

A distributional approach to the geometry of $2D$ dislocations at the mesoscale

Part B: The case of a countable family of dislocations

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May 8, 2009

Keywords: dislocations, single crystals, multi-scale analysis, homogenisation, distribution theory, multivalued functions

Abstract

This paper develops a geometrical model of dislocations and disclinations in single crystals at the mesoscopic scale. In the continuation of previous work the distribution theory is used to represent concentrated effects in the defect lines which in turn form the branching lines of the multiple-valued elastic displacement and rotation fields. Fundamental identities relating the incompatibility tensor to the dislocation and disclination densities are proved in the case of countably many parallel defect lines, under global $2D$ strain assumptions relying on the geometric measure theory. Our theory provides the appropriate objective internal variables and the required mathematical framework for a rigorous homogenization from mesoscopic to macroscopic scale.

1 Introduction

1.1 Preliminaries and principal hypotheses

The present paper provides a mathematical theory of the geometry of crystal dislocations and disclinations in continuation of the work of Van Goethem & Dupret (2009a) where the general context of this research is detailed and which will be referred to as Part A in the sequel. In summary, the objective of these investigations is to develop a rigorous mathematical framework for the treatment of line defects in single crystals at the mesoscopic scale. As explained in Part A, concentrated effects in the defect lines and their neighbourhood have to be considered at this scale and this requires to make

use of the distribution theory (Schwartz 1957) to handle the related fields (dislocation and disclination densities, contortion, incompatibility, etc.) and their relationships. Moreover, in view of the incompatibility of the elastic strain tensor in the presence of line defects, the associated rotation and displacement are multiple-valued fields whose branching lines are precisely the defect lines. The combined treatment of distributions and multivalued functions was addressed in Part A, where our theory was applied to the case of a set of isolated parallel, moving or not, line defects under the hypothesis of a $2D$ elastic strain field.

In this second paper, the case of countably many parallel defect lines is investigated. Therefore, instead of analysing the regularity of the elastic strain near an assumed isolated defect line, a more general abstract approach is selected with a view to defining the appropriate functional space to validate the main theorem relating the strain incompatibility to the defect densities.

Let us remark that our mesoscopic setting will be able to treat fine and complex dislocation structures since accumulation lines or points in the defective set will be allowed (such as typically the structures appearing in the work of Cantor (1915) on transfinite numbers, see figure 1). This feature represents a key ingredient of our theory since a tending to infinity number of defect lines unavoidably appears in the homogenization process from meso- to macro-scale. Moreover, when the defect lines exhibit a clustered mesoscopic structure, even if their actual number will always remain finite across any bounded area, it is much more convenient mathematically to consider a model where this structure may be infinitely refined.

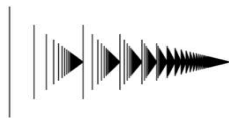


Figure 1: ω^2 dislocation structure, where ω is the first ordinal transfinite number (Cantor 1915).

Before any further development, precise definitions and assumptions are required. In general, the functional spaces used consist of distributions, Radon measures (Ambrosio et al. 2000, Mattila 1995), functions, etc., which can be considered as continuous functionals over a set of test-functions whose regularity determines the functional space properties. However particular care has to be given to avoiding undesirable boundary effects.

Definition 1.1 (functional spaces used) *The crystal domain is an open connected set Ω , on which some mathematical elements (distributions, Radon measures, locally summable functions, etc.) are defined as linear functionals over an associated set of test-functions whose support is a compact subset of Ω . Henceforth, the qualification “on Ω ” for these elements will mean in addition that extensions of these elements exist as functionals over all the test-functions having the desired properties and whose support is a compact subset of \mathbb{R}^3 .*

In other words, a distribution, Radon measure, locally summable function on Ω will always be constrained to also be the restriction to Ω of a mathematical element

of the same type defined on the whole \mathbb{R}^3 . A simple 1D example can illustrate this constraint. The function

$$f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \quad x \longmapsto f(x) = 1/x \quad (x > 0) \quad (1.1)$$

is locally summable on \mathbb{R}_0^+ in the classical sense but cannot be extended as a locally summable function over \mathbb{R} , this resulting from its behaviour for $x \rightarrow 0^+$. According to definition 1.1, in the present paper this function will not be called locally summable on \mathbb{R}_0^+ . Indeed, in a context where locally summable functions are needed and where the physical domain of interest is \mathbb{R}_0^+ , no shift to the right of f as defined by (1.1) can provide a locally summable function on \mathbb{R}_0^+ (with $g(x) = f(x - a)$, $x > a > 0$) whatever the definition of g is for $0 < x \leq a$. So, if to be acceptable the locally summable functions on \mathbb{R}_0^+ are requested to exhibit the same properties near the origin as in the vicinity of their interior points (to avoid peculiar boundary effects for $x \rightarrow 0^+$), freely translating these functions to move the origin onto an interior point should be allowed, and hence f cannot be accepted as locally summable on \mathbb{R}_0^+ if g cannot. This justifies definition 1.1.

Similarly, the same function f defined by (1.1) can be considered as the density of a Radon measure on \mathbb{R}_0^+ in the classical sense (as acting against continuous test-functions whose support is compact and contained in \mathbb{R}_0^+). However, again no shift to the right of f can provide a Radon measure on \mathbb{R}_0^+ since no extension to \mathbb{R} of f as a Radon measure in the classical sense exists. So, in our work f will not be considered as a Radon measure on \mathbb{R}_0^+ according to definition 1.1.

Considering now f as a distribution on \mathbb{R}_0^+ (as acting against \mathcal{C}^∞ test-functions whose support is a compact subset of \mathbb{R}_0^+), f can be prolonged as a distribution on \mathbb{R} by defining for example the prolonged f as the pseudo-function $Pf.(1/x)$, which is the distributional derivative of the function $F(x) := \log|x|$ and acts again a test-function by taking the Hadamard finite part of the resulting diverging integral. So, here f can be considered as a distribution on \mathbb{R}_0^+ in the sense of definition 1.1.

It should be noted that the spaces generated from definition 1.1 are non-closed subspaces of classical spaces (the usual distributions, Radon measures, ... on Ω) and hence cannot share all their properties (completeness, etc.).

In the crystal domain Ω , the meso-scale physics will then be represented by a nowhere dense set of defect lines which in 2D are parallel to each other.

Definition 1.2 (2D mesoscopic defect lines) *At the meso-scale, a 2D set of dislocations and/or disclinations $\mathcal{L} \subset \Omega$ is a closed set of Ω (this meaning the intersection with Ω of a closed set of \mathbb{R}^3) formed by a countable union of parallel lines $L^{(i)}$, $i \in \mathcal{I} \subset \mathbb{N}$, whose adherence is itself a countable union of lines and where the linear elastic strain is singular. In the sequel, these lines will be assumed as parallel to the z -axis.*

The present mesoscopic theory will be completely developed from the sole linear strain – which itself could be defined from the stress field (although the stress-strain relationship is not used in the sequel) and therefore is an objective internal field.

Assumption 1.1 (2D mesoscopic elastic strain) *The linear strain \mathcal{E}_{mn}^* is a given symmetric $L_{loc}^1(\Omega)$ tensor (in the sense of definition 1.1) prolonged by 0 on the dislocation set \mathcal{L} , and such that $\partial_z \mathcal{E}_{mn}^* = 0$. Moreover, \mathcal{E}_{mn}^* is assumed as compatible on $\Omega_{\mathcal{L}} := \Omega \setminus \mathcal{L}$ in the sense that the incompatibility tensor defined by*

$$INCOMPATIBILITY: \quad \eta_{kl}^* := \epsilon_{kpm} \epsilon_{lqn} \partial_p \partial_q \mathcal{E}_{mn}^*, \quad (1.2)$$

where derivation is intended in the distribution sense, vanishes everywhere on $\Omega_{\mathcal{L}}$. Equivalently, in 2D there are real numbers K, a_α and b such that the following equalities hold on $\Omega_{\mathcal{L}}$:

$$\begin{cases} \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\partial_\alpha\partial_\beta\mathcal{E}_{\gamma\delta}^* = 0, \\ \epsilon_{\alpha\beta}\partial_\alpha\mathcal{E}_{\beta z}^* = K, \\ \mathcal{E}_{zz}^* = a_\alpha x_\alpha + b. \end{cases} \quad (1.3)$$

In Part A, key tensor fields were derived from the mesoscopic elastic strain field as order-1 distributions (acting on C_c^1 test-functions with compact support).

Definition 1.3 (Frank and Burgers tensors)

FRANK TENSOR: $\overline{\partial}_m \omega_k^* := \epsilon_{kpq} \partial_p \mathcal{E}_{qm}^*$, (1.4)

BURGERS TENSOR: $\overline{\partial}_l b_k^* := \mathcal{E}_{kl}^* + \epsilon_{kpq} (x_p - x_{0p}) \overline{\partial}_l \omega_q^*$. (1.5)

Line integration of the Frank and Burgers tensors in $\Omega_{\mathcal{L}}$ (i.e., outside the defect set) provides the index-1 multivalued rotation and Burgers vector fields ω_k^* and b_k^* (with “index-1” meaning that their first derivatives in $\Omega_{\mathcal{L}}$, denoted by $\partial_j^{(s)} \omega_k^*$ and $\partial_j^{(s)} b_k^*$, are single-valued). These properties are summarized in the following theorem, whose proof is classical.

