Well-posedness and existence of bound states for a coupled Schrödinger-gKdV system

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Abstract

We derive some new results concerning the Cauchy problem and the existence of bound states for a class of coupled nonlinear Schrödinger-gKdV systems. In particular, we obtain the existence of strong global solutions for initial data in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, generalizing previous results obtained in [26], [9] and [14] for the nonlinear Schrödinger-KdV system.

1 Introduction

The generalized KdV (gKdV) equation

$$v_t + v_{xxx} + a(v)v_x = 0, \quad a \in C^\infty(\mathbb{R}),$$

was introduced in [29] as a model for the propagation of nonlinear waves in an anharmonic lattice. In recent years, many authors treated the case where the function $a$ is a polynomial: see for instance [17],[5] for well-posedness, [6] for the stability of solitary waves and [8],[25],[1] for the study of the dispersion rates of the solutions to (1). In contrast, little results exist in the literature concerning more general nonlinearities (see on this subject [28], [23] and [16]).

In the present paper, we will consider the coupled system

$$
\begin{align*}
\begin{cases}
i u_t + u_{xx} = \alpha uv + \beta |u|^2 u & \quad (a) \\
v_t + v_{xxx} + a(v)v_x = \gamma(|u|^2)_x & \quad (b)
\end{cases}
\end{align*}
$$

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with initial data
\[
\begin{align*}
  u(x, 0) &= u_0(x), \\
  v(x, 0) &= v_0(x),
\end{align*}
\]
where \( \alpha, \beta, \) and \( \gamma \) are real constants, \( u \) is a complex-valued function, \( v \) is a real-valued function and \( a \) is a \( C^\infty \) real-valued function satisfying some growth conditions that will be explicited later.

This system is a natural extension of the coupled nonlinear Schrödinger-KdV system
\[
\begin{align*}
  iu_t + u_{xx} &= \alpha uv + \beta |u|^2 u, \\
  v_t + v_{xxx} + vv_x &= \gamma (|u|^2)_x
\end{align*}
\] (4)
first studied in [26] and then in [3] and [9] for the Cauchy problem, and in [14] for the existence of positive bound states. This class of coupled systems, namely when the second equation is a linear (or quasilinear) transport equation, was introduced by D.J.Benney in [4] as a universal model for the interaction of short and long waves and was fully studied by several authors in [11], [12], [13], [15], [27] and [10].

This paper is organized as follows:
In Section 2 we extend the results obtained in [26] concerning the Cauchy problem in the case \( a(v) = v \) for initial data \( (u_0, v_0) \in \mathcal{H}^5(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}) \). By an adaptation of the proofs in [26] we easily obtain (see Theorem 1) a global existence and uniqueness result.

In Section 3, by a regularization method, we extend the global existence obtained in the previous section for initial data in \( \mathcal{H}^1(\mathbb{R}) \times \mathcal{H}^1(\mathbb{R}) \). In [9] very sharp results on the Cauchy problem associated to (4) were obtained, but the techniques employed do not seem suitable to handle the more general nonlinearity \( a(v)v_x \) in (2).

Finally, in Section 4, we extend to (2) the results obtained in [14] for the Schrödinger-KdV system (4) concerning the existence of positive non-trivial bound state solutions. We will treat the case where \( a(v) = C v^k \), where \( k = 1, 2 \) and \( C > 0 \) (for convenience we will put \( C = (k + 2)(k + 1) \)).

We will study the existence of such solutions for the system
\[
\begin{align*}
-\varphi'' + c^* \varphi &= -\beta \varphi^3 - \alpha \varphi \psi, \\
-\psi'' + c \psi &= (k + 2)\psi^{k+1} - \frac{\alpha}{2} \varphi^2,
\end{align*}
\] (5)

In [2], the existence of solutions for a system similar to (5) is derived in the special case where \( k = 2 \). However the method employed cannot be exploited for \( k = 1 \).

We finish this Section by introducing a few notations:
In what follows, \( \{U_S(t)\}_{t \in \mathbb{R}} \) and \( \{U_K(t)\}_{t \in \mathbb{R}} \) will be, respectively, the free Schrödinger and KdV evolution groups defined by
\[
U_S(t) = e^{it \frac{\partial^2}{\partial x^2}} = \mathcal{F}_x^{-1} e^{-it \xi^2} \mathcal{F}_x,
\]
and
\[
U_K(t) = e^{-t \frac{\partial^3}{\partial x^3}} = \mathcal{F}_x^{-1} e^{it \xi^3} \mathcal{F}_x, \quad t \in \mathbb{R}.
\]
Here, \( \mathcal{F}_x \) and \( \mathcal{F}_x^{-1} \) denote the Fourier and the inverse Fourier transform in \( x \). Also, for \( s \geq 0 \), we put
\[
D_s^x = \mathcal{F}_x^{-1} |\xi|^s \mathcal{F}_x.
\]
Finally, for \( 1 \leq p, q \leq +\infty \) and \( T > 0 \), we will consider the spaces \( L^p(\mathbb{R}; L^q([-T; T])) \) and \( L^q([-T; T]; L^p(\mathbb{R})) \) endowed respectively with the norms
\[
\| f \|_{L^p L^q_T} = \left( \int_{-T}^T \left( \int_{\mathbb{R}} |f(x,t)|^q dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}}
\]
and
\[
\| f \|_{L^q_T L^p_x} = \left( \int_{-T}^T \left( \int_{\mathbb{R}} |f(x,t)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}.
\]

