Well-posedness and existence of bound states for a coupled Schrödinger-gKdV system

João-Paulo Dias(1), Mário Figueira(1) and Filipe Oliveira(2)

(1) CMAF/UL and FCUL, Av. Prof. Gama Pinto, 2
1649-003 Lisboa-Portugal
(2) Centro de Matemática e Aplicações, FCT-UNL
Monte da Caparica-Portugal

Abstract

We derive some new results concerning the Cauchy problem and the existence of bound states for a class of coupled nonlinear Schrödinger-gKdV systems. In particular, we obtain the existence of strong global solutions for initial data in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, generalizing previous results obtained in [26], [9] and [14] for the nonlinear Schrödinger-KdV system.

1 Introduction

The generalized KdV (gKdV) equation

$$v_t + v_{xxx} + a(v)v_x = 0, \quad a \in C^\infty(\mathbb{R}), \tag{1}$$

was introduced in [29] as a model for the propagation of nonlinear waves in an anharmonic lattice. In recent years, many authors treated the case where the function a is a polynomial: see for instance [17],[5] for well-posedness, [6] for the stability of solitary waves and [8],[25],[1] for the study of the dispersion rates of the solutions to (1). In contrast, little results exist in the literature concerning more general nonlinearities (see on this subject [28], [23] and [16]).

In the present paper, we will consider the coupled system

$$\begin{cases} iu_t + u_{xx} = \alpha uv + \beta|u|^2u & (a) \\ v_t + v_{xxx} + a(v)v_x = \gamma(|u|^2)_x & (b) \end{cases} \tag{2}$$

email adresses:
dias@ptmat.fc.ul.pt(J.P.Dias); figueira@ptmat.fc.ul.pt(M.Figueira); fso@fct.unl.pt(F.Oliveira)

with initial data

$$\begin{array}{l} u(x,0) = u_o(x), \\ v(x,0) = v_o(x), \quad x \in \mathbb{R}, \end{array} \tag{3}$$

where α, β and γ are real constants, u is a complex-valued function, v is a real-valued function and a is a C^∞ real-valued function satisfying some growth conditions that will be explicited later.

This system is a natural extension of the coupled nonlinear Schrödinger-KdV system

$$\begin{cases} iu_t + u_{xx} = \alpha uv + \beta |u|^2 u \\ v_t + v_{xxx} + vv_x = \gamma(|u|^2)_x \end{cases} \tag{4}$$

first studied in [26] and then in [3] and [9] for the Cauchy problem, and in [14] for the existence of positive bound states. This class of coupled systems, namely when the second equation is a linear (or quasilinear) transport equation, was introduced by D.J.Benney in [4] as a universal model for the interaction of short and long waves and was fully studied by several authors in [11], [12], [13], [15], [27] and [10].

This paper is organized as follows:
In Section 2 we extend the results obtained in [26] concerning the Cauchy problem in the case $a(v) = v$ for initial data $(u_o, v_o) \in H^{\frac{5}{2}}(\mathbb{R}) \times H^2(\mathbb{R})$. By an adaptation of the proofs in [26] we easily obtain (see Theorem 1) a global existence and uniqueness result.

In Section 3, by a regularization method, we extend the global existence obtained in the previous section for initial data in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. In [9] very sharp results on the Cauchy problem associated to (4) were obtained, but the techniques employed do not seem suitable to handle the more general nonlinearity $a(v)v_x$ in (2).

Finally, in Section 4, we extend to (2) the results obtained in [14] for the Schrödinger-KdV system (4) concerning the existence of positive non-trivial bound state solutions. We will treat the case where $a(v) = Cv^k$, where $k = 1, 2$ and $C > 0$ (for convenience we will put $C = (k+2)(k+1)$).
We will study the existence of such solutions for the system

$$\begin{cases} -\phi'' + c^*\phi = -\beta\phi^3 - \alpha\phi\psi \\ -\psi'' + c\psi = (k+2)\psi^{k+1} - \frac{\alpha}{2}\phi^2, \end{cases} \tag{5}$$

In [2], the existence of solutions for a system similar to (5) is derived in the special case where $k = 2$. However the method employed cannot be exploited for $k = 1$.

We finish this Section by introducing a few notations:

In what follows, $\{U_S(t)\}_{t\in\mathbb{R}}$ and $\{U_K(t)\}_{t\in\mathbb{R}}$ will be, respectively, the free Schrödinger and KdV evolution groups defined by

$$U_S(t) = e^{it\frac{\partial^2}{\partial x^2}} = \mathcal{F}_x^{-1} e^{-it\xi^2} \mathcal{F}_x, \tag{6}$$

and

$$U_K(t) = e^{-t\frac{\partial^3}{\partial x^3}} = \mathcal{F}_x^{-1} e^{it\xi^3} \mathcal{F}_x, \quad t \in \mathbb{R}. \tag{7}$$

Here, $\mathcal{F}_x$ and $\mathcal{F}_x^{-1}$ denote the Fourier and the inverse Fourier transform in x. Also, for $s \geq 0$, we put

$$D_x^s = \mathcal{F}_x^{-1} |\xi|^s \mathcal{F}_x.$$

Finally, for $1 \leq p, q \leq +\infty$ and $T > 0$, we will consider the spaces $L^p(\mathbb{R};\, L^q(]-T;T[))$ and $L^q(]-T;T[;\, L^p(\mathbb{R}))$ endowed respectively with the norms

$$\|f\|_{L_x^p L_T^q} = \left(\int_{\mathbb{R}} \left(\int_{-T}^{T} |f(x,t)|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \text{ and } \|f\|_{L_T^q L_x^p} = \left(\int_{-T}^{T} \left(\int_{\mathbb{R}} |f(x,t)|^p dx \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}.$$

2 Global existence for initial data in $H^{\frac{5}{2}}(\mathbb{R}) \times H^2(\mathbb{R})$

Following [26], we introduce the function space

$$W_T^2(\mathbb{R}\times[-T;T];\mathbb{K}) = \{f \,:\mathbb{R}\times[-T;T] \to \mathbb{K}\,;\, \partial_x^{k-1} f \in L_x^2 L_T^\infty \text{ and } \partial_x^{k+1} f \in L_x^\infty L_T^2,\, k = 1,2\},$$

$\mathbb{K} = \mathbb{R}, \mathbb{C}$, which is a Banach space when endowed with the norm

$$\|f\|_{W_T^2} = \sum_{k=1}^{2} \|\partial_x^{k-1} f\|_{L_x^2 L_T^\infty} + \|\partial_x^{k+1} f\|_{L_x^\infty L_T^2}.$$