Theorem 1.1 (multiple-valued displacement field) *From a symmetric smooth linear strain \mathcal{E}_{ij}^* on $\Omega_{\mathcal{L}}$ and a point x_0 where displacement and rotation are given, a multivalued displacement field u_i^* of index 2 (whose second derivatives are single-valued) can be constructed on $\Omega_{\mathcal{L}}$ such that the symmetric part of the distortion $\partial_j^{(s)} u_i^*$ is the single-valued strain tensor \mathcal{E}_{ij}^* while its skew-symmetric part is the multivalued rotation tensor $\omega_{ij}^* := -\epsilon_{ijk} \omega_k^*$. Moreover, inside $\Omega_{\mathcal{L}}$ the gradient $\partial_j^{(s)}$ of the rotation and Burgers fields ω_k^* and $b_k^* = u_k^* - \epsilon_{klm} \omega_l^* (x_m - x_{0m})$ coincides with the Frank and Burgers tensors.*

From this result, the Frank and Burgers vectors can be defined as invariants of any isolated defect line $L^{(i)}$ of \mathcal{L} .

Definition 1.4 (Frank and Burgers vectors) *The Frank vector of the isolated defect line $L^{(i)}$ is the invariant*

$$\Omega_k^{*(i)} := [\omega_k^*]^{(i)}, \quad (1.6)$$

while its Burgers vector is the invariant

$$B_k^{*(i)} := [b_k^*]^{(i)} = [u_k^*]^{(i)}(x) - \epsilon_{klm} \Omega_l^{*(i)} (x_m - x_{0m}), \quad (1.7)$$

with $[\omega_k^*]^{(i)}, [b_k^*]^{(i)}$ and $[u_k^*]^{(i)}$ denoting the jumps of ω_k^*, b_k^* and u_k^* around $L^{(i)}$.

The case of non-isolated defect-lines represents a major issue of this work and will be resolved at a later stage. Besides their relationship with the multivalued rotation, Burgers and displacement fields, the Frank and Burgers tensors can be directly related to the strain incompatibility by use of (1.2), (1.4) and (1.5).

Theorem 1.2 *The distributional curls of the Frank and Burgers tensors are*

$$\epsilon_{ilj} \partial_l \overline{\partial}_j \omega_k^* = \eta_{ik}^*, \quad (1.8)$$

$$\epsilon_{ilj} \partial_l \overline{\partial}_j b_k^* = \epsilon_{kpq} (x_p - x_{0p}) \eta_{iq}^*, \quad (1.9)$$

with η_{ik}^* the incompatibility tensor.

From this theorem it results that single-valued rotation and Burgers fields ω_k^* and b_k^* can be integrated on Ω if, and only if the incompatibility tensor vanishes.

The dislocation and disclination densities are the basic physical tools used to model defect density at the meso-scale. In $2D$ (here meaning that $\partial_z \mathcal{E}_{mn}^* = 0$) the defect lines are parallel to the z -axis and hence only some components of the defect densities do not vanish.

Definition 1.5 (2D defect densities ¹)

$$\text{DISCLINATION DENSITY:} \quad \Theta_z^* := \sum_{i \in \mathcal{I}} \Omega_z^{*(i)} \delta_{L^{(i)}}, \quad (1.10)$$

$$\text{DISLOCATION DENSITY:} \quad \Lambda_k^* := \sum_{i \in \mathcal{I}} B_k^{*(i)} \delta_{L^{(i)}} \quad (k = 1 \cdots 3). \quad (1.11)$$

In general, the complete defect density tensors are denoted by Θ_{ij}^* and Λ_{ij}^* with the indices i and j referring to the local defect line direction and Frank or Burgers vector, respectively. So in $2D$,

$$\Theta_{k\kappa}^* = \Theta_{\kappa k}^* = 0, \quad \Theta_{zz}^* = \Theta_z^* \quad (1.12)$$

$$\Lambda_{\kappa k}^* = 0, \quad \Lambda_{zk}^* = \Lambda_k^*. \quad (1.13)$$

with $k = 1 \cdots 3$ and $\kappa = 1, 2$. The vanishing of $\Theta_{k\kappa}^*$ was shown in Part A.

An additional important defect density tensor called the contortion was introduced by Nye (1953) from the work of Kondo (1952).

Definition 1.6 (2D mesoscopic contortion)

$$\text{CONTORTION:} \quad \kappa_{ij}^* := \delta_{iz} \alpha_j^* - \frac{1}{2} \alpha_z^* \delta_{ij} \quad (i, j = 1 \cdots 3), \quad (1.14)$$

where

$$\alpha_k^* := \Lambda_k^* - \delta_{k\alpha} \epsilon_{\alpha\beta} \Theta_z^* (x_\beta - x_{0\beta}). \quad (1.15)$$

Here, α_k^* is an auxiliary defect density measure associated with a general $3D$ tensor $\alpha_{ij}^* = \Lambda_{ij}^* + \epsilon_{jlm} \Theta_{il}^* (x_m - x_{0m})$ such that in $2D$

$$\alpha_{\kappa k}^* = 0, \quad \alpha_{zk}^* = \alpha_k^*, \quad (1.16)$$

with $k = 1 \cdots 3$ and $\kappa = 1, 2$. The general $3D$ contortion is $\kappa_{ij}^* = \alpha_{ij}^* - \frac{1}{2} \alpha_{mm}^* \delta_{ij}$, in such a way that in $2D$, $\kappa_{\alpha j}^*$ vanishes if $\alpha \neq j$.

1.2 Objective of this work

The principal objective of this work is to identify key distributional fields at the meso-scale and to demonstrate their relationship in a rigorous functional analysis context, with a further view to providing the required framework for the homogenization of these fields and their relations to the macroscopic scale.

In §2, the main theorem of Part A (expressing the elastic strain incompatibility in terms of the defect densities and their gradients) will be extended to the case of a countable ensemble of parallel defect lines. To this end, besides the strain assumption 1.1 an additional assumption is made on the Frank tensor (1.4).

Assumption 1.2 (mesoscopic strain assumption) *The 2D strain \mathcal{E}_{mn}^* belongs to $L^1_{loc}(\Omega)$ (in the sense of definition 1.1) and is compatible on $\Omega_{\mathcal{L}}$. Moreover, the (m, z) components of the Frank tensor $\bar{\partial}_m \omega_z^* = \epsilon_{\alpha\beta} \partial_\alpha \mathcal{E}_{\beta m}^*$ ($1 \leq m \leq 3$) are Radon measures on Ω , whose singular parts with respect to the Lebesgue measure are purely concentrated on \mathcal{L} while their absolutely continuous parts have a 2D curl which itself is a Radon measure on Ω .*

Remark 1.1 *No similar assumption could be made on the complete Frank tensor without contradicting the edge and screw dislocation examples of Part A. Moreover the absolutely continuous part of $\bar{\partial}_m \omega_z^*$ cannot be required to be of bounded variation without contradicting the wedge disclination example of Part A. On the other hand, it will be seen that the sharp assumption 1.2 is required to establish our theory in the general case of countably many dislocations.*

Among several equivalent formulations, our main theorem then takes the following form.

Main theorem:
incompatibility decomposition for a countable set of 2D defect lines.

Under assumptions 1.1 and 1.2, incompatibility as defined by (1.2) is the vectorial first order distribution

$$\eta_{ik}^* = \eta_{ki}^* = \Theta_{ik}^* + \epsilon_{ilj} \partial_l \kappa_{kj}^*. \quad (1.17)$$

Let us also introduce here the main intermediate proposition needed for the proof of the above representation theorem. Under assumptions 1.1 and 1.2, the following decomposition theorem will be proved in the 2D case.

Theorem: 2D strain decomposition.

Let the 2D strain tensor \mathcal{E}_{mn}^ be a compatible L^1_{loc} tensor on $\Omega_{\mathcal{L}}$. Then the following decomposition holds:*

$$\mathcal{E}_{mn}^* = e_{mn}^* + E_{mn}^*, \quad (1.18)$$

where e_{mn}^* is everywhere compatible, whereas E_{mn}^* is a sum,

$$E_{mn}^* = \sum_{i \in \mathcal{I}} E_{mn}^{*(i)}, \quad (1.19)$$

where each $E_{mn}^{*(i)}$ ($i \in \mathcal{I}$) is analytically known, compatible and smooth on $\Omega \setminus L^{(i)}$, while E_{mn}^* is singular on \mathcal{L} and compatible and smooth on $\Omega_{\mathcal{L}}$.

From our main theorem, §3 will then be devoted to introducing new mesoscopic distributional fields, namely the completed Frank and Burgers tensors, which will represent the appropriate objective internal variables after homogenization to the macro-scale, and to reformulate the main theorem in their terms. Conclusions will be drawn in §4.

2 Distributional analysis of incompatibility for a countable set of parallel dislocations

To capture the macro-scale physics, homogenization must be performed on a set of dislocation lines whose number tends to infinity in order to define diffuse defect density tensors. Therefore, assumption 1.2 was introduced in a functional formulation that can be extended in some way from a set of defect lines (at the mesoscopic level) to a diffuse defect density (at the macroscopic level).