2 Global existence for initial data in \( H^{\frac{5}{2}}(\mathbb{R}) \times H^2(\mathbb{R}) \)

Following [26], we introduce the function space
\[
W^2_T(\mathbb{R} \times [-T; T]; \mathbb{K}) = \{ f : \mathbb{R} \times [-T; T] \to \mathbb{K} ; \partial_x^{k-1} f \in L^2_T L^\infty_x \text{ and } \partial_x^{k+1} f \in L^\infty_T L^2_x, \ k = 1, 2 \},
\]
\( \mathbb{K} = \mathbb{R}, \mathbb{C} \), which is a Banach space when endowed with the norm
\[
\| f \|_{W^2_T} = \sum_{k=1}^2 \| \partial_x^{k-1} f \|_{L^2_T L^\infty_x} + \| \partial_x^{k+1} f \|_{L^\infty_T L^2_x}.
\]
Furthermore, we set
\[
X^2_T = C([-T; T]; H^{\frac{5}{2}}(\mathbb{C}) \cap W^2_T(\mathbb{R} \times [-T; T]; \mathbb{C})),
\]
\[
X^2_T = C([-T; T]; H^2(\mathbb{R}) \cap W^2_T(\mathbb{R} \times [-T; T]; \mathbb{R})),
\]
\[
Z^2_T = X^2_T \times Y^2_T,
\]
with norms, respectively,
\[
\| f \|_{X^2_T} = \sup_{t \in [-T; T]} \| f \|_{H^{\frac{5}{2}}} + \| f \|_{W^2_T},
\]
\[
\| f \|_{Y^2_T} = \sup_{t \in [-T; T]} \| f \|_{H^2} + \| f \|_{W^2_T}
\]
and
\[
\|(u, v)\|_{Z^2_T} = \| u \|_{X^2_T} + \| v \|_{Y^2_T}.
\]
In what follows, we will make the following assumption on the function \( a(v) \):


Assumption 2.1 There exists constants \( c > 0 \) and \( q > 0 \) such that for all \( v \in \mathbb{R} \),
\[
|a(v)| \leq c|v| \quad \text{and} \quad |a^{(n)}(v)| \leq c(1 + |v|^q), \quad n \in \{1, 2, 3, 4\}.
\]
Examples of such nonlinearities include, for instance, \( a(v) = v^k \sin(v^p) \) and \( a(v) = ve^{-v^2} \), where \( k \in \{0, 1\} \) and \( p \in \mathbb{N} \).

Under this assumption, the next result can easily be shown by adapting the proof of Theorem 1 in [26]:

Theorem 1 Under the hypothesis stated in Assumption 2.1 and for initial data \((u_o, v_o) \in H^{5/2}(\mathbb{R}) \times H^{2}(\mathbb{R})\), there exists \( T > 0 \) and a unique solution \((u,v) \in Z^{2}_T\) satisfying (2) and (3) in the strong sense, that is, for all \( t \in [-T,T] \),
\[
u(t) = U_S(t)u_o + \int_0^t U_S(t-s)F_1(u,v)(s)ds \quad \text{(8)}
\]
\[
v(t) = U_K(t)v_o + \int_0^t U_K(t-s)F_2(u,v)(s)ds, \quad \text{(9)}
\]
where
\[
F_1(u,v) = -i\alpha vu - i\beta |u|^2 u \quad \text{and} \quad F_2(u,v) = -a(v)v_x + \gamma |u|^2 x.
\]

Now, before passing to the study of the global well-posedness, we need to state some conservation laws for the local solution \((u,v)\).

Lemma 2.2 Let \((u,v) \in Z^{2}_T\) as in Theorem 1. Setting
\[
f(v) = \int_0^v a(\xi)d\xi \quad \text{and} \quad F(v) = \int_0^v f(\xi)d\xi,
\]
the quantities
\[
\mathcal{I}(t) = \int_{\mathbb{R}} |u(x,t)|^2 \, dx, \quad \text{(10)}
\]
\[
\mathcal{E}(t) = \int_{\mathbb{R}} \left( \alpha\gamma v(x,t)|u(x,t)|^2 + \gamma |u_x(x,t)|^2 + \frac{\alpha}{2}(v_x)^2(x,t) - \alpha F(v)(x,t) + \frac{\beta\gamma}{2} |u(x,t)|^4 \right) \, dx \quad \text{(11)}
\]
and
\[
\mathcal{M}(t) = \int_{\mathbb{R}} \left( \alpha v^2(x,t) + 2\gamma Im(u(x,t)\overline{u_x}(x,t)) \right) \, dx \quad \text{(12)}
\]
are conserved in time, that is, for all \( t \in [-T,T] \),
\[
\frac{d}{dt} \mathcal{I}(t) = \frac{d}{dt} \mathcal{E}(t) = \frac{d}{dt} \mathcal{M}(t) = 0.
\]
Proof of Lemma 2.2:

The lemma follows from standard arguments. To prove the conservation of \( \mathcal{I}(t) \), we multiply (2)-(a) by \( \bar{v} \) and integrate the imaginary part. To prove the conservation of \( \mathcal{E}(t) \), we multiply (2)-(a) by \( \bar{v}_t \) and integrate the real part. Finally, to get the conservation of the momentum \( \mathcal{M}(t) \), we multiply (2)-(b) by \( v \) and integrate. In all three cases, the result is obtained after a few integrations by parts.

Note that the regularity of \((u, v)\) stated in Theorem 1 validates these computations. ■

Since by Assumption 2.1 we have \( |F(v)| \leq |v|^3 \), by application of the Gagliardo-Nirenberg inequalities we can derive from Lemma 2.2 the following result:

**Lemma 2.3** Let \((u, v) \in Z^2_T\) as in Theorem 1. Assuming that
\[ \alpha \gamma > 0, \quad (13) \]
we have, for all \( t \in [-T, T] \),
\[ \|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq c(\|u_o\|_{H^1}, \|v_o\|_{H^1}) \]
whith \( c \) depending continuously of its arguments (and independent of \( T \)).

We will assume from now on that condition (13) is satisfied. From Lemma 13 we obtain a bound for the \( H^2 \) norms of \( u \) and \( v \):

**Lemma 2.4** Let \((u, v) \in Z^2_T\) as in Theorem 1. Assuming that (13) holds, for all \( t \in [-T, T] \),
\[ \|u(t)\|_{H^2} + \|v(t)\|_{H^2} \leq c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2}) \]
whith \( c \) depending continuously of its arguments.