Furthermore, we set

$$X_T^2 = C([-T;T]; H^{\frac{5}{2}}(\mathbb{C}) \cap W_T^2(\mathbb{R} \times [-T;T];\mathbb{C})),$$

$$X_T^2 = C([-T;T]; H^2(\mathbb{R}) \cap W_T^2(\mathbb{R} \times [-T;T];\mathbb{R})),$$

$$Z_T^2 = X_T^2 \times Y_T^2,$$

with norms, respectively,

$$\|f\|_{X_T^2} = \sup_{t\in[-T;T]} \|f\|_{H^{\frac{5}{2}}} + \|f\|_{W_T^2},$$

$$\|f\|_{Y_T^2} = \sup_{t\in[-T;T]} \|f\|_{H^2} + \|f\|_{W_T^2}$$

and

$$\|(u,v)\|_{Z_T^2} = \|u\|_{X_T^2} + \|v\|_{Y_T^2}.$$

In what follows, we will make the following assumption on the function $a(v)$:

Assumption 2.1 *There exists constants* $c > 0$ *and* $q > 0$ *such that for all* $v \in \mathbb{R}$*,*

$$|a(v)| \leq c|v| \quad \text{and} \quad |a^{(n)}(v)| \leq c(1 + |v|^q), \quad n \in \{1, 2, 3, 4\}.$$

Examples of such nonlinearities include, for instance, $a(v) = v^k \sin(v^p)$ and $a(v) = ve^{-v^{2p}}$, where $k \in \{0, 1\}$ and $p \in \mathbb{N}$.

Under this assumption, the next result can easily be shown by adapting the proof of Theorem 1 in [26]:

Theorem 1 *Under the hypothesis stated in Assumption 2.1 and for initial data*

$$(u_o, v_o) \in H^{\frac{5}{2}}(\mathbb{R}) \times H^2(\mathbb{R}),$$

there exists $T > 0$ *and a unique solution* $(u, v) \in Z_T^2$ *satisfying (2) and (3) in the strong sense, that is, for all* $t \in [-T, T]$*,*

$$u(t) = U_S(t)u_o + \int_0^t U_S(t-s)F_1(u, v)(s)ds \tag{8}$$

and

$$v(t) = U_K(t)v_o + \int_0^t U_K(t-s)F_2(u, v)(s)ds, \tag{9}$$

where

$$F_1(u, v) = -i\alpha vu - i\beta|u|^2u \qquad \text{and} \qquad F_2(u, v) = -a(v)v_x + \gamma|u|_x^2.$$

Now, before passing to the study of the global well-posedness, we need to state some conservation laws for the local solution (u, v).

Lemma 2.2 *Let* $(u, v) \in Z_T^2$ *as in Theorem 1. Setting*

$$f(v) = \int_0^v a(\xi)d\xi \qquad \text{and} \qquad F(v) = \int_0^v f(\xi)d\xi,$$

the quantities

$$\mathcal{I}(t) = \int_{\mathbb{R}} |u(x, t)|^2 dx, \tag{10}$$

$$\mathcal{E}(t) = \int_{\mathbb{R}} \left(\alpha\gamma v(x, t)|u(x, t)|^2 + \gamma|u_x(x, t)|^2 + \frac{\alpha}{2}(v_x)^2(x, t) - \alpha F(v)(x, t) + \frac{\beta\gamma}{2}|u(x, t)|^4 \right) dx \tag{11}$$

and

$$\mathcal{M}(t) = \int_{\mathbb{R}} \left(\alpha v^2(x, t) + 2\gamma Im(u(x, t)\overline{u}_x(x, t)) \right) dx \tag{12}$$

are conserved in time, that is, for all $t \in [-T, T]$*,*

$$\frac{d}{dt}\mathcal{I}(t) = \frac{d}{dt}\mathcal{E}(t) = \frac{d}{dt}\mathcal{M}(t) = 0.$$

Proof of Lemma 2.2:

The lemma follows from standard arguments. To prove the conservation of $\mathcal{I}(t)$, we multiply (2)-(a) by $\overline{u}$ and integrate the imaginary part. To prove the conservation of $\mathcal{E}(t)$, we multiply (2)-(a) by $\overline{u}_t$ and integrate the real part. Finally, to get the conservation of the momentum $\mathcal{M}(t)$, we multiply (2)-(b) by v and integrate. In all three cases, the result is obtained after a few integrations by parts.
Note that the regularity of (u, v) stated in Theorem 1 validates these computations. ■

Since by Assumption 2.1 we have $|F(v)| \leq |v|^3$, by application of the Gagliardo-Nirenberg inequalities we can derive from Lemma 2.2 the following result:

Lemma 2.3 *Let $(u, v) \in Z_T^2$ as in Theorem 1. Assuming that*

$$\alpha\gamma > 0, \tag{13}$$

we have, for all $t \in [-T, T]$,

$$\|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq c(\|u_o\|_{H^1}, \|v_o\|_{H^1})$$

whith c depending continuously of its arguments (and independent of T).

We will assume from now on that condition (13) is satisfied. From Lemma 13 we obtain a bound for the H^2 norms of u and v:

Lemma 2.4 *Let $(u, v) \in Z_T^2$ as in Theorem 1. Assuming that (13) holds, for all $t \in [-T, T]$,*

$$\|u(t)\|_{H^2} + \|v(t)\|_{H^2} \leq c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2})$$

whith c depending continuously of its arguments.

Proof of Lemma 2.4:

As in the proof of Lemma 11 in [26], it is easy to obtain, for $t \in [0, T]$,

$$\|u_{xx}(t)\|_{L^2}^2 \leq C(T)\left(1 + \int_0^t \|v_{xx}(s)\|_{L^2}^2 ds\right). \tag{14}$$

It is also not difficult to derive from (2)

$$\frac{d}{dt}\int_{\mathbb{R}}\left((v_{xx})^2 - \frac{5}{3}a(v)(v_x)^2 + 2\gamma|u|^2 v_{xx}\right)dx \tag{15}$$

$$\begin{aligned} &= \int_{\mathbb{R}}\left(-\frac{1}{3}a''(v)(v_x)^3 v_{xx} + 5a(v)a'(v)(v_x)^3 + \frac{10}{3}(a(v))^2(v_x)^2 v_{xx}\right)dx \\ &+ \gamma\int_{\mathbb{R}}\left(\frac{1}{3}a'(v)(u_x)^2(|u|^2)_{xx} + \frac{4}{3}a(v)v_{xx}(|u|^2)_{xx} - \frac{5}{3}a'(v)(v_x)^2(|u|^2)_x\right)dx \\ &+ \gamma\int_{\mathbb{R}}\left(4Im(u_{xx}\overline{u})v_{xx} + 2((|u|^2)_{xx})^2\right)dx. \end{aligned}$$