The extension of our theory to a countable number of defect lines poses several technical problems. A first difficulty arises from the different kinds of convergence that could be required. Typically, considering a series of Dirac masses on $l_0 = \mathcal{L} \cap \{z = z_0\}$, then its convergence as a measure implies that the sum of the weights must converge absolutely, but this is no longer the case if a (coarser) distributional convergence is required. A second example is provided by those distributions that are the gradient of a summable function. If these distributions are concentrated on isolated points, they must be the sum of Dirac masses, whereas this property might fail on a countable set. More generally, it is known (Schwartz 1957) that a concentrated first-order distribution on isolated points is a sum of weighted Dirac masses and Dirac mass derivatives, while a concentrated measure on a countable set is a sum of weighted Dirac masses. However, it is false to claim that a concentrated first-order distribution on a countable set is a sum of Dirac masses and Dirac mass derivatives, as $1D$ counter-examples can show. In general, a more complex mathematics governs the accumulation points of l_0 , and appropriate tools are required to extend the representation theorems of Part A to a countable set of defects.

2.1 General strain decomposition property

In general any vector field can be decomposed into a solenoidal and an irrotational part, and this property can be easily extended to distributional fields. In this paper, the similar decomposition of any symmetric tensor field into a compatible and a solenoidal part will be used to extend the main theorem of Part A from isolated to countable dislocations². In what follows, we will first give a proof of the decomposition existence in the general distributional case and then investigate its regularity in the $2D$ case. The main theorem will be extended in a further section.

Theorem 2.1 (standard decomposition of a symmetric tensor) *Any symmetric 2^{nd} -order distribution tensor E (or E_{ij}) can be decomposed into a compatible and a solenoidal symmetric part:*

$$E = E^c + E^s, \quad (2.1)$$

with

$$\nabla \times E^c \times \nabla = 0 \quad (\text{compatible } E^c), \quad (2.2)$$

and

$$\nabla \cdot E^s = 0 \quad (\text{solenoidal } E^s). \quad (2.3)$$

²Kröner (1980) first observed that this decomposition provides a link between the dislocation density in a medium and the associated strain tensor incompatibility.

Proof. \blacktriangle Any tensor E^s defined by the relation

$$E^s = \nabla \times F \times \nabla \quad (2.4)$$

is symmetric and solenoidal if F is a symmetric tensor distribution. Then the remainder $E^c = E - E^s$ is compatible provided, after some calculations, F satisfies the relation

$$\Delta \Delta F_{ij} + \partial_i \partial_j \partial_k \partial_l F_{kl} - \Delta (\partial_j \partial_k F_{ik}) - \Delta (\partial_i \partial_k F_{jk}) = \epsilon_{ikl} \epsilon_{jmn} \partial_k \partial_m E_{ln}, \quad (2.5)$$

with Δ the Laplacian operator ($\Delta = \partial_i \partial_i$). If in addition the gauge condition

$$\nabla \cdot F = 0 \quad (2.6)$$

is imposed, then (2.4) reduces to the elliptic equation

$$\Delta \Delta F = \nabla \times E \times \nabla. \quad (2.7)$$

\blacktriangle Therefore, to find the searched decomposition (2.1), (2.2), (2.3), it is sufficient to solve (2.7) for F with the gauge condition (2.6). If E is sufficiently regular, F will simply be found by solving (2.7) with, among others, the 6 boundary conditions $\nabla \cdot F = 0$ and $\partial(\nabla \cdot F)/\partial n = 0$. As a matter of fact, a solution exists because the operator $\Delta \Delta$ is elliptic, and this solution is divergence-free because taking the divergence of (2.7) provides the relation $\Delta \Delta (\nabla \cdot F) = 0$ which, together with the boundary conditions implies that $\nabla \cdot F$ itself vanishes.

If E is not sufficiently regular, E can be approximated as a distribution by a family of C^∞ functions E_ϵ ($\epsilon > 0$) with $E_\epsilon \rightarrow E$ for $\epsilon \rightarrow 0^+$ (Schwartz 1957). The family of equations obtained by replacing E by E_ϵ in (2.6), (2.7) provides a family of solutions F_ϵ which tends to a suitable F when $\epsilon \rightarrow 0^+$. \square

2.2 First representation theorem of a 2D incompatible strain

The previous section has shown that a distributional decomposition of the symmetric strain $\mathcal{E}^* \in L^1_{loc}(\Omega)$ into compatible and solenoidal distributional parts \mathcal{E}^{*c} and \mathcal{E}^{*s} always exists, with the right-hand side of (2.7) showing to be the incompatibility tensor. However, more regular solutions exist in the 2D case. Before proving them, the following result will be needed.

Lemma 2.1 *Let $\delta^{(i)}$ stand for the Dirac measure at $\hat{x}^{(i)} \in l_0$ and $\sum_{i \in \mathcal{I}} C^{(i)} \delta^{(i)}$ be a Radon measure on $\Omega_{z_0} = \Omega \setminus \{z = z_0\}$ in the sense of definition 1.1. Then the sum of the weights $C^{(i)}$ is locally absolutely convergent, this meaning its absolute convergence on any bounded subset $\{\hat{x}^{(i)}, i \in \mathcal{I}'\}$ of l_0 .*

Proof. Since $\sum_{i \in \mathcal{I}} C^{(i)} \delta^{(i)}$ is a Radon measure, then $\sum_{i \in \mathcal{I}'} C^{(i)} \delta^{(i)}$ is a finite Radon measure and the sum can be indifferently carried out on every permutation of \mathcal{I}' . Hence, taking a test-function which equals 1 on l_0 , the sum of the weights converges for every permutation of \mathcal{I}' and is absolutely convergent. \square

Remark 2.1 If $\sum_{i \in \mathcal{I}} C^{(i)} \delta^{(i)}$ were assumed to be a general distribution instead of a Radon measure, no such statement on the absolute convergence of the sum of the weights could be proved as the following simple 1D counter-example shows: selecting $\hat{x}^{(i)} = 1/i$ ($i \in \mathcal{I} = \mathbb{N}_0$) and $C^{(i)} = (-1)^{i+1}(1/i + 1/(1+i))$ provides a distributionally convergent series $\sum_{i \in \mathcal{I}} C^{(i)} \delta^{(i)}$, since it is the derivative of the L^1_{loc} converging series $1 - \sum_{i \in \mathcal{I}} C^{(i)}(1 - H^{(i)})$ with $H^{(i)} = H(x - \hat{x}^{(i)})$ and H the step function, whereas the sum $\sum_{i \in \mathcal{I}} |C^{(i)}|$ does not converge.

Notations 2.1 Henceforth $\{\hat{x}^{(i)}, i \in \mathcal{I}\}$ will denote the set of points defining l_0 .

Theorem 2.2 (regularity of the strain decomposition) Let the strain and the Frank tensor satisfy assumptions 1.1 and 1.2, and the dislocation set be defined according to definition 1.2. Then the following decomposition holds:

$$\mathcal{E}_{mn}^* = \mathcal{E}_{mn}^{*c} + \mathcal{E}_{mn}^{*s}, \quad (2.8)$$

where $\mathcal{E}_{mn}^{*c} \in L^1_{loc}(\Omega)$ is compatible, whereas $\mathcal{E}_{mn}^{*s} \in L^1_{loc}(\Omega)$ is solenoidal.

Proof. Consider any 2D cut Ω_{z_0} of Ω and assume first that Ω_{z_0} is bounded (extension to unbounded sets is direct). Since the strain is independent of z , in 2D it suffices to solve (2.7) and (2.6) for F on Ω_{z_0} with $E_{ij} = \mathcal{E}_{ij}^*$. This will be achieved by solving an associated problem on Ω_{z_0} by means of complex (but not necessarily analytic) functions of two real variables. To this end, (2.7) is first expressed in block matrix notation:

$$\begin{aligned} & \left[\begin{array}{c|c} \Delta \Delta F_{\alpha\beta} & \Delta \Delta F_{\alpha z} \\ \Delta \Delta F_{z\alpha} & \Delta \Delta F \end{array} \right] = \left[\begin{array}{c|c} \eta_{\alpha\beta}^* & \eta_{\alpha z}^* \\ \eta_{z\alpha}^* & \eta_{zz}^* \end{array} \right] \\ = & \left[\begin{array}{cc|c} \partial_y^2 \mathcal{E}_{zz}^* & -\partial_x \partial_y \mathcal{E}_{zz}^* & \partial_y (\partial_x \mathcal{E}_{yz}^* - \partial_y \mathcal{E}_{xz}^*) \\ -\partial_x \partial_y \mathcal{E}_{zz}^* & \partial_x^2 \mathcal{E}_{zz}^* & -\partial_x (\partial_x \mathcal{E}_{yz}^* - \partial_y \mathcal{E}_{xz}^*) \\ \hline \partial_y (\partial_x \mathcal{E}_{yz}^* - \partial_y \mathcal{E}_{xz}^*) & -\partial_x (\partial_x \mathcal{E}_{yz}^* - \partial_y \mathcal{E}_{xz}^*) & \partial_x (\partial_x \mathcal{E}_{yy}^* - \partial_y \mathcal{E}_{xy}^*) \\ -\partial_y \mathcal{E}_{xz}^* & -\partial_y \mathcal{E}_{xz}^* & \partial_y (\partial_x \mathcal{E}_{xy}^* - \partial_y \mathcal{E}_{xx}^*) \end{array} \right] \quad (2.9) \end{aligned}$$

and every block equation is separately solved.