**Proof of Lemma 2.4:**

As in the proof of Lemma 11 in [26], it is easy to obtain, for \( t \in [0, T] \),
\[ \|u_{xx}(t)\|_{L^2}^2 \leq C(T) \left( 1 + \int_0^t \|v_{xx}(s)\|_{L^2}^2 ds \right). \quad (14) \]

It is also not difficult to derive from (2)
\[ \frac{d}{dt} \int_\mathbb{R} \left( (v_{xx})^2 - \frac{5}{3} \frac{a(v)}{a'}(v_x)^2 + 2 \gamma |u|^2 v_{xx} \right) dx \]
\[ = \int_\mathbb{R} \left( -\frac{1}{3} a''(v)(v_x)^3 v_{xx} + 5 a(v) a'(v)(v_x)^3 + \frac{10}{3} (a(v))^2 (v_x)^2 v_{xx} \right) dx \]
\[ + \gamma \int_\mathbb{R} \left( \frac{1}{3} a''(v)(u_x)^2 (|u|^2)_{xx} + \frac{4}{3} a(v) v_{xx} (|u|^2)_{xx} - \frac{5}{3} a'(v)(v_x)^2 (|u|^2)_x \right) dx \]
\[ + \gamma \int_\mathbb{R} \left( 4 \text{Im}(u_{xx} \bar{v}) v_{xx} + 2 ((|u|^2)_{xx})^2 \right) dx. \]
Since, by the Gagliardo-Nirenberg inequality,
\[ \|v_x\|_{L^2}^3 \|v_{xx}\|_{L^2} \leq c \|v_x\|_{L^2}^2 \|v_{xx}\|_{L^2} \leq c \|v_{xx}\|_{L^2}^2, \]
we derive from the Assumption 2.1 and (15) that
\[ \|v_{xx}(t)\|_{L^2}^2 \leq C(1 + \|v_{xx}(t)\|_{L^2}^2) + ct + \int_0^t (\|u_{xx}(s)\|_{L^2}^2 + \|v_{xx}(s)\|_{L^2}^2) ds. \] (16)
The result follows by (14), (16) and the Gronwall inequality. ■

Now we fix \( T > 0 \) and \((u_o, v_o) \in H^{5/2}(\mathbb{R}) \times H^2(\mathbb{R})\).
Applying the smoothing properties of \( \{U_S(t)\}_{t \in \mathbb{R}} \) and \( \{U_K(t)\}_{t \in \mathbb{R}} \) derived in [18], [19] and [20] to (8) and (9), we obtain by Lemma 2.4 for \( t \in [-T, T] \)
\[ \|D_x^2 u_{xx}(t)\|_{L^2} \leq C\|u_o\|_{H^{5/2}} + c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2})(1 + \|u\|_{L^2 L^\infty} + \|v\|_{L^2 L^\infty}) \leq C\|u_o\|_{H^{5/2}} + c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2}), \] (17)
where \( c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2}) \) depends continuously on its arguments.

Finally, by (17) and following the proof of Lemmas 7 and 8 in [26], we get
\[ \|(u, v)\|_{Z^2_T} \leq c(T, \|u_o\|_{H^2}, \|v_o\|_{H^2}), \]
where \( c \) can be extended continuously to \( \mathbb{R}^3_+ \).

Hence, we can state the following result:

**Theorem 2.5** Let \((u_o, v_o) \in H^{5/2}(\mathbb{R}) \times H^2(\mathbb{R})\) and \( T > 0 \).
Assuming Assumption 2.1 and (13), there exists a unique strong solution \((u, v) \in Z^2_T\) of the Cauchy problem (2),(3).

### 3 Global existence for initial data in \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \)

In this section we prove a global existence theorem in \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R}) \) for (2),(3) using an approximation method. A similar technique was applied by T. Kato ([16]) to prove a global existence result in \( H^1 \) for the gKdV equation.

In what follows we will use the notation \( C_w(I, X) \) for the space of all weakly continuous functions on \( I \) with values on a Banach space \( X \).

**Theorem 3.1** Let \((u_o, v_o) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\).
Assume Assumption 2.1 and (13). Then, for every \( T > 0 \), there exists a solution \((u, v)\) to (2),(3) such that
\[ (u, v) \in C([-T; T]; H^1(\mathbb{R})) \times C([-T; T]; H^1(\mathbb{R})). \]
Furthermore, putting as in Lemma 2.2
\[ F(v) = \int_0^v f(\xi) \, d\xi \quad \text{and} \quad f(y) = \int_0^y a(\xi) \, d\xi, \]
this solution satisfies for all \( t \in [-T; T] \) the conservation laws
\[ \frac{d}{dt} \mathcal{I}(t) = \frac{d}{dt} \mathcal{E}(t) = \frac{d}{dt} \mathcal{M}(t) = 0, \tag{18} \]
where \( \mathcal{I}, \mathcal{E} \) and \( \mathcal{M} \) are given by (10), (11) and (12).

Proof of Theorem 3.1:

Let \( (u^{(n)}_o, v^{(n)}_o) \in H^{5/2} \times H^2 \) such that
\[ \|(u^{(n)}_o, v^{(n)}_o) - (u_o, v_o)\|_{H^1 \times H^1} \longrightarrow_n 0 \]
and let \( (u^{(n)}, v^{(n)}) \) be the solution of (2),(3) with initial data \( (u^{(n)}_o, v^{(n)}_o) \), which exists in \( C([-T; T]; H^{5/2}) \times C([-T; T]; H^2) \) and satisfies
\[
\begin{align*}
    iu^{(n)}_t + u^{(n)}_{xx} &= \alpha u^{(n)} v^{(n)} + \beta |u^{(n)}|^2 u^{(n)}, \\
v^{(n)}_t + v^{(n)}_{xxx} + a(v^{(n)}) v^{(n)} &= \gamma (|u^{(n)}|^2)_x
\end{align*}
\]
and the conservation laws
\[ \frac{d}{dt} \mathcal{I}^{(n)}(t) = \frac{d}{dt} \mathcal{E}^{(n)}(t) = \frac{d}{dt} \mathcal{M}^{(n)}(t) = 0, \]
where
\[ \mathcal{I}^{(n)}(t) = \int_\mathbb{R} |u^{(n)}(x, t)|^2 \, dx, \quad \mathcal{M}^{(n)}(t) = \int_\mathbb{R} \left( \alpha (v^{(n)})^2 (x, t) + 2\gamma I m(u^{(n)}(x, t)u^{(n)}_x(x, t)) \right) \, dx, \]
and
\[ \mathcal{E}^{(n)}(t) = \int_\mathbb{R} \left( \alpha \gamma v^{(n)}(x, t)|u^{(n)}(x, t)|^2 + \gamma |u^{(n)}_x(x, t)|^2 \\
+ \frac{\alpha}{2} (v^{(n)}_x)^2 (x, t) - \alpha F(v^{(n)})(x, t) + \frac{\beta \gamma}{2} |u^{(n)}(x, t)|^4 \right) \, dx. \tag{19} \]
By the Gagliardo-Nirenberg inequality, it follows that that the sequences \( u^{(n)}(t) \) and \( v^{(n)}(t) \) are uniformly bounded in the \( H^1 \)-norm:
\[
\begin{align*}
    \|u^{(n)}(t)\|_{H^1} &\leq C_1(\|u_o\|_{H^1}, \|v_o\|_{H^1}) \\
    \|v^{(n)}(t)\|_{H^1} &\leq C_1(\|u_o\|_{H^1}, \|v_o\|_{H^1}), \quad t \in [-T; T].
\end{align*}
\]
Since we have the bounded sequences
\[ u(n) \in L^\infty([-T; T]; H^1), \quad v(n) \in L^\infty([-T; T]; H^1), \]
\[ u_i(n) \in L^\infty([-T; T]; H^{-1}) \quad \text{and} \quad v(n) \in L^\infty([-T; T]; H^{-2}), \]
we obtain
\[ u(n) \rightharpoonup u \text{ in } L^\infty([-T; T]; H^1) \text{ weak * and a.e.,} \]
\[ v(n) \rightharpoonup v \text{ in } L^\infty([-T; T]; H^1) \text{ weak * and a.e.,} \]
which implies that \( u \in C([-T; T]; H^{-1}), \ v \in C([-T; T]; H^{-2}) \) and therefore \( u(0) = u_0 \) and \( v(0) = v_0 \).

Now, from (20) and (21), we easily infer that
\[ u(n)v(n) \rightharpoonup uv \text{ in } L^\infty([-T; T]; H^1) \text{ weak *}, \]
\[ a(v(n))v(n) \rightharpoonup a(v)v_x \text{ in } L^\infty([-T; T]; L^2) \text{ weak *}, \]
\[ (|u(n)|^2)_x \rightharpoonup (|u|^2)_x \text{ in } L^\infty([-T; T]; L^2) \text{ weak *}, \]
so that \( (u, v) \in (L^\infty([-T; T]; H^1))^2 \) is a strong solution of (2), (3) in the space
\( C([-T; T]; H^{-1}) \times C([-T; T]; H^{-2}) \).

Next, let \( U_S(t) \) and \( U_K(t) \) be the free Schrödinger and KdV unitary groups already defined in (6) and (7). Then the solution
\[ (u, v) \in (L^\infty([-T; T]; H^1))^2 \cap (C([-T; T]; H^{-1}) \times C([-T; T]; H^{-2})) \]
satisfies the integral system
\[ u(t) = U_S(t)u_0 + \int_0^t U_S(t-s)F_1(u(s), v(s))ds \]
\[ v(t) = U_K(t)v_0 + \int_0^t U_K(t-s)F_2(u(s), v(s))ds, \]
where \( F_1(u, v) = -i\alpha vu - i\beta |u|^2 u \) and \( F_2(u, v) = -a(v)v_x + \gamma(|u|^2)_x \).

A standard computation in semi-group theory leads to the expressions
\[ u(t + h) - u(t) = (U_S(h) - I)u(t) + \int_t^{t+h} U_S(t + h - s)F_1(u(s), v(s))ds, \]
\[ v(t + h) - v(t) = (U_K(h) - I)v(t) + \int_t^{t+h} U_K(t + h - s)F_2(u(s), v(s))ds, \]
for all \( h \in \mathbb{R} \). Since
\[ \| F_1(u, v) \|_{H^1} \leq C_3(\| u_0 \|_{H^1}, \| v_0 \|_{H^1}) \quad \text{and} \quad \| F_2(u, v) \|_{H^1} \leq C_4(\| u_0 \|_{H^1}, \| v_0 \|_{H^1}), \]
we deduce that
\[
\|u(t + h) - u(t)\|_{H^1} \xrightarrow{h \to 0} 0, \quad \|v(t + h) - v(t)\|_{L^2} \xrightarrow{h \to 0} 0,
\]
and therefore \( u \in C([-T, T]; H^1) \) and \( u \in C([-T, T]; L^2) \).

On the other hand, from (21), we easily see that the sequence \( t \to v(n)(t) \) is weakly equicontinuous. Hence, it follows from the Ascoli-Arzelà theorem that the weak limit function is weakly continuous: \( v \in C_w([-T, T]; H^1) \).

We now prove that the solution \((u, v) \in (C([-T, T]; H^1)) \times (C([-T; T]; L^2) \cap C_w([-T; T]; H^1))\)
satisfies the conservation laws (18).

First of all, since \( u(t) \in H^1 \), we remark that
\[
\frac{d}{dt} \|u(t)\|_2^2 = Im(u_{xx}(t), u(t)) = 0
\]
and so \( \|u(t)\|_2^2 = \|u_0\|_2^2 \). Since
\[
\|u^{(n)}(t)\|_2 = \|u^{(n)}(0)\|_2 \xrightarrow{n \to \infty} \|u_0\|_2 = \|u(t)\|_{L^2}
\]
and \( u^{(n)}(t) \rightharpoonup u(t) \) weakly in \( H^1 \) we conclude that for all \( t \in [-T; T], u^{(n)}(t) \to u(t) \)
strongly in \( L^2 \).