Since, by the Gagliardo-Nirenberg inequality,

$$\|v_x\|_{L^2}^3\|v_{xx}\|_{L^2} \leq (c\|v_x\|_{L^2}^2\|v_{xx}\|_{L^2})\|v_{xx}\|_{L^2} \leq c\|v_{xx}\|_{L^2}^2,$$

we derive from the Assumption 2.1 and (15) that

$$\|v_{xx}(t)\|_{L^2}^2 \leq C(1+\|v_{xx}(t)\|_{L^2}) + ct + \int_0^t (\|u_{xx}(s)^2_{L^2} + \|v_{xx}(s)\|_{L^2}^2)ds. \tag{16}$$

The result follows by (14), (16) and the Gronwall inequality. ■

Now we fix $T > 0$ and $(u_o, v_o) \in H^{\frac{5}{2}}(\mathbb{R}) \times H^2(\mathbb{R})$.
Applying the smoothing properties of $\{U_S(t)\}_{t\in\mathbb{R}}$ and $\{U_K(t)\}_{t\in\mathbb{R}}$ derived in [18], [19] and [20] to (8) and (9), we obtain by Lemma 2.4 for $t \in [-T, T]$

$$\begin{aligned} \|D^{\frac{1}{2}}u_{xx}(t)\|_{L^2} &\leq C\|u_o\|_{H^{\frac{5}{2}}} + c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2})(1 + \|u\|_{L^2_xL^\infty_T} + \|v\|_{L^2_xL^\infty_T}) \\ &\leq C\|u_o\|_{H^{\frac{5}{2}}} + c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2}), \end{aligned} \tag{17}$$

where $c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2})$ depends continuously on its arguments.

Finally, by (17) and following the proof of Lemmas 7 and 8 in [26], we get

$$\|(u,v)\|_{Z_T^2} \leq c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2}),$$

where c can be extended continuously to $\mathbb{R}^3_+$.

Hence, we can state the following result:

Theorem 2.5 *Let $(u_o, v_o) \in H^{\frac{5}{2}}(\mathbb{R}) \times H^2(\mathbb{R})$ and $T > 0$.*
Assuming Assumption 2.1 and (13), there exists a unique strong solution $(u,v) \in Z_T^2$ of the Cauchy problem (2),(3).

3 Global existence for initial data in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$

In this section we prove a global existence theorem in $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ for (2),(3) using an approximation method. A similar technique was applied by T. Kato ([16]) to prove a global existence result in H^1 for the gKdV equation.

In what follows we will use the notation $\mathcal{C}_w(I, X)$ for the space of all weakly continuous functions on I with values on a Banach space X.

Theorem 3.1 *Let $(u_o, v_o) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$.*
Assume Assumption 2.1 and (13). Then, for every $T > 0$, there exists a solution (u,v) to (2),(3) such that

$$(u,v) \in C([-T;T]; H^1(\mathbb{R})) \times C([-T;T]; H^1(\mathbb{R})).$$

Furthermore, putting as in Lemma 2.2

$$F(v)=\int_0^v f(\xi)dx \qquad \text{and} \qquad f(y)=\int_0^y a(\xi)d\xi,$$

this solution satisfies for all $t\in[-T;T]$ the conservation laws

$$\frac{d}{dt}\mathcal{I}(t)=\frac{d}{dt}\mathcal{E}(t)=\frac{d}{dt}\mathcal{M}(t)=0, \tag{18}$$

where $\mathcal{I}$, $\mathcal{E}$ and $\mathcal{M}$ are given by (10), (11) *and* (12).

Proof of Theorem 3.1:

Let $(u_o^{(n)},v_o^{(n)})\in H^{\frac{5}{2}}\times H^2$ such that

$$\|(u_o^{(n)},v_o^{(n)})-(u_o,v_o)\|_{H^1\times H^1}\xrightarrow[n\to\infty]{}0$$

and let $(u^{(n)},v^{(n)})$ be the solution of (2),(3) with initial data $(u_o^{(n)},v_o^{(n)})$, which exists in $C([-T;T];H^{\frac{5}{2}})\times C([-T;T];H^2)$ and satisfies

$$\begin{cases} iu_t^{(n)}+u_{xx}^{(n)}=\alpha u^{(n)}v^{(n)}+\beta|u^{(n)}|^2u^{(n)} \\ v_t^{(n)}+v_{xxx}^{(n)}+a(v^{(n)})v_x^{(n)}=\gamma(|u^{(n)}|^2)_x \end{cases}$$

and the conservation laws

$$\frac{d}{dt}\mathcal{I}^{(n)}(t)=\frac{d}{dt}\mathcal{E}^{(n)}(t)=\frac{d}{dt}\mathcal{M}^{(n)}(t)=0,$$

where

$$\mathcal{I}^{(n)}(t)=\int_{\mathbb{R}}|u^{(n)}(x,t)|^2dx, \qquad \mathcal{M}^{(n)}(t)=\int_{\mathbb{R}}\left(\alpha\left(v^{(n)}\right)^2(x,t)+2\gamma Im(u^{(n)}(x,t)\overline{u_x^{(n)}}(x,t)\right)dx,$$

and

$$\mathcal{E}^{(n)}(t)=\int_{\mathbb{R}}\Big(\alpha\gamma v^{(n)}(x,t)|u^{(n)}(x,t)|^2+\gamma|u_x^{(n)}(x,t)|^2 \\ +\frac{\alpha}{2}\left(v_x^{(n)}\right)^2(x,t)-\alpha F(v^{(n)})(x,t)+\frac{\beta\gamma}{2}|u^{(n)}(x,t)|^4\Big)dx. \tag{19}$$

By the Gagliardo-Nirenberg inequality, it follows that that the sequences $u^{(n)}(t)$ and $v^{(n)}(t)$ are uniformly bounded in the H^1-norm:

$$\|u^{(n)}(t)\|_{H^1}\le C_1(\|u_o\|_{H^1},\|v_o\|_{H^1})$$

$$\|v^{(n)}(t)\|_{H^1}\le C_1(\|u_o\|_{H^1},\|v_o\|_{H^1}), \qquad t\in[-T;T].$$

Since we have the bounded sequences

$$\begin{array}{ll} u^{(n)} \in L^\infty([-T;T];H^1), & v^{(n)} \in\in L^\infty([-T;T];H^1), \\ u_t^{(n)} \in L^\infty([-T;T];H^{-1}) \quad \text{and} & v^{(n)} \in L^\infty([-T;T];H^{-2}), \end{array} \tag{20}$$

we obtain

$$\begin{array}{l} u^{(n)} \rightharpoonup u \text{ in } L^\infty([-T;T];H^1) \text{ weak * and a.e.}, \\ v^{(n)} \rightharpoonup v \text{ in } L^\infty([-T;T];H^1) \text{ weak * and a.e.}, \end{array} \tag{21}$$

which implies that $u \in C([-T;T];H^{-1})$, $v \in C([-T;T];H^{-2})$ and therefore $u(0) = u_o$ and $v(0) = v_o$.