▲ First block ($\eta_{\alpha\beta}^*$). By the compatibility condition (1.3) outside l_0 , it results that \mathcal{E}_{zz}^* is linear on the open and connected defect-free region $\Omega_{z_0} \setminus l_0$ and can be prolonged by a linear function on Ω_{z_0} . Since $F_{\alpha\beta} = 0$ is a solution of $\Delta \Delta F_{\alpha\beta} = 0$ on Ω_{z_0} , by (2.9) an admissible \mathcal{E}_{zz}^{*s} is

$$\mathcal{E}_{zz}^{*s} = 0. \quad (2.10)$$

▲ Second block ($\eta_{\alpha z}^*$). As $\bar{\partial}_z \omega_z^* = \partial_x \mathcal{E}_{yz}^* - \partial_y \mathcal{E}_{xz}^*$, it is convenient to seek a solution of

$$\Delta \Delta (F_{xz} + iF_{yz}) = -i(\partial_x + i\partial_y) \bar{\partial}_z \omega_z^*. \quad (2.11)$$

According to assumption 1.2, $\bar{\partial}_z \omega_z^*$ decomposes as follows:

$$\bar{\partial}_z \omega_z^* = \sum_{i \in \mathcal{I}} c^{(i)} \delta^{(i)} + K, \quad (2.12)$$

where the absolutely continuous part of the now finite Radon measure $\bar{\partial}_z \omega_z^*$ shows to be the constant K appearing in the compatibility condition (1.3), while its singular part is purely concentrated. Moreover, the sum of the weights $c^{(i)}$ is absolutely convergent by lemma 2.1 when Ω_{z_0} is bounded. So (2.11) develops as

$$\Delta \Delta (F_{xz} + iF_{yz}) = -i(\partial_x + i\partial_y) \sum_{i \in \mathcal{I}} c^{(i)} \delta^{(i)}. \quad (2.13)$$

Then, since $\Delta \Delta$ rewrites as $(\partial_x + i\partial_y)^2 (\partial_x - i\partial_y)^2$, it suffices to solve

$$(\partial_x - i\partial_y)(F_{xz} + iF_{yz}) = -i\mathcal{F}, \quad (2.14)$$

with \mathcal{F} a solution of

$$\Delta \mathcal{F} = \sum_{i \in \mathcal{I}} c^{(i)} \delta^{(i)}. \quad (2.15)$$

To solve this system, observe that (2.14) develops as

$$\begin{cases} \partial_x F_{yz} - \partial_y F_{xz} &= -\mathcal{F}, \\ \partial_x F_{xz} + \partial_y F_{yz} &= 0, \end{cases} \quad (2.16)$$

where an acceptable $W^{1,1}(\Omega_{z_0})$ field \mathcal{F} satisfying (2.15) is

$$\mathcal{F}(\chi) = \sum_{i \in \mathcal{I}} \frac{c^{(i)}}{2\pi} \log r^{(i)}, \quad (2.17)$$

using the notations $\chi = (x, y)$, $l_0 = \{\hat{\chi}^{(i)} = (x^{(i)}, y^{(i)}), i \in \mathcal{I}\}$ and $r^{(i)} := |\chi - \hat{\chi}^{(i)}|$. Hence, by (2.4) an admissible solenoidal strain belonging to $L^1(\Omega_{z_0})$ is

$$\mathcal{E}_{xz}^{*s} = \partial_y (\partial_x F_{yz} - \partial_y F_{xz}) = -\partial_y \mathcal{F}, \quad (2.18)$$

$$\mathcal{E}_{yz}^{*s} = -\partial_x (\partial_x F_{yz} - \partial_y F_{xz}) = \partial_x \mathcal{F}. \quad (2.19)$$

▲ Third block (η_{zz}^*). Recall first that $\bar{\partial}_\beta \omega_z^* = \epsilon_{\alpha\gamma} \partial_\alpha \mathcal{E}_{\gamma\beta}^*$ and that

$$\eta_{zz}^* = \partial_\alpha (\partial_\alpha \mathcal{E}_{\beta\beta}^* - \partial_\beta \mathcal{E}_{\alpha\beta}^*) = \epsilon_{\alpha\beta} \partial_\alpha \bar{\partial}_\beta \omega_z^*,$$

in such a way that the problem is to solve

$$\Delta \Delta F = \partial_x \bar{\partial}_y \omega_z^* - \partial_y \bar{\partial}_x \omega_z^* = \Re\{i(\partial_x + i\partial_y)(\bar{\partial}_x \omega_z^* - i\bar{\partial}_y \omega_z^*)\} \quad (2.20)$$

where, according to assumption 1.2, $\bar{\partial}_\alpha \omega_z^*$ develops as

$$\bar{\partial}_\alpha \omega_z^* = \sum_{i \in \mathcal{I}} c_\alpha^{(i)} \delta^{(i)} + f_\alpha \quad (2.21)$$

with absolutely convergent sums of the weights $c_\alpha^{(i)}$ by lemma 2.1 when Ω_{z_0} is bounded and for some functions f_α whose curl is here a finite Radon measure, which must be concentrated on l_0 to ensure the compatibility of $\mathcal{E}_{\alpha\beta}^*$ outside the defect set. So, (2.20) rewrites as

$$\Delta \Delta F = \Re \left\{ i(\partial_x + i\partial_y) \left(\sum_{i \in \mathcal{I}} (c_x^{(i)} - ic_y^{(i)}) \delta^{(i)} \right) \right\} + \partial_x f_y - \partial_y f_x.$$

Now, in view of the properties of f_α resulting from assumption 1.2, the last terms write as

$$\partial_x f_y - \partial_y f_x = \sum_{i \in \mathcal{I}} C^{(i)} \delta^{(i)}, \quad (2.22)$$

where the sum of the weights is absolutely convergent. Eventually, using the distributional identity

$$\partial_x \left(\frac{\delta x^{(i)}}{r^{(i)2}} \right) - \partial_y \left(\frac{\delta y^{(i)}}{r^{(i)2}} \right) = 2\pi \delta^{(i)},$$

with the notation $\delta x^{(i)} = x - \hat{x}^{(i)}$, $\delta y^{(i)} = y - \hat{y}^{(i)}$, (2.20) can be written in the form

$$\Delta \Delta F = \Re \left\{ i(\partial_x + i\partial_y) \left(\sum_{i \in \mathcal{I}} (c_x^{(i)} - ic_y^{(i)}) \delta^{(i)} - \sum_{i \in \mathcal{I}} \frac{C^{(i)}}{2\pi r^{(i)2}} (y^{(i)} + ix^{(i)}) \right) \right\}. \quad (2.23)$$

A particular solution of (2.23) is provided by solving

$$\begin{aligned} (\partial_x + i\partial_y)(\partial_x - i\partial_y)^2(F + iH) &= i \left(\sum_{i \in \mathcal{I}} (c_x^{(i)} - ic_y^{(i)}) \delta^{(i)} \right. \\ &\quad \left. - \sum_{i \in \mathcal{I}} \frac{C^{(i)}}{2\pi r^{(i)2}} (y^{(i)} + ix^{(i)}) \right), \end{aligned} \quad (2.24)$$

with H an additional unknown. This latter equation is equivalent to the system

$$(\partial_x - i\partial_y)(F + iH) = \mathcal{G} \quad \text{on } \Omega_{z_0}, \quad (2.25)$$

$$\Delta \mathcal{G} = i(\bar{\partial}_x \omega_z^* - i\bar{\partial}_y \omega_z^*) \quad \text{on } \Omega_{z_0}, \quad (2.26)$$

which can be easily solved. In a first step, a particular solution of (2.26) is given by

$$\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2,$$

$$\mathcal{G}_1 = \sum_{i \in \mathcal{I}} (c_y^{(i)} + ic_x^{(i)}) \frac{\log r^{(i)}}{2\pi}, \quad (2.27)$$

$$\mathcal{G}_2 = \sum_{i \in \mathcal{I}} C^{(i)} (\delta x^{(i)} - i\delta y^{(i)}) \frac{\log r^{(i)}}{4\pi}, \quad (2.28)$$

with both \mathcal{G}_1 and \mathcal{G}_2 belonging to $W^{1,1}(\Omega_{z_0})$. In a second step, (2.25) is simply rewritten as

$$\partial_x F + \partial_y H = \Re\{\mathcal{G}\} \quad \text{and} \quad \partial_x H - \partial_y F = \Im\{\mathcal{G}\}, \quad (2.29)$$

whose solution $F = F_1 + F_2$ and $H = H_1 + H_2$ is given by

$$F_1 = \partial_x \psi_1 + \partial_y \varphi_1 \quad \text{and} \quad H_1 = \partial_y \psi_1 - \partial_x \varphi_1, \quad (2.30)$$

$$F_2 = \partial_x \psi_2 + \partial_y \varphi_2 \quad \text{and} \quad H_2 = \partial_y \psi_2 - \partial_x \varphi_2, \quad (2.31)$$

for some gauge fields $\psi_1, \varphi_1, \psi_2, \varphi_2$ satisfying the equations

$$\Delta \psi_1 = \Re\{\mathcal{G}_1\} \quad \text{and} \quad \Delta \varphi_1 = -\Im\{\mathcal{G}_1\}, \quad (2.32)$$

$$\Delta \psi_2 = \Re\{\mathcal{G}_2\} \quad \text{and} \quad \Delta \varphi_2 = -\Im\{\mathcal{G}_2\}. \quad (2.33)$$