Next, to prove the conservation of \( M \), we note that
\[
\int_{\mathbb{R}} u^{(n)}(t)\bar{u}^{(n)}_x(t)dx \to \int_{\mathbb{R}} u(t)\bar{u}_x(t)dx,
\]
since \( u^{(n)}(t) \to u(t) \) in \( L^2 \) and \( u^{(n)}_x(t) \rightharpoonup u_x(t) \) weakly in \( L^2 \). Using the lower semi-continuity of the \( L^2 \) norm, we deduce that for all \( t \in [-T; T], \)
\[
M(t) \leq \lim M^{(n)}(t) = M(0).
\]
From the reversibility in time of equations (2), we obtain \( M(t) = M(0) \).

As a consequence, we derive
\[
\|v^{(n)}(t)\|_{L^2} \xrightarrow{n \to \infty} \|v(t)\|_{L^2}, \quad t \in [-T; T],
\]
and since \( v^{(n)}(t) \rightharpoonup v(t) \) weakly in \( H^1 \), we also conclude \( v^{(n)}(t) \to v(t) \) strongly in \( L^2 \).
Finally, we prove the conservation of the energy $\mathcal{E}(t)$.
We start by noticing that
\[ |F(v^{(n)}(t)) - F(v(t))| \leq C|v^{(n)}(t) - v(t)||v^{(n)}(t)^2 + v(t)v^{(n)}(t) + v(t)^2|, \]
and so
\[
\left| \int_{\mathbb{R}} (F(v^{(n)}(t)) - F(v(t))) \, dx \right| \leq C\|v^{(n)}(t) - v(t)\|_{L^2} \|v^{(n)}(t)^2 + v(t)v^{(n)}(t) + v(t)^2\|_{L^2} \\
\leq C\|v^{(n)}(t) - v(t)\|_{L^2} \xrightarrow{n \to \infty} 0.
\]
Hence,
\[
\int_{\mathbb{R}} F(v^{(n)}(t)) \, dx \xrightarrow{n \to \infty} \int_{\mathbb{R}} F(v(t)) \, dx.
\]
In a similar way, we have
\[
\left| \int_{\mathbb{R}} ((u^{(n)}(t))^4 - (u(t))^4) \, dx \right| \leq C\|u^{(n)}(t) - u(t)\|_{L^2} \xrightarrow{n \to \infty} 0.
\]
Now, using the semi-continuity of the $H^1$ norm, we obtain for all $t \in [-T, T],$
\[ \mathcal{E}(t) \leq \lim \mathcal{E}^{(n)}(t) = \mathcal{E}(0). \]
Once again, by the reversibility of equations (2),
\[ \mathcal{E}(t) = \mathcal{E}(0). \]
To prove the continuity of the map $t \to (u(t), v(t))$ in $H^1 \times H^1$ it remains to prove that $v$ is strongly continuous on $H^1$. From the conservation of the energy $\mathcal{E}(t)$, it follows that $t \to \|u(t)\|_{H^1}$ is continuous.
Since it has already been proved that $v \in C_w([-T; T]; H^1)$, this yields the strong continuity of $v(t)$ and therefore the proof of Theorem 2.5 is complete.

4 Existence of bound states

In this Section we look for bound-state solutions to (2) of the form
\[ (u(x, t), v(x, t)) = (e^{i\omega t + ikx} \phi(x - ct), \psi(x - ct)). \]
in the case where $F(v) = v^p$, where $F$ is defined in the Lemma 2.2 and corresponds to the case $a(v) = p(p-1)v^{p-2}$.

We will prove the following result:
**Theorem 4.1** Let \( F(v) = v^p, \gamma = \frac{\alpha}{2} \) and \( \alpha, \beta < 0 \).

For \( p = 3, \alpha < -3 \) or \( p = 4, \beta < -4 \) there exists a family

\[
(u_n(x, t), v_n(x, t)) = (e^{i\omega_n t + i\kappa_n x} \phi_n(x - c_n t), \psi_n(x - c_n t))
\]

(23)

of non trivial bound-state solutions to (2) with \( \lim_{n \to \infty} c_n = +\infty \).

Here, \( \phi_n, \psi_n \) are smooth positive functions which decay exponentially at infinity.

This result is an extension of the Theorem proved in [14] in the frame of the Schrödinger-KdV system (4). The method relies on the concentration-compactness theorem by P.L. Lions ([21],[22]) and an estimate for the Lagrange multipliers associated to a minimization problem.

By setting \( c^* := k^2 + \omega \) and putting \( c = 2k \), we obtain the system

\[
\begin{align*}
-\phi'' + c^* \phi &= -\beta \phi^3 - \alpha \phi \psi \\
-\psi'' + c \psi &= p \psi^{p-1} - \frac{\alpha}{2} \phi^2
\end{align*}
\]

(24)

For \( \mu \geq 0 \) we set \( X_\mu = \{(u, v) \in H^1 \times H^1 : \|u\|_{L^2}^2 + \|v\|_{L^2}^2 = \mu \} \) (\( u, v \) real-valued) and consider the minimizing problem

\[
I(\mu) = \inf \{ E(u, v) : (u, v) \in X_\mu \},
\]

(25)

where

\[
E(u, v) = \int_{\mathbb{R}} (u')^2 + (v')^2 + \alpha u^2 v - 2v^p + \frac{\beta}{2} u^4.
\]

Note that \( E = \frac{2}{\alpha} E \), where \( E \) is the energy given by (11).

We will use the concentration-compactness method to prove the existence of minimizers to \( I \).