Now, from (20) and (21), we easily infer that

$$\begin{array}{lll} u^{(n)}v^{(n)} & \rightharpoonup & uv \quad \text{in } L^\infty([-T;T];H^1) \text{ weak *}, \\ a(v^{(n)})v_x^{(n)} & \rightharpoonup & a(v)v_x \quad \text{in } L^\infty([-T;T];L^2)\text{weak *}, \\ (|u^{(n)}|^2)_x & \rightharpoonup & (|u|^2)_x \quad \text{in } L^\infty([-T;T];L^2) \text{ weak *}, \end{array}$$

so that $(u,v) \in \left(L^\infty([-T;T];H^1)\right)^2$ is a strong solution of (2), (3) in the space

$$C([-T;T];H^{-1}) \times C([-T;T];H^{-2}).$$

Next, let $U_S(t)$ and $U_K(t)$ be the free Schrödinger and KdV unitary groups already defined in (6) and (7). Then the solution

$$(u,v) \in (L^\infty([-T,T];H^1))^2 \cap (C([-T;T];H^{-1}) \times C([-T;T];H^{-2}))$$

satisfies the integral system

$$u(t) = U_S(t)u_o + \int_0^t U_S(t-s)F_1(u(s),v(s))ds$$

$$v(t) = U_K(t)v_o + \int_0^t U_K(t-s)F_2(u(s),v(s))ds,$$

where $F_1(u,v) = -i\alpha vu - i\beta|u|^2u$ and $F_2(u,v) = -a(v)v_x + \gamma(|u|^2)_x$.

A standard computation in semi-group theory leads to the expressions

$$u(t+h) - u(t) = (U_S(h) - I)u(t) + \int_t^{t+h} U_S(t+h-s)F_1(u(s),v(s))ds,$$

$$v(t+h) - v(t) = (U_K(h) - I)v(t) + \int_t^{t+h} U_K(t+h-s)F_2(u(s),v(s))ds,$$

for all $h \in \mathbb{R}$. Since

$$\|F_1(u,v)\|_{H^1} \le C_3(\|u_o\|_{H^1}, \|v_o\|_{H^1}) \quad \text{and} \quad \|F_2(u,v)\|_{H^1} \le C_4(\|u_o\|_{H^1}, \|v_o\|_{H^1}),$$

we deduce that

$$\|u(t+h) - u(t)\|_{H^1} \xrightarrow[h\to 0]{} 0, \qquad \|v(t+h) - v(t)\|_{L^2} \xrightarrow[h\to 0]{} 0,$$

and therefore $u \in C([-T,T]; H^1)$ and $u \in C([-T,T]; L^2)$.

On the other hand, from (21), we easily see that the sequence $t \to v^{(n)}(t)$ is weakly equicontinuous. Hence, it follows from the Ascoli-Arzelà theorem that the weak limit function is weakly continuous: $v \in C_w([-T,T]; H^1)$.

We now prove that the solution

$$(u,v) \in (C([-T,T]; H^1)) \times (C([-T;T]; L^2) \cap C_w([-T;T]; H^1))$$

satisfies the conservation laws (18).

First of all, since $u(t) \in H^1$, we remark that

$$\frac{d}{dt}\|u\|_2^2 = Im(u_{xx}(t), u(t)) = 0$$

and so $\|u(t)\|_2^2 = \|u_o\|_2^2$. Since

$$\|u^{(n)}(t)\|_2 = \|u^{(n)}(0)\|_2 \xrightarrow[n\to\infty]{} \|u_o\|_2 = \|u(t)\|_{L^2}$$

and $u^{(n)}(t) \rightharpoonup u(t)$ weakly in H^1 we conclude that for all $t \in [-T;T]$, $u^{(n)}(t) \to u(t)$ strongly in L^2.

Next, to prove the conservation of $\mathcal{M}$, we note that

$$\int_{\mathbb{R}} u^{(n)}(t)\overline{u}_x^{(n)}(t)dx \to \int_{\mathbb{R}} u(t)\overline{u}_x(t)dx,$$

since $u^{(n)}(t) \to u(t)$ in L^2 and $u_x^{(n)}(t) \rightharpoonup u_x(t)$ weakly in L^2. Using the lower semi-continuity of the L^2 norm, we deduce that for all $t \in [-T;T]$,

$$\mathcal{M}(t) \le \lim \mathcal{M}^{(n)}(t) = \mathcal{M}(0).$$

From the reversibility in time of equations (2), we obtain $\mathcal{M}(t) = \mathcal{M}(0)$.

As a consequence, we derive

$$\|v^{(n)}(t)\|_{L^2} \xrightarrow[n\to\infty]{} \|v(t)\|_{L^2}, \qquad t \in [-T;T],$$

and since $v^{(n)}(t) \rightharpoonup v(t)$ weakly in H^1, we also conclude $v^{(n)}(t) \to v(t)$ strongly in L^2.

Finally, we prove the conservation of the energy $\mathcal{E}(t)$.
We start by noticing that

$$|F(v^{(n)}(t)) - F(v(t))| \leq C|v^{(n)}(t) - v(t)||(v^{(n)}(t))^2 + v(t)v^{(n)}(t) + v(t)^2|,$$

and so

$$\begin{aligned}\left|\int_{\mathbb{R}}(F(v^{(n)}(t)) - F(v(t)))dx\right| &\leq C\|v^{(n)}(t) - v(t)\|_{L^2}\||(v^{(n)}(t))^2 + v(t)v^{(n)}(t) + v(t)^2|\|_{L^2} \\ &\leq C\|v^{(n)}(t) - v(t)\|_{L^2} \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}} F(v^{(n)}(t))dx \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} F(v(t))dx.$$

In a similar way, we have

$$\left|\int_{\mathbb{R}}((u^{(n)}(t))^4 - (u(t))^4)dx\right| \leq C\|u^{(n)}(t) - u(t)\|_{L^2} \xrightarrow[n\to\infty]{} 0.$$

Now, using the semi-continuity of the H^1 norm, we obtain for all $t \in [-T, T]$,

$$\mathcal{E}(t) \leq \lim \mathcal{E}^{(n)}(t) = \mathcal{E}(0).$$

Once again, by the reversibility of equations (2),

$$\mathcal{E}(t) = \mathcal{E}(0).$$

To prove the continuity of the map $t \to (u(t), v(t))$ in $H^1 \times H^1$ it remains to prove that v is strongly continuous on H^1. From the conservation of the energy $\mathcal{E}(t)$, it follows that $t \to \|u(t)\|_{H^1}$ is continuous.
Since it has already been proved that $v \in C_w([-T; T]; H^1)$, this yields the strong continuity of $v(t)$ and therefore the proof of Theorem 2.5 is complete. ■

4 Existence of bound states

In this Section we look for bound-state solutions to (2) of the form

$$(u(x,t), v(x,t)) = (e^{i\omega t + ikx}\phi(x - ct), \psi(x - ct)). \tag{22}$$

in the case where $F(v) = v^p$, where F is defined in the Lemma 2.2 and corresponds to the case $a(v) = p(p-1)v^{p-2}$.