Particular solutions of (2.32) belonging to $W^{3,1}(\Omega_{z_0})$ are

$$\psi_1 = \sum_{i \in \mathcal{I}} c_y^{(i)} r^{(i)2} \frac{\log r^{(i)} - 1}{8\pi} \quad \text{and} \quad \varphi_1 = - \sum_{i \in \mathcal{I}} c_x^{(i)} r^{(i)2} \frac{\log r^{(i)} - 1}{8\pi},$$

in such a way that

$$F_1(x, y) = \sum_{i \in \mathcal{I}} \frac{(2 \log r^{(i)} - 1)}{8\pi} \left(c_y^{(i)} \delta x^{(i)} - c_x^{(i)} \delta y^{(i)} \right) \quad (2.34)$$

belongs to $W^{2,1}(\Omega_{z_0})$, thereby defining the solenoidal strain $\mathcal{E}_{\alpha\beta}^{*s,1}$:

$$[\mathcal{E}_{\alpha\beta}^{*s,1}] := \begin{bmatrix} \partial_y^2 F_1 & -\partial_x \partial_y F_1 \\ -\partial_x \partial_y F_1 & \partial_x^2 F_1 \end{bmatrix}, \quad (2.35)$$

which belongs to $L^1(\Omega_{z_0}) \cap C^\infty(\Omega_{z_0} \setminus l_0)$. Similarly, particular solutions of (2.33) belonging to $W^{3,1}(\Omega_{z_0})$ are given by

$$\psi_2 = \sum_{i \in \mathcal{I}} C^{(i)} \delta x^{(i)} r^{(i)2} \frac{\log r^{(i)} - 3/4}{32\pi}, \quad (2.36)$$

$$\varphi_2 = \sum_{i \in \mathcal{I}} C^{(i)} \delta y^{(i)} r^{(i)2} \frac{\log r^{(i)} - 3/4}{32\pi}, \quad (2.37)$$

and hence

$$F_2(x, y) = \sum_{i \in \mathcal{I}} \frac{C^{(i)}}{16\pi} r^{(i)2} \left(2 \log r^{(i)} - 1 \right) \quad (2.38)$$

also belongs to $W^{2,1}(\Omega_{z_0})$, defining the solenoidal strain $\mathcal{E}_{\alpha\beta}^{*s,2}$:

$$[\mathcal{E}_{\alpha\beta}^{*s,2}] := \begin{bmatrix} \partial_y^2 F_2 & -\partial_x \partial_y F_2 \\ -\partial_x \partial_y F_2 & \partial_x^2 F_2 \end{bmatrix}, \quad (2.39)$$

which belongs to $L^1(\Omega_{z_0}) \cap C^\infty(\Omega_{z_0} \setminus l_0)$.

▲ Summary. The solenoidal part of the strain is the tensor \mathcal{E}_{mn}^{*s} ,

$$\mathcal{E}_{mn}^{*s} = \mathcal{E}_{zz}^{*s} \delta_{mz} \delta_{nz} + \mathcal{E}_{\alpha z}^{*s} (\delta_{mz} \delta_{n\alpha} + \delta_{m\alpha} \delta_{nz}) + (\mathcal{E}_{\alpha\beta}^{*s,1} + \mathcal{E}_{\alpha\beta}^{*s,2}) \delta_{m\alpha} \delta_{n\beta}, \quad (2.40)$$

where \mathcal{E}_{zz}^{*s} , $\mathcal{E}_{\alpha z}^{*s}$, $\mathcal{E}_{\alpha\beta}^{*s,1}$ and $\mathcal{E}_{\alpha\beta}^{*s,2}$ are given by (2.10), (2.17), (2.18), (2.35) and (2.39), respectively, and all belong to $L^1(\Omega_{z_0})$, observing that the weights $c^{(i)}$, $c_\alpha^{(i)}$ and $C^{(i)}$ defining the intermediate expressions \mathcal{F} , F_1 and F_2 in (2.15), (2.34) and (2.38) are associated with absolutely convergent series:

$$\sum_{i \in \mathcal{I}} |c^{(i)}| < \infty, \quad \sum_{i \in \mathcal{I}} \|c_\alpha^{(i)}\| < \infty \quad \text{and} \quad \sum_{i \in \mathcal{I}} |C^{(i)}| < \infty. \quad (2.41)$$

Therefore \mathcal{E}_{mn}^{*s} belongs to $L^1(\Omega_{z_0})$, is solenoidal and satisfies (2.8). Extension of the proof to unbounded domains Ω_{z_0} is immediate. \square

The coefficients $c^{(i)}$, $c_\alpha^{(i)}$ and $C^{(i)}$ will show to be the Burgers and Frank vectors of screw and edge dislocations and wedge disclinations, respectively.

Remark 2.2 *The hypothesis provided by assumption 1.2 that $\overline{\partial}_m \omega_z^*$ has an absolutely continuous part whose curl is a Radon measure is a request to make the proof in the case of a countable set of line defects. Indeed, when the 2D defect set l_0 has accumulation points in Ω_{z_0} , a complex distributional behaviour can take place near these points which forbids getting the proof if a sufficiently strong hypothesis is not introduced to account for a possibly countable number of disclinations on the sole basis of the strain field properties. More tractable hypotheses on $\overline{\partial}_\alpha \omega_z^*$ itself (and not its curl) are currently under investigation.*

As a 1D example to illustrate the above difficulty, the function

$$F = \sum_{i \in \mathcal{I} = \mathbb{N}_0} C^{(i)} (H^0 - H^{(i)})$$

with $H^{(i)} = H(x - \hat{x}^{(i)})$, $\hat{x}^{(i)} = 1/i$, $H^0 = H(x)$ and H the step function, may correspond to an L_{loc}^1 converging series even if the sum of the weights $C^{(i)}$ diverges. To show this, it suffices to select appropriate $C^{(i)}$ such that the partial sums defining F are all enclosed between the L_{loc}^1 functions $G(x)$ and $-G(x)$, with $G(x) = \log((1+x)/x)$ for $x > 0$ and $G(x) = 0$ for $x \leq 0$. The Lebesgue dominated convergence theorem then shows that $F \in L_{loc}^1$, in such a way that the distributional derivative of F , which cannot be the diverging series $-\sum_{i \in \mathcal{I}} C^{(i)} \delta^{(i)}$, exhibits a special behaviour near the origin to recover convergence. Similar effects take place in 2D and appropriate assumptions are then necessary to obtain (2.22).

2.3 Second representation theorem of a 2D incompatible strain

This section provides a further decomposition of the strain, since the solenoidal part is itself decomposed into an everywhere compatible part and another smooth part outside from the defect set \mathcal{L} .

Theorem 2.3 (analysis of the singular part of the strain decomposition) *Let the strain and the Frank tensor satisfy assumptions 1.1 and 1.2, and the dislocation set be defined according to definition 1.2. Then the solenoidal component of the strain satisfies the following decomposition:*

$$\mathcal{E}_{mn}^{*s} = \mathcal{E}_{mn}^{\prime *c} + E_{mn}^*, \quad (2.42)$$

where $\mathcal{E}_{mn}^{\prime *c} \in L_{loc}^1(\Omega_{z_0})$ is compatible on Ω and where

$$E_{mn}^* = \sum_{i \in \mathcal{I}} E_{mn}^{*(i)} \in L_{loc}^1(\Omega), \quad (2.43)$$

with $E_{mn}^{*(i)} (i \in \mathcal{I})$ smooth and compatible on $\Omega \setminus L^{(i)}$. Moreover, the Frank tensor part $\epsilon_{kpn} \partial_p E_{mn}^*$ is smooth on $\Omega_{\mathcal{L}}$.

Proof. As in the previous proof, since the strain is independent of z , it suffices to work on the 2D domain Ω_{z_0} which is again assumed to be bounded without loss of generality.

▲ $\mathcal{E}_{\alpha z}^{*s}$ components. Following decomposition (2.8), write

$$\begin{aligned} \mathcal{E}_{xz}^{*s} &= \partial_y (\partial_x F_{yz} - \partial_y F_{xz}) = E_{xz}^* + \mathcal{E}_{xz}^{\prime *c}, \\ \mathcal{E}_{yz}^{*s} &= -\partial_x (\partial_x F_{yz} - \partial_y F_{xz}) = E_{yz}^* + \mathcal{E}_{yz}^{\prime *c}, \end{aligned}$$

with vanishing $\mathcal{E}'_{xz}{}^{*c}$ and $\mathcal{E}'_{yz}{}^{*c}$:

$$\mathcal{E}'_{xz}{}^{*c} = \mathcal{E}'_{yz}{}^{*c} = 0.$$

Hence, according to (2.18) and (2.19), E_{xz}^* and E_{yz}^* are sums of screw dislocations:

$$E_{xz}^* = - \sum_{i \in \mathcal{I}} \frac{c^{(i)}}{2\pi r^{(i)2}} \delta y^{(i)} \quad \text{and} \quad E_{yz}^* = \sum_{i \in \mathcal{I}} \frac{c^{(i)}}{2\pi r^{(i)2}} \delta x^{(i)}, \quad (2.44)$$

which from (2.41) belong to $L^1(\Omega_{z_0}) \cap C^\infty(\Omega_{z_0} \setminus l_0)$. The relations (2.44) also show the smoothness of the Frank tensor parts $\epsilon_{\alpha\beta} \partial_\alpha E_{z\beta}^*$ and $\epsilon_{\beta\gamma} \partial_\gamma E_{\alpha z}^*$ outside l_0 .

