,

$$\theta_k(s) = \frac{\mu}{C_7 k(\mu-1)} s^{\frac{\mu-1}{\mu}}, \qquad \eta_k(s) = \left(\frac{C_7 k(\mu-1)}{\mu} s \right)^{\frac{\mu}{\mu-1}}$$

and (29) holds provided

$$\theta := \frac{q-1}{q(\mu-1)}, \qquad (32) \quad \Rightarrow \quad \theta = \frac{(q-1)[q(N+\sigma)-N\sigma]}{q^2(2-\sigma)} \tag{38}$$

and

$$\varepsilon \leq \varepsilon_0 := \left\{ \frac{C_7(1-k)}{C_3} \left[\frac{k(\mu-1)}{\mu} \right]^{\frac{1}{\mu-1}} \right\}^{\frac{q-1}{q}}, \tag{39}$$

the latter for some $k \in (0,1)$, *e.g.* $k = \mu/2$. Second assertion is thus proved and this concludes the proof. $\square$

Remark 1. Theorem 3 still holds for unbounded domains Ω as far as the used Sobolev type inequalities hold.

Remark 2. We could also have considered non-homogeneous boundary conditions, say $\mathbf{u}_*$, on Γ_T. But then, in order to carry out the results of Theorem 3, we would have to assume the existence of a time $t_* > 0$ such that $\mathbf{u}_* = \mathbf{0}$ for all $t \geq t_*$ and $E(t_*) < \infty$. In the above proof we only would have to replace the time $t = 0$ by $t = t_*$.

Remark 3. For the 2D stationary version of the MEP problem (6)-(9), we are able to prove that its weak solutions have compact support in Ω. This is obtained by using the arguments of Antontsev *et al.* [3, 4] and reducing the original problem to a fourth-order non-linear one for the stream function, where the pressure term does not appear anymore.

6 Discussion and extensions

In the previous section, we have shown the weak solutions to the MEP problem (6)-(9) extinct in a finite time wether $2N/(N+2) \leq q < 2$ or $q \geq 2$. In both cases the diffusion term $|\nabla \mathbf{u}|^2$ does not play any role in the obtained results. Therefore, we may simplify the exposition by saying, at the beginning, that the extra stress tensor should satisfy to the q-coercivity condition (13), instead of (2). This is certainly the case if $\mathbf{F}$ is given by (2). In such situation, the energy relation (19) would come in the form

$$\frac{d}{dt}E(t) + \int_\Omega \left(C_1 |\nabla \mathbf{u}(t)|^q + \alpha |\mathbf{u}(t)|^\sigma\right) d\mathbf{x} \leq \int_\Omega \mathbf{u}(t) \cdot \mathbf{f}(t)\, d\mathbf{x} \quad \text{for a.a. } t \in [0,T]\,,$$

where now $C_1 = C_K^q C$, being C the constant resulting from (13). The analysis of the case when $2N/(N+2) \leq q < 2$ takes only into account the q-diffusion term $|\nabla \mathbf{u}|^q$. Not only the diffusion term, but also the absorption term $|\mathbf{u}|^\sigma$ is neglected. Therefore, in this case, the extinction in a finite time property holds for all fluid problems governed by the equations (3)-(5), with $2N/(N+2) \leq q < 2$, supplemented with the initial-boundary conditions (8)-(9). Moreover, once the diffusion term $|\nabla \mathbf{u}|^2$ is useless for the obtained properties, we may also consider the q-coercivity condition (13), instead of (2), and the energy relation (19) would come in the form

$$\frac{d}{dt}E(t) + C_1 \int_\Omega |\nabla \mathbf{u}(t)|^q\, d\mathbf{x} \leq \int_\Omega \mathbf{u}(t) \cdot \mathbf{f}(t)\, d\mathbf{x} \quad \text{for a.a. } t \in [0,T]\,.$$

We thus can say that for a pseudoplastic fluid the structure of the stress tensor is able to stop the flow in a finite time. For $q \geq 2$, the analysis of the previous case is no longer valid, because in (23) we have now $q/2 \geq 1$. If $\mathbf{f} = \mathbf{0}$, then (24) comes with $q/2 \geq 1$. An explicit integration of (24) between $t = 0$ and t leads us to

$$E(t) \leq \left(C_3 \frac{q-2}{2} t + E(0)^{-(q-2)/2} \right)^{-2/(q-2)} \quad \text{for a.a. } t \in [0,T], \quad q > 2.$$