We will prove the following result:

Theorem 4.1 *Let* $F(v) = v^p$, $\gamma = \dfrac{\alpha}{2}$ *and* $\alpha, \beta < 0$.
For $p = 3$, $\alpha < -3$ *or* $p = 4$, $\beta < -4$ *there exists a family*

$$(u_n(x,t), v_n(x,t)) = (e^{i\omega_n t + ik_n x}\phi_n(x - c_n t), \psi_n(x - c_n t)) \tag{23}$$

of non trivial bound-state solutions to (2) whith $\lim_{n\to\infty} c_n = +\infty$.
Here, ϕ_n, ψ_n *are smooth positive functions which decay exponentially at infinity.*

This result is an extension of the Theorem proved in [14] in the frame of the Schrödinger-KdV system (4). The method relies on the concentration-compactness theorem by P.L. Lions ([21],[22]) and an estimate for the Lagrange multipliers associated to a minimization problem.

By setting $c^* := k^2 + \omega$ and putting $c = 2k$, we obtain the system

$$\begin{cases} -\phi'' + c^*\phi = -\beta\phi^3 - \alpha\phi\psi \\ \\ -\psi'' + c\psi = p\psi^{p-1} - \frac{\alpha}{2}\phi^2. \end{cases} \tag{24}$$

For $\mu \geq 0$ we set $X_\mu = \{(u,v) \in H^1 \times H^1 \,:\, \|u\|_{L^2}^2 + \|v\|_{L^2}^2 = \mu\}$ (u, v real-valued) and consider the minimizing problem

$$\mathcal{I}(\mu) = inf\{E(u,v) \,:\, (u,v) \in X_\mu\}, \tag{25}$$

where

$$E(u,v) = \int_{\mathbb{R}} (u')^2 + (v')^2 + \alpha u^2 v - 2v^p + \frac{\beta}{2}u^4.$$

Note that $E = \dfrac{2}{\alpha}\mathcal{E}$, where $\mathcal{E}$ is the energy given by (11).

We will use the concentration-compactness method to prove the existence of minimizers to $\mathcal{I}$.

Lemma 4.2 *For all* $\mu > 0$, $\mathcal{I}(\mu) > -\infty$.

Proof of Lemma 4.2:
Let $(u,v) \in X_\mu$: $\|u\|_{L^2}^2 \leq \mu$ and $\|v\|_{L^2}^2 \leq \mu$. By the Gagliardo-Nirenberg inequality,

$$\|v\|_{L^p}^p \leq C_1 \|v'\|_{L^2}^{\frac{p-2}{2}} \|v\|_{L^2}^{\frac{p+2}{2}} \leq C_1 \mu^{\frac{p+2}{2}} \|v'\|_{L^2}^{\frac{p-2}{2}}$$

and

$$\|u\|_{L^4}^4 \leq C_2 \|u'\|_{L^2} \|u\|_{L^2}^3 \leq C_2 \mu^{\frac{3}{2}} \|u'\|_{L^2},$$

where C_j denote positive constants.
Also,

$$\int |v|u^2 \leq \frac{1}{2}\|v\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^4}^4 \leq \frac{\mu}{2} + C_2\frac{\mu^{\frac{3}{2}}}{2}\|u'\|_{L^2}.$$

Finally, we obtain

$$\begin{array}{rcl} E(u,v) & \geq & \|u'\|_{L^2}^2 + \|v'\|_{L^2}^2 + \frac{\beta}{2}\int u^4 - 2|v|^p - |\alpha|\int u^2|v| \\ & \geq & \|u'\|_{L^2}^2 + \|v'\|_{L^2}^2 - \frac{1}{2}C_2(|\alpha| + |\beta|)\mu^{\frac{3}{2}}\|u'\|_{L^2} - \frac{1}{3}C_1\mu^{\frac{p+2}{2}}\|v'\|_{L^2}^{\frac{p-2}{2}} - \frac{|\alpha|\mu}{2}, \end{array} \quad (26)$$

where $\frac{p-2}{2} < 2$. We deduce the existence of an inferior bound for $E(u,v)$ depending exclusively on μ. ■

Note that for large values of p ($p \geq 6$), $\mathcal{I}(\mu) = -\infty$ for $\mu > 0$.
Indeed, let $v \in H^1$ such that $\|v\|_{L^2}^2 = \mu$. Then, by setting $v_a(x) = a^2 v(ax)$, $\|v_a\|_{L^2}^2 = \mu$ and

$$E(0, v_a) = a^6 \int |v'|^2 - a^p \int v^p \xrightarrow[a \to +\infty]{} -\infty.$$

Proposition 4.3 *For all $\mu \geq 0$, $\mathcal{I}(\mu) \leq 0$. Also, there exists $\mu^* > 0$ such that for all $\mu > \mu^*$, $\mathcal{I}(\mu) \leq -A\mu^2$, where A is a positive constant independent of μ.*

Proof of Proposition 4.3:
Let $\mu \geq 0$ and $u \in H^1$ such that $\|u\|_{L^2}^2 = \mu$. Then $(u,0) \in X_\mu$ and $E(u,0) \leq \int (u')^2$. Noticing that $\inf\{\int (u')^2 \,:\, \|u\|_{L^2}^2 = \mu\} = 0$, we get $\mathcal{I}_\mu \leq 0$.