▲ \mathcal{E}_{zz}^{*s} and $\mathcal{E}_{\alpha\beta}^{*s,1}$ components. By (2.10) the expression

$$E_{zz}^* := \mathcal{E}_{zz}^{*s} = 0 \quad (2.45)$$

exhibits the form (2.43) and provides Frank tensor parts $\epsilon_{\alpha\gamma} \partial_\gamma E_{zz}^*$ which identically vanish on Ω_{z_0} . On the other hand, it can be checked that on $\Omega_{z_0} \setminus l_0$:

$$\begin{aligned} [\mathcal{E}_{\alpha\beta}^{*s,1}] &= \sum_{i \in \mathcal{I}} \frac{c_y^{(i)}}{4\pi r^{(i)2}} \begin{bmatrix} \delta x^{(i)} (1 - 2 \frac{\delta y^{(i)2}}{r^{(i)2}}) & -\delta y^{(i)} (1 - 2 \frac{\delta x^{(i)2}}{r^{(i)2}}) \\ -\delta y^{(i)} (1 - 2 \frac{\delta x^{(i)2}}{r^{(i)2}}) & \delta x^{(i)} (1 + 2 \frac{\delta y^{(i)2}}{r^{(i)2}}) \end{bmatrix} \\ &\quad - \sum_{i \in \mathcal{I}} \frac{c_x^{(i)}}{4\pi r^{(i)2}} \begin{bmatrix} \delta y^{(i)} (1 + 2 \frac{\delta x^{(i)2}}{r^{(i)2}}) & -\delta x^{(i)} (1 - 2 \frac{\delta y^{(i)2}}{r^{(i)2}}) \\ -\delta x^{(i)} (1 - 2 \frac{\delta y^{(i)2}}{r^{(i)2}}) & \delta y^{(i)} (1 - 2 \frac{\delta x^{(i)2}}{r^{(i)2}}) \end{bmatrix}. \end{aligned} \quad (2.46)$$

Define then in the decomposition (2.42), (2.43) of $\mathcal{E}_{\alpha\beta}^{*s,1}$ the components $E_{\alpha\beta}^{*1}$ and $\mathcal{E}'_{\alpha\beta}{}^{*c,1}$ as follows:

$$[E_{\alpha\beta}^{*1}] := - \sum_{i \in \mathcal{I}} \frac{c_y^{(i)}}{2\pi r^{(i)2}} \begin{bmatrix} \delta x^{(i)} & \delta y^{(i)} \\ \delta y^{(i)} & -\delta x^{(i)} \end{bmatrix} + \sum_{i \in \mathcal{I}} \frac{c_x^{(i)}}{2\pi r^{(i)2}} \begin{bmatrix} -\delta y^{(i)} & \delta x^{(i)} \\ \delta x^{(i)} & \delta y^{(i)} \end{bmatrix}, \quad (2.47)$$

$$\begin{aligned} [\mathcal{E}'_{\alpha\beta}{}^{*c,1}] &:= \sum_{i \in \mathcal{I}} \frac{c_y^{(i)}}{4\pi r^{(i)4}} \begin{bmatrix} \delta x^{(i)} (\delta y^{(i)2} + 3\delta x^{(i)2}) & \delta y^{(i)} (\delta y^{(i)2} + 3\delta x^{(i)2}) \\ \delta y^{(i)} (\delta y^{(i)2} + 3\delta x^{(i)2}) & \delta x^{(i)} (-\delta x^{(i)2} + \delta y^{(i)2}) \end{bmatrix} \\ &\quad - \sum_{i \in \mathcal{I}} \frac{c_x^{(i)}}{4\pi r^{(i)4}} \begin{bmatrix} \delta y^{(i)} (\delta x^{(i)2} - \delta y^{(i)2}) & \delta x^{(i)} (\delta x^{(i)2} + 3\delta y^{(i)2}) \\ \delta x^{(i)} (\delta x^{(i)2} + 3\delta y^{(i)2}) & \delta y^{(i)} (\delta x^{(i)2} + 3\delta y^{(i)2}) \end{bmatrix}. \end{aligned} \quad (2.48)$$

Calculations show that $\mathcal{E}'_{\alpha\beta}{}^{*c,1}$ is the difference between (2.46) and (2.47), is compatible, and also belongs to $L^1(\Omega_{z_0})$ by (2.41). Moreover $E_{\alpha\beta}^{*1}$ is a sum of edge dislocations, whose curl is vanishing and which are smooth on $\Omega_{z_0} \setminus \{(\hat{x}^{(i)}, \hat{y}^{(i)})\}$.

▲ $\mathcal{E}_{\alpha\beta}^{*s,2}$ components. Define

$$E_{\alpha\beta}^{*2} := \mathcal{E}_{\alpha\beta}^{*s,2} \quad (2.49)$$

with vanishing $\mathcal{E}'_{\alpha\beta}{}^{*c,2}$. Calculations show that

$$[E_{\alpha\beta}^{*2}] = \sum_{i \in \mathcal{I}} \frac{C^{(i)}}{4\pi} \begin{bmatrix} \log r^{(i)} + \frac{\delta y^{(i)2}}{r^{(i)2}} & -\frac{\delta x^{(i)} \delta y^{(i)}}{r^{(i)2}} \\ -\frac{\delta x^{(i)} \delta y^{(i)}}{r^{(i)2}} & \log r^{(i)} + \frac{\delta x^{(i)2}}{r^{(i)2}} \end{bmatrix}, \quad (2.50)$$

in such a way that $E_{\alpha\beta}^{*2}$ is a sum of wedge disclinations, which are smooth and of vanishing curl on $\Omega_{z_0} \setminus \{(\hat{x}^{(i)}, \hat{y}^{(i)})\}$ (cf. Part A).

▲ In summary, the following solenoidal strain decomposition has been proved:

$$\mathcal{E}_{mn}^{*s} = \mathcal{E}_{mn}^{I*c} + E_{mn}^*, \quad (2.51)$$

where

$$\mathcal{E}_{mn}^{I*c} := \mathcal{E}_{zz}^{I*c} \delta_{mz} \delta_{nz} + \mathcal{E}_{\alpha z}^{I*c} (\delta_{m\alpha} \delta_{nz} + \delta_{mz} \delta_{n\alpha}) + (\mathcal{E}_{\alpha\beta}^{I*c,1} + \mathcal{E}_{\alpha\beta}^{I*c,2}) \delta_{m\alpha} \delta_{n\beta} \quad (2.52)$$

is compatible on Ω . Moreover, E_{mn}^* writes as

$$E_{mn}^* = E_{zz}^* \delta_{mz} \delta_{nz} + E_{\alpha z}^* (\delta_{m\alpha} \delta_{nz} + \delta_{mz} \delta_{n\alpha}) + (E_{\alpha\beta}^{*1} + E_{\alpha\beta}^{*2}) \delta_{m\alpha} \delta_{n\beta}, \quad (2.53)$$

with E_{zz}^* , $E_{\beta z}^*$, $E_{\alpha\beta}^{*1}$ and $E_{\alpha\beta}^{*2}$ defined by (2.45), (2.44), (2.47) and (2.50), and hence E_{mn}^* together with the related Frank tensor $\epsilon_{kpn} \partial_p E_{mn}^*$ is smooth on $\Omega_{\mathcal{L}}$ thereby terminating the proof. Extension to unbounded sets Ω_{z_0} is straightforward. \square

2.4 Applications of the strain decomposition

In this §2.4, \mathcal{I}' refers to any bounded subset $\{\hat{x}^{(i)}, i \in \mathcal{I}'\}$ of l_0 .

• *Set of parallel screw dislocations.* Part A and (2.44) directly provide the equality $c_z^{(i)} = B_z^{*(i)}$ and a vanishing strain compatible part. Moreover according to (2.41) the following condition holds:

$$\sum_{i \in \mathcal{I}'} |B_z^{*(i)}| < \infty.$$

• *Set of parallel edge dislocations.* From Part A and (2.46)-(2.48), it turns out that $c_{yz}^{(i)} = B_y^{*(i)}$ and $c_{xz}^{(i)} = B_x^{*(i)}$, with according to (2.41) the following bounds:

$$\sum_{i \in \mathcal{I}'} |B_x^{*(i)}| < \infty \quad \text{and} \quad \sum_{i \in \mathcal{I}'} |B_y^{*(i)}| < \infty.$$

• *Set of parallel wedge disclinations.* The expression of $[\mathcal{E}_{ij}^*]$ given in Part A is

$$\frac{\Omega_z^*(1 - \nu^*)}{4\pi} \begin{bmatrix} 1 + \log r & 0 & 0 \\ 0 & 1 + \log r & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{\Omega_z^*(1 + \nu^*)}{8\pi} \begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which shows to be the sum of

$$\frac{\Omega_z^*}{4\pi} \begin{bmatrix} \log r + \sin^2 \theta & -\sin \theta \cos \theta & 0 \\ -\sin \theta \cos \theta & \log r + \cos^2 \theta & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and a compatible part (since the associated Frank tensor part vanishes). Therefore, according to (2.50), $C^{(i)} = \Omega_z^{*(i)}$ with by (2.41) the bounds

$$\sum_{i \in \mathcal{I}'} |\Omega_z^{*(i)}| < \infty.$$

2.5 Mesoscopic defect densities in 2D incompatible elasticity

The following theorem expresses the 2D mesoscopic incompatibility in terms of the defect invariants for a countable set of dislocations.