**Lemma 4.2** For all \( \mu > 0, I(\mu) > -\infty \).

**Proof of Lemma 4.2:**

Let \((u, v) \in X_\mu: \|u\|_{L^2}^2 \leq \mu \) and \( \|v\|_{L^2}^2 \leq \mu \). By the Gagliardo-Nirenberg inequality,

\[
\|v\|_{L^{p_2}} \leq C_1 \|v'\|_{L^2}^{\frac{p-2}{2}} \|v\|_{L^2}^{\frac{p+2}{2}} \leq C_1 \mu^{\frac{p+2}{2}} \|v'\|_{L^2}^{\frac{p-2}{2}}
\]

and

\[
\|u\|_{L^4} \leq C_2 \|u'\|_{L^2} \|u\|_{L^2}^{\frac{3}{2}} \leq C_2 \mu^{\frac{3}{2}} \|u'\|_{L^2},
\]
where $C_j$ denote positive constants.

Also,

$$\int |v|u^2 \leq \frac{1}{2} \|v\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^4}^4 \leq \frac{\mu}{2} + C_2 \frac{\mu^2}{2} \|u^r\|_{L^2}^2.$$  

Finally, we obtain

$$E(u,v) \geq \|u^r\|_{L^2}^2 + \|v^r\|_{L^2}^2 + \frac{\beta}{2} \int u^4 - 2|v|^p - |\alpha| \int u^2 |v|$$

$$\geq \|u^r\|_{L^2}^2 + \|v^r\|_{L^2}^2 - \frac{1}{2} C_2 (|\alpha| + |\beta|) \|u^r\|_{L^2}^2 - \frac{1}{2} C_1 \mu \frac{\mu^2}{L^2} \|v^r\|_{L^2}^2 - |\alpha| \mu \frac{\mu^2}{L^2},$$

where $\frac{p-2}{2} < 2$. We deduce the existence of an inferior bound for $E(u,v)$ depending exclusively on $\mu$.

Note that for large values of $p \ (p \geq 6)$, $\mathcal{I}(\mu) = -\infty$ for $\mu > 0$.

Indeed, let $v \in H^1$ such that $\|v\|_{L^2}^2 = \mu$. Then, by setting $v_a(x) = a^2 v(ax)$, $\|v_a\|_{L^2}^2 = \mu$ and

$$E(0,v_a) = a^6 \int |v'|^2 - a^p \int v^p \xrightarrow{a \to +\infty} -\infty.$$  

**Proposition 4.3** For all $\mu \geq 0$, $\mathcal{I}(\mu) \leq 0$. Also, there exists $\mu^* > 0$ such that for all $\mu > \mu^*$, $\mathcal{I}(\mu) \leq -A\mu^2$, where $A$ is a positive constant independent of $\mu$.

**Proof of Proposition 4.3:**

Let $\mu \geq 0$ and $u \in H^1$ such that $\|u\|_{L^2}^2 = \mu$. Then $(u,0) \in X_\mu$ and $E(u,0) \leq \int (u')^2$.

Noticing that $\inf \{ \int (u')^2 : \|u\|_{L^2}^2 = \mu \} = 0$, we get $\mathcal{I}_\mu \leq 0$.

We now consider $u \in H^1(\mathbb{R})$ such that $\|u\|_{L^2}^2 = 1$. Putting $u_\mu(x) = \mu^{\frac{1}{4}} u(x)$, $(u_\mu,0) \in X_\mu$.

Furthermore,

$$\mathcal{I}(\mu) \leq E(u_\mu,0) = \mu \int (u')^2 + \frac{\beta}{2} \mu^2 \int u^4 = \mu \left( \int (u')^2 - \frac{|\beta|}{4} \mu \int u^4 \right) - \frac{\mu^2 |\beta|}{4} \int u^4.$$  

By choosing $A = \frac{|\beta|}{4} \int u^4$ and $\mu^*$ such that $\int (u')^2 - \frac{|\beta|}{4} \mu^* \int u^4 \leq 0$ we get the result.

**Remark 4.4** It is well known that for $f \in H^1(\mathbb{R})$ real valued, $\|f\|_{L^2}^2 \leq \|f'\|_{L^2}$. Therefore, for every pair $(u,v) \in X_\mu$,

$$E(\|u\|, \|v\|) \leq E(u,v).$$  

Hence, there exists a minimizing sequence $(u_j, v_j)$ for problem (25) with $u_j, v_j \geq 0$.

**Lemma 4.5** Let $\mu > \mu^*$. For all $\theta > 1$, $\mathcal{I}(\theta \mu) < \theta \mathcal{I}(\mu)$.
Proof: Consider a positive minimizing sequence \((u_j, v_j) \in X_\mu\) for problem (25). We have

\[
E(\sqrt{\theta}u_j, \sqrt{\theta}v_j) = \theta E(u_j, v_j) - |\beta|/2 (\theta^2 - \theta) \int u_j^4 + (\theta^2 - \theta) \left( \alpha \int u_j^2 v_j - 2 \int v_j^p \right) \leq \theta E(u_j, v_j) + max\{\theta - \theta^2, \theta - \theta^2\} \left( |\beta|/2 \int u_j^4 + |\alpha| \int u_j^2 v_j + 2 \int v_j^p \right).
\]

Since \((u_j, v_j)\) is a minimizing sequence, \(\frac{|\beta|}{2} \int u_j^4 + |\alpha| \int u_j^2 v_j + 2 \int v_j^p \geq \delta\) for some \(\delta > 0\). Otherwise there would exist a subsequence - still denoted \((u_j, v_j)\) - such that \(\lim E(u_j, v_j) \geq 0\), which is absurd since \(I(\mu) < 0\). Hence

\[
E(\sqrt{\theta}u_j, \sqrt{\theta}v_j) \leq \theta E(u_j, v_j) - \delta(\theta^{max\{2, \frac{2}{p}\}} - \theta) \quad \text{for all } \theta > 1.
\]

Since \(\|\sqrt{\theta}u_j\|_2^2 + \|\sqrt{\theta}v_j\|_2^2 = \theta \|u_j\|_2^2 + \theta \|v_j\|_2^2 = \theta \mu\), we obtain \(I(\theta \mu) < \theta I(\mu)\).

From this result it is straightforward that \(I\) is a non-decreasing function of \(\mu\).