We now consider $u \in H^1(\mathbb{R})$ such that $\|u\|_{L^2} = 1$. Putting $u_\mu(x) = \mu^{\frac{1}{2}} u(x)$, $(u_\mu, 0) \in X_\mu$. Furthermore,

$$\mathcal{I}(\mu) \leq E(u_\mu, 0) = \mu \int (u')^2 + \frac{\beta}{2}\mu^2 \int u^4 = \mu\left(\int (u')^2 - \frac{|\beta|}{4}\mu \int u^4\right) - \frac{\mu^2|\beta|}{4}\int u^4.$$

By choosing $A = \frac{|\beta|}{4}\int u^4$ and μ^* such that $\int (u')^2 - \frac{|\beta|}{4}\mu^* \int u^4 \leq 0$ we get the result. ■

Remark 4.4 *It is well known that for $f \in H^1(\mathbb{R})$ real valued, $\||f|'\|_{L^2} \leq \|f'\|_{L^2}$. Therefore, for every pair $(u,v) \in X_\mu$,*

$$E(|u|, |v|) \leq E(u,v).$$

Hence, there exists a minimizing sequence (u_j, v_j) for problem (25) with $u_j, v_j \geq 0$.

Lemma 4.5 *Let $\mu > \mu^*$. For all $\theta > 1$, $\mathcal{I}(\theta\mu) < \theta\mathcal{I}(\mu)$.*

Proof: Consider a positive minimizing sequence $(u_j, v_j) \in X_\mu$ for problem (25). We have

$$\begin{aligned} E(\sqrt{\theta}u_j, \sqrt{\theta}v_j) &= \theta E(u_j, v_j) - \tfrac{|\beta|}{2}(\theta^2 - \theta)\int u_j^4 + (\theta^{\frac{p}{2}} - \theta)\left(\alpha \int u_j^2 v_j - 2\int v_j^p\right) \\ &\leq \theta E(u_j, v_j) + \max\{\theta - \theta^2, \theta - \theta^{\frac{p}{2}}\}\left(\frac{|\beta|}{2}\int u_j^4 + |\alpha|\int u_j^2 v_j + 2\int v_j^p\right). \end{aligned}$$

Since (u_j, v_j) is a minimizing sequence, $\dfrac{|\beta|}{2}\displaystyle\int u_j^4 + |\alpha|\int u_j^2 v_j + 2\int v_j^p \geq \delta$ for some $\delta > 0$. Otherwise there would exist a subsequence - still denoted (u_j, v_j) - such that $\lim E(u_j, v_j) \geq 0$, which is absurd since $\mathcal{I}(\mu) < 0$. Hence

$$E(\sqrt{\theta}u_j, \sqrt{\theta}v_j) \leq \theta E(u_j, v_j) - \delta(\theta^{\max\{2, \frac{p}{2}\}} - \theta) \quad \text{for all } \theta > 1.$$

Since $\|\sqrt{\theta}u_j\|_2^2 + \|\sqrt{\theta}v_j\|_2^2 = \theta\|u_j\|_2^2 + \theta\|v_j\|_2^2 = \theta\mu$, we obtain $\mathcal{I}(\theta\mu) < \theta\mathcal{I}(\mu)$. ■

From this result it is straightforward that $\mathcal{I}$ is a non-decreasing function of μ.

Therefore, there exists $\mu_1 \geq 0$ such that

$$\mu > \mu_1 \Leftrightarrow \mathcal{I}(\mu) < 0.$$

Arguing as in Lemma 2.3 of [24], we get the strict sub-additivity of $\mathcal{I}(\mu)$:

Corollary 4.6
Let $\mu > \mu_1$ and $0 < \Omega < \mu$. Then $\mathcal{I}(\mu) < \mathcal{I}(\Omega) + \mathcal{I}(\mu - \Omega)$.

Next, we prove the existence of minimizers for the minimization problem (25):

Proposition 4.7 *Let $\mathcal{M}_\mu = \{(u, v) \in X_\mu : \mathcal{I}(\mu) = E(u, v)\}$. For $\mu > \mu_1$, $\mathcal{M}_\mu \neq \emptyset$.*

Proof of Proposition 4.7:

Let us consider a positive minimizing sequence $(u_j, v_j) \in X_\mu$ for problem (25). We will apply the concentration-compactness lemma to the sequence $\rho_j = u_j^2 + v_j^2$. Using the notations in [21], we introduce the concentration function of ρ_j:

$$Q_j(t) = \sup_{y\in\mathbb{R}} \int_{y-t}^{y+t} \rho_j, \quad \text{and we set} \quad \Omega = \lim_{t\to\infty} Q(t).$$

There are three alternatives: vanishing ($\Omega = 0$), dichotomy ($0 < \Omega < \mu$) and compactness ($\Omega = \mu$). The latter implies the relative compactness of the sequence (u_j, v_j) up to translations.

First, we rule out vanishing. Indeed, if $\Omega = 0$,

$$\lim_{j\to\infty} \sup_{y\in\mathbb{R}} \int_{y-t}^{y+t} u_j^2 = \lim_{j\to\infty} \sup_{y\in\mathbb{R}} \int_{y-t}^{y+t} v_j^2 = 0.$$

Since (u_j) and (v_j) are bounded in $H^1(\mathbb{R})$ (as seen in the proof of Proposition 4.2), a classical lemma (see [22], Lemma I.1) yields $\|u_j\|_p \to 0$ and $\|v_j\|_p \to 0$ for all $p > 2$.

It results that

$$\mathcal{I}(\mu) = \lim_{j\to\infty} E(u_j, v_j) = \lim_{j\to\infty} \int {u_j'}^2 + \int {v_j'}^2 - \frac{|\beta|}{2} \int u_j^4 - 2\int v_j^p + \alpha \int u_j^2 v_j \geq 0,$$

which is absurd by Proposition 4.3.

We now rule out dichotomy: by Lemma III.1 in [21], for all $\epsilon > 0$ there exists constants $R_1 > 0$, $R_2 > R_1$, a sequence $\{y_j\}$ and cut-off functions $\eta_i \in C^\infty(\mathbb{R})$, $0 \leq \eta_i \leq 1$ such that
$\eta_1(x) = 1$ for $|x| \leq R_1$ and $\eta_1(x) = 0$ for $|x| \geq \frac{R_2}{2}$,
$\eta_1(x) = 1$ for $|x| \geq R_2$ and $\eta_1(x) = 0$ for $|x| \leq \frac{R_2}{2}$,
$|\eta_i(x)| \leq \epsilon$ for all $x \in \mathbb{R}$,
$\int_{R_1 \leq x - y_j \leq R_2} \rho_j(x) \leq \epsilon$,
$\left|\int \eta_1^2(x)\rho_j(x-y_j) - \Omega\right| < \epsilon$ and $\left|\int \eta_2^2(x)\rho_j(x-y_j) - (\mu - \Omega)\right| < \epsilon$.