If $q = 2$, then (24) is linear and the same integration procedure leads us to

$$E(t) \leq E(0) e^{-C_3 t} \quad \text{for a.a. } t \in [0,T].$$

Then $E(t)$ decays to zero at the rate $t^{-2/(q-2)}$ if $q > 2$ and with an exponential decay if $q = 2$. This means that dilatant fluid flows tend to extinct with the fractional time rate $t^{-2/(q-2)}$ and Newtonian fluid flows with the exponential time rate e^{-t}. If $\mathbf{f} \neq \mathbf{0}$, we obtain analogous results as for $\mathbf{f} = \mathbf{0}$ (see Bae [6], Guo and Zhu [11] and Nečasová and Penel [16]). As a consequence, for $q \geq 2$, if we want to obtain an analogous property as for the case $2N/(N+2) \leq q < 2$, we need to modify the momentum equation (5) by introducing there the absorption term $|\mathbf{u}|^{\sigma-2}\mathbf{u}$. As we have seen, in this case, the results are valid under the assumption that $\sigma < 2$ (see (32)-(33)). On the other hand for the well-posedness of the MEP problem (6)-(9), one needs to assume that $\sigma > 1$. In the limit case of $\sigma = 1$, the extinction in a finite time property remains valid, because it is still true that $\mu > 1$ (see (32)-(33)). If $\sigma = 2$, then the obtained ordinary differential inequalities become linear and using this framework we can only obtain an exponential decay.

These properties extend for the MEP problem (6)-(8) supplemented with the following slip boundary conditions:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{u} \cdot \tau = \beta^{-1} \mathbf{t} \cdot \tau \quad \text{on} \quad \Gamma_T;$$

where $\mathbf{n}$ and τ denote, respectively, unit normal and tangential vectors to the boundary $\partial\Omega$, $\mathbf{t} = \mathbf{n} \cdot \mathbf{T}$ is the stress vector and β is a coefficient with no defined sign. In this case, the energy relation (19) comes in the form

$$\frac{d}{dt} E(t) + \int_\Omega \left(C_0 |\nabla \mathbf{u}(t)|^2 + C_1 |\nabla \mathbf{u}(t)|^q + \alpha |\mathbf{u}(t)|^\sigma \right) d\mathbf{x} \leq$$
$$\int_\Omega \mathbf{u}(t) \cdot \mathbf{f}(t)\, d\mathbf{x} + \beta \int_{\partial\Omega} |\mathbf{u}(t)|^2\, d\mathbf{x} \quad \text{for a.a. } t \in [0,T].$$

The last term on the right-hand side is estimated by using an interpolation trace inequality (see Antontsev and Oliveira [2] and Ladyzhenskaya *et al.* [13]). Proceeding in an analogous way, we can prove that, regardless the sign of β, the weak solutions to this problem extinct in a finite time if $q < 2$ or if $1 < \sigma < 2$.

We can also consider the MEP problem (6)-(9) with an anisotropic absorption, *i.e.* if we replace the modified momentum equation (7) by the following one

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{div}\,\left[\left(\nu_0 + \nu_1|\mathbf{D}|^{q-2}\right)\mathbf{D}\right] + \left(\alpha_1|u_1|^{\sigma_1-2}u_1,\ldots,\alpha_N|u_N|^{\sigma_N-2}u_N\right)$$
$$= \mathbf{f} - \nabla p\,,$$

where α_1, ..., α_N are non-negative constants. For the weak solutions to this problem, we are able to obtain the same properties if, at most, $\alpha_i = 0$ for only one i and if the domain Ω is convex in that direction x_i, where $i \in \{1,\ldots,N\}$. If that direction is $i = N$, then the energy relation (19) comes in the form

$$\frac{d}{dt}E(t) + \int_\Omega \left(C_0|\nabla\mathbf{u}(t)|^2 + C_1|\nabla\mathbf{u}(t)|^q + \sum_{i=1}^{N-1}\alpha_i|u_i(t)|^{\sigma_i}\right)d\mathbf{x} \leq$$
$$\int_\Omega \mathbf{u}(t)\cdot\mathbf{f}(t)\,d\mathbf{x} \quad \text{for a.a. } t\in[0,T]\,.$$

We prove the extinction in a finite time property by using the same techniques of the previous section to estimate all the components u_i, with $i = 1,\ldots,N-1$, and the incompressibility condition (6) to estimate u_N (see Antontsev and Oliveira [1]).

The main results established in this paper can be generalized to non-homogeneous fluid flows. In this case, we only need to assume that the density function ρ is bounded as follows

$$\frac{1}{C_\rho} \leq \rho\,,\ \rho_0 \leq C_\rho\,,$$

where C_ρ is a positive constant and ρ_0 is the initial density. The energy function is now defined by

$$E(t) := \frac{1}{2}\int_\Omega \rho(t)|\mathbf{u}(t)|^2\,d\mathbf{x}$$

and the energy relation comes such as in (23) and in (34), with the constants appearing there depending also on C_ρ. See Antontsev *et al.* [5] where this property was obtained for fluid problems with the extra stress tensor satisfying to the q-coercivity condition (13) with $1 < q < 2$, *i.e.* pseudoplastic fluids.

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