Therefore, there exists \(\mu_1 \geq 0\) such that

\[
\mu > \mu_1 \iff I(\mu) < 0.
\]

Arguing as in Lemma 2.3 of [24], we get the strict sub-additivity of \(I(\mu)\):

**Corollary 4.6**

Let \(\mu > \mu_1\) and \(0 < \Omega < \mu\). Then \(I(\mu) < I(\Omega) + I(\mu - \Omega)\).

Next, we prove the existence of minimizers for the minimization problem (25):

**Proposition 4.7** Let \(M_\mu = \{(u,v) \in X_\mu : I(\mu) = E(u,v)\}\). For \(\mu > \mu_1\), \(M_\mu \neq \emptyset\).

**Proof of Proposition 4.7:**

Let us consider a positive minimizing sequence \((u_j, v_j) \in X_\mu\) for problem (25). We will apply the concentration-compactness lemma to the sequence \(\rho_j = u_j^2 + v_j^2\). Using the notations in [21], we introduce the concentration function of \(\rho_j\):

\[
Q_j(t) = \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} \rho_j, \quad \text{and we set} \quad \Omega = \lim_{t \to \infty} Q(t).
\]

There are three alternatives: vanishing (\(\Omega = 0\)), dichotomy (\(0 < \Omega < \mu\)) and compactness (\(\Omega = \mu\)). The latter implies the relative compactness of the sequence \((u_j, v_j)\) up to translations.
First, we rule out vanishing. Indeed, if $\Omega = 0$,

$$
\limsup_{j \to \infty} \int_{y-t}^{y+t} u_j^2 = \limsup_{j \to \infty} \int_{y-t}^{y+t} v_j^2 = 0.
$$

Since $(u_j)$ and $(v_j)$ are bounded in $H^1(\mathbb{R})$ (as seen in the proof of Proposition 4.2), a classical lemma (see [22], Lemma I.1) yields $\|u_j\|_p \to 0$ and $\|v_j\|_p \to 0$ for all $p > 2$.

It results that

$$
\mathcal{I}(\mu) = \lim_{j \to \infty} E(u_j, v_j) = \lim_{j \to \infty} \int u_j' \int u_j' - \frac{|\beta|}{2} \int u_j^4 - 2 \int v_j + \alpha \int u_j^2 v_j \geq 0,
$$

which is absurd by Proposition 4.3.

We now rule out dichotomy: by Lemma III.1 in [21], for all $\epsilon > 0$ there exists constants $R_1 > 0$, $R_2 > R_1$, a sequence \{y_j\} and cut-off functions $\eta_i \in C^\infty(\mathbb{R})$, $0 \leq \eta_i \leq 1$ such that

- $\eta_i(x) = 1$ for $|x| \leq R_1$ and $\eta_i(x) = 0$ for $|x| \geq \frac{R_2}{2}$,
- $\eta_i(x) = 1$ for $|x| \geq R_2$ and $\eta_i(x) = 0$ for $|x| \leq \frac{R_2}{2}$,
- $|\eta_i(x)| \leq \epsilon$ for all $x \in \mathbb{R}$,
- $\int_{R_1 \leq |x-y_j| \leq R_2} \eta_i(x) \leq \epsilon$,
- $|\int \eta_i^2(x) \rho_i(x-y_j) - \Omega| < \epsilon$ and $|\int \eta_i^2(x) \rho_i(x-y_j) - (\mu - \Omega)| < \epsilon$.

Putting $u_j^{(i)} = \eta_i(x-y_j)$ and $v_j^{(i)} = \eta_i(x-y_j)$,

$$
\|u_j'\|_2^2 - \|u_j^{(1)'}\|_2^2 - \|u_j^{(2)'}\|_2^2 =
\int_{R_1 \leq |x-y_j| \leq R_2} (u_j')^2 - \int_{R_1 \leq |x-y_j| \leq \frac{1}{2} R_2} (u_j^{(1)'})^2 - \int_{\frac{1}{2} R_2 \leq |x-y_j| \leq R_2} (u_j^{(2)'})^2 =
\int_{R_1 \leq |x-y_j| \leq \frac{1}{2} R_2} (u_j')^2 - (u_j^{(1)'})^2 + \int_{\frac{1}{2} R_2 \leq |x-y_j| \leq R_2} (u_j')^2 - (u_j^{(2)'})^2. \quad (27)
$$

Also,

$$
\int_{R_1 \leq |x-y_j| \leq \frac{1}{2} R_2} (u_j')^2 - (u_j^{(1)'})^2 =
\int_{R_1 \leq |x-y_j| \leq \frac{1}{2} R_2} (1 - \eta_1^2(x-y_j)) (u_j')^2 - (\eta_1'(x-y_j))^2 u_j^2 - 2\eta_1(x-y_j) \eta_1'(x-y_j) u_j u_j' \geq
-\epsilon^3 - C\epsilon^3.
$$
A similar estimate holds for the second term in the right-hand-side of (27), hence we obtain
\[ \|u'_j\|^2_2 - \|u_j^{(1)}\|^2_2 - \|u_j^{(2)}\|^2_2 \geq -2\epsilon^3 - 2C\epsilon^3. \]

By the same computations,
\[ \|v'_j\|^2_2 - \|v_j^{(1)}\|^2_2 - \|v_j^{(2)}\|^2_2 \geq -2\epsilon^3 - 2C\epsilon^3. \]

Moreover, it is straightforward to prove that
\[ \int \|v_j\|^p_p - \|v_j^{(1)}\|^p_p - \|v_j^{(2)}\|^p_p \leq C_1\epsilon^{p+2}, \quad \|u_j\|^4_4 - \|u_j^{(1)}\|^4_4 - \|u_j^{(2)}\|^4_4 \leq C_2\epsilon^3 \]
and
\[ \int u_j^2 v_j - \int u_j^{(1)} v_j^{(1)} - \int u_j^{(2)} v_j^{(2)} \leq C_3\epsilon^2. \]

Finally, we obtain
\[ E(u_j, v_j) \geq E(u_j^{(1)}, v_j^{(1)}) + E(u_j^{(2)}, v_j^{(2)}) - C\epsilon. \]

This leads to $I(\mu) \geq I(\Omega) + I(\mu - \Omega)$, which is in contradiction with Corollary 4.6.