Putting $u_j^{(i)} = \eta_i(x-y_j)$ and $v_j^{(i)} = \eta_i(x-y_j)$,

$$\|u_j'\|_2^2 - \|u_j^{(1)'}\|_2^2 - \|u_j^{(2)'}\|_2^2 =$$

$$\int_{R_1 \leq |x-y_j| \leq R_2} (u_j')^2 - \int_{R_1 \leq |x-y_j| \leq \frac{1}{2}R_2} (u_j^{(1)'})^2 - \int_{\frac{1}{2}R_2 \leq |x-y_j| \leq R_2} (u_j^{(2)'})^2 =$$

$$\int_{R_1 \leq |x-y_j| \leq \frac{1}{2}R_2} (u_j')^2 - (u_j^{(1)'})^2 + \int_{\frac{1}{2}R_2 \leq |x-y_j| \leq R_2} (u_j')^2 - (u_j^{(2)'})^2. \tag{27}$$

Also,

$$\int_{R_1 \leq |x-y_j| \leq \frac{1}{2}R_2} (u_j')^2 - (u_j^{(1)'})^2 =$$

$$\int_{R_1 \leq |x-y_j| \leq \frac{1}{2}R_2} (1 - \eta_1^2(x-y_j))(u_j')^2 - (\eta_1'(x-y_j))^2 u_j^2 - 2\eta_1(x-y_j)\eta_1'(x-y_j)u_j u_j' \geq$$

$$-\epsilon^3 - C\epsilon^{\frac{3}{2}}.$$

A similar estimate holds for the second term in the right-hand-side of (27), hence we obtain

$$\|u_j'\|_2^2 - \|u_j^{(1)'}\|_2^2 - \|u_j^{(2)'}\|_2^2 \geq -2\epsilon^3 - 2C\epsilon^{\frac{3}{2}}.$$

By the same computations,

$$\|v_j'\|_2^2 - \|v_j^{(1)'}\|_2^2 - \|v_j^{(2)'}\|_2^2 \geq -2\epsilon^3 - 2C\epsilon^{\frac{3}{2}}.$$

Moreover, it is straightforward to prove that

$$\|v_j\|_p^p - \|v_j^{(1)}\|_p^p - \|v_j^{(2)'}\|_p^p \leq C_1\epsilon^{\frac{p+2}{2}}, \qquad \|u_j\|_4^4 - \|u_j^{(1)}\|_4^4 - \|u_j^{(2)'}\|_4^4 \leq C_2\epsilon^3$$

and

$$\int u_j^2 v_j - \int u_j^{(1)^2} v_j^{(1)} - \int u_j^{(2)^2} v_j^{(2)} \leq C_3\epsilon^{\frac{5}{2}}.$$

Finally, we obtain

$$E(u_j, v_j) \geq E(u_j^{(1)}, v_j^{(1)}) + E(u_j^{(2)}, v_j^{(2)}) - C\epsilon.$$

This leads to $I(\mu) \geq I(\Omega) + I(\mu - \Omega)$, which is in contradiction with Corollary 4.6.

Hence, we have compactness: extracting once again a subsequence, there exists $\{y_j\}$ such that

$$(\tilde{u}_j = u_j(. - y_j), \tilde{v}_j = v_j(. - y_j)) \to (\phi, \psi) \text{ in } L^2(\mathbb{R}).$$

Furthermore, the sequence $(\tilde{u}_j, \tilde{v}_j)$ converges to (ϕ, ψ) in $H^1(\mathbb{R})$ weak. Hence, $(\tilde{u}_j, \tilde{v}_j) \to (\phi, \psi)$ in L^q for all $q \geq 2$: $\|\tilde{u}_j\|_4 \to \|\phi\|_4$, $\|\tilde{v}_j\|_p \to \|\psi\|_p$, $\int \tilde{u}_j^2 \tilde{v}_j \to \int \phi^2\psi$ and

$$\mathcal{I}(\mu) \leq E(\phi, \psi) \leq \underline{\lim} E(\tilde{u}_j, \tilde{v}_j) = \mathcal{I}(\mu).$$

Finally, $(\phi, \psi) \in \mathcal{M}_\mu \neq \emptyset$.

Note that we have obtained $\underline{\lim} \displaystyle\int (\tilde{u}_j')^2 + (\tilde{v}_j')^2 = \int (\phi')^2 + (\psi')^2$, hence the convergence takes place in H^1 strong: $\phi \geq 0$ and $\psi \geq 0$.

Also, it is clear that $\psi \neq 0$. We now show that $\phi \neq 0$.

Taking ψ such that $(0, \psi) \in X_\mu$, for all $\theta \in [0, 1]$, $(\theta^{\frac{1}{2}}\psi, (1-\theta)^{\frac{1}{2}}\psi) \in X_\mu$. A straightforward computation then leads to

$$E(\theta^{\frac{1}{2}}\psi, (1-\theta)^{\frac{1}{2}}\psi) = E(0, \psi) + f_{\alpha,\beta}(\theta),$$

where

$$f_{\alpha,\beta}(\theta) = \alpha\theta(1-\theta)^{\frac{1}{2}}\|\psi\|_{L^3}^3 + 2(1 - (1-\theta)^{\frac{p}{2}})\|\psi\|_{L^p}^p + \frac{\beta}{2}\theta^2\|\psi\|_{L^4}^4.$$

If $p = 3$, $f'_{\alpha,\beta}(0) = (3+\alpha)\|\psi\|^3_{L^3} < 0$ if $\alpha < -3$, hence, for small θ,

$$E(\theta^{\frac{1}{2}}\psi, (1-\theta)^{\frac{1}{2}}\psi) < E(0,\psi).$$

If $p = 4$, we notice that $f_{\alpha,\beta}(1) = \left(2 + \frac{\beta}{2}\right)\|\phi\|^4_{L^4} < 0$ if $\beta < -4$: for $1-\theta$ small, we get once again

$$E(\theta^{\frac{1}{2}}\psi, (1-\theta)^{\frac{1}{2}}\psi) < E(0,\psi).$$

■

End of the Proof of Theorem 4:

Let $(\phi,\psi) \in \mathcal{M}_\mu$, $\phi,\psi \geq 0$, $\phi \neq 0$, $\psi \neq 0$.
There exists a Lagrange multiplier $\lambda = \lambda(\mu) \in \mathbb{R}$ such that

$$\begin{cases} -\phi'' - \lambda\phi = -\beta\phi^3 - \alpha\phi\psi \\ \\ -\psi'' - \lambda\psi = p\psi^{p-1} - \frac{\alpha}{2}\phi^2. \end{cases}$$