Theorem 2.4 (main result) *For a countable set of parallel defect lines \mathcal{L} and under assumptions 1.1 and 1.2 and definition 1.2, incompatibility as defined by equations (1.2) and (1.3) is the vectorial first order distribution*

$$\eta_k^* = \delta_{kz}\eta_z^* + \delta_{k\kappa}\eta_\kappa^*, \quad (2.54)$$

where its out-of-plane component is

$$\eta_z^* = \sum_{i \in \mathcal{I}} \left(\Omega_z^{*(i)} \delta_{L^{(i)}} + \epsilon_{\alpha\gamma} \left(B_\gamma^{*(i)} + \epsilon_{\beta\gamma} (\hat{x}_\beta^{(i)} - x_{0\beta}) \Omega_z^{*(i)} \right) \partial_\alpha \delta_{L^{(i)}} \right) \quad (2.55)$$

while its in-plane components are

$$\eta_\kappa^* = \frac{1}{2} \epsilon_{\kappa\alpha} \sum_{i \in \mathcal{I}} B_z^{*(i)} \partial_\alpha \delta_{L^{(i)}}, \quad (2.56)$$

and where $x_0 \in \Omega$ is a selected reference point.

Proof. With use of theorems 2.2 and 2.3, the $L^1(\Omega)$ strain decomposes as:

$$\mathcal{E}_{mn}^* = e_{mn}^* + E_{mn}^*, \quad (2.57)$$

where e_{mn}^* is compatible on Ω and where

$$E_{mn}^* = \sum_{i \in \mathcal{I}} E_{mn}^{*(i)} \quad (2.58)$$

is smooth away from the defect set $\mathcal{L} = \bigcup_{i \in \mathcal{I}} L^{(i)}$.

Now, the local strain assumption of Part A is satisfied by each individual $E_{mn}^{*(i)}$. Then, the Frank and Burgers vectors of each isolated defect line $L^{(i)}$ are defined according to theorem 1.1 and definition 1.4. In a next step, the strain contributions $E_{mn}^{*(i)}$ associated with these isolated defect lines are removed from the decomposition of the strain \mathcal{E}_{mn}^* provided by (2.8), (2.42) and (2.43), as allowed by the absolute convergence (2.41) of all their weight series. This operation defines a strain reminder whose defect lines are the accumulation lines (or the so-called derived set) of the initial defective set \mathcal{L} , and the extraction procedure of isolated defect lines is repeated on this derived set, and then repeated again by transfinite recursion until the Frank and Burgers vectors are defined for each line of \mathcal{L} (and not only for the isolated lines). Finally, applying the main theorem of Part A to each $L^{(i)}$, summing the results on $\mathcal{I} \subset \mathbb{N}$ and recalling that both \mathcal{E}_{mn}^{*c} and $\mathcal{E}_{mn}'^{*c}$ provide vanishing contributions to incompatibility, the proof is achieved. \square

The remaining of this section consists in a reformulation of the above result in terms of the defect density tensors Θ_{ik}^* , Λ_{ik}^* , α_{ik}^* and κ_{ik}^* which, in 2D, simplify in the Θ_k^* , Λ_k^* , α_k^* and κ_{ij}^* tensors defined by (1.10)-(1.16). However, the sums are now

performed on a countable ensemble \mathcal{L} of rectilinear dislocations, and Θ_k^* and Λ_k^* are Radon measures in view of the inequalities

$$\sum_{i \in \mathcal{I}'} |\Omega_z^{*(i)}| < \infty, \quad \sum_{i \in \mathcal{I}'} \|B_k^{*(i)}\| < \infty, \quad (2.59)$$

where \mathcal{I}' refers to any bounded subset $\{\hat{x}^{(i)}, i \in \mathcal{I}'\}$ of l_0

Theorem 2.5 *For a countable set \mathcal{L} of parallel defect lines and under assumptions 1.1 and 1.2 and definition 1.2, the mesoscopic strain incompatibility writes as*

$$\eta_k^* = \delta_{zk} \Theta_z^* + \epsilon_{\alpha\beta} \partial_\alpha \kappa_{k\beta}^*, \quad (2.60)$$

or equivalently as

$$\eta_k^* = \delta_{zk} \Theta_z^* + \epsilon_{k\alpha l} \partial_\alpha \kappa_{zl}^*. \quad (2.61)$$

Proof. Consider any straight dislocation $L^{(i)} \in \mathcal{L}$ passing by $\hat{x}^{(i)} = (\hat{x}_\beta^{(i)}, z_0)$. From theorem 2.4, the associated incompatibility writes as

$$\begin{aligned} \eta_k^{*(i)} = \delta_{kz} \left(\Omega_z^{*(i)} \delta_{L^{(i)}} + \epsilon_{\alpha\gamma} \left(B_\gamma^{*(i)} + \epsilon_{\beta\gamma} (\hat{x}_\beta^{(i)} - x_{0\beta}) \Omega_z^{*(i)} \right) \partial_\alpha \delta_{L^{(i)}} \right) \\ + \delta_{k\kappa} \frac{1}{2} \epsilon_{\kappa\alpha} B_z^{*(i)} \partial_\alpha \delta_{L^{(i)}}. \end{aligned} \quad (2.62)$$

Taking into account (1.10), (1.11), (1.15) and (1.14) for a single line, and the relation

$$\partial_\alpha ((x_\beta - x_{0\beta}) \delta_{L^{(i)}}) = \partial_\alpha \left((\hat{x}_\beta^{(i)} - x_{0\beta}) \delta_{L^{(i)}} \right) = (\hat{x}_\beta^{(i)} - x_{0\beta}) \partial_\alpha \delta_{L^{(i)}},$$

it results that $\eta_k^{*(i)}$ can be written in the formulations (2.60) or (2.61), and the result follows after summation over \mathcal{I} . \square

3 Displacement and rotation fields in 2D incompatible elasticity at mesoscopic scale

3.1 Position of the problem

The principal objective of the present work is to pave the way for a mathematically rigorous treatment of dislocation homogenization from meso- to macro-scale (Van Goethem and Dupret 2009b). To this end, this section elucidates the link between incompatibility (expressed as a combination of the concentrated defect densities) and the multivalued displacement and rotation fields. For the sake of completeness, most results are expressed using complete 3D tensor components (with latin indices, cf. Part A) under the hypothesis of a 2D strain field (whose components do not depend on z).

On the one hand, the mesoscopic fields $\Theta_k^* = \Theta_{zk}^*$, $\Lambda_k^* = \Lambda_{zk}^*$, $\eta_k^* = \eta_{zk}^* = \eta_{kz}^*$ and the contortion κ_{ij}^* are concentrated distributions on the defect lines which, as shown in Part A and §2, provide all the information on the dislocation and disclination densities and the strain incompatibility.

On the other hand, by theorem 1.1 the rotation field is a multivalued function of index 1 obtained on $\Omega_{\mathcal{L}}$ by line integration of $\partial_l^{(s)} \omega_k^* = \bar{\partial}_l \omega_k^* = \epsilon_{kpq} \partial_p \mathcal{E}_{ql}^*$ (cf. (1.4))

while the displacement field u_k^* is a multivalued function of index-2 obtained on $\Omega_{\mathcal{L}}$ by recursive line integration of $\partial_j^{(s)} \partial_l^{(s)} u_k^* = \partial_j^{(s)} (\mathcal{E}_{kl}^* + \omega_{kl}^*)$. The Frank tensor was introduced as a distributional field $\bar{\partial}_l \omega_k^*$ defined on the entire Ω , which coincides with $\partial_l^{(s)} \omega_k^*$ on every defect-free region and to which additional distributional terms are added to let this tensor be related to the strain gradient by (1.4).

Direct analysis shows that displacement is not the most appropriate vector field to describe the dislocations since the Burgers field b_k^* defined from (1.5) by line integration on $\Omega_{\mathcal{L}}$ of $\partial_l^{(s)} b_k^* = \bar{\partial}_l b_k^*$ exhibits more useful properties. In particular, b_k^* is a multivalued field of index 1 (compared to the less tractable index-2 multivaluedness of displacement) and the properties of the Frank tensor and vector $\bar{\partial}_l \omega_k^*$ and ω_k^* , and of the Burgers tensor and vector $\bar{\partial}_l b_k^*$ and b_k^* , show a clear analogy.

Both the Burgers and the rotation field are defined by means of a Riemann foliation F (cf. Part A) in the sense that mappings of the following kind exist:

$$\Omega_{\mathcal{L}} \xleftarrow{\mathcal{P}} F \xrightarrow{\omega_k^*, b_k^*} \mathbb{R}^3$$

where $F := \{(x, \#C)\}$ for every $x \in \Omega_{\mathcal{L}}$ and every curve C joining x_0 to x , with $\#C$ the equivalence class of all curves homotopic to C in $\Omega_{\mathcal{L}}$ while \mathcal{P} is the projection of F onto $\Omega_{\mathcal{L}}$, in such a way that $\mathcal{P}(x, \#C) = x$. Accordingly, the relationships between the multivalued fields ω_k^* and b_k^* (defined on F together with the projection \mathcal{P}), and the distributional fields $\bar{\partial}_l \omega_k^*$ and $\bar{\partial}_l b_k^*$ (defined on Ω) are very similar.

Careful analysis however reveals an apparent contradiction between the expected meanings of the Frank and Burgers tensors and their mathematical properties. Theorem 2.4 first shows that in the absence of disclinations ($\Omega_z^{*(i)} = 0$) but in the presence of dislocations, the incompatibility η_k^* does not vanish (this resulting from non-vanishing coefficients multiplying the Dirac mass derivatives ($\partial_\alpha \delta_{L^{(i)}}$) in (2.55) and (2.56)). Since by (1.8) incompatibility is the curl of the Frank tensor, the latter is not curl-free and surprisingly cannot be the distributional gradient of a single-valued rotation field ω_k^* in the absence of rotational defects (a situation where ω_k^* is expected to exist in the entire domain Ω and not only on $\Omega_{\mathcal{L}}$).