Hence, we have compactness: extracting once again a subsequence, there exists \( \{y_j\} \) such that
\[ (\tilde{u}_j = u_j(. - y_j), \tilde{v}_j = v_j(. - y_j)) \to (\phi, \psi) \text{ in } L^2(\mathbb{R}). \]

Furthermore, the sequence \( (\tilde{u}_j, \tilde{v}_j) \) converges to \( (\phi, \psi) \) in \( H^1(\mathbb{R}) \) weak. Hence, \( (\tilde{u}_j, \tilde{v}_j) \to (\phi, \psi) \text{ in } L^q \) for all \( q \geq 2 \) : \( \|\tilde{u}_j\|_4 \to \|\phi\|_4, \|\tilde{v}_j\|_p \to \|\psi\|_p, \int \tilde{u}_j^2 \tilde{v}_j \to \int \phi^2 \psi \) and
\[ T(\mu) \leq E(\phi, \psi) \leq \lim E(\tilde{u}_j, \tilde{v}_j) = T(\mu). \]

Finally, \( (\phi, \psi) \in \mathcal{M}_\mu \neq \emptyset. \)

Note that we have obtained \( \lim \int (\tilde{u}_j')^2 + (\tilde{v}_j')^2 = \int (\phi')^2 + (\psi')^2 \), hence the convergence takes place in \( H^1 \) strong: \( \phi \geq 0 \) and \( \psi \geq 0 \).

Also, it is clear that \( \psi \neq 0 \). We now show that \( \phi \neq 0 \).

Taking \( \psi \) such that \( (0, \psi) \in X_\mu \), for all \( \theta \in [0, 1], (\theta^{\frac{1}{2}} \psi, (1 - \theta)^{\frac{1}{2}} \psi) \in X_\mu \). A straightforward computation then leads to
\[ E(\theta^{\frac{1}{2}} \psi, (1 - \theta)^{\frac{1}{2}} \psi) = E(0, \psi) + f_{\alpha, \beta}(\theta), \]
where
\[ f_{\alpha, \beta}(\theta) = \alpha \theta(1 - \theta)^{\frac{1}{2}} \|\psi\|_{L^3}^2 + 2(1 - (1 - \theta)^{\frac{1}{2}})\|\psi\|_{L^p}^p + \frac{\beta}{2}(1 - \theta)^{\frac{1}{2}} \|\psi\|_{L^4}^4. \]
If \( p = 3 \), \( f'_{\alpha,\beta}(0) = (3 + \alpha)\|\psi\|_{L^3}^3 < 0 \) if \( \alpha < -3 \), hence, for small \( \theta \),
\[
E(\theta^\frac{1}{2}\psi, (1 - \theta)^\frac{1}{2}\psi) < E(0, \psi).
\]

If \( p = 4 \), we notice that \( f_{\alpha,\beta}(1) = (2 + \frac{\beta}{2}) \|\phi\|_{L^4}^4 < 0 \) if \( \beta < -4 \); for \( 1 - \theta \) small, we get once again
\[
E(\theta^\frac{1}{2}\psi, (1 - \theta)^\frac{1}{2}\psi) < E(0, \psi).
\]

\[\blacksquare\]

End of the Proof of Theorem 4:

Let \((\phi, \psi) \in M_\mu, \phi, \psi \geq 0, \phi \neq 0, \psi \neq 0\).
There exists a Lagrange multiplier \( \lambda = \lambda(\mu) \in \mathbb{R} \) such that
\[
\begin{cases}
\phi'' - \lambda \phi = -\beta \phi^3 - \alpha \phi \psi \\
\psi'' - \lambda \psi = p \psi^{p-1} - \frac{\alpha}{2} \phi^2.
\end{cases}
\]

Multiplying these equations by \( \phi \) and \( \psi \) and integrating by parts leads to
\[
\begin{align*}
\int (\phi')^2 - \lambda \int \phi^2 &= -\beta \int \phi^4 - \alpha \int \phi^2 \psi \\
\int (\psi')^2 - \lambda \int \psi^2 &= p \int \psi^{p} - \frac{\alpha}{2} \int \phi^2 \psi.
\end{align*}
\]

hence
\[
\int (\phi')^2 + \int (\psi')^2 - \lambda \mu = -\beta \int \phi^4 + p \int \psi^{p} - \frac{3\alpha}{2} \int \phi^2 \psi.
\]
Since
\[
\mathcal{I}(\mu) = E(\phi, \psi) = \int (\phi')^2 + \int (\psi')^2 + \alpha \int \phi^2 \psi - 2 \int \psi^p + \frac{\beta}{2} \int \phi^4,
\]
we get
\[
\lambda \mu = \mathcal{I}(\mu) + \frac{\beta}{2} \int \phi^4 + (2 - p) \int \psi^{p} + \frac{\alpha}{2} \int \phi^2 \psi \leq \mathcal{I}(\mu) \leq -A\mu^2
\]
by Proposition 4.3.
Finally
\[
\lambda \leq -A\mu.
\]

The proof is complete by choosing a sequence \( \mu_n \to +\infty \) and setting \( c_n = -\lambda_n, k_n = -\frac{1}{2} \lambda_n \) and \( \omega_n = -\lambda_n - k_n^2 \).

Note that a classical bootstrap argument proves the regularity of \( \phi \) and \( \psi \) and that, since \( c^* = k_n^2 + \omega_n^2 = -\lambda_n > 0 \), the argument used in the proof of Theorem 8.1.1 in [7] easily shows the existence of \( \epsilon > 0 \) such that \( e^{\epsilon |x|} \phi, e^{\epsilon |x|} \psi \in L^\infty \), leading to the exponential decreasing of \( \phi \) and \( \psi \). \[\blacksquare\]

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