Multiplying these equations by ϕ and ψ and integrating bt parts leads to

$$\begin{cases} \int(\phi')^2 - \lambda\int\phi^2 = -\beta\int\phi^4 - \alpha\int\phi^2\psi \\ \\ \int(\psi')^2 - \lambda\int\psi^2 = p\int\psi^p - \frac{\alpha}{2}\int\phi^2\psi. \end{cases}$$

hence

$$\int(\phi')^2 + \int(\psi')^2 - \lambda\mu = -\beta\int\phi^4 + p\int\psi^p - \frac{3\alpha}{2}\int\phi^2\psi.$$

Since

$$\mathcal{I}(\mu) = E(\phi,\psi) = \int(\phi')^2 + \int(\psi')^2 + \alpha\int\phi^2\psi - 2\int\psi^p + \frac{\beta}{2}\int\phi^4,$$

we get

$$\lambda\mu = \mathcal{I}(\mu) + \frac{\beta}{2}\int\phi^4 + (2-p)\int\psi^p + \frac{\alpha}{2}\int\phi^2\psi \leq \mathcal{I}(\mu) \leq -A\mu^2$$

by Proposition 4.3.
Finally

$$\lambda \leq -A\mu.$$

The proof is complete by choosing a sequence $\mu_n \to +\infty$ and setting $c_n = -\lambda_n$, $k_n = -\frac{1}{2}\lambda_n$ and $\omega_n = -\lambda_n - k_n^2$.

Note that a classical bootstrap argument proves the regularity of ϕ and ψ and that, since $c^* = k_n^2 + \omega_n^2 = -\lambda_n > 0$, the argument used in the proof of Theorem 8.1.1 in [7] easily shows the existence of $\epsilon > 0$ such that $e^{\epsilon|x|}\phi, e^{\epsilon|x|}\psi \in L^\infty$, leading to the exponential decreasing of ϕ and ψ. ■

Acknowledgements: The research of J. P. Dias and M. Figueira was partially supported by FCT, Financiamento Base 2008-ISFL-1-209 and the research of F. Oliveira was partially supported by Financiamento Base 2008-ISFL-1-297.

References

[1] Ablowitz, M. J. and Segur, H., Asymptotic solutions of the Korteweg-deVries equation. Studies in Appl. Math., 57, 13-44 (1977).

[2] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, C. R. Acad. Sci. Paris, Ser. I 342, (2006).

[3] D.Bekiranov, T.Ogawa and G.Ponce, Weak solvability and well-posedness of a coupled Scrödinger-Korteweg-de Vries equation for capillary-gravity wave interaction, Proc. Am. Math. Soc., 125, 2907-2919 (1997).

[4] D.J. Benney, A general theory for interactions between short and long waves, Studies ina Appl. Math., 56, 81-94 (1977).

[5] B. Birnir, C. Kenig, G. Ponce, N. Svanstedt, L. Vega. On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrdinger equations. J. London Math. Soc. (2), 53, 551-559 (1996).

[6] J. Bona, P.E. Souganidis, W. Strauss. Stability and instability of solitary waves of Korteweg-de Vries type. Proc. R. Soc. Lond, 411 (1987), 395-412.

[7] T. Cazenave, *An Introduction to Nonlinear Schrödinger Equations*, Textos de Métodos Matemáticos, Vol. 22, Rio de Janeiro, 1989.

[8] M. Christ, M. Weinstein. Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal, 100, 87-109 (1991).

[9] A.L.Corcho and F.Linares, Well-posedness for the Schrödinger-Korteweg de Vries system, Trans. Am. Math. Soc., 359, 4089-4106 (2007).

[10] J.P.Dias and M. Figueira, Existence of weak solutions for a quasilinear version of Benney equations. J. Hyperbolic Differ. Equ. 4, no. 3, 555–563 (2007).

[11] J.P.Dias, M. Figueira and H.Frid, Vanishing viscosity with short wave long wave interactions for systems of conservation laws, to appear in Arch. Rat. Mech. and Analysis.

[12] J.P.Dias, M. Figueira and H.Frid, Vanishing viscosity with short wave long wave interactions for multi-d scalar conservation laws,to appear.

[13] J.P.Dias, M. Figueira and F.Oliveira, Existence of local strong solutions for a quasilinear Benney system. C. R. Math. Acad. Sci. Paris, 344, 493-496 (2007).

[14] J.P.Dias, M. Figueira and F.Oliveira, Existence of bound states for the coupled Schrödinger-KdV system with cubic nonlinearity, to appear.

[15] M.Tsutsumi and S.Hatano, Well-posedness of the Cauchy problem for Benney's first equations of long wave short wave interactions. Funkcial. Ekvac., 37, 289-316 (1994).

[16] T.Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Adv. Math. Suppl. Stud., 8, 93–128 (1983).

[17] C. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math, 46, 527-560 (1993).

[18] C. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations. Indiana Univ. Math. J. 40 (1991), no. 1, 33–69, (1991).

[19] C. Kenig, G. Ponce and L. Vega, Well-posedness of the initial value problem for the Korteweg-de Vries equation. J. Amer. Math. Soc. 4, 323–347 (1991).

[20] C. Kenig, G. Ponce and L. Vega, Small solutions to nonlinear Schrödinger equations. Ann. Inst. H. Poincaré Anal. Non Linéaire 10, 255–288 (1993).

[21] P. L. Lions, The concentration-compactness principle in the Calculus of Variations, Part 1, Ann. Inst. H. Poincaré, Vol. 1, Série C, 109-145, (1984).

[22] P. L. Lions, The concentration-compactness principle in the Calculus of Variations, Part 2, Ann. Inst. H. Poincaré, Vol .1, Série C, 223-283 (1984).

[23] Y. Martel and F. Merle, Asymptotic stability of solitons of the gKdV equations with general nonlinearity, Math. Ann. 341, 391-427 (2008).

[24] M. Ohta, Stability of Stationary states for the coupled Klein-Gordon-Schrödinger equations, Nonlinear An. TMA, Vol. 27, 455-461 (1996).

[25] G. Ponce, L. Vega. Non-linear small data scattering for the generalized Korteweg-de Vries equation. J. Funct. Anal, 90, 445-457 (1990).

[26] M.Tsutsumi, Well-posedness of the Cauchy Problem for a coupled Schrödinger-KdV Equation, Gakuto Internat. Ser. Math. Sci. Appl., 2, 513-528 (1993).

[27] M.Tsutsumi and S.Hatano, Well-posedness of the Cauchy problem for the long wave–short wave resonance equations. Nonlinear Anal., 22, 155-171 (1994).

[28] M.Tsutsumi and T.Mukasa, Parabolic Regularizations of the Generalized Korteweg-de Vries Equation, Funkcialaj Ekvacioj, 14, 89-110 (1971).

[29] N.J. Zabusky, A synergetic approach to problems of nonlinear dispersive wave propagation and interaction, Pr. of the Symp. of PDEs, Academic Press, (1967).