Secondly, the link between the Burgers tensor and vector has also to be clarified, but the situation is slightly different since no pure disclinations exist while there are pure dislocations. Indeed, it should be noticed that the Burgers vectors of the defect lines depend on the reference point x_0 in case their Frank tensor does not vanish. More precisely, when x_0 is changed to x'_0 , the following transformation rule applies on each defect line $L^{(i)}$ in the general 3D case:

$$B_k^{*'(i)} = B_k^{*(i)} + \epsilon_{klm} \Omega_l^{*(i)} (x'_{0m} - x_{0m}),$$

whereas the Frank vector $\Omega_k^{*(i)}$ will remain invariant together with its scalar product with the Burgers vector. So, if in 2D the Frank vector of a given defect line $L^{(i)}$ does not vanish ($\Omega_z^{*(i)} \neq 0$), an appropriate change of x_0 can always generate arbitrary values of the edge Burgers vector components $B_\alpha^{*(i)}$ for this particular line.

Suppose now x_0 can be selected in such a way that all the Burgers vectors $B_k^{*(i)}$ vanish while the Frank vectors do not ($\Omega_z^{*(i)} \neq 0$). Then by theorem 2.4, the incompatibility η_k^* does not vanish, and hence by (1.9) the Burgers tensor is not curl-free and cannot be the distributional gradient of a single-valued Burgers field b_k^* in the absence of translational defects for this particular reference point x_0 .

3.2 Completed Frank and Burgers tensors

In order to resolve the problem posed in §3.1, the tensors $\bar{\partial}_j \omega_k^*$ and $\bar{\partial}_l b_k^*$ are completed by appropriate concentrated effects within the defect lines, without however modifying their relationship with the multiple-valued Burgers and rotation fields on $\Omega_{\mathcal{L}}$. These tensors are called the completed Frank and Burgers tensors.

Definition 3.1

$$\text{COMPLETED FRANK TENSOR} \quad \bar{\partial}_j \omega_k^* := \bar{\partial}_j \omega_k^* - \kappa_{kj}^*, \quad (3.1)$$

$$\text{COMPLETED BURGERS TENSOR} \quad \bar{\partial}_j b_k^* := \mathcal{E}_{kj}^* + \epsilon_{kpq}(x_p - x_{0p}) \bar{\partial}_j \omega_q^*. \quad (3.2)$$

The following theorems justify the introduction of the completed Frank and Burgers tensors.

Theorem 3.1 *In 2D, the 2nd-order-tensor distribution $\bar{\partial}_j \omega_k^*$ verifies the relation:*

$$\text{DISCLINATION DENSITY} \quad \Theta_{ik}^* = \epsilon_{ilj} \partial_l \bar{\partial}_j \omega_k^*. \quad (3.3)$$

Proof. This statement is a mere consequence of (2.60) and the relation (1.8) expressing incompatibility as the curl of the Frank tensor. \square

Theorem 3.2 *In 2D, the 2nd-order-tensor distribution $\bar{\partial}_j b_k^*$ verifies the relation:*

$$\text{DISLOCATION DENSITY} \quad \Lambda_{ik}^* = \epsilon_{ilj} \partial_l \bar{\partial}_j b_k^*. \quad (3.4)$$

Proof. This statement directly follows from (1.9), (3.1) and (3.2). \square

From the above analysis, it results that the curls of the completed Frank or Burgers tensors vanish in Ω in the absence of rotational or translational line defects and that in these respective cases these tensors are equal to the gradients of existing single-valued rotation or Burgers vector fields. It is worth noting that the same concentrated contortion term κ_{kj}^* is subtracted from the Frank tensor in (3.1) and (3.2) in order to provide the completed Frank and Burgers tensors.

3.3 Integral relations and Stokes' theorem

Returning to 2D tensor notations, the following result restates the main theorem 2.4 or 2.5 in terms of the completed Frank and Burgers tensors (3.1) and (3.2) and the associated multivalued rotation and Burgers vector fields.

Theorem 3.3 *Under assumptions 1.1 and 1.2, the mesoscopic strain incompatibility for a countable set \mathcal{L} of rectilinear dislocations writes as*

$$\eta_k^* = \delta_{kz} \epsilon_{\alpha\beta} \partial_\alpha \bar{\partial}_\beta \omega_k^* + \epsilon_{\alpha\beta} \partial_\alpha \left(\delta_{kz} \epsilon_{\gamma\tau} \partial_\gamma \bar{\partial}_\tau b_\beta^* - \frac{1}{2} \delta_{k\beta} \epsilon_{\gamma\tau} \partial_\gamma \bar{\partial}_\tau b_z^* \right),$$

with in $\Omega_{\mathcal{L}}$, the index-1 multivalued fields ω_k^* and b_k^* given by

$$\omega_k^*(x) = \omega_{k0}^* + \int_{x_0}^x \bar{\partial}_\beta \omega_k^* dx_\beta \quad \text{and} \quad b_k^*(x) = b_{k0}^* + \int_{x_0}^x \bar{\partial}_\beta b_k^* dx_\beta.$$

Proof. This proposition directly results from the introduction of (3.1) and (3.2) in the main theorem. \square

Line integration of the completed Frank and Burgers tensors in $\Omega_{\mathcal{L}}$ therefore provides the rotation and Burgers fields. When this integration is carried out on a loop enclosing a corresponding 2D area, the dislocation and disclination densities can themselves be integrated on this area.

Theorem 3.4 (Stokes' theorem for the completed defect tensors) *Consider under assumptions 1.1 and 1.2 a countable set of dislocations and/or disclinations and a 2D open set $S \subset \Omega_{z_0}$ perpendicular to the defect lines and bounded by the counter clockwise-oriented Jordan curve $C \subset \Omega_{\mathcal{L}}$ which encloses once the defect subset $\mathcal{L}_C := \{L^{(i)}, L^{(i)} \cap S \neq \emptyset, i \in \mathcal{I}\}$. Then the following equalities hold:*

$$\int_C \bar{\partial}_\beta \omega_k^* dx_\beta = \int_S \epsilon_{\alpha\beta} \partial_\alpha \bar{\partial}_\beta \omega_k^* dS = \int_S \Theta_k^* dS = \sum_{L^{(i)} \in \mathcal{L}_C} \Omega_z^{*(i)} \delta_{zk}, \quad (3.5)$$

$$\int_C \bar{\partial}_\beta b_k^* dx_\beta = \int_S \epsilon_{\alpha\beta} \partial_\alpha \bar{\partial}_\beta b_k^* dS = \int_S \Lambda_k^* dS = \sum_{L^{(i)} \in \mathcal{L}_C} B_k^{*(i)}. \quad (3.6)$$

Proof. Since (2.59) results from the assumptions, the dislocation and disclination densities are Radon measures on Ω_{z_0} and hence can be integrated on S . Then (3.5) and (3.6) directly result from (3.3), (1.10), (1.16) and (1.11). \square

Remark 3.1 *The vector $\bar{\partial}_\beta \omega_z^*$ does not verify Stokes' theorem, neither in the classical sense, since $\epsilon_{\alpha\beta} \partial_\alpha \bar{\partial}_\beta \omega_z^*$ is singular at $\hat{x}^{(i)}$, nor in a measure theoretical sense, since $\epsilon_{\alpha\beta} \partial_\alpha \bar{\partial}_\beta \omega_z^*$ is not a measure but a first-order distribution. The same remark can be made about the Burgers tensor. Nonetheless, as often observed in the literature, even in an inappropriate context the formal use of Stokes' theorem here gives a correct final result.*

4 Concluding remarks

This paper is part of a work devoted to the development of a mathematical theory to analyse dislocated single crystals at the meso-scale by combining distributions with multiple-valued kinematic fields. The distributions are concentrated along the defect lines which in turn form the branching lines of the multivalued fields. From this analysis, a basic theorem relating the incompatibility tensor to the Burgers and Frank vectors of the dislocations and disclinations has been established in the case of countably many defect lines, under precise hypotheses on the distributional elastic strain gradient (via the Frank tensor). Quite surprisingly the sums of the norms of the Burgers and Frank vectors of the defect lines – which can be derived from the elastic strain – are required to be locally bounded to obtain the proof, thereby providing a fundamental defect norm for a further homogenization of the medium properties from meso- to macro-scale. This latter problem is addressed in Van Goethem & Dupret (2009b).

Moreover, in addition to the elastic strain, two key objective internal fields (the completed Frank and Burgers tensors) have been identified to represent the medium defective state independently of the selection of the reference configuration. While the curls of these two first-order distributional tensors are precisely the disclination and

dislocation densities, their recursive line integration in the defect-free region provides the multiple valued rotation and displacement fields.

After homogenization from meso- to macro-scale, no concentrated effects will remain present anymore in the macroscopic model, which will consist of a set of evolution PDE's governing the tensorial defect densities in the framework of elasto- or viscoplasticity (cf. e.g. Kratochvil & Dillon 1969). More precisely, the thermo-mechanical macroscopic model will govern the homogenized elastic strain and completed Frank and Burgers second-order tensors. Let us also mention that the non-vanishing mesoscopic elastic strain incompatibility will generate a macroscopic plastic strain which cannot be defined independently of the choice of the reference configuration. This property simply shows to be a reminiscence of the mesoscopic displacement and rotation multivaluedness.

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